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ON RAINBOW ANTIMAGIC COLORING OF WHEEL-RELATED GRAPHS

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ABSTRACT

For an undirected simple connected $G = (V(G), E(G))$ with p-vertices and q-edges, a bijective mapping $\varsigma : V(G) \to \{1,2,3,\cdots,|V(G)|\}$ assigns antimagic labelling to the vertices of G with edge weights defined by $W_\varsigma(xy) = \varsigma(x) + \varsigma(y)$, in order to form a rainbow path. The Rainbow Antimagic Connection or rac number represents the minimum number of colors required for a graph G to exhibit rainbow connectivity through a rainbow antimagic labeling of G. This paper investigates the study of rac number of some distinctive wheel-related graphs such as, sunlet graph SL_p, helm graph H_p, closed helm graph CH_p and web graph WB_p.

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KEYWORDS

Rainbow antimagic coloring, Wheel-related graphs, Helm graph, Sunlet graph, Closed Helm graph, Web graph.

1. INTRODUCTION

In this paper, we consider a simple, finite, connected, and undirected graph represented as G, with $V(G)$ and $E(G)$ denoting its vertices and edges respectively. The cardinality of vertices and edges in G are represented by p and q. Additionally, we define the minimum and maximum degree of G as $\delta(G)$ and $\Delta(G)$ respectively. The rainbow antimagic connection number of a graph G is a graph parameter defined as the minimum number of colors required to color the edges of the graph such that no two edges of the same color meet at a common vertex, and for every pair of vertices there exists atleast one rainbow path. A rainbow path in a graph is a path in which all the edges have distint colors. The concept of rainbow path is used in the study of rainbow connectivity of graphs. In the

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context of graph coloring, the main goal is to achieve the minimum number of colors required to make sure that every pair of vertices in the graph is connected by a rainbow path.

Chartrand et al. (2016) introduced the concept of rainbow coloring and the graph definitions used in this study are based on Chartrand et al. (2016). Our research focuses on the integration of the concepts of rainbow coloring and antimagic labeling of graphs, referred to as rainbow antimagic coloring. This paper examines the rainbow antimagic coloring of wheel-related graphs such as sunlet, helm, closed helm and web graphs and also investigates their lower bound. We have reviewed some of the fundamental definitions and results from both fields to establish a comprehensive understanding.

The minimum number of colors needed to make a graph G rainbow-connected is known as the rainbow connection number of G, denoted by $rc(G)$. A bijective function ς maps each vertex of $V(G)$ to the cardinality of vertices of G. For any vertices $u, v \in G$, each edge $uv \in E(G)$ is assigned a weight, $W_\varsigma(uv) = \varsigma(u) + \varsigma(v)$. This bijection ς is called a rainbow antimagic vertex labeling if no two edges along a rainbow $u - v$ path have the same weight under ς i.e, $W_\varsigma(uv) \neq W_\varsigma(vw)$ for $u, v, w \in V(G)$. The rainbow antimagic connection number of G, denoted by $rac(G)$, is the minimum number of colors needed to achieve this rainbow coloring through antimagic vertex labeling of G (Dafik et al, 2021).

Arumugam et al. (2017) introduced the concept of pairing graph labeling and coloring, which inspired Dafik et al. (2021) to propose the concept of rainbow antimagic coloring, combining the principles of antimagic labeling and rainbow coloring. The lower bounds of the rac number for any connected graph G was determined by Septory et al. (2021). Further developments in the field of rainbow antimagic coloring can be found in works such as (Dafik et al., 2021; Budi et al., 2021; Joedo et al., 2022; Septory et al., 2021 and Sulistiyono et al., 2020). Recent studies have focused on exploring some special graphs including the friendship graph, firecraker graph, ladder and triangular ladder graphs, book and triangular book graphs, lemon graph, jahangir graph, broom graph, and the comb product of friendship graph and tree.

2. PRELIMINARIES

Definition 1: *The p-sunlet graph SL_p is formed by attaching a pendant edge to each node of a cycle C_p, resulting in a graph with a sun-like structure, where the cycle forms the center of the sun and the pendant edges resemble the sun rays. It has a maximum degree 3 and a minimum degree 1.*

Definition 2: *The helm graph H_p is formed by attaching a pendant edge to each node of the cycle in a p-wheel graph. It has a hub vertex with maximum degree p; p vertices of degree 3, and p vertices of minimum degree 1.*

Definition 3: *A closed helm CH_p is formed by connecting each pendant vertex of a helm H_p to form a cycle. It consists of three types of vertices, namely the hub vertex with maximum degree p; p vertices of degree 4, and p vertices of minimum degree 3.*

Definition 4: *The web graph WB_p is formed by attaching a pendant edge to each node of a closed helm graph CH_p. It has a hub vertex with maximum degree p; p vertices of degree 3; p vertices of degree 4, and p vertices of minimum degree 1.*

