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MODIFIED UNIT EXPONENTIAL DISTRIBUTION

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ABSTRACT

In this paper, we introduce a new unit distribution named the Modified Unit Exponential (MUE) distribution, derived via a transformation of the classical Exponential distribution. We derive and analyze key properties including the probability density function, survival function, hazard function and moments. For parameter estimation, six methods are considered: Maximum Likelihood Estimation (MLE), Bayesian Estimation, Least Squares (LS), Weighted Least Squares (WLS), Anderson-Darling (AD) and Cramer-von Mises (CVM). A comprehensive Monte Carlo simulation compares the estimators using MSE and BIAS metrics. Moreover, the performance of the MUE distribution is illustrated through real data applications, where it is shown to outperform other common unit distributions such as Beta, Kumaraswamy, Topp-Leone, Unit Weibull and Unit Lindley. Our results indicate that the MUE distribution offers a flexible and powerful alternative for modeling proportional data in (0,1).

KEYWORDS

Unit Distribution, Exponential distribution, Modified Unit Exponential, Parameter Estimates.

1. INTRODUCTION

In studies involving phenomena or events with values confined to the unit interval (0,1), it is essential to have continuous distributions that are bounded within this range to accurately represent and model such data. Several well-known distributions have been developed for modeling data in the unit interval.

Among them, the Beta distribution has been one of the most widely used due to its flexibility. However, the Beta distribution may not always provide a good fit in all practical cases, which has prompted researchers to propose alternative bounded distributions.

Some alternative models that have been proposed include the Johnson SB distribution [3], the Topp-Leone distribution [18], and the Kumaraswamy distribution [6]. Additionally, several unit distributions have been introduced through transformation methods, such as the Unit-Logistic distribution [22], Unit-Gamma distribution [4], Unit-Weibull distribution [20], and Unit-Lindley distribution [21].

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In recent years, numerous new bounded and unit distributions have been introduced to enhance flexibility and improve fit to real data. For instance, a new two-parameter mixture family of generalized distributions was proposed in [15], offering strong practical performance. In [13], the authors utilized the Phi-4 transformation to construct a new flexible bounded distribution. The unit Zeghdoudi distribution introduced in [25] has shown promise in modeling proportional data. Furthermore, the inverse power X-Lindley distribution developed in [23] and the asymmetric version of the power half-logistic distribution proposed in [11] have demonstrated effectiveness across various applications. These developments indicate an ongoing trend in the literature toward constructing new bounded distributions with enhanced properties, motivating the proposal of the Modified Unit Exponential (MUE) distribution in this study.

In this work, we propose a new two-parameter unit distribution called the Modified Unit Exponential (MUE) distribution. This distribution is derived by applying the transformation $Y=(1-\operatorname{sech}(x))^{1/\alpha}$ to the standard exponential distribution. The proposed model offers more flexibility in the unit interval compared to the classical exponential distribution. We investigate several statistical properties of the distribution, including the survival function, hazard function, quantile function and moments. Additionally, we employ six different methods—Maximum Likelihood Estimation (MLE), Bayesian estimation, Least Squares (LS), Weighted Least Squares (WLS), Anderson-Darling (AD) and Cramer-von Mises (CVM)—to estimate the model parameters.

2. MODIFIED UNIT EXPONENTIAL DISTRIBUTION

Let X be a random variable following an Exponential distribution with parameter β, and let its distribution function be denoted by $G(x)$. A new probability distribution, referred to as the Modified Unit Exponential (MUE) distribution, is defined by applying the transformation $Y=(1-\operatorname{sech}(X))^{\frac{1}{\alpha}}$. The cumulative distribution and related functions derived from this transformation are given as follows:

$$\begin{aligned} F(y)=P\left(Y\le y\right)&=P\left\{\left[\left[1-\operatorname{sech}(x)\right)^{\frac{1}{\alpha}}\le y\right]\right\}\\ &=P\left\{x\le \operatorname{arcsech}\left(1-y^{\alpha}\right)\right\}\\ &=G\left\{\operatorname{arcsech}\left(1-y^{\alpha}\right)\right\} \end{aligned}$$

Given that the cumulative distribution function (CDF) of the Exponential distribution is $G(x)=1-\exp(-\beta x)$ for $x>0$ and $\beta>0$, the CDF of the transformed random variable Y is expressed as:

$$F(y\mid\alpha,\beta)=1-\exp\left\{-\beta\left[\operatorname{arcsech}\left(1-y^{\alpha}\right)\right]\right\},\quad y\in(0,1) \tag{1}$$

Here, β and α denote the scale and shape parameters of the distribution respectively. The probability density function (PDF) of the random variable Y is then given by:

$$f(y\,|\,\alpha,\beta)=\frac{\beta\alpha y^{(\alpha-1)}\exp\left\{-\beta\left[\operatorname{arcsech}\left(1-y^{\alpha}\right)\right]\right\}}{\sqrt{y^{\alpha}-y^{2\alpha}}\sqrt{2-3y^{\alpha}+y^{2\alpha}}}$$

The graphs of the probability density function (PDF) for various parameter values of the proposed distribution are presented in Figure 1. It is observed that the PDF of the Modified Unit Exponential (MUE) distribution can exhibit increasing, decreasing and right-skewed shapes depending on the values of the parameters.

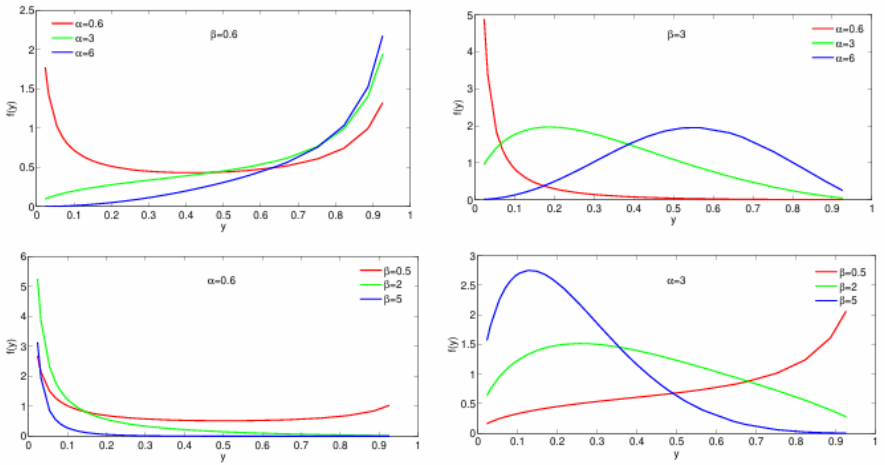


Fig. 1: Probability Density Function of MUE Distribution for Different Values of Parameters α and β

The survival and hazard functions of the MUE distribution are respectively given by:

$$S(y\,|\,\alpha,\beta)=\exp\left\{-\beta\left[\operatorname{arcsech}\left(1-y^{\alpha}\right)\right]\right\},$$

$$h(y\,|\,\alpha,\beta)=\frac{\beta\alpha y^{(\alpha-1)}}{\sqrt{y^{\alpha}-y^{2\alpha}}\sqrt{2-3y^{\alpha}+y^{2\alpha}}}$$

Graphs of the hazard function for different parameter values are shown in 2.

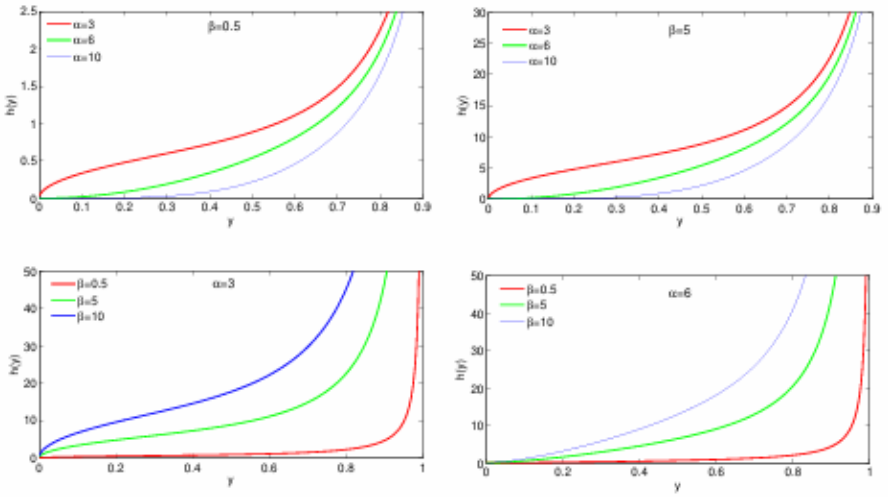


Fig. 2: Hazard Function for Different Values of Parameters α and β

2.1 Quantile and Moment

This section presents an analysis of the quantile function and the moments of the MUE distribution.

2.1.1 Quantile

The quantile function can be used to generate random variates from the MUE distribution and to compute specific quantiles, such as the median. Let X be a random variable that follows the MUE distribution with the CDF defined in Eq. 2. Then, the quantile function, denoted by $x = F^{-1}(y,\beta,\alpha)$, is given by:

$$x = \left\{1 - \operatorname{sech}\left[\frac{1}{\beta}\ln\left(1-U\right)\right]\right\}^{\frac{1}{\alpha}}$$

Here U is a random variable from the Uniform (0, 1) distribution. The pth quantile function and the median of the MUE distribution are given as follows respectively:

$$x_p = x = \left\{1 - \operatorname{sech}\left[\frac{1}{\beta}\ln\left(1-p\right)\right]\right\}^{1/\alpha}$$

$$x_{0.5} = x = \left\{1 - \operatorname{sech}\left[\frac{1}{\beta}\ln\left(1-0.5\right)\right]\right\}^{1/\alpha}$$

2.1.2 Moment

Let X be a random variable following the MUE distribution. The r th moment of Y is given by:

$$\mu_r = \int_0^1 y^r f(y)dy$$
$$= \int_0^1 y^r \frac{\beta\alpha y^{(\alpha-1)} \exp\left\{-\beta\left[\operatorname{arcsech}\left(1-y^{\alpha}\right)\right]\right\}}{\sqrt{y^{\alpha}-y^{2\alpha}}\sqrt{2-3y^{\alpha}+y^{2\alpha}}}dy$$

The mean, variance, skewness, kurtosis and moment values of the MUE distribution for various values of α and β are presented in Tables 1 and 2.

Table 1
Expected Value, Variance, Skewness and Kurtosis of the MUE Distribution for Different Values of α and β

α	β	**Mean**	**Variance**	**Skewness**	**Kurtosis**
0.5	2	0.5960	0.0993	-0.2806	1.7615
	4	0.7673	0.0520	-0.9363	2.7619
	6	0.8122	0.0409	-1.2061	3.7904
2	2	0.3243	0.0587	0.6522	2.4265
	4	0.4472	0.0551	0.3652	2.1328
	6	0.6394	0.0390	-0.1693	1.9668
3	2	0.1800	0.0239	1.0364	3.8251
	4	0.3867	0.0373	0.3269	2.6464
	6	0.5254	0.0317	-0.2369	2.3931

Table 2
First Four Moments of the MUE Distribution for Different Values of α and β

α	β	μ_1	μ_2	μ_3	μ_4
0.3	**2**	**0.7395**	**0.6380**	**0.5760**	**0.5321**
	4	**0.8406**	**0.7496**	**0.6858**	**0.6375**
	6	**0.8963**	**0.8258**	**0.7734**	**0.7323**
1.7	**2**	**0.3660**	**0.1911**	**0.1185**	**0.0821**
	4	**0.4944**	**0.3045**	**0.2115**	**0.1581**
	6	**0.6480**	**0.4625**	**0.3516**	**0.2791**
3	**2**	**0.2135**	**0.0793**	**0.0373**	**0.0199**
	4	**0.3902**	**0.1933**	**0.1102**	**0.0691**
	6	**0.5463**	**0.3322**	**0.2158**	**0.1470**

As the value of α increases, the expected value and kurtosis tend to increase, while the variance and skewness decrease. In contrast, increasing β results in higher skewness and lower mean, variance and kurtosis.

