Pak. J. Statist.
2026 Vol. 42(1), 79-104

EFFICIENT ESTIMATION OF THE MEAN IN COMPLEX SITUATIONS WITH NONRESPONSE AND MEASUREMENT ERROR

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ABSTRACT

This paper develops a generalized estimator for the finite population mean under simple random sampling (SRS) in the presence of nonresponse and measurement error in both study and auxiliary variables. The estimator integrates ratio-type and regression-type corrections with a power component, which makes it adaptable to both linear and nonlinear relationships. First-order bias and mean squared error (MSE) are derived and optimized with respect to tuning constants. Theoretical comparisons show improved efficiency when the correlation between the study and auxiliary variable is strong. A Monte Carlo study based on four simulated populations demonstrates that the proposed estimator attains the highest percent relative efficiency (PRE) and generally the smallest MSE among competing estimators across different nonresponse and subsampling levels.

KEYWORDS

Measurement error; Nonresponse; Auxiliary Variable; Ratio Estimator; Mean Square Error.

Mathematics Subject Classification: 62D05, 26A33, 42A38

1. INTRODUCTION

Surveys inherently face various technical challenges, as achieving a flawless survey in all aspects is rarely possible. Survey statistics are generally subject to errors. In classical survey sampling, two primary types of errors consistently arise even with meticulously designed and executed surveys. The first type, sampling error, refers to the difference between sample-based estimates and the actual population parameters. These errors are an inevitable part of the sampling process and cannot be eliminated, although they can be minimized through appropriate sample size and design. The second type, nonsampling errors, stem from numerous sources, such as inadequate sampling frames, poorly designed questionnaires, inaccurate sampling procedures, measurement method errors, coding and decoding mistakes, respondent refusals, inaccessibility of respondents, respondents not

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being at home, and other factors. Nonsampling errors can be further classified into response errors and nonresponse errors based on their underlying causes.

Response error, also known as measurement error, occurs when the reported values deviate from the true values. In survey research, measurement error refers as the difference between the recorded value and the true value of the variable being measured. For instance, in a study investigating students' cumulative grade point average (CGPA), students may report inaccurate or rounded figures instead of their actual CGPA. This discrepancy between the reported CGPA and the true CGPA signifies measurement error. Extensive research has been conducted to estimate unknown parameters in the presence of measurement error, with significant contributions from Cochran (1977), Sabir and Sanaullah (2022), Shalabh (1997), Shukla et al. (2012a), and Shukla et al. (2012b). Their work contributed immensely to the insights and techniques for reducing the impact of measurement error on parameter estimation in survey research.

On the other hand, nonresponse errors stem from incomplete coverage of sample units due to communication barriers such as language problems, unavailability of respondents, or even censorship. In statistical practice, to ignore the randomness associated with nonresponse can lead to biased data representations. For instance, bias may result as a consequence of a survey relies on obsolete information. Consider a situation in which a survey is distributed to a group of people who attended an event two years ago. Their email addresses may be stored in your customer relationship management system but haven't been updated since the time of the event. Accordingly, the survey may experience extremely poor delivery and response rates. This suggests that the attendees may have changed their email addresses and no longer access the accounts recorded in your database.

To minimize the impact of nonresponse, statisticians normally use subsampling techniques and various estimation methods. Unlike sampling errors, which usually decrease with large samples, in most cases, the size of nonsampling errors may increase in magnitude as sample sizes grows. One such methodology, namely Hansen and Hurwitz (1946) sub-sampling technique, which comprises drawing a second sample to reduce the nonresponse in some of the questions in a survey. Other researchers have explored the use of auxiliary information together with complete responses to enhance the response and minimize nonresponse bias. Some other significant contributions by Ahmad et al. (2022), Hussain et al. (2022), Kalton (1986), Singh and Karpe (2009), Singh et al. (2010), Yunusa and Kumar (2014) provide insight into ways of dealing with nonresponses. At the same time, these approaches enable researchers to account for nonresponse and produce reliable findings.

Thus, the methodology aims to estimate the mean of the finite population in the existence of measurement errors and nonresponses. Former studies have recognized nonresponse while ignoring measurement error, whereas others focused on specially measurement errors without considering nonresponse. Many noteworthy attempts have addressed the challenges of nonresponse and measurement error in estimating the population mean. Researchers such as Irfan et al. (2018), Kumar, (2016), Azeem and Hanif, (2017), and Rao, (1987) viewed the impact of nonresponse and measurement error in the estimation of population mean. Their work was restricted to simple random sampling with a single auxiliary variable. The present study extends this course of research by introducing

a generalized class of linear and nonlinear estimators to enhance the efficiency of current estimators for population mean estimation within the context of simple random sampling.

This paper is mainly related to nonlinear and positive relationships between the study variable and the auxiliary variable in the setting of simple random sampling. In this paper, we construct a class of estimators for the population mean that are more versatile and suitable for handling simultaneous nonresponse and measurement errors in both variables of interest and auxiliary variables. An empirical study has been conducted to establish the better performance and efficiency of the proposed estimators against others in the existing literature of estimation methods.

Figure 1 illustrates the scenario of the Hansen and Hurwitz nonresponse scheme with measurement error, while Figure 2 depicts the situation of nonresponse without measurement error.

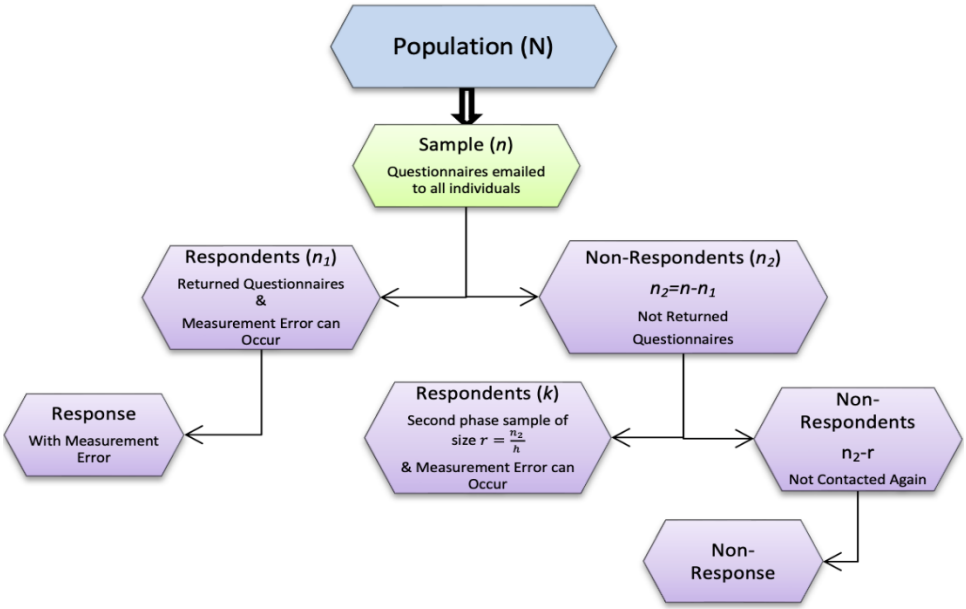

Figure 1: The Hansen and Hurwitz Nonresponse Scheme with Measurement Error

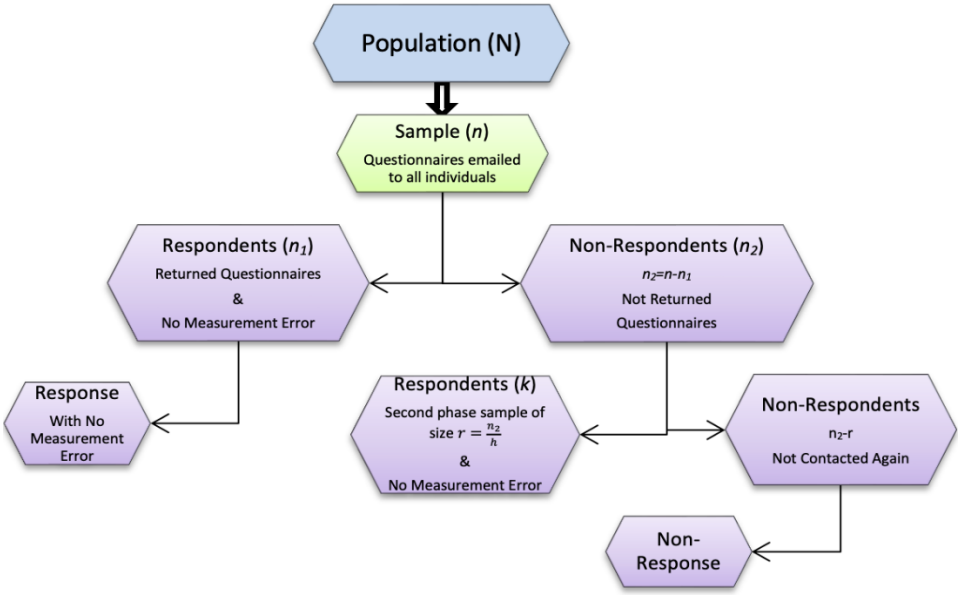

Figure 2: The Hansen and Hurwitz Nonresponse Scheme without Measurement Error

2. NOTATIONS AND SAMPLING PROCEDURE

A sample of size n is obtained from the population of size N using a simple random sampling without replacement (SRSWOR) design following the condition $n < N$. The finite population consisting of N units is categorized into two mutually exclusive groups: N_1 (respondents) and N_2 (nonrespondents). Let (y_i, x_i) and (Y_i, X_i) be the reported values and the true values, respectively, of the two variables (y, x) for individual $ith(i = 1,2, \dots, n)$.

The questionnaires were emailed to all n individuals. Let n_1 and n_2 refer to the number of respondents and nonrespondents in the sample, respectively, and $r = \frac{n_2}{h}; h > 1$ denotes a subsample of nonrespondents who responded when contacted again through personal interviews or telephone communication.