Preposition 1: *For any connected graph G, $rac(G) \geq rc(G)$ (Septory et al., 2021).*

Lemma 1: *For a connected graph G, $rac(G) \geq max\{rc(G), \Delta(G)\}$ where $rc(G)$ denotes the rainbow connection number of G and $\Delta(G)$ denotes the maximum degree of G (Septory et al., 2021).*

By Proposition 1, we have $rac(G) \geq rc(G)$. By the property of antimagic labelling, $rac(G) \geq \Delta(G)$, then altogether we have $rac(G) \geq max\{rc(G), \Delta(G)\}$

Theorem 1: *Let P_p be a path of order p with $p \geq 3$. The rac number of P_p is $p-1$ (Sulistiyono et al., 2020).*

Theorem 2: *The rainbow antimagic connection number of a cycle C_p for $p \geq 3$ is,*

$$rac(C_p) = \begin{cases} 3, & for\ p = 4 \\ \left\lceil \frac{p}{2} \right\rceil, & for\ p \equiv 1,2(mod4) \end{cases}$$

$$\lceil \tfrac{p}{2} \rceil \leq rac(C_p) \leq \lceil \tfrac{p}{2} \rceil + 1\ for\ p \equiv 0,3(mod4); p \neq 4$$ *(Dafik et al., 2021).*

Theorem 3: *The rainbow antimagic connection number of a wheel graph W_p for $p \geq 3$ is,*

$$rac(W_p) = \begin{cases} 5, & for\ p = 3,4 \\ p, & for\ p \geq 5 \end{cases}$$ *(Dafik et al., 2021).*

3. RAINBOW ANTIMAGIC COLORING OF WHEEL-RELATED GRAPHS

Theorem 4: *For any positive integer, the rainbow antimagic connection number of a $p \geq 3$, sunlet graph is,*

$$rac(SL_p) = \begin{cases} \frac{3p}{2}, & for\ p \equiv 0,2(mod4) \\ \frac{5(p-1)}{4} + 3, & for\ p \equiv 1(mod4) \\ \frac{5(p+1)}{4}, & for\ p \equiv 3(mod4) \end{cases}$$

Proof: The *p-sunlet* graph can be constructed by connecting p pendant edges to a cycle graph C_p. The vertex set of sunlet graph be $V(SL_p) = \{x_t, y_t : 1 \leq t \leq p\}$ assosiated with the edge set $E(SL_p) = \{x_t x_{t+1} : 1 \leq t \leq p-1\} \cup \{x_p x_1\} \cup \{x_t y_t : 1 \leq t \leq p\}$. The cardinality of vertices and and edges are $|V(SL_p)| = |E(SL_p)| = 2p$. To prove the theorem, we have considered the following three cases.

Case 1: For $p \equiv 0,2(mod4)$

In order to establish the rainbow antimagic connection number, we must confirm both the lower and upper bounds. From Theorem-2, it is evident that for even cycles C_p, the rac number varies between $\lceil \tfrac{p}{2} \rceil \leq rac(C_p) \leq \lceil \tfrac{p}{2} \rceil + 1$. Since there is one copy of cycle

C_p with p-pendent edges and considering the minimum rac number $\lceil \frac{p}{2} \rceil$, we get the lower bound as $\lceil \frac{p}{2} \rceil + p$. Thus $rac(SL_p) \geq \lceil \frac{p}{2} \rceil + p$. Now to establish the upper bound we define an antimagic labelling $\varsigma_1 : V(SL_p) \to \{1, 2, 3, \cdots, 2p\}$ given by:

$$\varsigma_1(x_t) = \begin{cases} 2t-1, & for\ p \equiv 0(mod4); t = 1, 3, 5, \cdots, \frac{p-2}{2} \\ & for\ p \equiv 2(mod4); t = 1, 3, 5, \cdots, \frac{p}{2} \\ 2t, & for\ p \equiv 0(mod4); t = 2, 4, 6, \cdots, \frac{p}{2} \\ & for\ p \equiv 2(mod4); t = 2, 4, 6, \cdots, \frac{p-2}{2} \\ 2t-p, & for\ p \equiv 0(mod4); t = \frac{p+2}{2}, \frac{p+6}{2}, \frac{p+10}{2}, \cdots, p-1 \\ & for\ p \equiv 2(mod4); t = \frac{p+2}{2}, \frac{p+6}{2}, \frac{p+10}{2}, \cdots, p \\ 2t-p-1, & for\ p \equiv 0(mod4); t = \frac{p+4}{2}, \frac{p+8}{2}, \frac{p+12}{2}, \cdots, p \\ & for\ p \equiv 2(mod4); t = \frac{p+4}{2}, \frac{p+8}{2}, \frac{p+12}{2}, \cdots, p-1 \end{cases}$$