3. PARAMETER ESTIMATION

In this section, we will use 6 different estimators to estimate the model parameters of the MUE distribution. These estimation methods are; Maximum likelihood method, Bayesian estimation, Least Squares method, Weighted Least Squares method, Anderson Darling and Cramer-von Misas estimators.

3.1 Maximum Likelihood Estimate

The likelihood function of the MUE distribution is constructed based on the joint distribution of independent and identically distributed observations. Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from the MUE distribution. The log-likelihood function is given by:

$$L(\alpha,\beta) = \prod_{i=1}^{n} \frac{\beta\alpha y_i^{(\alpha-1)} \exp\left\{-\beta\left[\operatorname{arcsech}\left(1-y_i^{\alpha}\right)\right]\right\}}{\sqrt{y_i^{\alpha}-y_i^{2\alpha}}\sqrt{2-3y_i^{\alpha}+y_i^{2\alpha}}}$$

$$= \beta^n \alpha^n \exp\left\{(\alpha-1)\sum_{i=1}^{n}\log(y_i)\right\}\exp\left\{-\beta\left[\sum_{i=1}^{n}\operatorname{arcsech}\left(1-y_i^{\alpha}\right)\right]\right\} \qquad (2)$$

$$\exp\left\{-\frac{1}{2}\log\sum_{i=1}^{n}\left(y_i^{\alpha}-y_i^{2\alpha}\right)\right\}\exp\left\{-\frac{1}{2}\log\sum_{i=1}^{n}\left(2-3y_i^{\alpha}+y_i^{2\alpha}\right)\right\}$$

$$\ell(\alpha,\beta) = n\log(\beta)+n\log(\alpha)+(\alpha-1)\sum_{i=1}^{n}\log(y_i)-\sum_{i=1}^{n}\left\{\beta\left[\operatorname{arcsech}\left(1-y_i^{\alpha}\right)\right]\right\}$$

$$-\frac{1}{2}\sum_{i=1}^{n}\log(y_i^{\alpha}-y_i^{2\alpha})-\frac{1}{2}\sum_{i=1}^{n}\log(2-3y_i^{\alpha}+y_i^{2\alpha}) \qquad (3)$$

The maximum likelihood estimates are obtained by solving the nonlinear equations derived from setting the first derivatives of the log-likelihood with respect to α and β equal to zero. Since the equations are analytically intractable, the estimates are computed using iterative methods such as Newton-Raphson in the R software environment.

To ensure that the maximum likelihood estimators (MLEs) truly correspond to a global maximum rather than a local one, we performed several diagnostic steps. First, the log-likelihood surface was analyzed numerically over a dense grid of parameter values. This provided visual evidence that the maximum identified by the optimization algorithm corresponds to the global peak of the likelihood function. Additionally, multiple starting values for the parameters α and β were tested in the iterative optimization to confirm convergence toward the same point. These initial values were chosen based on method-of-moments estimates and empirical observations from the data. The convergence of the optimization algorithm and the consistency of the results across starting values suggest that the obtained estimates are indeed global maxima.

Furthermore, the Hessian matrix at the estimated parameters was evaluated and found to be negative definite, indicating a local concavity and confirming that the estimates correspond to a maximum of the likelihood function. These checks support the robustness of the MLE procedure used for the Modified Unit Exponential distribution.

3.2 Bayesian Estimation

It can be treated either as an unknown constant for the probability density function (PDF) of the MUE distribution discussed in Eq. 2, or as a random variable with its own probability distribution. In Bayesian inference, to estimate unknown parameters, we must assign a prior distribution to each parameter that is not directly specified by the model itself.

In the Bayesian framework, we consider the sample $Y_1, Y_2, \ldots, Y_n$ consisting of random variables with parameters (α, β), as drawn from a conditional distribution. The likelihood function of the sample is given in Eq. 1, and it is combined with the prior distribution $\pi(\alpha, \beta)$.

Assuming independence between the parameters, for all $\alpha \in (0, \infty)$ and $\beta \in (0, \infty)$, we assume $\alpha \sim Gamma(a, b)$ and $\beta \sim Gamma(c, d)$ Therefore, the joint prior distribution of the parameters $\theta = (\alpha, \beta)$ is given by:

$$\pi(\Theta) = \alpha^{a-1} \exp(-b\alpha) \beta^{c-1} \exp(-d\beta)$$

According to Bayes' theorem, the probability density function of the conditional distribution—that is, the posterior distribution—of (α, β) is given as follows:

$$\begin{aligned} g(\theta \mid y) &= \frac{L(y \mid \theta)\pi(\theta)}{\int_0^\infty \int_0^\infty L(y \mid \gamma)\pi(\theta) d\theta} \\ &= \frac{L(\alpha, \beta)\alpha^{a-1} \exp(-b\alpha)\beta^{c-1} \exp(-d\beta)}{\int_{\alpha=0}^\infty \int_{\beta=0}^\infty L(\alpha, \beta)\alpha^{a-1} \exp(-b\alpha)\beta^{c-1} \exp(-d\beta) d\alpha d\beta} \end{aligned} \tag{4}$$

The marginal posterior densities of the parameters α and β are obtained by integrating the joint posterior distribution, that is: $g(\alpha \mid y) = \int_\beta g(\theta \mid y) d\beta$ and $g(\beta \mid y) = \int_\alpha g(\theta \mid y) d\alpha.$

In Bayesian estimation, the risk function is used to estimate unknown parameters. The choice of an appropriate risk function depends on the context of the problem, although there is no strict rule for selecting one. Risk functions are generally classified into two types: symmetric and asymmetric. A risk function is considered symmetric if it assigns equal weight to deviations above and below the estimate; otherwise, it is asymmetric.

The unknown parameters α and β are associated with a loss incurred when we estimate them as $\hat{\alpha}$ and $\hat{\beta}$ respectively [24]. The value of a decision is measured by the expected loss, which is referred to as the posterior risk.

Accordingly, the Bayes estimator of θ is obtained as follows:

$$E(\theta \mid y) = \int_\theta \theta g(\theta \mid y) d\theta$$

The Bayesian estimator is defined as the value that minimizes the risk function.

In Eq. 3, we employ the Markov Chain Monte Carlo (MCMC) technique and the Metropolis-Hastings algorithm as an alternative approach, since the probability density function of the posterior distribution has a complex form and is analytically intractable.

We begin by constructing a reversible Markov chain. Then, the following steps are repeated:

Algorithm

1. At iteration N, Generate $\theta^* \sim q(\theta_N \mid \theta)$ from a proposal distribution
2. calculating the probability of acceptance

$$p = \min\left\{\frac{L(\theta^*)\pi(\theta)}{L(\theta_N)\pi(\theta)}.\frac{Q(\theta^* \mid \theta_N)}{Q(\theta_N \mid \theta^*)},1\right\}.$$

3. Generate $U \sim Uniform(0,1)$
4. if $u \le p$

$$\theta_{(N+1)} = \theta^*$$

Otherwise

$$\theta_{(N+1)} = \theta_{(N)}$$

5. $N = N + 1$
6. The approximate values $\hat{\alpha}$ and $\hat{\beta}$ are:

$$\hat{\alpha} = \frac{1}{N}\sum_{i=1}^{N}\alpha_i g(\alpha_i \mid y), \quad \hat{\beta} = \frac{1}{N}\sum_{i=1}^{N}\beta_i g(\beta_i \mid y)$$

7. Note that after step 3, M observations can be discarded as the burn-in observations to get more precise estimates of the parameters, i.e.,

$$\hat{\alpha} = \frac{1}{N-M}\sum_{i=M+1}^{N}\alpha_i g(\alpha_i \mid y), \quad \hat{\beta} = \frac{1}{N-M}\sum_{i=1}^{N}\beta_i g(\beta_i \mid y)$$

For the implementation of the Bayesian estimation procedure, the initial values of the parameters α and β used in the Markov Chain Monte Carlo (MCMC) algorithm were chosen based on the corresponding maximum likelihood estimates (MLEs). This approach provides a reasonable and informed starting point that improves the convergence behavior of the Metropolis-Hastings algorithm. The use of MLE-based initial values ensures that the posterior chain starts in a high-density region of the parameter space, enhancing the efficiency and stability of the simulation process.

3.3 Least Squares Method

The Least Squares (LS) estimator is obtained by minimizing the sum of squared differences between the theoretical cumulative distribution function (CDF) and the empirical CDF. This method assigns equal weights to all observations, regardless of their position in the distribution.

Let $y_{(1)} < y_{(2)} < \ldots < y_{(n)}$ denote the ordered observed values (i.e., order statistics) drawn from a population following the MUE distribution. The LS estimation method for the parameters β and α based on minimizing the following expression:

$$F(y_{(i)}) = 1 - \exp\left\{-\beta\left[\operatorname{arcsech}(1 - y_{(i)}^{\alpha})\right]\right\}, \quad i = 1, 2, \ldots \tag{4}$$

When the empirical distribution function is substituted for $F(y_{(i)})$ in 4, the following function is used to obtain the Least Squares (LS) estimates of the parameters β and α.

$$\begin{aligned} LS(\alpha,\beta) &= \sum_{i=1}^{n}\left[F(y_{(i)}) - \frac{i}{n+1}\right]^2 \\ &= \sum_{i=1}^{n}\left[1 - \exp\left\{-\beta\left[\operatorname{arcsech}(1 - y_{(i)}^{\alpha})\right]\right\} - \frac{i}{n+1}\right]^2 \end{aligned} \tag{5}$$

The LS estimates of the parameters β and α can be obtained by minimizing the LS function given in Eq. 5. The partial derivatives of the LS function with respect to β and α are given as follows:

$$\frac{\partial LS(\alpha,\beta)}{\partial \beta} = \sum_{i=1}^{n}\left[1 - \exp\left\{-\beta\left[\operatorname{arcsech}(1 - y_{(i)}^{\alpha})\right]\right\} - \frac{i}{n+1}\right]\eta_1(y_{(i)}) = 0$$

$$\frac{\partial LS(\alpha,\beta)}{\partial \alpha} = \sum_{i=1}^{n}\left[1 - \exp\left\{-\beta\left[\operatorname{arcsech}(1 - y_{(i)}^{\alpha})\right]\right\} - \frac{i}{n+1}\right]\eta_2(y_{(i)}) = 0$$

where η_1 and η_2 are as follows:

$$\eta_1(y_{(i)}) = \operatorname{arcsech}(1 - y_{(i)}^{\alpha}) \exp\left\{-\beta\left[\operatorname{arcsech}(1 - y_{(i)}^{\alpha})\right]\right\} \tag{6}$$

$$\eta_2(y_{(i)}) = \frac{\beta y^{\alpha} \ln(y_{(i)}) \exp\left\{-\beta\left[\operatorname{arcsech}(1 - y_{(i)}^{\alpha})\right]\right\}}{\sqrt{y_{(i)}^{\alpha} - y_{(i)}^{2\alpha}}\sqrt{2 - 3y_{(i)}^{\alpha} + y_{(i)}^{2\alpha}}} \tag{7}$$

Since the obtained equations cannot be solved numerically, they will be solved with the help of iterative methods.

3.4 Weighted Least Squares Method

WLS is a refinement of LS in which weights are introduced to account for variance heterogeneity across different quantiles. Observations in the tails typically receive less weight, and those near the center more, making WLS more robust in certain cases. The method provides better performance when the underlying distribution exhibits non-uniform behavior across its domain.

Let the random variables $y_{(1)} < y_{(2)} < ... < y_{(n)}$ represent the ordered observations from a population governed by the MUE distribution. The WLS estimates of β and α are obtained by minimizing the following weighted sum:

$$WLS(\beta,\alpha) = \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F(y_{(i)}) - \frac{i}{n+1} \right]^2$$

$$= \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[1 - \exp\left\{ -\beta \left[\operatorname{arcsech}(1-y_{(i)}^{\alpha}) \right] \right\} - \frac{i}{n+1} \right]^2$$

The WLS estimates of the parameters β and α obtained by minimizing the WLS function. The partial derivatives of the WLS function with respect to β and α are given as follows:

$$\frac{\partial WLS(\beta,\alpha)}{\partial \beta} = \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[1 - \exp\left\{ -\beta \left[\operatorname{arcsech}(1-y_{(i)}^{\alpha}) \right] \right\} - \frac{i}{n+1} \right] \eta_1(y_{(i)}) = 0$$

$$\frac{\partial WLS(\beta,\alpha)}{\partial \alpha} = \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[1 - \exp\left\{ -\beta \left[\operatorname{arcsech}(1-y_{(i)}^{\alpha}) \right] \right\} - \frac{i}{n+1} \right] \eta_2(y_{(i)}) = 0$$

Here, η_1 and η_2 are defined in Eqs. 6 and 7 respectively. The WLS estimates of the parameters β and α are obtained by solving the above system of equations. Since the resulting equations do not have closed-form solutions, they must be solved numerically, typically using iterative methods.