Let $S_Y^2 = \frac{1}{N-1}\sum_{i=1}^{N} (Y_i - \bar{Y})^2$ and $S_X^2 = \frac{1}{N-1}\sum_{i=1}^{N} (X_i - \bar{X})^2$ be the population variances of the variables Y and X, respectively. Let $S_{Y(2)}^2 = \frac{1}{N_2-1}\sum_{i=1}^{N_2} (Y_i - \bar{Y})^2$ and $S_{X(2)}^2 = \frac{1}{N_2-1}\sum_{i=1}^{N_2} (X_i - \bar{X})^2$ be the variances of the variables Y and X, respectively, for the nonresponding part of the population.

Let the measurement errors be $U_i = y_i - Y_i$ and $V_i = x_i - X_i$. (U_i, V_i) are random in nature, and both are uncorrelated with mean zero and variance.

Let $S_U^2 = \frac{1}{N-1}\sum_{i=1}^{N}(U_i - \bar{U})^2$ and $S_V^2 = \frac{1}{N-1}\sum_{i=1}^{N}(V_i - \bar{V})^2$ be the population variances associated with the measurement errors of variables Y and X, respectively, for the responding part of the population.

Let $S_{U(2)}^2 = \frac{1}{N_2-1}\sum_{i=1}^{N_2}(U_i - \bar{U})^2$ and $S_{V(2)}^2 = \frac{1}{N_2-1}\sum_{i=1}^{N_2}(V_i - \bar{V})^2$ be the population variances associated with the measurement errors of variables Y and X, respectively, for the nonresponding part of the population. Let C_Y and C_X be the coefficients of variation of variables Y and X, respectively, for the respondents in the population, and ρ_{XY} be the coefficient of correlation between variables Y and X for the respondents in the population. Let $C_{Y(2)}$ and $C_{X(2)}$ be the coefficients of variation for the group of nonrespondents in a population, and $\rho_{XY(2)}$ be the correlation coefficients for the group of nonrespondents in a population.

3. FEW EXISTING METHODS OF ESTIMATION

3.1. **Sample Mean Estimator based on Hansen and Hurwitz (1946) Subsampling**

A sample mean estimator based on Hansen and Hurwitz (1946) subsampling assuming the simultaneous presence of nonresponse and measurement error occurs to estimate the population mean is given by,

$$\bar{y}^* = \left(\frac{n_1}{n}\right)\bar{y}_{n1} + \left(\frac{n_2}{n}\right)\bar{y}_{nk}, \tag{1}$$

where $\bar{y}_{n1} = \frac{1}{n_1}\sum_{i=1}^{n_1} y_i$ and $\bar{y}_i = \frac{1}{k}\sum_{i=1}^{k} y_i$

The variance $\bar{y}^*$ is given by

$$MSE(\bar{y}^*) = \lambda_2(S_Y^2 + S_U^2) + \theta\left(S_{Y(2)}^2 + S_{U(2)}^2\right), \tag{2}$$

where $\lambda_2 = \frac{1}{n} - \frac{1}{N}, \theta = \frac{N_2(h-1)}{nN}$.

Equation (2) shows that the MSE of $\bar{y}^*$consists of two components: a sampling component (through λ_2) and an additional component due to nonresponse (through θ). Both components increase when measurement error variances S_U^2 and $S_{U(2)}^2$ increase, which confirms that measurement error inflates uncertainty in mean estimation.

3.2. **Cochran's (1977) Estimator**

A modified version of Cochran (1977) ratio estimator assuming the presence of nonresponse and measurement errors is given by:

$$t_{cr} = \frac{\bar{y}^*}{\bar{x}^*}\bar{X}. \tag{3}$$

The mean square error (MSE) of t_{cr} assuming the joint involvement of nonresponse and measurement error is given as:

$$\begin{aligned} \mathrm{MSE}(t_{cr}) = {} & \lambda_2(S_Y^2 + S_U^2) + \theta\left(S_{Y(2)}^2 + S_{U(2)}^2\right) \\ & + R^2\left\{\lambda_2(S_X^2 + S_V^2) + \theta\left(S_{X(2)}^2 + S_{V(2)}^2\right)\right\} \\ & - 2R\left\{\left(\lambda_2\rho_{YX}S_YS_X + \theta\rho_{YX(2)}S_{Y(2)}S_{X(2)}\right)\right\}. \end{aligned} \tag{4}$$

Expression (4) indicates that the ratio-type estimator benefits from strong positive correlation between Y and X , but its MSE increases with measurement error in the auxiliary variable through S_V^2 and $S_{V(2)}^2$. Hence, auxiliary measurement error may reduce the gain obtained from using ratio adjustment.

3.3. **Bahl and Tuteja (1991) estimator**

Taking the joint existence of nonresponse and measurement errors, a Bahl and Tuteja's (1991) exponential estimator is modified to the given situation, and form the modified estimator is:

$$t_{bt} = \bar{y}^* exp\left(\frac{\bar{X} - \bar{x}^*}{\bar{X} + \bar{x}^*}\right), \tag{5}$$

The MSE of t_{bt} in the presence of nonresponse and measurement error is given as:

$$\begin{aligned} \text{MSE}\,(t_{bt}) = {} & \{\lambda_2(S_Y^2 + S_U^2) + \theta(S_{Y(2)}^2 + S_{U(2)}^2)\} \\ & + \frac{1}{4}R\{\lambda_2(S_X^2 + S_V^2) + \theta(S_{X(2)}^2 + S_{V(2)}^2)\} \\ & - R(\lambda_2\rho_{YX}S_YS_X + \theta\rho_{YX(2)}S_{Y(2)}S_{X(2)}). \end{aligned} \tag{6}$$

3.4. **Regression estimator**

A regression estimation modified to the assumption of the joint existence of the nonresponse and measurement errors is given by,

$$t_{lr} = \bar{y}^* + b(\bar{X} - \bar{x}^*). \tag{7}$$

The MSE of the modified regression estimation (t_{lr}) is given as:

$$\begin{aligned} \text{MSE}^*\,(t_{lr}) = {} & \lambda_2(S_Y^2 + S_U^2) + \theta(S_{Y(2)}^2 + S_{U(2)}^2) \\ & + b_0^2\{\lambda(S_X^2 + S_V^2) + \theta(S_{X(2)}^2 + S_{V(2)}^2)\} \\ & - 2b_0\{\lambda_2\rho_{YX}S_YS_X + \theta\rho_{YX(2)}S_{Y(2)}S_{X(2)}\}. \end{aligned} \tag{8}$$

where $b_0 = \frac{\lambda_2\rho_{YX}S_YS_X + \theta\rho_{YX(2)}S_{Y(2)}S_{X(2)}}{\lambda_2(S_X^2 + S_V^2) + \theta(S_{X(2)}^2 + S_{V(2)}^2)}$.

3.5. **Singh and Kumar estimator**

Singh and Kumar (2008) provided a chain-type ratio estimator based on a single auxiliary variable, is modified to the assumption of the joint existence of the nonresponse and measurement errors is given by,

$$t_{sk} = \bar{y}^*\left(\frac{\bar{X}}{\bar{x}^*}\right)\left(\frac{\bar{X}}{\bar{x}}\right). \tag{9}$$

The MSE of the modified estimator t_{sk} is expressed by an expression given as,

$$\begin{aligned} \text{MSE}(t_{sk}) = {} & \lambda_2[S_Y^2 + 4R^2S_X^2 - 4R\rho_{YX}S_XS_Y] \\ & + \theta[S_{Y(2)}^2 + R^2S_{X(2)}^2 - 2R\rho_{YX(2)}S_{Y(2)}S_{X(2)}] \\ & + \lambda_2[S_U^2 + 4RS_V^2] + \theta[S_{U(2)}^2 + R^2S_{V(2)}^2]. \end{aligned} \tag{10}$$

3.6. **Azeem and Hanif estimators**

Following the assumption of joint existence of nonresponse and measurement errors, Azeem and Hanif (2015) introduced three different forms of estimators for estimating the population mean. The three estimators are taking the forms respectively given as,

3.6.1. $t_{AH1} = \bar{y}^* \left(\frac{\bar{x}^*}{\bar{X}}\right)\left(\frac{\bar{x}'^*}{\bar{x}^*}\right)$. (11)

with the MSE given by,

$$\begin{aligned}\mathrm{MSE}(t_{AH1}) = \lambda_2 &\left[S_Y^2 + \left(\frac{N+n}{N-n}\right)^2 R^2 S_X^2 - 2\left(\frac{N+n}{N-n}\right)\rho_{YX} R S_Y S_X\right] \\ &+\theta\left[S_{Y(2)}^2 + \left(\frac{N+n}{N-n}\right)^2 R^2 S_{X(2)}^2 - 2R\left(\frac{N+n}{N-n}\right)\rho_{YX(2)} S_{Y(2)} S_{X(2)}\right] \\ &+\lambda_2\left[S_U^2 + R^2\left(\frac{N+n}{N-n}\right)^2 S_V^2\right] + \theta\left[S_{U(2)}^2 + R^2\left(\frac{N+n}{N-n}\right)^2 S_{V(2)}^2\right].\end{aligned} \quad (12)$$

3.6.2. $t_{AH2} = \bar{y}^* \left(\frac{\bar{x}'^*}{\bar{X}}\right) \exp\left(\frac{\bar{x}'^* - \bar{x}^*}{\bar{x}'^* + \bar{x}}\right)$. (13)

with MSE

$$\begin{aligned}\mathrm{MSE}(t_{AH2}) = \lambda_2 &\left[S_Y^2 + \frac{1}{4}\left(\frac{N+2n}{N-n}\right)^2 R^2 S_X^2 - \left(\frac{N-2n}{N-n}\right)\rho_{YX} R S_Y S_X\right] \\ &+\theta\left[S_{Y(2)}^2 + \frac{1}{4}\left(\frac{N+2n}{N-n}\right)^2 S_{X(2)}^2 R^2 - \left(\frac{N+2n}{N-n}\right)\rho_{YX(2)} R S_{Y(2)} S_{X(2)}\right] \\ &+\lambda_2\left[S_U^2 + \frac{1}{4}\left(\frac{N+2n}{N-n}\right)^2 R^2 S_V^2\right] + \theta\left[S_{U(2)}^2 + \frac{1}{4}\left(\frac{N+2n}{N-n}\right)^2 R^2 S_{V(2)}^2\right].\end{aligned} \quad (14)$$

3.6.3. $t_{AH3} = \bar{y}^*\left[\alpha \exp\left(\frac{\bar{X}-\bar{x}^*}{\bar{X}+\bar{x}^*}\right) + (1-\alpha)\exp\left(\frac{\bar{x}'^*-\bar{x}^*}{\bar{x}'^*+\bar{x}^*}\right)\right]$ (15)

where α is a constant to be suitably chosen, and $\bar{x}'^* = \frac{N\bar{X}-n\bar{x}^*}{N-n}$.