$$\varsigma_1(y_t) = \begin{cases} 2t+p-1, & for\ p \equiv 0(mod4); t = 1, 3, 5, \cdots, \frac{p-2}{2} \\ & for\ p \equiv 2(mod4); t = 1, 3, 5, \cdots, \frac{p}{2} \\ 2t+p, & for\ p \equiv 0(mod4); t = 2, 4, 6, \cdots, \frac{p}{2} \\ & for\ p \equiv 2(mod4); t = 2, 4, 6, \cdots, \frac{p-2}{2} \\ 2t, & for\ p \equiv 0(mod4); t = \frac{p+2}{2}, \frac{p+6}{2}, \frac{p+10}{2}, \cdots, p-1 \\ & for\ p \equiv 2(mod4); t = \frac{p+2}{2}, \frac{p+6}{2}, \frac{p+10}{2}, \cdots, p \\ 2t-1, & for\ p \equiv 0(mod4); t = \frac{p+4}{2}, \frac{p+8}{2}, \frac{p+12}{2}, \cdots, p \\ & for\ p \equiv 2(mod4); t = \frac{p+4}{2}, \frac{p+8}{2}, \frac{p+12}{2}, \cdots, p-1 \end{cases}$$

Based on the above antimagic labelling, we can develop the following edge weight sets for $p \equiv 0(mod4)$:

$W_{\varsigma_1}^1(x_t x_{t+1}) = W_{\varsigma_1}^1\left(x_{\frac{p}{2}+t} x_{\frac{p+2}{2}+t}\right) = 4t+1$ for $1 \leq t \leq \frac{p-2}{2}$

$W_{\varsigma_1}^2(x_p x_1) = p\ \forall p \in N$

$W_{\varsigma_1}^3\left(x_{\frac{p}{2}} x_{\frac{p}{2}+1}\right) = p+2\ \forall p \in N$

$W_{\varsigma_1}^4(x_t y_t) = 4t+p-2$ for $t = 1, 3, 5, \cdots, \frac{p-2}{2}$

$W_{\varsigma_1}^5(x_t y_t) = 4t+p$ for $t = 2, 4, 6, \cdots, \frac{p}{2}$

$W_{\varsigma_1}^6(x_t y_t) = 4t-p$ for $t = \frac{p+2}{2}, \frac{p+6}{2}, \frac{p+10}{2}, \cdots, p-1$

$W_{\varsigma_1}^7(x_t y_t) = 4t-p-2$ for $t = \frac{p+4}{2}, \frac{p+8}{2}, \frac{p+12}{2}, \cdots, p$

By clearly examining the above sets, we can determine the exact count of distinct edge weights. Furthermore, $W_{\varsigma_1}^1$, $W_{\varsigma_1}^4$, $W_{\varsigma_1}^5$, $W_{\varsigma_1}^6$ and $W_{\varsigma_1}^7$ compose an arithmetic sequence of the form:

$$W_{\varsigma_1}^1 = \{5, 9, 13, \cdots, 2p-3\}$$

$$W_{\varsigma_1}^4 = \{p+2, p+10, p+18, \cdots, 3p-6\}$$

$$W_{\varsigma_1}^5 = \{p+8, p+16, p+24, \cdots, 3p\}$$

$$W_{\varsigma_1}^6 = \{p+4, p+12, p+20, \cdots, 3p-4\}$$

$$W_{\varsigma_1}^7 = \{p+6, p+14, p+22, \cdots, 3p-2\}$$

Utilizing the *N - term* formula for an arithmetic sequence, we can effortlessly calculate the following:

Weight Function	$\boldsymbol{U_N = a + (N-1)d}$	$\boldsymbol{N}$
$W_{\varsigma_1}^1$	$2p-3 = 5 + (N-1)(4)$	$\frac{p-2}{2}$
$W_{\varsigma_1}^4$	$3p-6 = p+2 + (N-1)(8)$	$\frac{p}{4}$
$W_{\varsigma_1}^5$	$3p = p+8 + (N-1)(8)$	$\frac{p}{4}$
$W_{\varsigma_1}^6$	$3p-4 = p+4 + (N-1)(8)$	$\frac{p}{4}$
$W_{\varsigma_1}^7$	$3p-2 = p+6 + (N-1)(8)$	$\frac{p}{4}$

Notably, the sets $W_{\varsigma_1}^2$ and $W_{\varsigma_1}^3$ exhibits a cardinality of 1. As the edge weight of $W_{\varsigma_1}^3$ has already been included in $W_{\varsigma_1}^4$, it is deemed superfluous and, therefore, it is excluded. Therefore, $\left|W_{\varsigma_1}^2\right| = 1$ is considered. Consequently, the total number of unique edge weights is calculated as: $\frac{p-2}{2} + \frac{p}{4} + \frac{p}{4} + \frac{p}{4} + \frac{p}{4} + 1 = \frac{3p}{4}$. In effect with this, $\varsigma_1 : V(SL_p) \to \{1, 2, 3, \cdots, 2p\}$ induces a rainbow antimagic coloring with $\frac{3p}{2}$ colors.