3.5 Anderson-Darling Method

This estimator is based on the Anderson–Darling (AD) goodness-of-fit statistic, originally introduced by Anderson and Darling (1952) [1]. Detailed information on the Anderson–Darling method can be found in [5]. The AD estimates of the parameters β and α are obtained by minimizing the following function:

$$AD(\alpha,\beta) = -n - \sum_{i=1}^{n} \frac{2i-1}{n} \left\{ \log\left[F(y_{(i)}) \right] + \log\left[S(y_{(i)}) \right] \right\}$$

$$= -n - \sum_{i=1}^{n} \frac{2i-1}{n} \left[\begin{array}{l} \log\left\{ 1 - \exp\left[-\beta \operatorname{arcsech}(1-y_{(i)}^{\alpha}) \right] \right\} \\ + \log\left\{ \exp\left[-\beta \operatorname{arcsech}(1-y_{(i)}^{\alpha}) \right] \right\} \end{array} \right]$$

Derivatives of AD function according to β and α parameters are as follows;

$$\frac{\partial AD(\alpha,\beta)}{\partial \beta} = \sum_{i=1}^{n} (2i-1) \left[\begin{array}{c} \dfrac{\eta_1(y_{(i)})}{1 - \exp\left\{ -\beta \operatorname{arcsech}(1-y_{(i)}^{\alpha}) \right\}} \\ - \dfrac{\eta_1(y_{(n+1-i)})}{\exp\left\{ -\beta \operatorname{arcsech}(1-y_{(i)}^{\alpha}) \right\}} \end{array} \right]$$

$$\frac{\partial AD(\alpha,\beta)}{\partial \alpha} = \sum_{i=1}^{n}(2i-1)\left[\frac{\eta_2(y_{(i)})}{1-\exp\left\{-\beta\,\mathrm{arcsech}(1-y_{(i)}^{\alpha})\right\}} - \frac{\eta_2(y_{(n+1-i)})}{\exp\left\{-\beta\,\mathrm{arcsech}(1-y_{(i)}^{\alpha})\right\}}\right]$$

where the η_1 and η_2 are given in equations 6 and 7. AD estimates of parameters β and α are obtained by the solution of the above system of equations. Since the obtained equations cannot be solved numerically, they can be solved with the help of iterative methods.

3.6 Cramer-von Mises Method

Cramer-von Mises (CVM) estimator is estimator based on minimum distances. Detailed information about the Cramer-von Mises method can be available in [5]. Cramer-von Mises (CVM) estimates of β and α parameters are obtained by minimizing the following CVM function.

$$CVM(\alpha,\beta) = \frac{1}{12} + \sum_{i=1}^{n}\left\{F(y_{(i)}) - \frac{2i-1}{2n}\right\}^2$$

$$= \frac{1}{12} + \sum_{i=1}^{n}\left\{1-\exp\left[-\beta\,\mathrm{arcsech}(1-y_{(i)}^{\alpha})\right] - \frac{2i-1}{2n}\right\}^2$$

Derivatives according to β and α parameters of CVM function are as follows;

$$\frac{\partial CVM(\alpha,\beta)}{\partial \beta} = \sum_{i=1}^{n}\left[1-\exp\left\{-\beta\left[\mathrm{arcsech}(1-y_{(i)}^{\alpha})\right]\right\} - \frac{2i-1}{2n}\right]^2 \eta_1(y_{(i)}) = 0$$

$$\frac{\partial CVM(\alpha,\beta)}{\partial \alpha} = \sum_{i=1}^{n}\left[1-\exp\left\{-\beta\left[\mathrm{arcsech}(1-y_{(i)}^{\alpha})\right]\right\} - \frac{2i-1}{2n}\right]^2 \eta_2(y_{(i)}) = 0$$

where the η_1 and η_2 are given in equations 6 and 7. CVM estimates of parameters β and α are obtained by the solution of the above system of equations. Since the obtained equations cannot be solved numerically, they can be solved with the help of iterative methods.

4. SIMULATION EXPERIMENTS

In this section, the performance of the MLE, LS, WLS, AD and CVM estimators for the unknown parameters of the MUE distribution is evaluated and compared based on mean squared error (MSE) and bias criteria. The analysis is carried out through a Monte Carlo simulation study. In addition, Bayesian estimation is performed using the Markov Chain Monte Carlo (MCMC) method with the Metropolis–Hastings algorithm. The simulation results are presented in Tables 4-14 for various combinations of the parameters

β and α. Each scenario is replicated 5000 times, and the sample sizes considered are $n = 50,100,150,200,250,300,350,\ 400,450,500$.

Table 3
BIAS Estimates of for the Estimators of MLE and Bayes α and β Parameters $(\alpha = 0.9, \beta = 0.7)$

n	MLE		Bayes	
	α	β	α	β
50	0.10349	0.04564	0.06824	0.10728
100	0.08800	0.03634	0.05436	0.10202
150	0.07791	0.02805	0.05128	0.04736
200	0.07536	0.02556	0.05070	0.03390
250	0.07184	0.02409	0.03334	0.03178
300	0.07095	0.02325	0.03067	0.01884
350	0.06952	0.02230	0.02461	0.00915
400	0.06893	0.02175	0.02374	0.00902
450	0.06782	0.02071	0.02041	0.00899
500	0.06717	0.01970	0.01972	0.00874

Table 4
BIAS Estimates of for the Estimators of LS, WLS, AD and CVM α and β Parameters $(\alpha = 0.9, \beta = 0.7)$

n	LS		WLS		AD		CVM	
	α	β	α	β	α	β	α	β
50	0.05366	0.01911	0.10833	0.07071	0.08997	0.06671	0.05399	0.01210
100	0.04873	0.00891	0.10432	0.06381	0.08851	0.04365	0.04894	0.00907
150	0.04597	0.00852	0.10187	0.05978	0.08241	0.03584	0.04614	0.00815
200	0.04306	0.00834	0.09897	0.05529	0.07522	0.02665	0.04310	0.00803
250	0.04089	0.00805	0.09601	0.05375	0.07311	0.02552	0.04087	0.00802
300	0.04083	0.00729	0.09011	0.05383	0.07061	0.02273	0.04081	0.00726
350	0.03868	0.00774	0.08913	0.05192	0.06892	0.02199	0.03866	0.00772
400	0.03792	0.00697	0.07798	0.05081	0.06750	0.02146	0.03789	0.00693
450	0.03705	0.00621	0.07623	0.04778	0.06724	0.02064	0.03710	0.00623
500	0.03571	0.00604	0.06573	0.04756	0.06713	0.01994	0.03570	0.00602

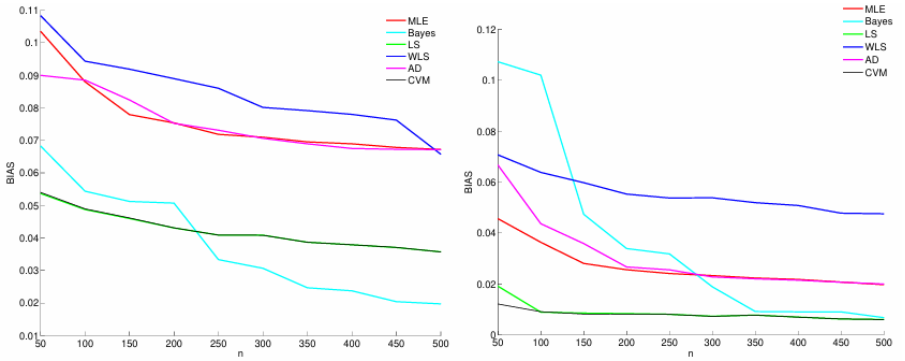


Fig. 3: BIAS Estimates for α and β Parameters respectively $(\alpha = 0.9, \beta = 0.7)$

Table 5
MSE Estimates of for the Estimators of MLE and Bayes α and β Parameters $(\alpha = 0.9, \beta = 0.7)$

n	MLE		Bayes	
	α	β	α	β
50	0.04179	0.01901	0.04092	0.01452
100	0.02294	0.00955	0.02135	0.01402
150	0.01588	0.00582	0.02049	0.01368
200	0.01337	0.00454	0.00847	0.00986
250	0.01087	0.00354	0.00711	0.00762
300	0.00986	0.00304	0.00419	0.00390
350	0.00877	0.00258	0.00244	0.00288
400	0.00827	0.00230	0.00151	0.00095
450	0.00777	0.00208	0.00084	0.00081
500	0.00735	0.00186	0.00079	0.00057

Table 6
MSE Estimates of for the Estimators of LS, WLS, AD and CVM α and β Parameters $(\alpha = 0.9, \beta = 0.7)$

n	LS		WLS		AD		CVM	
	α	β	α	β	α	β	α	β
50	0.00852	0.00480	0.02121	0.01677	0.03975	0.01573	0.00852	0.00482
100	0.00595	0.00241	0.01797	0.01326	0.03782	0.01054	0.00598	0.00241
150	0.00481	0.00157	0.01658	0.01156	0.02485	0.00914	0.00484	0.00157
200	0.00428	0.00120	0.01606	0.01041	0.02052	0.00874	0.00428	0.00120
250	0.00367	0.00097	0.01562	0.00993	0.01102	0.00801	0.00367	0.00097
300	0.00348	0.00078	0.01522	0.00947	0.00976	0.00695	0.00348	0.00078
350	0.00310	0.00069	0.01502	0.00905	0.00883	0.00561	0.00309	0.00069
400	0.00291	0.00057	0.01451	0.00861	0.00841	0.00534	0.00289	0.00057
450	0.00275	0.00053	0.01418	0.00815	0.00794	0.00417	0.00276	0.00534
500	0.00259	0.00043	0.01391	0.00802	0.00775	0.00408	0.00259	0.00004

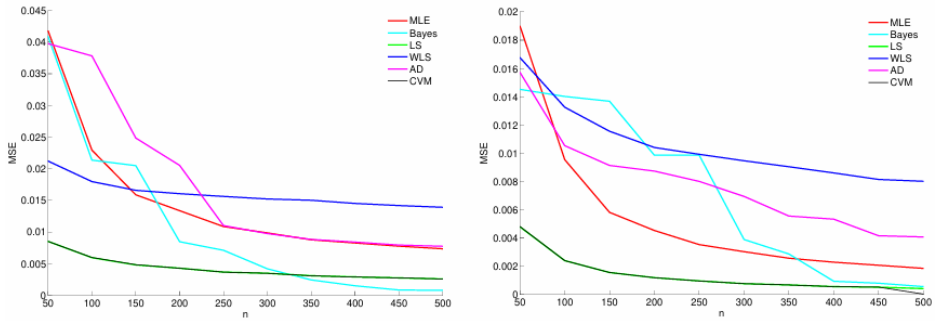


Fig. 4: MSE Estimates for α and β Parameters respectively $(\alpha = 0.9, \beta = 0.7)$

Figure 5 presents the MCMC trace plots and mean estimates for the parameter β and α under the setting $(\alpha = 0.9, \beta = 0.7)$