The MSE is

$$\begin{aligned}\mathrm{MSE}(t_{AH3}) = \lambda_2 &\left[S_Y^2 + \frac{1}{4}\mu_0^2 R^2 S_X^2 - \mu_0 \rho_{YX} R S_Y S_X\right] \\ &+\theta\left[S_{Y(2)}^2 + \frac{1}{4}\mu_0^2 R^2 S_{X(2)}^2 - \mu_0 \rho_{YX(2)} R S_{Y(2)} S_{X(2)}\right] \\ &+\lambda_2\left[S_U^2 + \tfrac{1}{4}\mu_0^2 R^2 S_V^2\right] + \theta\left[S_{U(2)}^2 + \tfrac{1}{4}\mu_0^2 R^2 S_{V(2)}^2\right].\end{aligned} \quad (16)$$

The optimum value of μ is given by:

$$\mu_0 = \frac{2\left[\lambda_2 \rho_{YX} C_Y C_X + \theta \rho_{YX(2)} C_{Y(2)} C_{X(2)}\right]}{\lambda_2\left[\frac{S_X^2 + S_V^2}{\bar{X}^2}\right] + \theta\left[\frac{S_{X(2)}^2 + S_{V(2)}^2}{\bar{X}^2}\right]}.$$

3.7. **Kumar estimator**

Kumar et al. (2015) suggested an estimator using a linear combination of Bahl and Tuteja (1996) type exponential-ratio and exponential-product estimators to improve efficiencies. Their proposed estimator is modified for the purpose of mean estimation assuming the situation when there are nonresponse and measurement errors concerning jointly. The form of the modified estimator is given by,

$$t_k = \bar{y}^* \frac{\bar{x}^*}{\bar{X}} \left\{ \alpha \exp\left(\frac{\bar{x}^* - \bar{x}'^*}{\bar{x}^* + \bar{x}'^*} \right) + (1-\alpha) exp\left(\frac{\bar{x}^* - \bar{x}'^*}{\bar{x}^* + \bar{x}'^*} \right) \right\}. \tag{17}$$

An expression of the minimum MSE of the modified estimator is shown as,

$$\text{min. MSE}\,(t_k) = \lambda_2 (S_Y^2 + S_U^2) + \theta \left(S_{Y(2)}^2 + S_{U(2)}^2 \right) - \frac{\left\{ \lambda_2 \rho_{YX} S_Y S_X + \theta \rho_{YX(2)} S_{Y(2)} S_{X(2)} \right\}^2}{\lambda_2 (S_X^2 + S_V^2) + \theta \left(S_{X(2)}^2 + S_{V(2)}^2 \right)}. \tag{18}$$

3.8. **Irfan's estimator**

Irfan et al. (2018) introduced a more generalized class of estimators in order to estimate the mean of a finite population in the presence of nonresponse and measurement errors. Form of the estimator is modified estimator is given as,

$$t_{Ir} = \omega_1 \bar{y}^* \left(\frac{\bar{Z}}{\bar{z}} \right)^g + \omega_2 \bar{y}^* exp\left(\frac{\delta(\bar{Z} - \bar{z})}{(\bar{Z} + \bar{z})} \right), \tag{19}$$

where $\bar{Z} = a\bar{X} + b$ and $\bar{z} = a\bar{x}^* + b$, (ω_1, ω_2) are the weights and (g, δ) are constants.

The MSE of the estimator t_{Ir} is given as,

$$\text{MSE}(t_{Ir}) = \bar{Y} + \omega_1 A + \omega_2 B - \omega_1 C - \omega_2 D + \omega_1 \omega_2 E\,. \tag{20}$$

where

$$A = [\bar{Y} - q_1 \bar{Y}^2 \tau^2 n^2 g + q_2 + 4 q_3 \bar{Y} \tau n g]$$

$$B = \left[\bar{Y} + \frac{1}{2} q_1 \bar{Y}^2 \tau^2 \delta (1+\delta) + q_2 + 2 q_3 \bar{Y} \tau \delta \right]$$

$$C = [2\bar{Y} - q_1 \bar{Y}^2 \tau^2 n^2 g (1+g) + 2 q_3 \bar{Y} \tau n g]$$

$$D = \left[2\bar{Y} + \frac{1}{2} q_1 \bar{Y}^2 \tau^2 \delta \left(1 + \frac{\delta}{2} \right) + q_3 \bar{Y} \tau \delta \right]$$

$$E = \left[2\bar{Y} + q_1 \bar{Y}^2 \tau^2 \left\{ \frac{\delta}{2} \left(1 + \frac{\delta}{2} \right) - n^2 g (1+g) + \delta n g \right\} + 2 q_2 + 4 q_3 \bar{Y} \tau \left\{ \frac{\delta}{2} + n g \right\} \right]$$

with optimal weights $\omega_1 = \frac{2BC - DE}{E^2 - 4AB}$ and $\omega_2 = \frac{2AD - CE}{E^2 - 4AB}$.

4. PROPOSED ENHANCED GENERALIZED ESTIMATOR OF THE POPULATION MEAN

This study suggests a more generalized and efficient estimator under SRSWOR to address the problems of estimating the mean of finite population when nonresponse and measurement errors are existing simultaneously. A form of the advised estimator is given by:

$$t_g = \lambda\left[\bar{y} + k_1\bar{y}\left(\frac{\bar{z}}{\alpha\bar{z} + (1-\alpha)\bar{Z}}\right) + k_2(\bar{Z} - \bar{z})\right] * \left(\frac{\bar{z}}{\alpha\bar{z} + (1-\alpha)\bar{Z}}\right)^g, \qquad (21)$$

where $\bar{Z} = a\bar{X} + b$ and $\bar{z} = a\bar{x} + b$, and the constants $a \neq 0$ & b are considered to be either any real numbers or any functions of some known parameters of the auxiliary variable. For instance, some known parameters derived from the population of an auxiliary variable may include a median Q_2, standard deviation S_x, quartile deviation, coefficient of skewness $B_1(X)$, coefficient of variation C_x, and coefficient of kurtosis $B_2(X)$. (k_1, k_1) are the contents that needs to be optimizing such that the MSE of the proposed class of estimators decreases.(λ, α, g) are constants that take values of (0,1) for designing the different estimators. The proposed estimator can be molded into different classes of estimators or special estimators by assigning different possible values to the constants $(a, b, \lambda, \alpha, g, k_1, \& k_2)$. A few examples are shown in Tables 1-5. Tables 1-5 show the forms of the estimators, along with their MSE expressions, for ease of reading.

The proposed class in (21) combines ratio-type adjustment, regression-type correction, and a flexible power term g. This structure allows the estimator to adapt to linear as well as nonlinear relationships between the study and auxiliary variables, while also accommodating nonresponse and measurement error through $\bar{y}^*$and $\bar{x}^*$.

To obtain the MSE of the suggested more generalized and efficient class of estimators, let us consider,

$$\omega_Y^* = \sum_{i=1}^n (Y_i^* - \bar{Y}),\ \omega_U^* = \sum_{i=1}^n U_i^*,\ \text{and}\omega_X^* = \sum_{i=1}^n (X_i^* - \bar{X}),\ \omega_V^* = \sum_{i=1}^n V_i^*.$$

By adding ω_Y^* and ω_U^* and dividing by n, we obtain

$$\frac{1}{n}(\omega_Y^* + \omega_U^*) = \frac{1}{n}\sum_{i=1}^n y_i^* - \bar{Y}.$$

Therefore, $\bar{y}^* = \bar{Y} + \frac{1}{n}(\omega_Y^* + \omega_U^*)$. Similarly, let us consider, $\bar{x}^* = \bar{X} + \frac{1}{n}(\omega_X^* + \omega_V^*)$, such that,

$$E\left(\frac{\omega_X^2 + \omega_V^2}{n}\right)^2 = \lambda_2(S_X^2 + S_V^2) + \theta\left(S_{X(2)}^2 + S_{V(2)}^2\right) = q_1,$$
$$E\left(\frac{\omega_Y^* + \omega_U^*}{n}\right)^2 = \lambda_2(S_Y^2 + S_U^2) + \theta\left(S_{Y(2)}^2 + S_{U(2)}^2\right) = q_2, \text{ and}$$
$$E\left[\left(\frac{\omega_Y^* + \omega_U^*}{n}\right)\left(\frac{\omega_X^* + \omega_V^*}{n}\right)\right] = \lambda_2\rho_{YX}S_YS_X + \theta\rho_{YX(2)}S_{Y(2)}S_{X(2)} = q_3.$$

The MSE of t_g to the degree of first order (O^{-1}) of approximation is derived and given by as,

$$\text{MSE}(t_g) = \lambda^2[k_1^2A - 2k_1k_2B + 2k_1C + k_2^2D - 2k_2E + F] - 2\lambda[k_1G + k_2H + I] + \bar{Y}^2. \qquad (22)$$

where, $A = \bar{Y}^2 - 4\alpha\Omega(g+1)q_3 + \Omega^2\alpha^2(2g + g^2 + 1)q_1 + q_2$,
$B = -\alpha a\Omega q_1(2g+1) + aq_3$.
$C = \bar{Y}^2 - 2\alpha\Omega(2g+1)q_3 + \alpha^2\Omega^2(g^2+g)q_1 + q_2$.
$D = a^2q_1$.
$E = -2g\alpha a\Omega q_1 + aq_3$.

$$F = \bar{Y}^2 - 4g\alpha\Omega q_3 + g^2\alpha^2\Omega^2 q_1 + q_2.$$
$$G = \bar{Y}^2 - \alpha\Omega q_3(g+1).$$
$$H = a\alpha g\Omega q_1.$$
$$I = \bar{Y}^2 - g\alpha\Omega q_3.$$