Next we develop the following edge weight sets for $p \equiv 2(mod4)$:

$$W_{\varsigma_1}^8(x_t x_{t+1}) = W_{\varsigma_1}^1\left(x_{\frac{p}{2}+t} x_{\frac{p+2}{2}+t}\right) = 4t+1 \text{ for } 1 \le t \le \frac{p-2}{2}$$

$$W_{\varsigma_1}^9(x_p x_1) = W_{\varsigma_1}^2\left(x_{\frac{p}{2}} x_{\frac{p}{2}+1}\right) = p+1 \; \forall p \in N$$

$$W_{\varsigma_1}^{10}(x_t y_t) = 4t + p - 2 \text{ for } t = 1, 3, 5, \cdots, \frac{p}{2}$$

$$W_{\varsigma_1}^{11}(x_t y_t) = 4t + p \text{ for } t = 2, 4, 6, \cdots, \frac{p-2}{2}$$

$$W_{\varsigma_1}^{12}(x_t y_t) = 4t - p \text{ for } t = \frac{p+2}{2}, \frac{p+6}{2}, \frac{p+10}{2}, \cdots, p$$

$$W_{\varsigma_1}^{13}(x_t y_t) = 4t - p - 2 \text{ for } t = \frac{p+4}{2}, \frac{p+8}{2}, \frac{p+12}{2}, \cdots, p-1$$

By clearly examining the above sets, we can determine the exact count of distinct edge weights. Furthermore, $W_{\varsigma_1}^8$, $W_{\varsigma_1}^{10}$, $W_{\varsigma_1}^{11}$, $W_{\varsigma_1}^{12}$, $W_{\varsigma_1}^{13}$ compose an arithmetic sequence of the form:

$$W_{\varsigma_1}^8 = \{5, 9, 13, \cdots, 2p-3\}$$
$$W_{\varsigma_1}^{10} = \{p+2, p+10, p+18, \cdots, 3p-2\}$$
$$W_{\varsigma_1}^{11} = \{p+8, p+16, p+24, \cdots, 3p-4\}$$
$$W_{\varsigma_1}^{12} = \{p+4, p+12, p+20, \cdots, 3p\}$$
$$W_{\varsigma_1}^{13} = \{p+6, p+14, p+22, \cdots, 3p-6\}$$

Utilizing the *N - term* formula for an arithmetic sequence, we can effortlessly calculate the following:

Weight Function	$\boldsymbol{U_N = a + (N-1)d}$	$\boldsymbol{N}$
$W_{\varsigma_1}^8$	$2p-3 = 5 + (N-1)(4)$	$\frac{p-2}{2}$
$W_{\varsigma_1}^{10}$	$3p-2 = p+2 + (N-1)(8)$	$\frac{p+2}{4}$
$W_{\varsigma_1}^{11}$	$3p-4 = p+8 + (N-1)(8)$	$\frac{p-2}{4}$
$W_{\varsigma_1}^{12}$	$3p = p+4 + (N-1)(8)$	$\frac{p+2}{4}$
$W_{\varsigma_1}^{13}$	$3p-6 = p+6 + (N-1)(8)$	$\frac{p-2}{4}$

Notably, the set $W_{\varsigma_1}^9$ exhibits a cardinality of 1. Thus, $\left|W_{\varsigma_1}^9\right| = 1$ is considered. Consequently, the total number of unique edge weights is calculated as: $\frac{p-2}{2} + \frac{p+2}{4} + \frac{p-2}{4} + \frac{p+2}{4} + \frac{p-2}{4} + 1 = \frac{3p}{2}$. In effect with this, $\varsigma_1 : V(SL_p) \to \{1, 2, 3, \cdots, 2p\}$ induces a rainbow antimagic coloring with $\frac{3p}{2}$ colors.

As a result, $rac(SL_p) \leq \frac{3p}{2}$. After taking into account both the lower and upper bounds, it is clear that the rainbow antimagic connection number falls within $\lceil \frac{p}{2} \rceil + p \leq rac(SL_p) \leq \frac{3p}{2}$. However, since in this case , the lower bound $\lceil \frac{p}{2} \rceil + p \simeq \frac{3p}{2}$, it implies $rac(SL_p) = \frac{3p}{2}$.