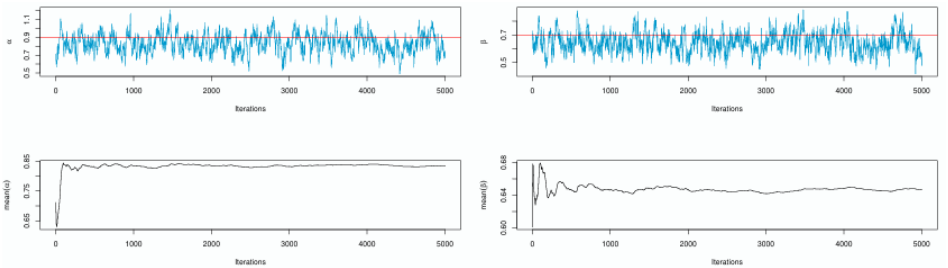


Fig. 5: MCMC Iterations Plot for $(\alpha = 0.9, \beta = 0.7)$ α **and** β **Parameters**

Table 7
BIAS Estimates of for the Estimators of MLE and Bayes α and β Parameters $(\alpha = 1.5, \beta = 1)$

n	MLE		Bayes	
	α	β	α	β
50	0.04536	0.05053	0.17450	0.12350
100	0.03104	0.02015	0.15960	0.10604
150	0.02233	0.01814	0.09344	0.08899
200	0.02022	0.01367	0.04470	0.04508
250	0.01641	0.00992	0.02973	0.04086
300	0.01468	0.00978	0.00817	0.01711
350	0.01506	0.01012	0.00725	0.01377
400	0.01284	0.00882	0.00695	0.00982
450	0.00970	0.00574	0.00590	0.00758
500	0.00869	0.00461	0.00547	0.00678

Table 8
BIAS Estimates of for the Estimators of LS, WLS, AD and CVM α and β Parameters $(\alpha = 1.5, \beta = 1)$

n	LS		WLS		AD		CVM	
	α	β	α	β	α	β	α	β
50	0.07798	0.02973	0.11249	0.08627	0.05642	0.06297	0.07830	0.02968
100	0.06672	0.02799	0.10676	0.07436	0.03347	0.02407	0.06660	0.02872
150	0.06024	0.02749	0.09265	0.06313	0.02376	0.01861	0.06018	0.02767
200	0.05647	0.01715	0.08987	0.06260	0.02422	0.01640	0.05666	0.01722
250	0.05405	0.01633	0.08958	0.05808	0.01245	0.00547	0.05404	0.01629
300	0.05099	0.01587	0.08764	0.05453	0.01672	0.00519	0.05097	0.01579
350	0.04660	0.01563	0.08723	0.05246	0.01660	0.01039	0.04676	0.01571
400	0.04573	0.01452	0.08408	0.05227	0.01560	0.00954	0.04573	0.01451
450	0.04504	0.01369	0.08354	0.04814	0.01264	0.00886	0.04515	0.01374
500	0.00402	0.01018	0.07912	0.04713	0.01300	0.00670	0.04404	0.01012

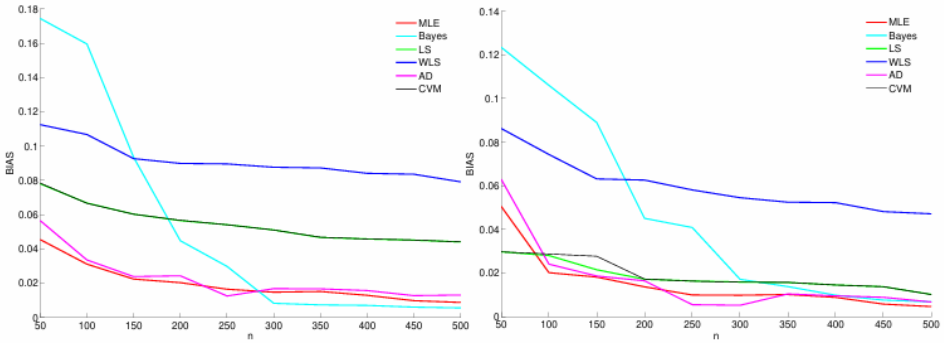


Fig. 6: BIAS Estimates for α and β Parameters respectively $(\alpha = 1.5, \beta = 1)$

Table 9
MSE Estimates of for the Estimators of MLE and Bayes α and β Parameters $(\alpha = 1.5, \beta = 1)$

n	MLE		Bayes	
	α	β	α	β
50	0.04705	0.01988	0.05432	0.03676
100	0.03571	0.01861	0.04645	0.02126
150	0.02312	0.01166	0.00825	0.01282
200	0.01744	0.00836	0.00271	0.00829
250	0.01347	0.00653	0.00137	0.00213
300	0.01152	0.00551	0.00092	0.00085
350	0.00947	0.00473	0.00078	0.00059
400	0.00816	0.00411	0.00062	0.00058
450	0.00725	0.00353	0.00056	0.00045
500	0.00667	0.00327	0.00052	0.00038

Table 10
MSE Estimates of for the Estimators of LS, WLS, AD and CVM α and β parameters $(\alpha=1.5,\beta=1)$

n	LS		WLS		AD		CVM	
	α	β	α	β	α	β	α	β
50	0.01849	0.01013	0.04322	0.03246	0.04541	0.02927	0.01859	0.01014
100	0.01222	0.00552	0.03706	0.02462	0.03256	0.01793	0.01221	0.00554
150	0.00940	0.00414	0.03378	0.02045	0.02335	0.01173	0.00939	0.00413
200	0.00771	0.00333	0.03268	0.01871	0.01777	0.00838	0.00770	0.00333
250	0.00692	0.00281	0.03116	0.01725	0.01578	0.00747	0.00690	0.00281
300	0.00611	0.00249	0.03038	0.01615	0.01165	0.00557	0.00610	0.00248
350	0.00527	0.00221	0.03016	0.01530	0.00975	0.00484	0.00530	0.00221
400	0.00520	0.001981	0.03004	0.01491	0.00886	0.00417	0.00526	0.00197
450	0.00473	0.00176	0.02886	0.01370	0.00790	0.00362	0.00475	0.00176
500	0.00455	0.00170	0.02829	0.01319	0.00682	0.00348	0.00455	0.00170

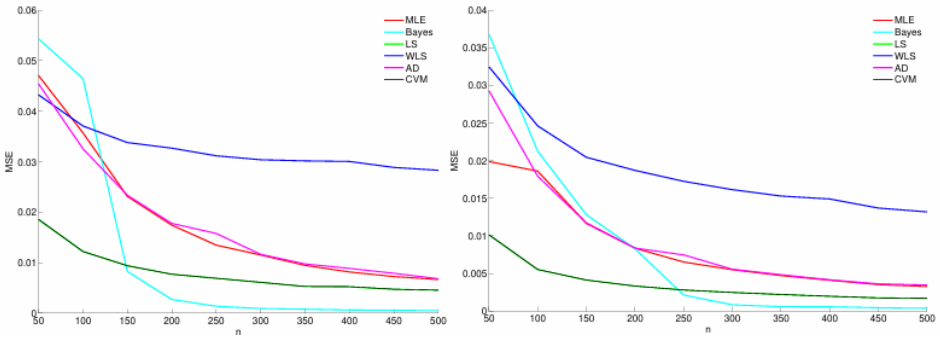


Fig. 7: MSE Estimates for α and β Parameters respectively $(\alpha=1.5,\beta=1)$

Figure 8 presents the MCMC trace plots and mean estimates for the parameter β and α under the setting $(\alpha=1.5,\beta=1)$

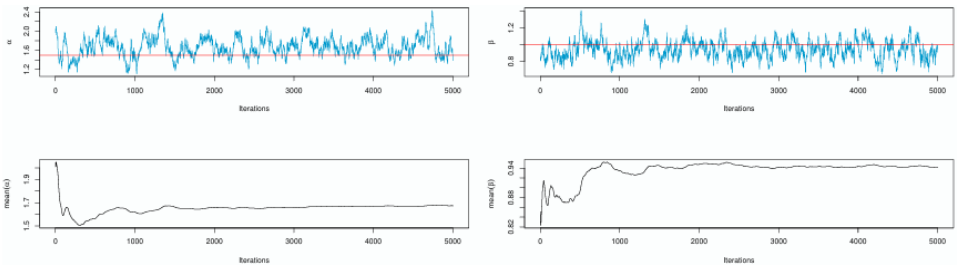


Fig. 8: MCMC Iterations Plot for $(\alpha=1.5,\beta=1)$ α and β Parameters

Table 11
BIAS Estimates of for the Estimators of MLE and Bayes α and β Parameters $(\alpha = 3, \beta = 2)$

n	MLE		Bayes	
	α	β	α	β
50	0.09288	0.06421	0.08377	0.11909
100	0.04553	0.05918	0.08199	0.10026
150	0.03405	0.04167	0.07867	0.06903
200	0.02349	0.03300	0.07224	0.05814
250	0.01630	0.02251	0.04325	0.04630
300	0.01341	0.01846	0.03688	0.03633
350	0.01166	0.01625	0.00455	0.02860
400	0.01061	0.01373	0.00249	0.01508
450	0.00815	0.01239	0.00213	0.00952
500	0.00871	0.01222	0.00154	0.00437

Table 12
BIAS Estimates of for the Estimators of LS, WLS, AD and CVM α and β parameters $(\alpha = 3, \beta = 2)$

n	LS		WLS		AD		CVM	
	α	β	α	β	α	β	α	β
50	0.11389	0.05641	0.12327	0.11339	0.11962	0.12435	0.11522	0.05611
100	0.09437	0.05336	0.11776	0.10476	0.09457	0.09519	0.09459	0.05316
150	0.08510	0.05240	0.11441	0.08925	0.08564	0.08642	0.08488	0.05295
200	0.07706	0.04991	0.10251	0.08918	0.07367	0.05674	0.07714	0.05199
250	0.07408	0.04594	0.09206	0.07804	0.06729	0.05506	0.07404	0.04309
300	0.07095	0.03932	0.09127	0.06994	0.05846	0.04752	0.07081	0.03912
350	0.06897	0.03804	0.08874	0.04296	0.04317	0.04738	0.06905	0.03809
400	0.06769	0.02652	0.08835	0.04020	0.04270	0.04335	0.06763	0.03643
450	0.06369	0.02636	0.08434	0.03654	0.03984	0.04287	0.06354	0.02550
500	0.06448	0.02471	0.08215	0.02190	0.01877	0.03167	0.06245	0.02474

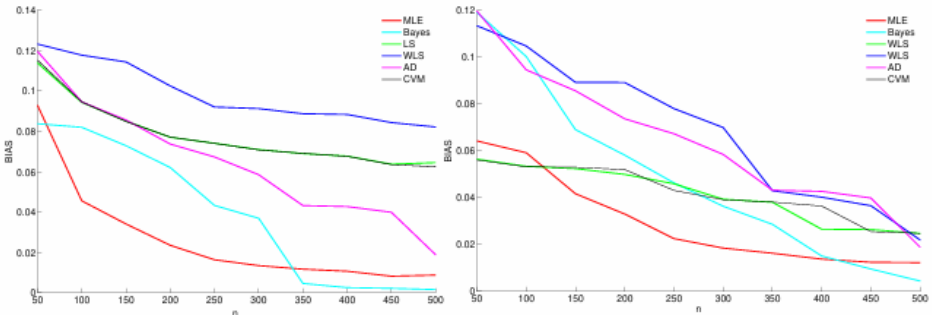


Fig. 9: BIAS Estimates for α and β Parameters respectively $(\alpha = 3, \beta = 2)$

Table 13
MSE Estimates of for the Estimators of MLE and Bayes α and β Parameters $(\alpha = 3, \beta = 2)$

n	MLE		Bayes	
	α	β	α	β
50	0.09786	0.22192	0.04441	0.04182
100	0.09527	0.09357	0.02554	0.04144
150	0.06306	0.05826	0.02430	0.02801
200	0.04657	0.04339	0.01804	0.01977
250	0.03749	0.03435	0.00968	0.01809
300	0.03038	0.02814	0.00794	0.01364
350	0.02619	0.02367	0.00098	0.00371
400	0.02281	0.02051	0.00081	0.00116
450	0.02025	0.01827	0.00043	0.00075
500	0.01877	0.01651	0.00036	0.00022