By differentiating MSE (t_g) with respect to k_1 and k_2, we obtain the optimum values.

$$k_1 = \frac{1/\lambda(GD+HB) - (CD-BE)}{(AD-B^2)},$$
$$k_2 = \frac{1/\lambda(GD+HA) - (CD-AE)}{(AD-B^2)}.$$

By setting the values of k_1 and k_2, we obtain MSE (t_g):

$$\text{MSE}(t_g) = \frac{1}{(AD-B^2)^2}[\lambda^2 J_1 - 2\lambda J_2 + J_3 + (AD-B^2)^2\bar{Y}^2]. \quad (23)$$

To obtain the optimum value of λ by differentiating MSE (t_g) with respect to λ, we obtain

$$\lambda = \frac{J_2}{J_1}.$$

Using $\lambda = \frac{J_2}{J_1}$, the minimum MSE of t_g is

$$\min \cdot \text{MSE}(t_g) = \frac{1}{(AD-B^2)^2}\left[J_3 - \frac{J_2^2}{J_1} + \bar{Y}^2(AD-B^2)^2\right], \quad (24)$$

where

$$J_1 = (CD-BE)^2A + 2(CD-BE)(CB-AB)B - 2(CD-BE)(AD-B^2)C + (CB-AB)^2D + 2(CB-AB)(AD-B^2) + (AD-B^2)^2F,$$

$$J_2 = (GD+HB)(CD-BE)A + 2(GD+HB)(CB-AB)B + 2(GB+HA)(CD-BE)B - 2(GD+HB)(AD-B^2)C + 2(GB+HA)(CB-AB)D + 2(GB+HA)(AD-B^2)E - 2(CD-BE)(AD-B^2)G + 2(CB-AB)(AD-B^2)H - 2(AD-B^2)^2I,$$

$$J_3 = (\text{GD} + \text{HB})^2A - 2(GD+HB)(GB+HA)B - 2(GD+HB)(AD-B^2)G + 2(GB+HA)(AD-B^2)H + (GB+HA)^2D.$$

The minimum MSE in (24) highlights how the efficiency of the proposed estimator depends on the covariance term q_3 and the auxiliary variability term q_1. When the association between $Y$and X is strong (large q_3), the reduction in MSE becomes more pronounced.

Table 1
First Class of Estimators obtained from the Generalized Estimator t_g

Values of $(k_1, k_2, \lambda, \alpha, g)$					Estimator	MSE
k_1	k_2	λ	α	g		
0	0	1	0	1	$t_g^1 = \bar{y}$	$\text{MSE}\left(t_g^1\right) = q_2$
0	0	1	1	1	$t_g^2 = \bar{y} * \left(\frac{\bar{Z}}{\bar{z}}\right)$	$\text{MSE}\left(t_g^2\right) = \Omega^2 q_1 - 2\Omega q_3 + q_2 \text{y}$
0	0	1	1	0	$t_g^3 = \bar{y}$	$\text{MSE}\left(t_g^3\right) = q_2$
0	0	1	0	0	$t_g^4 = \bar{y}$	$\text{MSE}\left(t_g^4\right) = q_2$
1	1	1	0	1	$t_g^5 = 2\bar{y} + (\bar{Z} - \bar{z})$	$\text{MSE}\left(t_g^5\right) = \bar{Y}^2 + a^2 q_1 + 4q_2 - 4aq_3$
1	1	1	1	1	$t_g^6 = \left[\bar{y} + \bar{y}\left(\frac{\bar{Z}}{\bar{z}}\right) + (\bar{Z} - \bar{z})\right]^* \left(\frac{\bar{Z}}{\bar{z}}\right)$	$MSE\left(t_g^6\right) = \bar{Y}^2 - 18\Omega q_3 + 9\Omega^2 q_1 - 4aq_3 + 4q_2 + a^2 q_1 + 8a\Omega q_1$
1	1	1	1	0	$t_g^7 = \left[\bar{y} + \bar{y}\left(\frac{\bar{Z}}{\bar{z}}\right) + (\bar{Z} - \bar{z})\right]$	$MSE\left(t_g^7\right) = \bar{Y}^2 - 6\Omega q_3 + \Omega^2 q_1 + 2a\Omega q_1 - 4aq_3 - 2\Omega q_3 + a^2 q_1 + 4q_2$
1	1	1	0	0	$t_g^8 = 2\bar{y} + (\bar{Z} - \bar{z})$	$MSE\left(t_g^8\right) = \bar{Y}^2 + a^2 q_1 + 4q_2 - 4aq_3$
1	0	1	0	1	$t_g^9 = 2\bar{y}$	$MSE\left(t_g^9\right) = \bar{Y}^2 + 4q_2$
1	0	1	1	1	$t_g^{10} = \left[\bar{y} + \bar{y}\left(\frac{\bar{Z}}{\bar{z}}\right)\right]^* \left(\frac{\bar{Z}}{\bar{z}}\right)$	$MSE\left(t_g^{10}\right) = 3\bar{Y}^2 - 20\Omega q_3 + 9\Omega^2 q_1 + 4q_2$
1	0	1	1	0	$t_g^{11} = \left[\bar{y} + \bar{y}\left(\frac{\bar{Z}}{\bar{z}}\right)\right]^* \left(\frac{\bar{Z}}{\bar{z}}\right)$	$MSE\left(t_g^{11}\right) = \bar{Y}^2 - 6\Omega q_3 + 3\Omega^2 q_1 + 4q_2$
1	0	1	0	0	$t_g^{12} = 2\bar{y}$	$MSE\left(t_g^{12}\right) = \bar{Y}^2 + 4q_2$
0	1	1	0	1	$t_g^{13} = [\bar{y} + (\bar{Z} - \bar{z})]$	$MSE\left(t_g^{13}\right) = a^2 q_1 - 2aq_3 + q_2$
0	1	1	1	1	$t_g^{14} = [\bar{y} + (\bar{Z} - \bar{z})]^* \left(\frac{\bar{Z}}{\bar{z}}\right)$	$MSE\left(t_g^{14}\right) = a^2 q_1 + 2a\Omega q_1 - 2\Omega q_3 + \Omega^2 q_1 - 2aq_3 + q_2$
0	1	1	1	0	$t_g^{15} = [\bar{y} + (\bar{Z} - \bar{z})]$	$MSE\left(t_g^{15}\right) = a^2 q_1 - 2aq_3 + q_2$
0	1	1	0	0	$t_g^{16} = [\bar{y} + (\bar{Z} - \bar{z})]$	$MSE\left(t_g^{16}\right) = a^2 q_1 - 2aq_3 + q_2$

Table 2
Second class of Estimators obtained from the Generalized Estimator t_g

k_1	k_2	λ	α	g	Estimator	MSE
0	k_2	1	0	1	$t_g^{17} = [(\bar{y} + k_2(\bar{Z} - \bar{z})]$	$MSE(t_g^{17}) = k_2^2 a^2 q_1 - 2k_2 a q_3 + q_2$
0	k_2	1	1	1	$t_g^{18} = [\bar{y} + k_2(\bar{Z} - \bar{z})] * \left(\frac{\bar{Z}}{\bar{z}}\right)$	$\text{MSE}\left(t_g^{18}\right) = k_2^2 a^2 q_1 + 2k_2 a\Omega q_1 - 2k_2 a q_3 - 2\Omega q_3 + \Omega^2 q_1 + q_2$
0	k_2	1	1	0	$t_g^{19} = [(\bar{y} + k_2(\bar{Z} - \bar{z})]$	$\text{MSE}\left(t_g^{19}\right) = k_2^2 a^2 q_1 - 2k_2 a q_3 + q_2$
0	k_2	1	0	0	$t_g^{20} = [(\bar{y} + k_2(\bar{Z} - \bar{z})]$	$\text{MSE}\left(t_g^{20}\right) = k_2^2 a^2 q_1 - 2k_2 a q_3 + q_2$
k_1	0	1	0	1	$t_g^{21} = \bar{y} + k_1 \bar{y}$	$\text{MSE}\left(t_g^{21}\right) = k_1^2 \bar{Y}^2 + k_1^2 q_2 + 2k_1 q_2 + q_2$
k_1	0	1	1	1	$t_g^{22} = \left[\bar{y} + k_1 \bar{y} * \left(\frac{\bar{Z}}{\bar{z}}\right)\right] * \left(\frac{\bar{Z}}{\bar{z}}\right)$	$\text{MSE}\left(t_g^{22}\right) = k_1^2 \bar{Y}^2 - 8k_1^2 \Omega q_3 + 4k_1^2 \Omega^2 q_1 + k_1^2 q_2 - 8k_1 \Omega q_3 + 4k_1 \Omega^2 q_1 + 2k_1 q_2 - 4\Omega q_3 + \Omega^2 q_1 + 2\Omega q_3 + q_2$
k_1	0	1	1	0	$t_g^{23} = \left[\bar{y} + k_1 \bar{y} * \left(\frac{\bar{Z}}{\bar{z}}\right)\right]$	$\text{MSE}\left(t_g^{23}\right) = k_1^2 \bar{Y}^2 - 4k_1^2 \Omega q_3 + k_1^2 \Omega^2 q_1 + k_1^2 q_2 - 2k_1 \Omega q_3 + 2k_1 q_2 + q_2$
k_1	0	1	0	0	$t_g^{24} = \bar{y} + k_1 \bar{y}$	$\text{MSE}\left(t_g^{24}\right) = k_1^2 \bar{Y}^2 + k_1^2 q_2 + 2k_1 q_2 + q_2$
1	k_2	1	0	1	$t_g^{25} = 2\bar{y} + k_2(\bar{Z} - \bar{z})$	$\text{MSE}\left(t_g^{25}\right) = \bar{Y}^2 - 4k_2 a q_3 + k_2^2 a^2 q_1 + 4q_2$
1	k_2	1	1	1	$t_g^{26} = \left[\bar{y} + \bar{y}\left(\frac{\bar{Z}}{\bar{z}}\right) + k_2(\bar{Z} - \bar{z})\right] * \left(\frac{\bar{Z}}{\bar{z}}\right)$	$\text{MSE}\left(t_g^{26}\right) = \bar{Y}^2 - 18\Omega q_3 + 9\Omega^2 q_1 + 8k_2 a\Omega q_1 + 4q_2 - 4k_2 a q_3 + k_2^2 a^2 q_1$
1	k_2	1	1	0	$t_g^{27} = \left[\bar{y} + \bar{y}\left(\frac{\bar{Z}}{\bar{z}}\right) + k_2(\bar{Z} - \bar{z})\right]$	$\text{MSE}\left(t_g^{27}\right) = \bar{Y}^2 - 6\Omega q_3 + \Omega^2 q_1 + 4q_2 + k_2^2 a^2 q_1 + 2k_2 a\Omega q_1 - 4k_2 a q_3$
1	k_2	1	0	0	$t_g^{28} = [2\bar{y} + k_2(\bar{Z} - \bar{z})]$	$\text{MSE}\left(t_g^{28}\right) = \bar{Y}^2 - 4k_2 a q_3 + k_2^2 a^2 q_1 + 4q_2$