Case 2: For $p \equiv 1(mod 4)$

In order to establish the rainbow antimagic connection number, we must confirm both the lower and upper bounds. From Theorem - 2, it is evident that for odd cycles C_p, the rac number is $rac(C_p) = \lceil \frac{p}{2} \rceil$. Since there is one copy of cycle C_p with *p*-pendent edges and considering the repetition of colors, we take rac number of cycle as $\lceil \frac{p+5}{4} \rceil$ and we get the lower bound as $\lceil \frac{p+5}{4} \rceil + p$. Thus, $rac(SL_p) \geq \lceil \frac{p+5}{4} \rceil + 4$. Now to establish the upper bound we define an antimagic labelling $\varsigma_2 : V(SL_p) \to \{1, 2, 3, \cdots, 2p\}$ given by:

$$\varsigma_2(x_t) = \begin{cases} 2t-1, & for\ t = 1,3,5,\cdots,\frac{p-3}{2} \\ 2t, & for\ t = 2,4,6,\cdots,\frac{p-1}{2} \\ 2t-p+1, & for\ t = \frac{p+1}{2},\frac{p+5}{2},\frac{p+9}{2},\cdots,p-2 \\ 2t-p, & for\ t = \frac{p+3}{2},\frac{p+7}{2},\frac{p+11}{2},\cdots,p-1\ and\ t=p \end{cases}$$

$$\varsigma_2(y_t) = \begin{cases} 2t+p-1, & for\ t = 1,3,5,\cdots,\frac{p-3}{2} \\ 2t+p, & for\ t = 2,4,6,\cdots,\frac{p-1}{2} \\ 2t+1, & for\ t = \frac{p+1}{2},\frac{p+5}{2},\frac{p+9}{2},\cdots,p-2 \\ 2t, & for\ t = \frac{p+3}{2},\frac{p+7}{2},\frac{p+11}{2},\cdots,p-1\ and\ t=p \end{cases}$$

Based on the above antimagic labelling, we can develop the following edge weight sets:

$W^1_{\varsigma_2}(x_t x_{t+1}) = W^1_{\varsigma_2}\left(x_{\frac{p-1}{2}+t}x_{\frac{p+1}{2}+t}\right) = 4t+1$ for $1 \le t \le \frac{p-3}{2}$

$W^2_{\varsigma_2}(x_p x_1) = W^2_{\varsigma_2}\left(x_{\frac{p-1}{2}}x_{\frac{p+1}{2}}\right) = p+1\ \forall p \in N$

$W^3_{\varsigma_2}(x_{p-1}x_p) = 2p-2\ \forall p \in N$

$W^4_{\varsigma_2}(x_t y_t) = 4t+p-2$ for $t = 1,3,5,\cdots,\frac{p-3}{2}$

$W^5_{\varsigma_2}(x_t y_t) = 4t+p$ for $t = 2,4,6,\cdots,\frac{p-1}{2}$

$W^6_{\varsigma_2}(x_t y_t) = 4t-p+2$ for $t = \frac{p+1}{2},\frac{p+5}{2},\frac{p+9}{2},\cdots,p-2$

$W^7_{\varsigma_2}(x_t y_t) = 4t-p$ for $t = \frac{p+3}{2},\frac{p+7}{2},\frac{p+11}{2},\cdots,p-1$

$W^8_{\varsigma_2}(x_t y_t) = 3p$ for $t = p$

By clearly examining the above sets, we can determine the exact count of distinct edge weights. Furthermore, $W^1_{\varsigma_2}$, $W^4_{\varsigma_2}$, $W^5_{\varsigma_2}$, $W^6_{\varsigma_2}$ and $W^7_{\varsigma_2}$ compose an arithmetic sequence of the form:

$W^1_{\varsigma_2} = \{5,9,13,\cdots,p\}$

$W^4_{\varsigma_2} = \{p+2, p+10, p+18,\cdots,3p-8\}$

$W^5_{\varsigma_2} = \{p+8, p+16, p+24,\cdots,3p-2\}$

$W^6_{\varsigma_2} = \{p+4, p+12, p+20,\cdots,3p-6\}$

$W^7_{\varsigma_2} = \{p+6, p+14, p+22,\cdots,3p-4\}$

Utilizing the *N - term* formula for an arithmetic sequence, we can effortlessly calculate the following:

Weight Function	$U_N = a + (N-1)d$	N
$W_{\varsigma_2}^1$	$p = 5 + (N-1)(4)$	$\frac{p-1}{4}$
$W_{\varsigma_2}^4$	$3p - 8 = p + 2 + (N-1)(8)$	$\frac{p-1}{4}$
$W_{\varsigma_2}^5$	$3p - 2 = p + 8 + (N-1)(8)$	$\frac{p-1}{4}$
$W_{\varsigma_2}^6$	$3p - 6 = p + 4 + (N-1)(8)$	$\frac{p-1}{4}$
$W_{\varsigma_2}^7$	$3p - 4 = p + 6 + (N-1)(8)$	$\frac{p-1}{4}$