Table 14
MSE Estimates of for the Estimators of LS, WLS, AD and CVM α and β Parameters $(\alpha = 3, \beta = 2)$

n	LS		WLS		AD		CVM	
	α	β	α	β	α	β	α	β
50	0.04365	0.03221	0.12948	0.09415	0.20926	0.09786	0.04418	0.03208
100	0.02558	0.01862	0.10803	0.06681	0.10297	0.08661	0.02560	0.01860
150	0.01962	0.01490	0.09733	0.05696	0.06469	0.05908	0.01956	0.01479
200	0.01539	0.01274	0.08998	0.05087	0.04745	0.04736	0.01543	0.01273
250	0.01369	0.01123	0.08907	0.04821	0.03792	0.03566	0.01368	0.01120
300	0.01231	0.01027	0.08503	0.04299	0.03059	0.02752	0.01229	0.01023
350	0.01113	0.00975	0.08409	0.04296	0.02621	0.02413	0.01113	0.00974
400	0.01057	0.00872	0.07970	0.04020	0.02333	0.02305	0.01022	0.00870
450	0.00924	0.00778	0.07198	0.03842	0.02069	0.01912	0.00925	0.00776
500	0.00916	0.00775	0.07074	0.03643	0.01919	0.01861	0.00917	0.00767

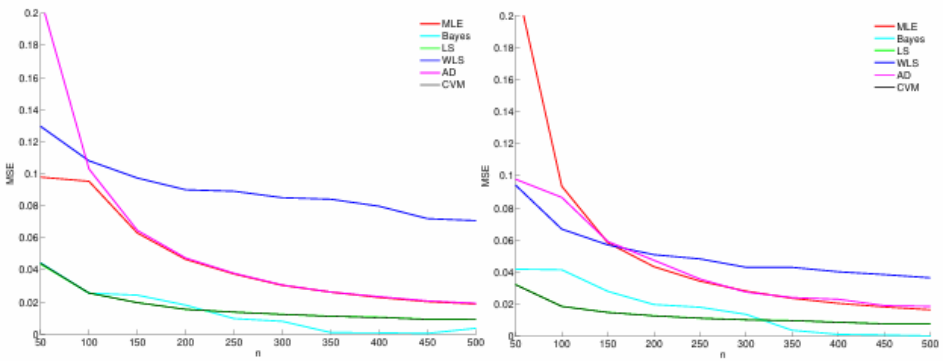


Fig. 10: MSE Estimates for α and β Parameters respectively $(\alpha = 3, \beta = 2)$

Figure 11 presents the MCMC trace plots and mean estimates for the parameter β and α under the setting $(\alpha = 3, \beta = 2)$

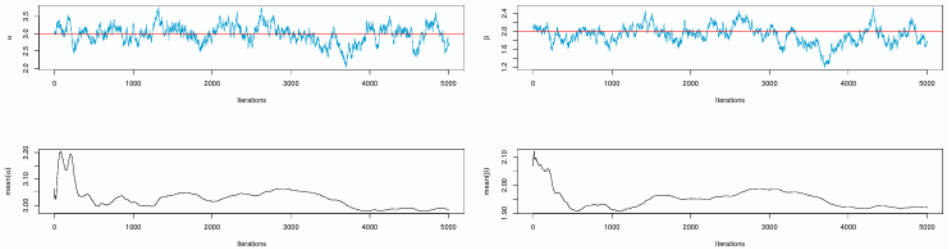


Fig. 11: MCMC Iterations Plot for $(\alpha = 3, \beta = 2)$ α and β Parameters

The performance results of the MLE, Bayesian, LS, WLS, AD and CVM estimators for the parameters (α, β) of the Modified Unit Exponential (MUE) distribution are presented in the figures and tables above. These results are based on a full-sample simulation study consisting of 10,000 Monte Carlo replications for each parameter combination, and are evaluated according to the mean squared error (MSE) and bias criteria.

As observed from the simulation outcomes, both the bias and MSE values tend to decrease as the sample size increases or for various combinations of parameter values. The simulation design included diverse parameter settings to examine the robustness of the estimators under different distributional shapes. Among the methods, Bayesian estimation generally provides smaller mean bias and MSE values compared to the other estimators. However, the differences between the estimators are relatively minor, and all methods yield sufficiently small estimation errors.

Therefore, it can be concluded that all six estimation methods considered in this study are suitable and reliable for estimating the parameters of the MUE distribution. Among them, the Bayesian and MLE approaches may be particularly advantageous in practice due to their relatively higher accuracy, especially in small or moderate sample sizes.

5. APPLICATIONS

In this section, we compare the performance of the proposed MUE distribution with several well-known bounded distributions from the literature, including the Beta, Kumaraswamy, Topp–Leone, Unit Weibull and Unit Lindley distributions. The probability density functions (PDFs) of these distributions are provided below for reference

-Beta Distribution:

$$f(x \mid \alpha, \beta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha) + \Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \ \alpha, \beta > 0$$

-Kumaraswamy (KW) Distribution:

$$f(x \mid \alpha, \beta) = \alpha\beta x^{\alpha-1}(1-x^{\alpha})^{\beta-1}, \ \alpha, \beta > 0$$

-Topp Leone (TL) Distribution:

$$f(x|\beta) = 2\beta x^{\beta-1}(1-x)(2-x)^{\beta-1},\ \beta > 0$$

-Unit Weibull (UW) Distribution:

$$f(x|\alpha,\beta) = \frac{1}{x}\beta\alpha\left(-\log(x)\right)^{\alpha-1}\exp\left(-\beta\left(-\log(x)\right)^{\alpha}\right),\ \alpha,\beta > 0$$

-Unit Lindley (Ulind) Distribution:

$$f(x|\beta) = \frac{\beta^2}{1+\beta}(1+x)^{-3}\exp\left(-\frac{\beta x}{1-x}\right),\ \beta > 0$$

In order to determine the most appropriate model, several goodness-of-fit measures are calculated, including the log-likelihood value $\hat{\ell}$, Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Corrected Akaike Information Criterion (CAIC), Hannan–Quinn Information Criterion (HQIC), Kolmogorov–Smirnov (KS), Anderson–Darling (AD) and Cramer–von Mises (CVM) statistics.

Some of the information criteria are defined as follows:

$$AIC = -2\ell + 2k,$$

$$BIC = -2\ell + k\log(n),$$

$$CAIC = -2\ell + \frac{2kn}{n-k-1},$$

$$HQIC = -2\ell + 2k\log[\log(n)].$$

5.1 First Actual Data

The initial dataset comprises 30 observations of the tensile strength of polyester fibers. This dataset was first introduced by Quesenberry and Hales (1980) and was subsequently analyzed in detail by Mazucheli, Menezes and Dey (2019). For more information, see [16, 19].

Table 15
Data Set 1

Data Set 1
0.023, 0.032, 0.054, 0.069, 0.081, 0.094, 0.105, 0.127, 0.148, 0.169, 0.188, 0.216, 0.255, 0.277, 0.311, 0.361, 0.376, 0.395, 0.432, 0.463, 0.481, 0.519, 0.529, 0.567, 0.642, 0.674, 0.752, 0.823, 0.887, 0.926

Table 16 presents the parameter estimates obtained using the MLE, LS, WLS, AD and CVM methods for the first empirical dataset.

Table 16
Parameter Estimates for Data Set 1

		Model					
		MUE	Beta	KW	TL	UW	ULind
MLE	$\hat{\alpha}$	**2.1801**	0.9664	0.9687	-	1.3688	-
	$\hat{\beta}$	**1.6092**	1.6204	1.6083	1.109	0.5717	1.05
Bayes	$\hat{\alpha}$	**2.0021**	0.8848	0.9662	-	1.3751	-
	$\hat{\beta}$	**1.4994**	1.5581	1.5857	1.1081	0.5659	0.9478
LS	$\hat{\alpha}$	**2.1886**	1.6934	0.9675	-	1.4264	-
	$\hat{\beta}$	**1.6657**	0.9599	1.6982	1.1265	0.5596	1.9015
WLS	$\hat{\alpha}$	**2.1953**	1.6934	0.9675	-	1.4535	-
	$\hat{\beta}$	**1.6307**	0.9599	1.6982	1.165	0.5605	1.7906
AD	$\hat{\alpha}$	**2.1865**	1.6198	0.9691	-	1.4574	-
	$\hat{\beta}$	**1.6103**	0.9678	1.6857	1.487	0.5706	1.8079
CVM	$\hat{\alpha}$	**2.1886**	1.6934	0.9691	-	1.4264	-
	$\hat{\beta}$	**1.6657**	0.9592	1.6512	1.1265	0.5596	1.9015

The fit statistics for the first dataset, as computed by the MLE and Bayesian methods, are reported in Table 17.

Table 17
Goodness of Fit Statistics obtained by MLE and Bayesian Methods for Data Set 1

		Model					
		MUE	Beta	KW	TL	UW	ULind
MLE	*AIC*	**-3.1586**	-2.6101	-2.6220	-1.8077	-3.4345	181.3202
	BIC	**-0.3562**	0.1922	0.1803	0.9946	-0.6321	182.7214
	CAIC	**1.6437**	2.1922	2.1803	2.9946	1.3678	183.7214
	HQIC	**-2.2621**	-1.7136	-1.7255	-0.9112	-2.5379	181.7685
	$(\hat{\ell})$	**3.5793**	3.3050	3.3110	2.9038	3.7172	-89.6601
	KS	**0.0559**	0.0669	0.0649	0.0664	0.0576	0.2723
	(p)	**(0.9999)**	(0.9979)	(0.9987)	(0.9981)	(0.9998)	(0.0186)
	AD	**0.1248**	0.1702	0.1633	0.3277	0.1259	6.0694
	(p)	**(0.9997)**	(0.9966)	(0.9975)	(0.9153)	(0.9997)	(0.0009)
	CVM	**0.01526**	0.0220	0.0207	0.0332	0.01630	0.9799
	(p)	**(0.9996)**	(0.9953)	(0.9968)	(0.9153)	(0.9994)	(0.0024)
Bayes	*KS*	**0.0544**	0.0595	0.0701	0.0662	0.0595	6.8151
	(p)	**(0.9999)**	(0.9997)	(0.9961)	(0.9982)	(0.9997)	(0.0004)
	AD	**0.1098**	0.1633	0.1824	0.3275	0.1273	7.6545
	(p)	**(0.9999)**	(0.9974)	(0.9946)	(0.9946)	(0.9997)	(0.0010)
	CVM	**0.0110**	0.0144	0.0257	0.0331	0.0164	1.2778
	(p)	**(1.000)**	(0.9998)	(0.9893)	(0.9669)	(0.9994)	(0.0004)

As shown in Table 17, the proposed distribution provides a good fit to the test statistic measurements in Dataset 1. The comparison of the fitted distributions with the empirical distribution, based on the results obtained using the MLE and Bayesian methods, is illustrated in Figures 12 and 13 respectively.

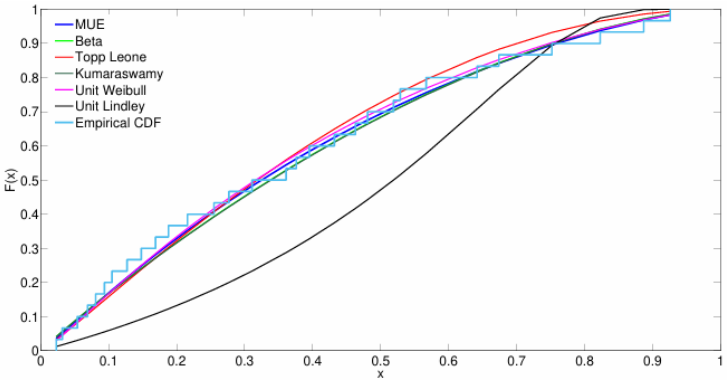


Fig. 12: Empirical and Comparative Distributions by MLE Estimation

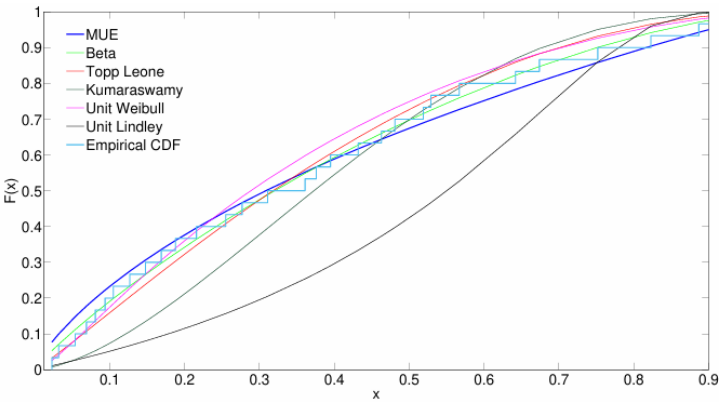


Fig. 13: Empirical and Comparative Distributions by Bayesian Estimation

Table 18 presents the fit statistics obtained using the LS and WLS methods for the first dataset.