Table 3
Third class of Estimators obtained from the Generalized Estimator t_g

k_1	k_2	λ	α	g	Estimator	MSE
k_1	1	1	0	1	$t_g^{29} = [\bar{y} + k_1\bar{y} + (\bar{Z} - \bar{z})]$	$\text{MSE}\left(t_g^{29}\right) = k_1^2\bar{Y}^2 + k_1^2q_2 - 2k_1aq_3 + 2k_1q_2 + a^2q_1 - 2aq_3 + q_2$
k_1	1	1	1	1	$t_g^{30} = \left[\bar{y} + k_1\bar{y}\left(\frac{\bar{Z}}{\bar{z}}\right) + (\bar{Z} - \bar{z})\right] * \left(\frac{\bar{Z}}{\bar{z}}\right)$	$\text{MSE}\left(t_g^{30}\right) = k_1^2\bar{Y}^2 - 8k_1^2\Omega q_3 + 4k_1^2\Omega^2q_1 + k_1^2q_2 + 6k_1a\Omega q_1 - 2k_1aq_3 - 8k_1\Omega q_3 + 4k_1\Omega^2q_1 + 2k_1q_2 + a^2q_1 + 2a\Omega q_1 - 2aq_3 - 2\Omega q_3 + \Omega^2q_1 + q_2$
k_1	1	1	1	0	$t_g^{31}\left[\bar{y} + k_1\bar{y}\left(\frac{\bar{Z}}{\bar{z}}\right) + (\bar{Z} - \bar{z})\right] * \left(\frac{\bar{Z}}{\bar{z}}\right)$	$\text{MSE}\left(t_g^{31}\right) = k_1^2\bar{Y}^2 - 4k_1^2\Omega q_3 + k_1^2\Omega^2q_1 + k_1^2q_2 + 2k_1a\Omega q_1 - 2k_1aq_3 + 2k_1q_2 + a^2q_1 - 2aq_3 - 2k_1\Omega q_3 + q_2$
k_1	1	1	0	0	$t_g^{32} = [\bar{y} + k_1\bar{y} + (\bar{Z} - \bar{z})]$	$\text{MSE}\left(t_g^{32}\right) = k_1^2\bar{Y}^2 + k_1^2q_2 - 2k_1aq_3 + 2k_1q_2 + a^2q_1 - 2aq_3 + q_2$
k_1	k_2	1	0	0	$t_g^{33} = [(1 + k_1)\bar{y} + k_2(\bar{Z} - \bar{z})]$	$\text{MSE}\left(t_g^{33}\right) = k_1^2\bar{Y}^2 + k_1^2q_2 - 2k_1k_2aq_3 + 2k_1q_2 + k_2^2a^2q_1 - 2k_2aq_3 + q_2$

Table 4: 4th class of Estimators obtained from the Generalized Estimator t_g

k_1	k_2	λ	α	g	Estimator	MSE
k_1	1	1	0	1	$t_g^{29} = [\bar{y} + k_1\bar{y} + (\bar{Z} - \bar{z})]$	$\text{MSE}\left(t_g^{29}\right) = k_1^2\bar{Y}^2 + k_1^2q_2 - 2k_1aq_3 + 2k_1q_2 + a^2q_1 - 2aq_3 + q_2$
k_1	1	1	1	1	$t_g^{30} = \left[\bar{y} + k_1\bar{y}\left(\frac{\bar{Z}}{\bar{z}}\right) + (\bar{Z} - \bar{z})\right] * \left(\frac{\bar{Z}}{\bar{z}}\right)$	$\text{MSE}\left(t_g^{30}\right) = k_1^2\bar{Y}^2 - 8k_1^2\Omega q_3 + 4k_1^2\Omega^2q_1 + k_1^2q_2 + 6k_1a\Omega q_1 - 2k_1aq_3 - 8k_1\Omega q_3 + 4k_1\Omega^2q_1 + 2k_1q_2 + a^2q_1 + 2a\Omega q_1 - 2aq_3 - 2\Omega q_3 + \Omega^2q_1 + q_2$
k_1	1	1	1	0	$t_g^{31}\left[\bar{y} + k_1\bar{y}\left(\frac{\bar{Z}}{\bar{z}}\right) + (\bar{Z} - \bar{z})\right] * \left(\frac{\bar{Z}}{\bar{z}}\right)$	$\text{MSE}\left(t_g^{31}\right) = k_1^2\bar{Y}^2 - 4k_1^2\Omega q_3 + k_1^2\Omega^2q_1 + k_1^2q_2 + 2k_1a\Omega q_1 - 2k_1aq_3 + 2k_1q_2 + a^2q_1 - 2aq_3 - 2k_1\Omega q_3 + q_2$
k_1	1	1	0	0	$t_g^{32} = [\bar{y} + k_1\bar{y} + (\bar{Z} - \bar{z})]$	$\text{MSE}\left(t_g^{32}\right) = k_1^2\bar{Y}^2 + k_1^2q_2 - 2k_1aq_3 + 2k_1q_2 + a^2q_1 - 2aq_3 + q_2$
k_1	k_2	1	0	0	$t_g^{33} = [(1 + k_1)\bar{y} + k_2(\bar{Z} - \bar{z})]$	$\text{MSE}\left(t_g^{33}\right) = k_1^2\bar{Y}^2 + k_1^2q_2 - 2k_1k_2aq_3 + 2k_1q_2 + k_2^2a^2q_1 - 2k_2aq_3 + q_2$

Table 5: 5th Class of Estimators obtained from the Generalized Estimator t_g

k_1	k_2	λ	α	g	Estimator	MSE
k_1	k_2	1	1	0	$t_g^{34} = \left[\bar{y} + k_1\bar{y}\left(\frac{\bar{Z}}{\bar{z}}\right) + k_2(\bar{Z} - \bar{z})\right]$	$\text{MSE}\left(t_g^{34}\right) = k_1^2\bar{Y}^2 - 4k_1^2\Omega q_3 + k_1^2\Omega^2q_1 + k_1^2q_2 + 2k_1k_2a\Omega q_1 - 2k_1k_2aq_3 - 2k_1\Omega q_3 + 2k_1q_2 + k_2^2a^2q_1 - 2k_2aq_3 + q_2$
k_1	k_2	1	0	1	$t_g^{35} = [(1 + k_1)\bar{y} + k_2(\bar{Z} - \bar{z})]$	$\text{MSE}\left(t_g^{35}\right) = k_1^2\bar{Y}^2 + k_1^2q_2 - 2k_1k_2aq_3 + 2k_1q_2 + k_2^2a^2q_1 - 2k_2aq_3 + q_2$
k_1	k_2	1	1	1	$t_g^{36} = \left[\bar{y} + k_1\bar{y}\left(\frac{\bar{Z}}{\bar{z}}\right) + k_2(\bar{Z} - \bar{z})\right] * \left(\frac{\bar{Z}}{\bar{z}}\right)$	$\text{MSE}\left(t_g^{36}\right) = k_1^2\bar{Y}^2 - 8k_1^2\Omega q_3 + 4k_1^2\Omega^2q_1 + k_1^2q_2 + 6k_1k_2a\Omega q_1 - 2k_1k_2aq_3 - 8k_1\Omega q_3 + 4k_1\Omega^2q_1 + 2k_1q_2 + k_2^2a^2q_1 + 2k_2a\Omega q_1 - 2k_2aq_3 - 2\Omega q_3 + \Omega^2q_1 + q_2$

5. EFFICIENCY COMPARISON

To compare the efficiencies of the various existing estimators considered in this paper with those of the proposed estimator t_g and the other estimators

$\bar{y}^*, t_{cr}, t_{bt}, t_{lr}, t_{s\&k}, t_{AH1}, t_{AH2}, t_{AH3}, t_{s1}, t_{s2}$ and t_{Ir}.