Only colors upto p are included in $W_{\varsigma_2}^1$, while the remaining ones are already included in $W_{\varsigma_2}^5$ and $W_{\varsigma_2}^6$. Notably, the set $W_{\varsigma_2}^2$, $W_{\varsigma_2}^3$ and $W_{\varsigma_2}^8$ exhibits a cardinality of 1, i.e, $|W_{\varsigma_2}^2| = |W_{\varsigma_2}^3| = |W_{\varsigma_2}^8| = 1$ is considered. Consequently, the total number of unique edge weights is calculated as: $\frac{p-1}{4} + \frac{p-1}{4} + \frac{p-1}{4} + \frac{p-1}{4} + \frac{p-1}{4} + 3 = \frac{5(p-1)}{4} + 3$. In effect with this, $\varsigma_2: V(SL_p) \to \{1, 2, 3, \cdots, 2p\}$ induces a rainbow antimagic coloring with $\frac{5(p-1)}{4} + 3$ colors. As a result, $rac(SL_p) \leq \frac{5(p-1)}{4} + 3$. After taking into account both the lower and upper bounds, it is clear that the rainbow antimagic connection number falls within $\lceil\frac{p+5}{4}\rceil + p \leq rac(SL_p) \leq \frac{5(p-1)}{4} + 3$. However, since in this case $\lceil\frac{p+5}{4}\rceil + p \simeq \frac{5(p-1)}{4} + 3$, it implies $\lceil\frac{p+5}{4}\rceil + p = 3$.

Case 3: For $p \equiv 3(mod 4)$

In order to establish the rainbow antimagic connection number, we must confirm both the lower and upper bounds. From Theorem-2, it is evident that for odd cycles C_p, the rac number varies between $\lceil\frac{p}{2}\rceil + rac(C_p) \leq \frac{p}{2} + 1$. Since there is one copy of cycle C_p with p-pendent edges and considering the repetition of colors, we take rac number of cycle as $\lceil\frac{p+5}{4}\rceil$. Thus we get the lower bound as $\lceil\frac{p+5}{4}\rceil + p$. Now to establish the upper bound we define an antimagic labelling $\varsigma_3 : V(SL_p) \to \{1, 2, 3, \cdots, 2p\}$ given by:

$$\varsigma_3(x_t) = \begin{cases} 2t-1, & for\ t = 1, 3, 5, \cdots, \frac{p-1}{2} \\ 2t, & for\ t = 2, 4, 6, \cdots, \frac{p-3}{2}; p \geq 7 \\ 2t-p+1, & for\ t = \frac{p+1}{2}, \frac{p+5}{2}, \frac{p+9}{2}, \cdots, p-1 \\ 2t-p, & for\ t = \frac{p+3}{2}, \frac{p+7}{2}, \frac{p+11}{2}, \cdots, p \end{cases}$$

$$\varsigma_3(y_t) = \begin{cases} 2t+p-1, & for\ t = 1, 3, 5, \cdots, \frac{p-1}{2} \\ 2t+p, & for\ t = 2, 4, 6, \cdots, \frac{p-3}{2}; p \geq 7 \\ 2t+1, & for\ t = \frac{p+1}{2}, \frac{p+5}{2}, \frac{p+9}{2}, \cdots, p-1 \\ 2t, & for\ t = \frac{p+3}{2}, \frac{p+7}{2}, \frac{p+11}{2}, \cdots, p \end{cases}$$

Based on the above antimagic labelling, we can develop the following edge weight sets:

$W_{\varsigma_3}^1(x_t x_{t+1}) = 4t+1$ for $1 \le t \le \frac{p-3}{2}; p \ge 7$

$W_{\varsigma_3}^{1a}\left(x_{\frac{p-1}{2}+t} x_{\frac{p+1}{2}+t}\right) = 4t+1$ for $1 \le t \le \frac{p-1}{2}$

$W_{\varsigma_3}^2(x_p x_1) = p+1 \; \forall p \in N$

$W_{\varsigma_3}^3\left(x_{\frac{p-1}{2}} x_{\frac{p+1}{2}}\right) = p \; \forall p \in N$

$W_{\varsigma_3}^4(x_2 x_3) = 5$ for $p = 3$

$W_{\varsigma_3}^5(x_t y_t) = 4t+p-2$ for $t = 1, 3, 5, \cdots, \frac{p-1}{2}$

$W_{\varsigma_3}^6(x_t y_t) = 4t+p$ for $t = 2, 4, 6, \cdots, \frac{p-3}{2}; p \ge 7$

$W_{\varsigma_3}^7(x_t y_t) = 4t-p+2$ for $t = \frac{p+1}{2}, \frac{p+5}{2}, \frac{p+9}{2}, \cdots, p-1$

$W_{\varsigma_3}^8(x_t y_t) = 4t-p$ for $t = \frac{p+3}{2}, \frac{p+7}{2}, \frac{p+11}{2}, \cdots, p$

By clearly examining the above sets, we can determine the exact count of distinct edge weights. Furthermore, $W_{\varsigma_3}^1, W_{\varsigma_3}^{1a}, W_{\varsigma_3}^5, W_{\varsigma_3}^6, W_{\varsigma_3}^7$ and $W_{\varsigma_3}^8$ compose an arithmetic sequence of the form:

$W_{\varsigma_3}^1 = \{5, 9, 13, \cdots, p-2\}$

$W_{\varsigma_3}^{1a} = \{5, 9, 13, \cdots, p-2\}$

$W_{\varsigma_3}^5 = \{p+2, p+10, p+18, \cdots, 3p-4\}$

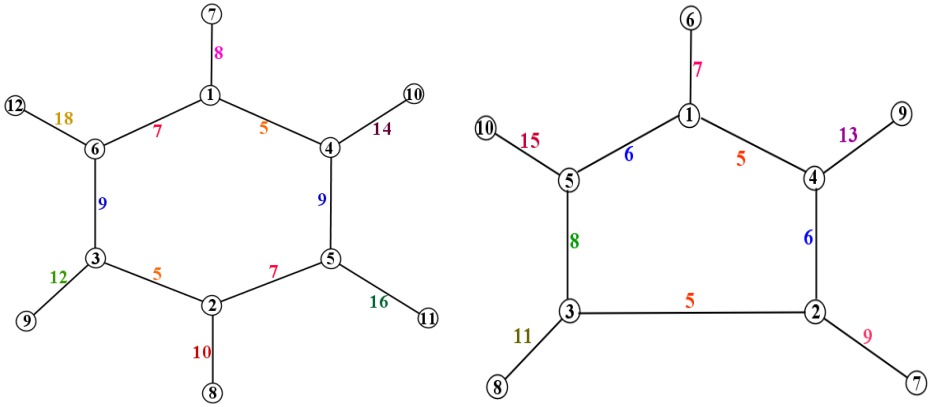
$W_{\varsigma_3}^6 = \{p+8, p+16, p+24, \cdots, 3p-6\}$

$W_{\varsigma_3}^7 = \{p+4, p+12, p+20, \cdots, 3p-2\}$

$W_{\varsigma_3}^8 = \{p+6, p+14, p+22, \cdots, 3p\}$

Utilizing the *N - term* formula for an arithmetic sequence, we can effortlessly calculate the following:

Weight Function	$\boldsymbol{U_N = a + (N-1)d}$	$\boldsymbol{N}$
$W_{\varsigma_3}^1 = W_{\varsigma_3}^{1a}$	$p-2 = 5 + (N-1)(4)$	$\frac{p-3}{4}$
$W_{\varsigma_3}^5$	$3p-4 = p+2 + (N-1)(8)$	$\frac{p+1}{4}$
$W_{\varsigma_3}^6$	$3p-6 = p+8 + (N-1)(8)$	$\frac{p-3}{4}$
$W_{\varsigma_3}^7$	$3p-2 = p+4 + (N-1)(8)$	$\frac{p+1}{4}$
$W_{\varsigma_3}^8$	$3p-4 = p+6 + (N-1)(8)$	$\frac{p+1}{4}$



(a) $rac(SL_6) = 9$ **(b) $rac(SL_5) = 8$**

Figure 1: Illustration of Rainbow Antimagic Connection Number of Sunlet Graph

Only colors upto $p-2$ are considered in $W_{\varsigma_3}^1$ and $W_{\varsigma_3}^{1a}$, whereas the remaining colors are already added in $W_{\varsigma_3}^5$ and $W_{\varsigma_3}^8$. To ensure consistency, we consider any one of $W_{\varsigma_3}^1$ and $W_{\varsigma_3}^{1a}$, since they share the same colors, thus resulting in $U_N^{W_{\varsigma_3}^1} = U_N^{W_{\varsigma_3}^{1a}}$. Notably, the sets $W_{\varsigma_3}^2$, $W_{\varsigma_3}^3$ and $W_{\varsigma_3}^4$ exhibits a cardinality of 1. As the edge weight of $W_{\varsigma_3}^4$ has already been included in $W_{\varsigma_3}^5$, it is deemed superfluous and, therefore, it is excluded. Therefore, $\left|W_{\varsigma_3}^2\right| = \left|W_{\varsigma_3}^3\right| = 1$ is considered. Consequently, the total number of unique edge weights is calculated as: $\frac{p-3}{4} + \frac{p+1}{4} + \frac{p-3}{4} + \frac{p+1}{4} + \frac{p+1}{4} + 2 = \frac{5(p+1)}{4}$. In effect with this, $\varsigma_3 : V(SL_p) \to \{1, 2, 3, \cdots, 2p\}$ induces a rainbow antimagic coloring with $\frac{5(p+1)}{4}$ colors. As a result, $rac(SL_p) \leq \frac{5(p+1)}{4}$. After taking into account both the lower and upper bounds, it is clear that the rainbow antimagic connection number falls within $\lceil\frac{p+5}{4}\rceil + p \leq rac(SL_p) \leq \frac{5(p+1)}{4}$. However, since in this case $p \equiv 3(mod4)$, the lower bound $\lceil\frac{p+5}{4}\rceil + p \simeq \frac{5(p+1)}{4}$, it implies $rac(SL_p) = \frac{5(p+1)}{4}$.