Table 18
Goodness of Fit Statistics Using LS and WLS Methods for Data Set 1

Model	LS			WLS		
	KS (p)	*AD* (p)	*CVM* (p)	*KS* (p)	*AD* (p)	*CVM* (p)
MUE	**0.0550 (0.9999)**	**0.1267 (0.9997)**	**0.0144 (0.9998)**	**0.0562 (0.9999)**	**0.1274 (0.9997)**	**0.0157 (0.9996)**
BETA	0.0531 (1.0000)	0.1581 (0.9980)	0.0154 (0.9997)	0.0531 (1.0000)	0.1581 (0.9980)	0.0154 (0.9997)
KW	0.0543 (1.0000)	0.1599 (0.9978)	0.0156 (0.9996)	0.0647 (0.9987)	0.1661 (0.9971)	0.0201 (0.9974)
TL	0.0711 (0.9953)	0.3347 (0.9093)	0.0348 (0.9602)	0.0814 (0.9790)	0.3702 (00.8768)	0.0414 (0.9286)
UW	0.0653 (0.9985)	0.1783 (0.9953)	0.0233 (0.9936)	0.0723 (0.9943)	0.2205 (0.9838)	0.0285 (0.9823)
ULind	0.1281 (0.6616)	3.2609 (0.0204)	0.1182 (0.5050)	0.1256 (0.6842)	2.9927 (0.0279)	0.1296 (0.4602)

s

As shown in Table 18, the proposed distribution provides a good fit to the test statistic measurements in Dataset 1. The empirical distribution and the fitted distributions based on the LS and WLS methods are illustrated in Figures 14 and 15 respectively.

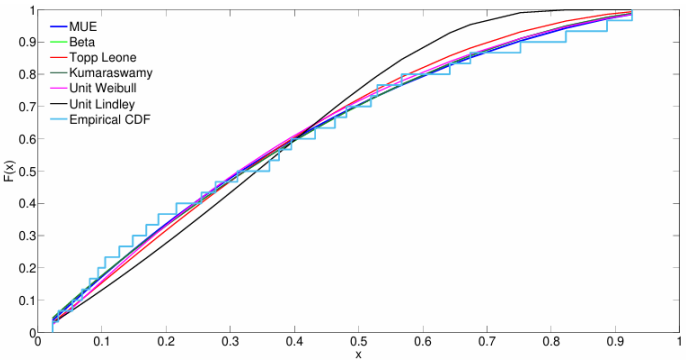


Fig. 14: Empirical and Comparative Distributions by LS Estimation

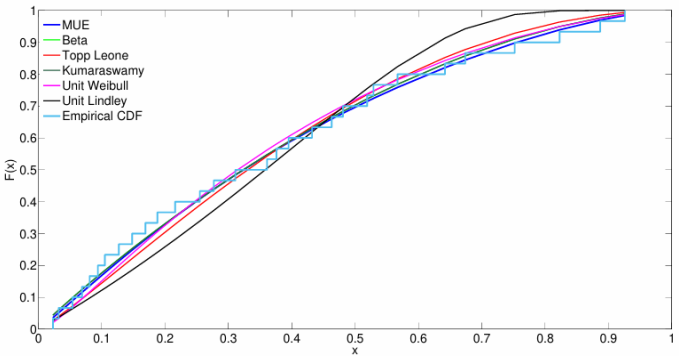


Fig. 15: Empirical and Comparative Distributions by WLSE Estimation

The fit statistics for the first dataset, calculated via the AD and CVM methods, are summarized in Table 19.

Table 19
Goodness of Fit Statistics Using AD and CVM Methods for Data Set 1

Model	AD			CVM		
	KS (p)	*AD* (p)	*CVM* (p)	*KS* (p)	*AD* (p)	*CVM* (p)
MUE	**0.0560 (0.9999)**	**0.1254 (0.9997)**	**0.0153 (0.9996)**	**0.0550 (0.9999)**	**0.1267 (0.9997)**	**0.0144 (0.9998)**
BETA	0.0531 (1.0000)	0.1581 (0.9980)	0.0154 (0.9997)	0.0531 (1.0000)	0.1581 (0.9980)	0.0154 (0.9997)
KW	0.0541 (1.0000)	0.1578 (0.9974)	0.1552 (0.9996)	0.0534 (1.0000)	0.1599 (0.9978)	0.0156 (0.9996)
TL	0.0874 (0.9792)	0.3265 (0.9124)	0.0374 (0.9662)	0.0711 (0.9953)	0.3347 (0.9093)	0.0348 0.9602
UW	0.0725 (0.9945)	0.2052 (0.9820)	0.0280 (0.9912)	0.0653 (0.9985)	0.1783 (0.9953)	0.0233 (0.9936)
ULind	0.1297 (0.6617)	3.0578 (0.0248)	0.1193 (0.4875)	0.1281 (0.6616)	3.2609 (0.0204)	0.1182 (0.5050)

As shown in Table 19, the proposed distribution provides a good fit to the test statistic measurements in Dataset 1. The empirical distribution and the fitted distributions based on the AD and CVM methods are illustrated in Figures 16 and 17 respectively.

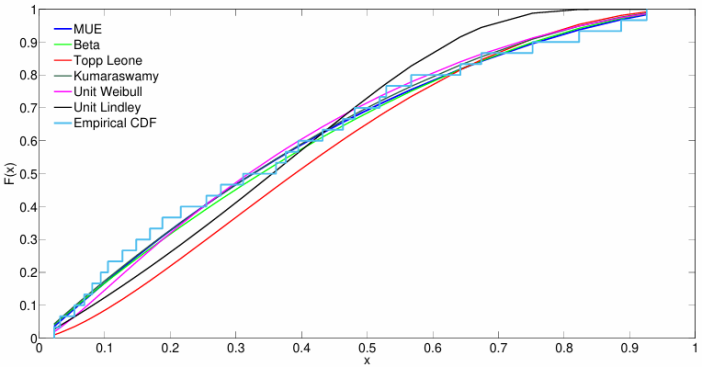


Fig. 16: Empirical and Comparative Distributions by AD Estimation

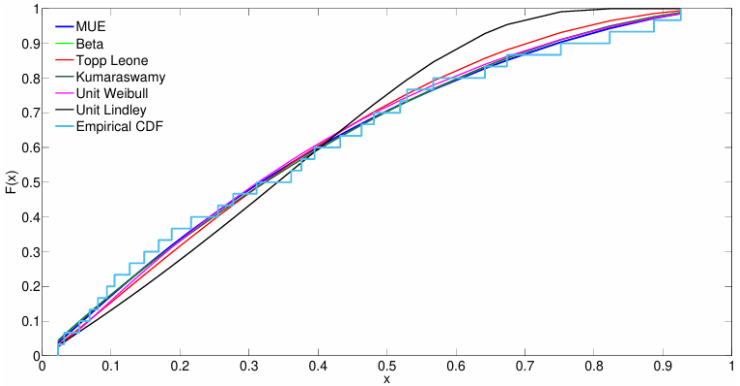


Fig. 17: Empirical and Comparative Distributions by CVM Estimation

5.2 Second Actual Data

The second dataset pertains to the novel coronavirus (COVID-19) pandemic in Turkey and comprises 25 observations of the daily recovery rate, calculated as the ratio of the number of daily recoveries to the total number of confirmed cases. This dataset was compiled by Gündüz and Mustafa (2020) and Mazucheli, Bapat and Menezes (2020) [9, 21], based on data provided by the Turkish Ministry of Health.

Table 20
Data Set 2

Data Set 2
0.007371007, 0.009456904, 0.011325639, 0.014962594, 0.017958761, 0.021238599, 0.022883926, 0.023134649, 0.032840311, 0.038494218, 0.043882583, 0.046380721, 0.048291739, 0.050659855, 0.051521402, 0.056836698, 0.060502844, 0.064816787, 0.073704904, 0.081767351, 0.095548098, 0.109884654, 0.126966197, 0.138762079, 0.14761486

Table 21 presents the parameter estimates obtained using the MLE, LS, WLS, AD and CVM methods for the second empirical dataset.

Table 21
Parameter Estimates for Data Set 2

		Model					
		MUE	Beta	KW	TL	UW	ULind
MLE	$\hat{\alpha}$	**2.8599**	1.7469	1.4158	-	0.0054	-
	$\hat{\beta}$	**37.9621**	29.5368	50.8868	0.3972	4.1559	17.2468
Bayes	$\hat{\alpha}$	**2.8601**	1.7476	1.416	-	4.1546	-
	$\hat{\beta}$	**37.9628**	29.5364	50.8632	0.4193	0.0063	17.1543
LS	$\hat{\alpha}$	**2.8553**	1.784	1.415	-	0.0046	-
	$\hat{\beta}$	**38.7149**	30.5768	50.8867	0.2855	4.3551	15.1221
WLS	$\hat{\alpha}$	**2.8616**	1.7615	1.4058	-	0.0051	-
	$\hat{\beta}$	**38.0592**	29.898	50.7784	0.2845	4.1921	14.904
AD	$\hat{\alpha}$	**2.8611**	1.7587	1.4575	-	0.0056	-
	$\hat{\beta}$	**38.0174**	30.2847	51.8473	0.2853	4.1875	16.4702
CVM	$\hat{\alpha}$	**2.8553**	1.784	1.4158	-	0.0046	-
	$\hat{\beta}$	**38.7149**	30.5768	50.8868	0.2855	4.3565	15.1221

The fit statistics for the second dataset, as computed by the MLE and Bayesian methods, are reported in Table 22.

Table 22
Goodness of fit statistics obtained by MLE and Bayesian methods for Data Set 2

		Model					
		MUE	Beta	KW	TL	UW	ULind
MLE	*AIC*	**-94.6689**	-94.8859	-94.6843	-59.4423	-93.1361	-76.7707
	BIC	**-92.2311**	-92.8859	-92.2465	-58.2234	-90.6984	-75.5518
	CAIC	**-90.2311**	-90.4482	-90.2465	-57.2234	-88.6984	-74.5518
	HQIC	**-93.9928**	-94.2098	-94.0082	-59.1042	-92.4600	-76.4326
	$(\hat{\ell})$	**49.3344**	49.44298	49.3421	30.7211	48.5608	39.3853
	KS (p)	**0.1020** **(0.9337)**	0.1024 (0.9315)	0.1016 (0.9335)	0.4025 (0.0003)	0.1360 (0.6936)	0.1272 (0.7671)
	AD (p)	**0.2110** **(0.9872)**	0.2041 (0.9893)	0.2125 (0.9867)	5.3408 (0.0020)	0.3582 (0.8879)	0.6149 (0.6323)
	CVM (p)	**0.0275** **(0.9854)**	0.0271 (0.9863)	0.0277 (0.9847)	1.0832 (0.0013)	0.0604 (0.8157)	0.0913 (0.6329)
Bayes	*KS* (p)	**0.1021** **(0.9337)**	0.1027 (0.9303)	0.1017 (0.9347)	0.4194 (0.0001)	0.1134 (0.8691)	0.1304 (0.7409)
	AD (p)	**0.2111** **(0.9871)**	0.2044 (0.9892)	0.2127 (0.9866)	5.7777 (0.0012)	0.4819 (0.7637)	0.6025 (0.6126)
	CVM (p)	**0.0275** **(0.9853)**	0.0271 (0.9998)	0.0278 (0.9845)	1.2003 (0.0006)	0.0509 (0.8749)	0.0885 (0.6079)

As shown in Table 22, our proposed distribution provides a good fit to Dataset 2 based on the goodness-of-fit statistics. The estimated distribution curves obtained using the MLE and Bayesian methods are compared with the empirical distribution in Figures 18 and 19 respectively.