The following are the MSE/min. MSEs of the listed estimators,

i. $\text{MSE}(\bar{y}^*) = \lambda_2(S_Y^2 + S_U^2) + \theta\left(S_{Y(2)}^2 + S_{U(2)}^2\right) = q_2.$

ii. $\text{MSE}\ (t_{cr}) = q_2 + \frac{\bar{Y}^2}{\bar{X}^2} q_1 - 2\frac{\bar{Y}}{\bar{X}} q_3.$

iii. $\text{MSE}\ (t_{bt}) = q_2 + \frac{\bar{Y}^2}{4\bar{X}^2} q_1 - \frac{\bar{Y}}{\bar{X}} q_3.$

iv. $\text{MSE}\ (t_{lr}) = q_2 + b_0^2 q_1 - 2b_0 q_3.$

v. $\text{MSE}\ (t_{s\&k}) = q_2 + \frac{\bar{Y}^2}{\bar{X}^2} q_1 + 3\frac{\bar{Y}^2}{\bar{X}^2}\lambda_2(S_X^2 + S_V^2) - 2\frac{\bar{Y}}{\bar{X}} q_3 - 2\frac{\bar{Y}}{\bar{X}}\lambda_2 \rho_{YX} S_X S_Y.$

vi. $\text{MSE}\ (t_{AH1}) = q_2 + \left(\frac{N+n}{N-n}\right)^2 \frac{\bar{Y}^2}{\bar{X}^2} q_1 - 2\left(\frac{N+n}{N-n}\right)\frac{\bar{Y}}{\bar{X}} q_3.$

vii. $\text{MSE}\ (t_{AH2}) = q_2 + \frac{1}{4}\left(\frac{N+2n}{N-n}\right)^2 \frac{\bar{Y}^2}{\bar{X}^2} q_1 - \left(\frac{N+2n}{N-n}\right)\frac{\bar{Y}}{\bar{X}} q_3.$

viii. $\text{MSE}\ (t_{AH3}) = q_2 + \frac{1}{4}\mu_0^2 \frac{\bar{Y}^2}{\bar{X}^2} q_1 - \mu_0 \frac{\bar{Y}}{\bar{X}} q_3.$

ix. $\text{MSE}(t_k) = q_2 + \frac{\bar{Y}^2}{\bar{X}^2}\left\{\frac{1}{4}\left(\frac{N+2n}{N-n}\right)^2 + \alpha^2\left(\frac{N}{N-n}\right)^2 - \alpha\left(\frac{N}{N-n}\right)\left(\frac{N+2n}{N-n}\right)\right\} q_1 - \frac{\bar{Y}}{\bar{X}}\left(\frac{N+2n-2nN}{N-n}\right) q_3.$

x. $\text{MSE}\ (t_{Ir}) = \bar{Y} - \frac{(C'D'E' - B'C'^2 - A'D'^2)}{(E'^2 - 4A'B')}.$

xi. $\min \cdot \text{MSE}\left(t_g\right) = \frac{J_4}{(AD-B^2)^2}$, where $J_4 = \left[J_3 - \frac{J_2^2}{J_1} + \bar{Y}^2(AD - B^2)^2\right].$

5.1. $MSE(\bar{y}^*) - min.MSE\left(t_g\right) = q_2 - \frac{J_4}{(AD-B^2)^2} \geq 0,$ (25)

or, $(AD - B^2)^2 q_2 - J_4 \geq 0$ if $(AD - B^2)^2 q_2 \geq J_4.$

5.2. $\text{MSE}\ (t_{cr}) - \min \cdot \text{MSE}\left(t_g\right) = q_2 + \frac{\bar{Y}^2}{\bar{X}^2} q_1 - 2\frac{\bar{Y}}{\bar{X}} q_3 - \frac{J_4}{(AD-B^2)^2} \geq 0$ (26)

or

$(AD - B^2)^2[q_2 + R^2 q_1 - 2Rq_3] - J_4 \geq 0$ if $(AD - B^2)^2[q_2 + R^2 q_1 - 2Rq_3] \geq J_4$

5.3. $\text{MSE}\ (t_{bt}) - \min \cdot \text{MSE}\left(t_g\right) = q_2 + \frac{\bar{Y}^2}{4\bar{X}^2} q_1 - \frac{\bar{Y}}{\bar{X}} q_3 - \frac{J_4}{(AD-B^2)^2} \geq 0,$ (27)

or

$(AD - B^2)^2\left[q_2 + \frac{1}{4}R^2 q_1 - Rq_3\right] - J_4 \geq 0$ if $(AD - B^2)^2\left[q_2 + \frac{1}{4}R^2 q_1 - Rq_3\right] \geq J_4$

5.4. $\text{MSE}(t_{lr}) - \min \cdot \text{MSE}\left(t_g\right) = q_2 + b_0^2 q_1 - 2b_0 q_3 - \frac{J_4}{(AD-B^2)^2} \geq 0,$ (28)

or

$(AD - B^2)^2[q_2 + b_0^2 q_1 - 2b_0 q_3] - J_4 \geq 0$ if $(AD - B^2)^2[q_2 + b_0^2 q_1 - 2b_0 q_3] \geq J_4.$

5.5. $\text{MSE}(t_{s\&k}) - \min \cdot \text{MSE}\left(t_g\right) = q_2 + \frac{\bar{Y}^2}{\bar{X}^2} q_1 + 3\frac{\bar{Y}^2}{\bar{X}^2}\lambda_2(S_X^2 + S_V^2) - 2\frac{\bar{Y}}{\bar{X}} q_3$
$-2\frac{\bar{Y}}{\bar{X}}\lambda_2 \rho_{YX} S_X S_Y - \frac{J_4}{(AD-B^2)^2} \geq 0,$ (29)

$$(AD-B^2)^2\begin{bmatrix} q_2+R^2q_1+3R^2\lambda_2(S_X^2+S_V^2) \\ -2Rq_3-2R\lambda_2\rho_{YX}S_XS_Y \end{bmatrix}-J_4\geq 0,$$

$$\text{if } (AD-B^2)^2[q_2+R^2q_1+3R^2\lambda_2(S_X^2+S_V^2)-2Rq_3-2R\lambda_2\rho_{YX}S_XS_Y]\geq J_4.$$

5.6. $$\text{MSE}\,(t_{AH1})-MSE(t_g)=q_2+\left(\frac{N+n}{N-n}\right)^2\frac{\bar{Y}^2}{\bar{X}^2}q_1-2\left(\frac{N+n}{N-n}\right)\frac{\bar{Y}}{\bar{X}}q_3-\frac{J_4}{(AD-B^2)^2}\geq 0. \quad (30)$$

$$(AD-B^2)^2\left[q_2+\left(\frac{N+n}{N-n}\right)^2R^2q_1-2\left(\frac{N+n}{N-n}\right)Rq_3\right]-J_4\geq 0,$$

$$\text{if } (AD-B^2)^2\left[q_2+\left(\frac{N+n}{N-n}\right)^2R^2q_1-2\left(\frac{N+n}{N-n}\right)Rq_3\right]\geq J_4.$$

5.7. $$\text{MSE}\,(t_{AH2})-\text{MSE}\,(t_g)=q_2+\frac{1}{4}\left(\frac{N+2n}{N-n}\right)^2\frac{\bar{Y}^2}{\bar{X}^2}q_1-\left(\frac{N+2n}{N-n}\right)\frac{\bar{Y}}{\bar{X}}q_3-\frac{J_4}{(AD-B^2)^2}\geq 0, \quad (31)$$

$$\text{or } (AD-B^2)^2\left[q_2+\frac{1}{4}\left(\frac{N+2n}{N-n}\right)^2R^2q_1-\left(\frac{N+2n}{N-n}\right)Rq_3\right]-J_4\geq 0,$$

$$\text{if } (AD-B^2)^2\left[q_2+\frac{1}{4}\left(\frac{N+2n}{N-n}\right)^2R^2q_1-\left(\frac{N+2n}{N-n}\right)Rq_3\right]\geq J_4.$$

5.8. $$\text{MSE}\,(t_{AH3})-\text{MSE}\,(t_g)=q_2+\frac{1}{4}\mu_0^2\frac{\bar{Y}^2}{\bar{X}^2}q_1-\mu_0\frac{\bar{Y}}{\bar{X}}q_3-\frac{J_4}{(AD-B^2)^2}\geq 0, \quad (32)$$

$$\text{or } (AD-B^2)^2\left[q_2+\frac{1}{4}\mu_0^2R^2q_1-\mu_0Rq_3\right]-J_4\geq 0,$$

$$\text{if}(AD-B^2)^2\left[q_2+\frac{1}{4}\mu_0^2R^2q_1-\mu_0Rq_3\right]\geq J_4$$

5.9. $$\text{MSE}(t_k)-\text{MSE}(t_g)=q_2+\frac{\bar{Y}^2}{\bar{X}^2}\left\{\frac{1}{4}\left(\frac{N+2n}{N-n}\right)^2+\alpha^2\left(\frac{N}{N-n}\right)^2-\alpha\left(\frac{N}{N-n}\right)\left(\frac{N+2n}{N-n}\right)\right\}q_1-\frac{\bar{Y}}{\bar{X}}\left(\frac{N+2n-2nN}{N-n}\right)q_3-\frac{J_4}{(AD-B^2)^2}\geq 0, \quad (33)$$

$$\text{or } (AD-B^2)^2\left[q_2+R^2\left\{\frac{1}{4}\left(\frac{N+2n}{N-n}\right)^2+\alpha^2\left(\frac{N}{N-n}\right)^2-\alpha\left(\frac{N}{N-n}\right)\left(\frac{N+2n}{N-n}\right)\right\}q_1-R\left(\frac{N+2n-2nN}{N-n}\right)q_3\right]-J_4\geq 0,$$

$$\text{if } (AD-B^2)^2\left[q_2+R^2\left\{\frac{1}{4}\left(\frac{N+2n}{N-n}\right)^2+\alpha^2\left(\frac{N}{N-n}\right)^2-\alpha\left(\frac{N}{N-n}\right)\left(\frac{N+2n}{N-n}\right)\right\}q_1-R\left(\frac{N+2n-2nN}{N-n}\right)q_3\right]\geq J_4.$$

$$\text{MSE}\,(t_{Ir})-\text{MSE}\,(t_g)=\bar{Y}-\frac{(C'D'E'-B'C'^2-A'D'^2)}{(E^2-4A'B')}-\frac{J_4}{(AD-B^2)^2}\geq 0 \quad (34)$$

$$\text{or } (AD-B^2)^2\left[\bar{Y}-\frac{(C'D'E'-B'C'^2-A'D'^2)}{(E^2-4A'B')}\right]-J_4\geq 0,$$

$$\text{if } (AD-B^2)^2\left[\bar{Y}-\frac{(C'D'E'-B'C'^2-A'D'^2)}{(E^2-4A'B')}\right]\geq J_4.$$

As shown in Equations. (25)-(34), the proposed estimator demonstrates more efficient than the usual unbiased estimator, ratio estimator, and other estimators in the presence of nonresponse and measurement error on the study variable as well as auxiliary variables.

6. EMPIRICAL STUDY

In this section, we evaluate the performance of various estimators compared to the usual unbiased estimators by generating four populations from a normal distribution with different parameter choices. The auxiliary information for variable X was generated from an $N(5,10)$ distribution following the methodology of Azeem & Hanif (2016).