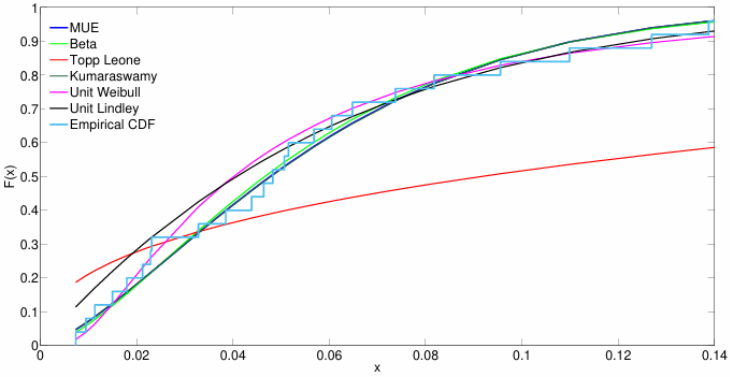


Fig. 18: Empirical and Comparative Distributions by MLE Estimation

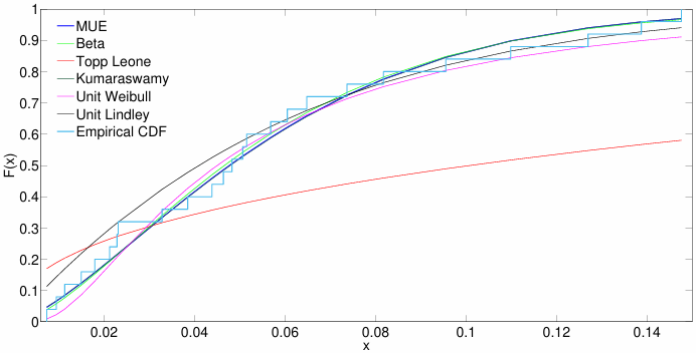


Fig. 19: Empirical and Comparative Distributions by Bayesian Estimation

Table 23 presents the fit statistics obtained using the LS and WLS methods for the second dataset.

Table 23
Goodness of Fit Statistics Using LS and WLS Methods Data Set 2

Model	LS			WLS		
	KS (p)	*AD* (p)	*CVM* (p)	*KS* (p)	*AD* (p)	*CVM* (p)
MUE	**0.0964 (0.9566)**	**0.2127 (0.9866)**	**0.0260 (0.9888)**	**0.1020 (0.9333)**	**0.2115 (0.9870)**	**0.0275 (0.9853)**
BETA	0.1025 (0.9312)	0.2203 (0.9830)	0.0286 (0.9823)	0.1029 (0.9292)	0.2095 (0.9877)	0.0276 (0.9850)
KW	0.0745 (0.9358)	0.2124 (0.9867)	0.0277 (0.9848)	0.1015 (0.9358)	0.2124 (0.9867)	0.0277 (0.9848)
TL	0.3094 (0.0126)	4.4420 (0.0055)	0.8128 (0.0062)	0.3085 (0.0130)	4.4249 (0.0055)	0.8139 (0.0062)
UW	0.1159 (0.8522)	0.4192 (0.8281)	0.0526 (0.8645)	0.1427 (0.6366)	0.3810 (0.8661)	0.0677 (0.7701)
ULind	0.0999 (0.9427)	0.5150 (0.7298)	0.0645 (0.7896)	0.0990 (0.9466)	0.5294 (0.7153)	0.0669 (0.7748)

As shown in Table 23, our proposed distribution provides a good fit to Dataset 2 based on the goodness-of-fit statistics. The estimated distribution curves obtained using the LS and WLS methods are compared with the empirical distribution in Figures 20 and 21 respectively.

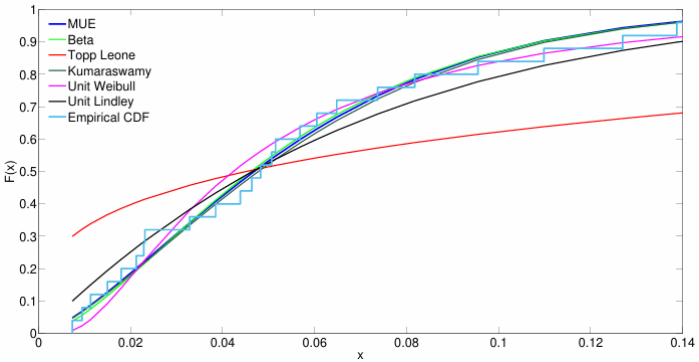


Fig. 20: Empirical and Comparative Distributions by LS Estimation

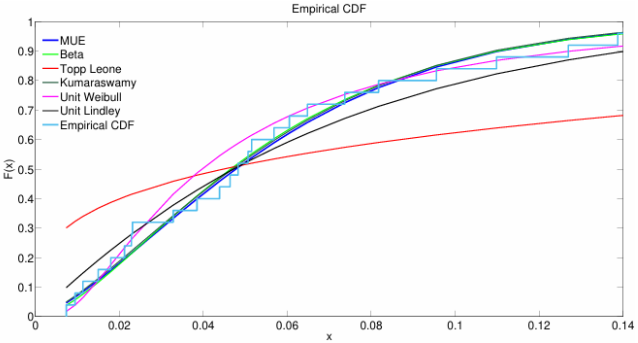


Fig. 21: Empirical and Comparative Distributions by WLSE Estimation

The fit statistics for the second dataset, calculated via the AD and CVM methods, are summarized in Table 24.

Table 24
Goodness of Fit Statistics Using AD and CVM Methods Data Set 2

Model	AD			CVM		
	KS (p)	*AD* (p)	*CVM* (p)	*KS* (p)	*AD* (p)	*CVM* (p)
MUE	**0.0927 (0.9457)**	**0.2117 (0.9871)**	**0.0270 (0.9853)**	**0.0964 (0.9566)**	**0.2127 (0.9866)**	**0.0260 (0.9888)**
BETA	0.1041 (0.9298)	0.2099 (0.9870)	0.0268 (0.9849)	0.1025 (0.9312)	0.2203 (0.9830)	0.0286 (0.9823)
KW	0.1015 (0.9627)	0.2098 (0.9790)	0.0249 (0.9848)	0.0946 (0.9630)	0.2088 (0.9879)	0.0255 (0.9899)
TL	0.3092 (0.0126)	4.4250 (0.0055)	0.8135 (0.0062)	0.3094 (0.0126)	4.4203 (0.0055)	0.8128 (0.0062)
UW	0.1587 (0.6052)	0.3978 (0.8678)	0.0568 (0.7917)	0.1159 (0.8522)	0.4192 (0.8274)	0.0526 (0.8642)
ULind	0.0999 (0.9448)	0.5163 (0.7145)	0.0668 (0.7791)	0.0999 (0.9427)	0.5150 (0.7298)	0.0645 (0.7896)

As shown in Table 24, our proposed distribution provides a good fit to Dataset 2 based on the goodness-of-fit statistics. The estimated distribution curves obtained using the AD and CVM methods are compared with the empirical distribution in Figures 22 and 23 respectively.

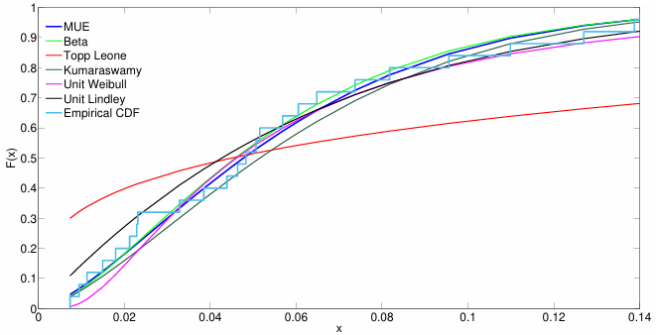


Fig. 22: Empirical and Comparative Distributions by AD Estimation

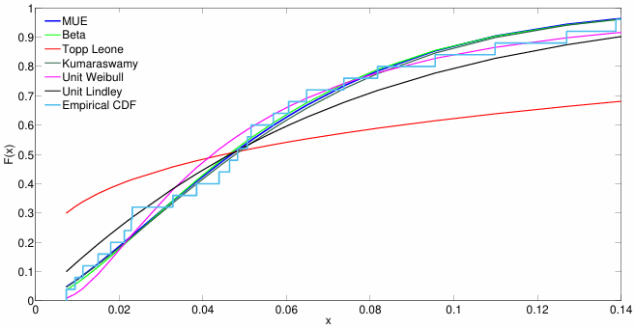


Fig. 23: Empirical and Comparative Distributions by CVM Estimation

5.3 Third Actual Data

The third dataset comprises 38 observations of the long-term unemployment rate. It was analyzed by Karakaya and Korkmaz (2022) [10], using data obtained from the OECD Statistics portal (https://stats.oecd.org/).

Table 25
Data Set 3

Data Set 3
0.0131, 0.0184, 0.0354, 0.0077, 0.0079, 0.0104, 0.0131, 0.0192, 0.0213, 0.0400, 0.0157, 0.1565, 0.0172, 0.0026, 0.0323, 0.0049, 0.0659, 0.0103, 0.0005, 0.0335, 0.0269, 0.0235, 0.0007, 0.0197, 0.0074, 0.0066, 0.0152, 0.0443, 0.0478, 0.0317, 0.0766, 0.0112, 0.0182, 0.0239, 0.0113, 0.0066, 0.0159, 0.1646

Table 26 presents the parameter estimates obtained using the MLE, LS, WLS, AD and CVM methods for the third empirical dataset.

Table 26
Parameter Estimates for Data Set 3

		Model					
		MUE	Beta	KW	TL	UW	ULind
MLE	$\hat{\alpha}$	**1.9181**	0.9798	0.9406	-	3.6134	-
	$\hat{\beta}$	**21.892**	33.2394	28.1414	0.2898	0.0041	33.5
Bayes	$\hat{\alpha}$	**1.9057**	0.9647	0.9515	-	3.6214	-
	$\hat{\beta}$	**22.0541**	34.2572	28.9954	0.2024	0.0045	34.2414
LS	$\hat{\alpha}$	**1.9032**	0.9628	0.9433	-	3.6503	-
	$\hat{\beta}$	**22.0421**	35.4181	29.7444	0.2087	0.004	37.8999
WLS	$\hat{\alpha}$	**1.8825**	1.0046	0.9706	-	3.7528	-
	$\hat{\beta}$	**23.9357**	33.1888	29.2581	0.2081	0.004	33.5515
AD	$\hat{\alpha}$	**1.9201**	0.9802	0.9446	-	3.6247	-
	$\hat{\beta}$	**21.9074**	33.2564	28.9532	0.2862	0.0041	33.9571
CVM	$\hat{\alpha}$	**2.4567**	0.9628	0.9433	-	3.6503	-
	$\hat{\beta}$	**21.2528**	35.4181	29.7444	0.2086	0.004	37.8999

The fit statistics for the third dataset, as computed by the MLE and Bayesian methods, are reported in Table 27.

Table 27
Goodness of Fit Statistics obtained by MLE and Bayesian Methods for Data Set 3

		Model					
		MUE	Beta	KW	TL	UW	ULind
MLE	*AIC*	**-190.7878**	-190.0578	-190.2959	-140.534	-188.4059	-178.1922
	BIC	**-187.5126**	-186.7826	-187.0207	-138.8964	-185.1307	-176.5546
	CAIC	**-185.5126**	-184.7826	-185.0207	-137.8964	-183.1307	-175.5546
	HQIC	**-185.5126**	-188.8925	-189.1306	-139.9514	-187.2406	-177.6096
	$(\hat{\ell})$	**97.3939**	97.0288	97.1476	71.2670	96.2029	90.0961
	KS (p)	**0.1166** **(0.6792)**	0.1223 (0.6200)	0.1169 (0.6764)	0.3918 (0.0000)	0.1374 (0.4649)	0.1350 (0.4928)
	AD (p)	**0.7842** **(0.4916)**	0.8569 (0.4409)	0.8367 (0.4545)	8.4408 (0.0000)	1.0251 (0.3439)	0.9810 (0.3668)
	CVM (p)	**0.1228** **(0.4859)**	0.1365 (0.4348)	0.1317 (0.4521)	1.7433 (0.0000)	0.1584 (0.3661)	0.1607 (0.3596)
Bayes	*KS* (p)	**0.1144** **(0.7020)**	0.1192 (0.6523)	0.1242 (0.6008)	0.5090 (0.0000)	0.1426 (0.4216)	0.1262 (0.5822)
	AD (p)	**0.7630** **(0.5075)**	0.8607 (0.4385)	0.9067 (0.4094)	14.6690 (0.0000)	1.0525 (0.3304)	0.8271 (0.4610)
	CVM (p)	**0.1141** **(0.5219)**	0.1368 (0.4337)	0.1519 (0.3851)	3.2425 (0.0000)	0.1664 (0.3441)	0.1225 (0.4871)

As shown in Table 27, based on the goodness-of-fit criteria, the MUE distribution provides the best fit for Dataset 3. The comparison between the fitted distributions and the empirical distribution, obtained using the MLE method, is illustrated in Figures 24 and 25 respectively.

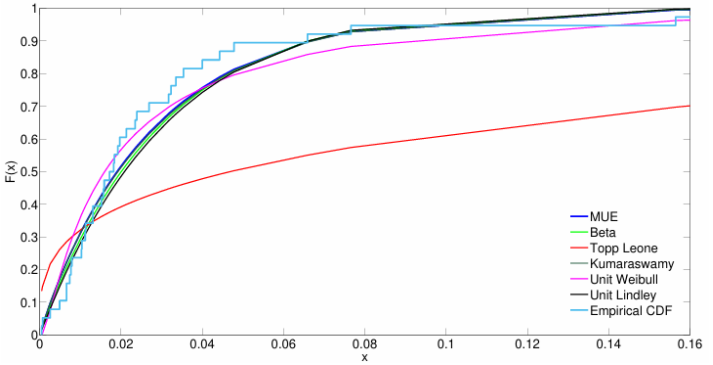


Fig. 24: Empirical and Comparative Distributions by MLE Estimation

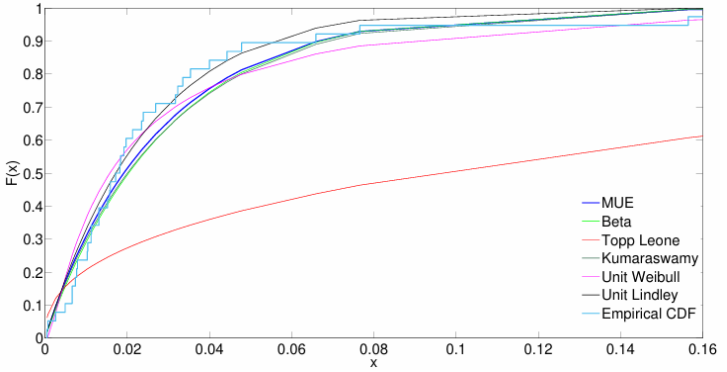


Fig. 25: Empirical and Comparative Distributions by Bayesian Estimation

Table 28 presents the fit statistics obtained using the LS and WLS methods for the third dataset.

Table 28
Goodness of Fit Statistics Using LS and WLS Methods Data Set 3

Model	LS			WLS		
	KS (p)	*AD* (p)	*CVM* (p)	*KS* (p)	*AD* (p)	*CVM* (p)
MUÜ	**0.0926 (0.8998)**	**0.6922 (0.5643)**	**0.0528 (0.8619)**	**0.11629 (0.6830)**	**0.8429 (0.4502)**	**0.0923 (0.6261)**
BETA	0.1191 (0.6533)	0.7478 (0.5192)	0.0987 (0.5935)	0.1344 (0.4984)	0.9489 (0.3846)	0.1615 (0.3575)
KW	0.1252 (0.5902)	0.7845 (0.4914)	0.1134 (0.5248)	0.1377 (0.4671)	1.0087 (0.3522)	0.1788 (0.3135)
TL	0.3016 (0.0019)	7.0081 (0.0003)	1.3331 (0.0003)	0.3070 (0.0015)	6.9704 (0.0003)	1.3293 (0.0003)
UW	0.1220 (0.6234)	0.9576 (0.3797)	0.1373 (0.4311)	0.1265 (0.5765)	1.096 (0.3102)	0.1461 (0.4029)
ULind	0.1223 (0.7240)	0.7510 (0.5167)	0.0843 (0.6696)	0.1344 (0.4982)	0.9753 (0.3699)	0.1592 (0.3639)

As shown in Table 28, based on the goodness-of-fit criteria, the MUE distribution provides the best fit for Dataset 3. The comparisons between the fitted distributions and the empirical distribution, obtained using the LS and WLS methods, are illustrated in Figures 26 and 27 respectively.

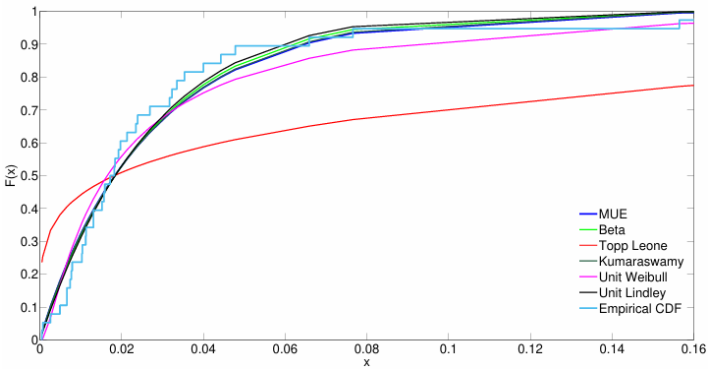


Fig. 26: Empirical and Comparative Distributions by LS Estimation

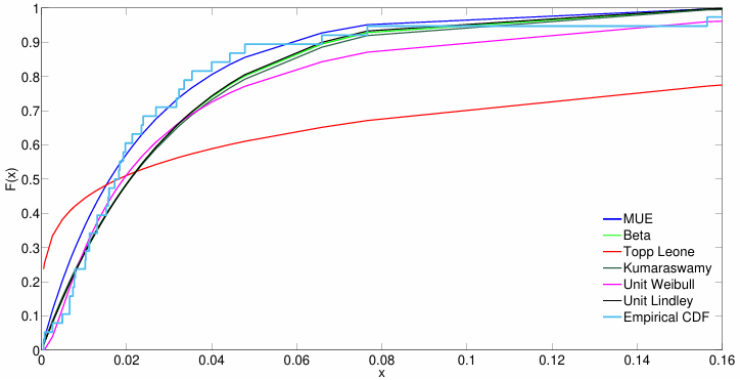


Fig. 27: Empirical and Comparative Distributions by WLSE Estimation

The fit statistics for the third dataset, calculated via the AD and CVM methods, are summarized in Table 29.

Table 29
Goodness of Fit Statistics Using AD and CVM Methods Data Set 3

Model	AD			CVM		
	KS (p)	*AD* (p)	*CVM* (p)	*KS* (p)	*AD* (p)	*CVM* (p)
MUÜ	**0.01032 (0.7034)**	**0.8128 (0.4409)**	**0.0931 (0.6124)**	**0.0933 (0.8949)**	**0.6613 (0.5909)**	**0.0536 (0.8569)**
BETA	0.1341 (0.4984)	0.6341 (0.3941)	0.1597 (0.3789)	0.1191 (0.6533)	0.7478 (0.5192)	0.0987 (0.5935)
KW	0.1258 (0.6011)	0.3578 (0.3220)	0.1250 (0.5205)	0.1252 (0.5902)	0.7845 (0.4914)	0.1134 (0.5248)
TL	0.3042 (0.0017)	7.0087 (0.0003)	1.3258 (0.0003)	0.3016 (0.0019)	7.0081 (0.0003)	1.1331 (0.0003)
UW	0.1232 (0.6008)	0.9914 (0.3043)	0.1462 (0.4054)	0.1220 (0.6234)	0.9576 (0.3797)	0.1376 (0.4311)
ULind	0.1268 (0.5292)	0.8647 (0.3949)	0.1278 (0.3374)	0.1223 (0.7240)	0.7510 (0.5167)	0.0843 (0.6696)

As shown in Table 29, based on the goodness-of-fit criteria, the MUE distribution provides the best fit for Dataset 3. The comparisons between the fitted distributions and the empirical distribution, obtained using the AD and CVM methods, are illustrated in Figures 28 and 29 respectively.

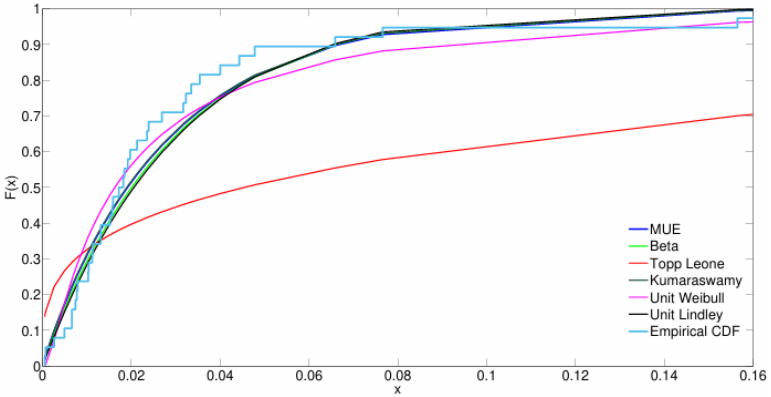


Fig. 28: Empirical and Comparative Distributions by AD Estimation

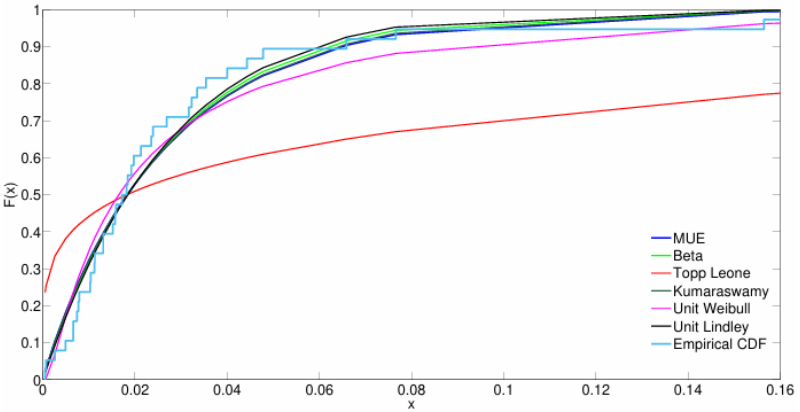


Fig. 29: Empirical and Comparative Distributions by CVM Estimation

6. CONCLUSION

In this study, a new two-parameter distribution called the Modified Unit Exponential (MUE) distribution has been proposed as a flexible alternative to existing unit distributions. Various statistical properties of the distribution—such as the cumulative distribution function (CDF), probability density function (PDF), quantile function, survival and hazard functions and moments—have been derived and discussed in detail.

To estimate the model parameters, five different methods were considered: maximum likelihood estimation (MLE), least squares (LS), weighted least squares (WLS), Anderson–Darling (AD) and Cramer–von Mises (CVM) estimation. The performance of these estimators was evaluated through a comprehensive Monte Carlo simulation study.

The practical utility of the proposed model was illustrated through three real-data applications. In terms of goodness-of-fit criteria, the MUE distribution was compared with several well-known unit distributions, including the Beta, Kumaraswamy, Topp–Leone,

Unit Weibull and Unit Lindley distributions. The results demonstrate that the MUE model provides a competitive—and in many cases, superior—fit for modeling proportional data.

Thanks to its analytical tractability and strong empirical performance, the MUE distribution presents itself as a promising tool for both theoretical exploration and practical implementation. Overall, this study contributes to the growing body of research on flexible bounded distributions capable of modeling various data shapes within the unit interval.

Future work may focus on multivariate or regression-based extensions of the MUE distribution, with potential applications in fields such as finance, biomedical sciences and environmental risk modeling.

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