Population I:

$X = N(5,10), Y = X + N(0,1)$
$y = Y + N(1,3), x = X + N(1,3)$
$N = 5000, \bar{Y} = 4.927167, \bar{X} = 4.924306, S_Y^2 = 102.0075, S_X^2 = 101.4117,$
$S_U^2 = 8.862114, S_V^2 = 9.001304, \rho_{YX} = 0.995059.$

N_1	N_2	$S_{Y(2)}^2$	$S_{X(2)}^2$	$S_{U(2)}^2$	$S_{V(2)}^2$	$\rho_{YX(2)}$
4500	500	99.99174	99.87471	9.150544	8.756592	0.994916
4000	1000	104.2711	103.2349	8.821278	8.339179	0.995472

Population II:

$X = N(5,10), Y = X + N(0,1)$
$y = Y + N(1,5), x = X + N(1,5)$
$N = 5000, \bar{Y} = 4.996681, \bar{X} = 5.013507, S_Y^2 = 97.12064, S_X^2 = 95.95803$
$S_U^2 = 23.96055, S_V^2 = 24.19283, \rho_{YX} = 0.994822.$

N_1	N_2	$S_{Y(2)}^2$	$S_{X(2)}^2$	$S_{U(2)}^2$	$S_{V(2)}^2$	$\rho_{YX(2)}$
4500	500	97.02783	94.54578	22.80557	25.43263	0.994546
4000	1000	96.09356	94.71923	24.42978	23.03076	0.99467

Population III:

$X = N(5,10), Y = X + N(0,1)$
$y = Y + N(2,3), x = X + N(2,3)$
$N = 5000, \bar{Y} = 4.730993, \bar{X} = 4.741928, S_Y^2 = 101.2633, S_X^2 = 100.2288$
$S_U^2 = 9.1025, S_V^2 = 9.052019, \rho_{YX} = 0.995187$

N_1	N_2	$S_{Y(2)}^2$	$S_{X(2)}^2$	$S_{U(2)}^2$	$S_{V(2)}^2$	$\rho_{YX(2)}$
4500	500	102.7504	101.2097	9.095136	8.8123	0.995045
4000	1000	105.4334	103.8947	9.277715	9.072151	0.995105

Population IV:

$X = N(5,10), Y = X + N(0,1)$
$y = Y + N(2,5), x = X + N(2,5)$
$N = 5000, \bar{Y} = 4.961081, \bar{X} = 4.96178, S_Y^2 = 102.2408, S_X^2 = 100.868$
$S_U^2 = 25.94111, S_V^2 = 25.03951, \rho_{YX} = 0.394221$

N_1	N_2	$S_{Y(2)}^2$	$S_{X(2)}^2$	$S_{U(2)}^2$	$S_{V(2)}^2$	$\rho_{YX(2)}$
4500	500	103.5361	102.1031	25.31099	22.84483	0.394622
4000	1000	100.1031	99.31665	25.80394	24.50468	0.394778

We computed the percent relative efficiencies (PREs) and MSEs of the estimators. The results of the PREs of all the estimators are shown in Tables (6-9), and the MSEs are presented in Tables 10-13.

Table 6
PREs of the Estimators with respect to $\bar{y}^*$ for Population I

N_1	N_2	Estimator	PRE with Measurement Error				PRE without Measurement Error			
			1/k				1/k			
			(1/2)	(1/3)	(1/4)	(1/5)	(1/2)	(1/3)	(1/4)	(1/5)
4500	500	$\bar{y}^*$	100	100	100	100	100	100	100	100
		t_{cr}	586.15	585.26	584.53	583.90	10106.99	10082.03	10061.27	10043.73
		t_{bt}	297.57	297.44	297.34	297.25	390.62	390.74	390.85	390.94
		t_{lr}	611.39	610.49	609.75	609.11	10115.23	10091.44	10071.7	10055.07
		$t_{s\&k}$	81.96	88.79	95.44	101.91	109.63	120.27	130.88	141.47
		t_{AH1}	388.98	388.52	388.14	387.82	1657.61	1653.98	1650.96	1648.41
		t_{AH2}	443.42	443.06	442.77	442.51	842.84	843.32	843.72	844.06
		t_{AH3}	611.39	610.49	609.75	609.11	10115.23	10091.44	10071.7	10055.07
		t_k	562.28	561.26	560.42	559.70	1104.79	1105.42	1105.95	1106.41
		t_{Ir}	710.22	721.75	734.18	747.64	10034.61	9999.23	9966.24	9936.72
		Proposed	790.87	813.95	838.94	865.99	10115.26	10091.43	10071.72	10055.07
4000	1000	$\bar{y}^*$	100	100	100	100	100	100	100	100
		t_{cr}	593.76	598.34	601.73	604.34	10297.7	10410.8	10494.91	10559.93
		t_{bt}	298.17	298.48	298.71	298.88	390.29	390.18	390.09	390.02
		t_{lr}	618.49	622.71	625.83	628.23	10302.55	10414.41	10497.72	10562.18
		$t_{s\&k}$	89.48	103.28	116.38	128.84	121.21	143.36	165.41	187.38
		t_{AH1}	390.97	391.96	392.69	393.25	1619.50	1591.48	1571.64	1556.85
		t_{AH2}	445.57	446.76	447.63	448.29	841.79	841.46	841.23	841.05
		t_{AH3}	618.48	622.71	625.83	628.23	10302.55	10414.41	10497.72	10562.18
		t_k	569.27	573.31	576.29	578.59	1120.99	1133	1141.96	1148.88
		t_{Ir}	735.71	774.74	818.09	867.60	10208.3	10290.9	10338.37	10358.18
		Proposed	830.20	898.43	977.48	1071.41	10302.79	10414.59	10497.76	10561.99

Table 7
PREs of the Estimators with respect to $\bar{y}^*$ for Population II

N_1	N_2	Estimator	PRE with Measurement Error				PRE without Measurement Error			
			1/k				1/k			
			(1/2)	(1/3)	(1/4)	(1/5)	(1/2)	(1/3)	(1/4)	(1/5)
4500	500	$\bar{y}^*$	100	100	100	100	100	100	100	100
		t_{cr}	246.80	246.53	246.31	246.13	9605.09	9556.54	9516.45	9482.78
		t_{bt}	219.32	219.30	219.28	219.27	384.33	383.89	383.51	383.20
		t_{lr}	273.00	272.82	272.68	272.56	9626.52	9582.73	9546.95	9517.16
		$t_{s\&k}$	60.79	65.21	69.41	73.43	112.82	123.94	135.03	146.10
		t_{AH1}	184.86	184.61	184.40	184.23	1763.95	1771.31	1777.49	1782.75
		t_{AH2}	260.26	260.17	260.10	260.04	816.44	814.57	813.02	811.71
		t_{AH3}	272.99	272.82	272.68	272.56	9626.52	9582.73	9546.95	9517.16
		t_k	219.18	219.21	219.23	219.25	474.61	472.60	470.93	469.54
		t_{Ir}	275.81	275.94	276.09	276.27	9604.61	9561.73	9522.05	9491.82
		Proposed	295.86	298.24	300.70	303.27	9624.53	9584.03	9546.66	9518.82
4000	1000	$\bar{y}^*$	100	100	100	100	100	100	100	100
		t_{cr}	247.53	247.81	248.02	248.17	9612.87	9576.92	9550.67	9530.65
		t_{bt}	219.17	219.05	218.96	218.89	384.68	384.55	384.45	384.37
		t_{lr}	273.28	273.33	273.37	273.40	9630.34	9595.30	9569.72	9550.23
		$t_{s\&k}$	65.33	73.67	81.28	88.26	123.74	145.69	167.55	189.30
		t_{AH1}	184.45	183.95	183.59	183.32	1626.25	1547.38	1494.12	1455.73
		t_{AH2}	260.25	260.17	260.12	260.07	817.86	817.25	816.81	816.47
		t_{AH3}	273.28	273.33	273.37	273.40	9630.34	9595.30	9569.72	9550.23
		t_k	218.97	218.84	218.75	218.68	479.32	480.89	482.05	482.95
		t_{Ir}	276.43	277.13	277.83	278.57	9610.18	9569.09	9537.57	9514.95
		Proposed	298.36	303.62	309.16	314.99	9632.11	9595.58	9568.90	9551.37

Table 8
PREs of the Estimators with respect to $\bar{y}^*$ for Population III

N_1	N_2	Estimator	PRE with Measurement Error				PRE without Measurement Error			
			1/k				1/k			
			(1/2)	(1/3)	(1/4)	(1/5)	(1/2)	(1/3)	(1/4)	(1/5)
4500	500	$\bar{y}^*$	100	100	100	100	100	100	100	100
		t_{cr}	579.64	580.78	581.73	582.53	10373.5	10346.93	10324.93	10306.43
		t_{bt}	294.10	294.13	294.16	294.17	386.65	386.48	386.34	386.23
		t_{lr}	601.59	602.58	603.3976	604.09	10382.13	10356.68	10335.67	10318.03
		$t_{s\&k}$	83.27	90.41	97.36	104.11	112.31	123.55	134.76	145.95
		t_{AH1}	390.39	391.24	391.95	392.55	1753.75	1756.10	1758.07	1759.73
		t_{AH2}	435.79	435.99	436.16	436.31	826.94	826.23	825.64	825.14
		t_{AH3}	601.59	602.58	603.39	604.09	10382.13	10356.68	10335.67	10318.03
		t_k	552.05	553.01	553.81	554.49	1094.16	1096.58	1098.59	1100.31
		t_{Ir}	700.00	715.05	730.93	747.70	10297.71	10261.1	10227.52	10195.75
		Proposed	782.23	809.73	839.17	870.87	10379.94	10355.78	10335.76	10318.92
4000	1000	$\bar{y}^*$	100	100	100	100	100	100	100	100
		t_{cr}	580.99	582.87	584.25	585.30	10370.02	10345.39	10327.57	10314.07
		t_{bt}	294.20	294.29	294.36	294.41	386.52	386.29	386.13	386.00
		t_{lr}	602.83	604.52	605.75	606.69	10379.7	10356.86	10340.41	10328.01
		$t_{s\&k}$	90.77	104.80	118.06	130.62	124.14	147.13	170.01	192.79
		t_{AH1}	385.26	382.51	380.54	379.05	1631.73	1559.20	1510.40	1475.32
		t_{AH2}	436.15	436.57	436.87	437.10	826.41	825.46	824.77	824.24
		t_{AH3}	602.83	604.52	605.75	606.69	10379.7	10356.86	10340.41	10328.01
		t_k	553.30	554.96	556.19	557.12	1096.47	1100.10	1102.75	1104.78
		t_{Ir}	716.25	750.09	788.55	832.99	10285.99	10233.02	10179.77	10121.55
		Proposed	811.71	875.31	950.17	1040.07	10381.45	10358.24	10341.39	10328.63

Table 9
PREs of the Estimators with respect to $\bar{y}^*$ for Population IV

N_1	N_2	Estimator	PRE with Measurement Error				PRE without Measurement Error			
			1/k				1/k			
			(1/2)	(1/3)	(1/4)	(1/5)	(1/2)	(1/3)	(1/4)	(1/5)
4500	500	$\bar{y}^*$	100	100	100	100	100	100	100	100
		t_{cr}	236.06	236.11	236.16	236.21	2604.88	2586.69	2571.65	2558.99
		t_{bt}	215.36	215.10	214.89	214.70	364.62	364.43	364.26	364.13
		t_{lr}	262.86	262.63	262.45	262.29	2610.59	2592.46	2577.47	2564.86
		$t_{s\&k}$	59.00	63.25	67.30	71.17	106.44	116.24	125.96	135.60
		t_{AH1}	177.17	177.42	177.63	177.81	1101.31	1097.45	1094.24	1091.52
		t_{AH2}	252.67	252.37	252.12	251.90	708.18	707.24	706.45	705.78
		t_{AH3}	262.86	262.63	262.45	262.29	2610.59	2592.46	2577.47	2564.86
		t_k	212.08	211.61	211.23	210.90	430.88	432.05	433.06	433.90
		t_{Ir}	264.16	264.12	264.06	264.03	25599.62	2580.21	2563.86	2550.94
		Proposed	275.25	276.26	277.33	278.46	2610.71	2592.35	2577.13	2565.37
4000	1000	$\bar{y}^*$	100	100	100	100	100	100	100	100
		t_{cr}	236.84	237.43	237.86	238.19	2670.37	2700.85	2723.56	2741.14
		t_{bt}	216.33	216.78	217.11	217.37	365.96	366.73	367.29	367.72
		t_{lr}	264.26	265.04	265.62	266.06	2677.17	2708.55	2731.97	2750.12
		$t_{s\&k}$	63.56	71.74	79.0	86.3	117.49	138.11	158.42	178.44
		t_{AH1}	176.74	176.66	176.59	176.54	1084.77	1070.60	1060.48	1052.89
		t_{AH2}	254.06	254.76	255.28	255.68	713.77	716.85	719.12	720.86
		t_{AH3}	264.26	265.04	265.62	266.06	2677.17	2708.55	2731.97	2750.12
		t_k	213.81	214.62	215.21	215.67	431.50	432.91	433.95	434.75
		t_{Ir}	265.80	266.95	267.89	268.70	2664.68	2693.03	2713.3	2728.18
		Proposed	278.52	282.51	286.44	290.38	2677.40	2708.59	2731.85	2749.86

Table 10
MSEs of Estimators using Population I

		$N_1 = 4500, N_2 = 500$				$N_1 = 4000, N_2 = 1000$			
		1/k				1/k			
a	b	(1/2)	(1/3)	(1/4)	(1/5)	(1/2)	(1/3)	(1/4)	(1/5)
C_X	ρ_{YX}	0.0248	0.02596	0.02691	0.02760	0.02596	0.02760	0.02820	0.02874
ρ_{YX}	C_X	0.03168	0.03436	0.03694	0.03942	0.03436	0.03942	0.04411	0.04839
1	C_X	0.03166	0.03433	0.03690	0.03938	0.03433	0.03938	0.04405	0.04832
1	ρ_{YX}	0.02797	0.02985	0.03156	0.03308	0.02985	0.03308	0.03559	0.03735
1	0	0.02004	0.02021	0.02001	0.01945	0.02021	0.01945	0.01719	0.01337
S_X	ρ_{YX}	0.02118	0.02159	0.02167	0.10899	0.02159	0.02142	0.01985	0.01685
ρ_{YX}	S_X	0.03596	0.03954	0.04311	0.04668	0.03954	0.04668	0.05380	0.06091
1	S_X	0.03166	0.03433	0.03690	0.03938	0.03954	0.04668	0.05380	0.06089

Table 11
MSEs of Estimators using Population II

		$N_1 = 4500, N_2 = 500$				$N_1 = 4000, N_2 = 1000$			
		1/k				1/k			
a	b	(1/2)	(1/3)	(1/4)	(1/5)	(1/2)	(1/3)	(1/4)	(1/5)
C_X	ρ_{YX}	0.07899	0.08573	0.09228	0.09861	0.08573	0.09861	0.11059	0.12166
ρ_{YX}	C_X	0.08501	0.09308	0.10107	0.10899	0.09301	0.10899	0.12459	0.13987
1	C_X	0.08491	0.09298	0.10097	0.10888	0.09305	0.10895	0.12454	0.13979
1	ρ_{YX}	0.08181	0.08919	0.09642	0.10351	0.08919	0.10351	0.11721	0.13028
1	0	0.07429	0.08003	0.08544	0.09051	0.08002	0.09051	0.09963	0.10734
S_X	ρ_{YX}	0.07534	0.08132	0.08701	0.09238	0.08139	0.09246	0.10221	0.11079
ρ_{YX}	S_X	0.08851	0.09734	0.10618	0.11501	0.09741	0.11508	0.13274	0.15040
1	S_X	0.08492	0.09298	0.10097	0.10888	0.09741	0.11507	0.13273	0.15039

Table 12
MSEs of Estimators using Population III

		$N_1 = 4500, N_2 = 500$				$N_1 = 4000, N_2 = 1000$			
		1/k				1/k			
a	b	(1/2)	(1/3)	(1/4)	(1/5)	(1/2)	(1/3)	(1/4)	(1/5)
C_X	ρ_{YX}	0.02466	0.02566	0.02639	0.02683	0.02566	0.02683	0.02684	0.02565
ρ_{YX}	C_X	0.03231	0.03501	0.03760	0.04009	0.03501	0.04010	0.04479	0.04908
1	C_X	0.03229	0.03498	0.03757	0.04006	0.03498	0.04006	0.04474	0.06161
1	ρ_{YX}	0.02823	0.03003	0.03163	0.03304	0.03003	0.03304	0.03526	0.03666
1	0	0.01961	0.01949	0.01896	0.01802	0.01949	0.01801	0.01485	0.00992
S_X	ρ_{YX}	0.02087	0.02103	0.02082	0.02022	0.02103	0.02022	0.01786	0.01387
ρ_{YX}	S_X	0.03651	0.04011	0.04370	0.04730	0.04011	0.04730	0.05447	0.04908
1	S_X	0.03229	0.03498	0.03757	0.04006	0.04010	0.04729	0.05446	0.06160

Table 13
MSEs of Estimators using Population IV

		$N_1 = 4500, N_2 = 500$				$N_1 = 4000, N_2 = 1000$			
		1/k				1/k			
a	b	(1/2)	(1/3)	(1/4)	(1/5)	(1/2)	(1/3)	(1/4)	(1/5)
C_X	ρ_{YX}	0.02089	0.02263	0.02431	0.02592	0.02263	0.02592	0.02894	0.03168
ρ_{YX}	C_X	0.02380	0.02615	0.02850	0.03084	0.02615	0.03084	0.03553	0.04021
1	C_X	0.02379	0.02614	0.02849	0.03083	0.02614	0.03083	0.03551	0.04019
1	ρ_{YX}	0.02280	0.02494	0.02706	0.02916	0.02494	0.02916	0.03329	0.03733
1	0	0.01742	0.01841	0.01925	0.01993	0.01841	0.01993	0.02081	0.02099
S_X	ρ_{YX}	0.01960	0.02106	0.02244	0.02371	0.02106	0.02371	0.02595	0.02777
ρ_{YX}	S_X	0.02391	0.02628	0.02865	0.03102	0.02628	0.03102	0.03576	0.04050
1	S_X	0.02379	0.02614	0.02849	0.03083	0.02677	0.03102	0.03576	0.04050

Tables 6–9 shows that the proposed estimator consistently achieves the largest PRE across the considered nonresponse levels. The gains are particularly significant when measurement error is absent, reflecting the strong contribution of auxiliary information when it is measured accurately. When measurement error is present, the proposed estimator still carries an advantage, indicating robustness to error defect.

Tables 10–13 confirm the PRE findings by showing that the proposed estimator yields the smallest MSE in most patterns. The MSE generally increases as k increases (i.e., as the subsampling fraction decreases), reflecting less information obtained from nonrespondents. The choice $a = 1$ and $b = 0$ produces the most stable and efficient performance across all populations.

7. CONCLUSION

This study develops a generalized class of estimators for the finite population mean under SRSWOR when nonresponse and measurement error jointly affect the study and auxiliary variables. The proposed form blends ratio-type adjustment, regression-type correction, and a power component within a single framework, with several existing estimators obtained as special cases. The first-order MSE expression makes the main drivers of efficiency explicit through (q_1, q_2, q_3): the estimator benefits when the study–auxiliary association (captured by q_3) is strong, whereas increased measurement error variances (contributing to q_1 and q_2) inflate uncertainty and limit the attainable MSE reduction.

Optimal choices of the tuning constants (k_1, k_2, λ) are obtained by minimizing the derived MSE, leading to a closed-form minimum MSE. Theoretical comparisons provide conditions under which this minimum MSE is smaller than that of the Hansen–Hurwitz estimator and other ratio, regression, exponential, and generalized competitors, essentially requiring that the covariance-driven reduction dominates the added variability from nonresponse and measurement error. The Monte Carlo study based on four simulated populations supports these implications: across different nonresponse levels and subsampling fractions, the proposed estimator consistently achieves the highest PRE and typically the smallest MSE. The results also show a clear deterioration in efficiency as the subsampling fraction decreases (larger k), and the setting $a = 1$ & $b = 0$ remains stable across the considered scenarios. Overall, the proposed class provides a flexible and technically grounded option for mean estimation in surveys where both nonresponse and measurement error are expected.

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