Theorem 5: *For any positive integer $p \geq 3$. the rainbow antimagic connection number of a helm graph is, $\lceil\frac{3p}{2}\rceil + 2$.*

Proof: The helm graph H_p is a graph with p-pendent edges on a wheel graph W_p, whose vertex set and edge set are defined by, $V(H_p) = \{h\} \cup \{x_t y_t : 1 \leq t \leq p\}$ and $E(H_p) = \{hx_t, x_t y_t : 1 \leq t \leq p\} \cup \{x_t x_{t+1} : 1 \leq t \leq p-1\} \cup \{x_p x_1\}$ respectively. The cardinality of vertices are, $|V(H_p)| = 2p+1$ vertices and edges are, $E|(H_p)| = 3p$ edges. In order to establish the rainbow antimagic connection number, we must confirm both the lower and upper bounds. From Theorem-3, it is evident that the rac number of wheel W_p is p. Further from p-pendant edges we have $\lceil\frac{p}{2}\rceil + 2$ new colors, so we obtain, $p + \lceil\frac{p}{2}\rceil + 2$. Thus $rac(H_p) \geq p + \lceil\frac{p}{2}\rceil + 2$. To prove the upper bound, let us divide the proof into two cases.

Case 1: For $p \equiv 0(mod2)$

Now to establish the upper bound we define an antimagic labelling $\varsigma_4 : V(H_p) \rightarrow \{1, 2, 3, \cdots, 2p+1\}$ given by:

$$\varsigma_4(h) = 2p+1 \qquad \forall p \in N$$

$$\varsigma_4(x_t) = \begin{cases} \frac{t+1}{2}, & for\ t = 1,3,5,\cdots,p-1 \\ p+1-\left(\frac{t}{2}\right), & for\ t = 2,4,6,\cdots,p \end{cases}$$

$$\varsigma_4(y_t) = \begin{cases} p+\left(\frac{t+1}{2}\right), & for\ t = 1,3,5,\cdots,p-1 \\ 2p-\left(\frac{t-2}{2}\right), & for\ t = 2,4,6,\cdots,p \end{cases}$$

Based on the above antimagic labelling, we can develop the following edge weight sets:

$W^1_{\varsigma_4}(hx_t) = 2p + \frac{t+3}{2}$ for $t = 1,3,5,\cdots,p-1$

$W^2_{\varsigma_4}(hx_t) = 3p + 2 - \frac{t}{2}$ for $t = 2,4,6,\cdots,p$

$W^3_{\varsigma_4}(x_t x_{t+1}) = p+1$ for $t = 1,3,5,\cdots,p-1$

$W^4_{\varsigma_4}(x_t x_{t+1}) = p+2$ for $t = 2,4,6,\cdots,p-2$

$W^5_{\varsigma_4}(x_p x_1) = \frac{p+4}{2}\ \forall p \in N$

$W^6_{\varsigma_4}(x_t y_t) = p+t+1$ for $t = 1,3,5,\cdots,p-1$

$W^7_{\varsigma_4}(x_t y_t) = 3p-t+2$ for $t = 2,4,6,\cdots,p$

By clearly examining the above sets, we can determine the exact count of distinct edge weights. Furthermore, $W^1_{\varsigma_4}, W^2_{\varsigma_4}, W^6_{\varsigma_4}$ and $W^7_{\varsigma_4}$ compose an arithmetic sequence of the form:

$$W^1_{\varsigma_4} = \left\{2p+2, 2p+3, 2p+4, \cdots, \frac{5p+2}{2}\right\}$$

$$W^2_{\varsigma_4} = \left\{3p+1, 3p, 3p-1, \cdots, \frac{5p+4}{2}\right\}$$

$$W^6_{\varsigma_4} = \{p+2, p+4, p+6, \cdots, 2p\}$$

$$W^7_{\varsigma_4} = \{3p, 3p-2, 3p-4, \cdots, 2p+2\}$$

Utilizing the *N - term* formula for an arithmetic sequence, we can effortlessly calculate the following:

Weight Function	$U_N = a + (N-1)d$	N
$W_{\varsigma_4}^1$	$\frac{5p+2}{2} = 2p + 2 + (N-1)(1)$	$\frac{p}{2}$
$W_{\varsigma_4}^2$	$\frac{5p+4}{2} = 3p + 1 + (N-1)(-1)$	$\frac{p}{2}$
$W_{\varsigma_4}^6$	$2p = p + 2 + (N-1)(2)$	$\frac{p}{2}$
$W_{\varsigma_4}^7$	$2p + 2 = 3p + (N-1)(-2)$	$\frac{p}{2}$

To ensure consistency, we exclude $W_{\varsigma_4}^7$, since all the colors of $W_{\varsigma_4}^7$ are already included in $W_{\varsigma_4}^1$ and $W_{\varsigma_4}^2$. Notably, the sets $W_{\varsigma_4}^3$, $W_{\varsigma_4}^4$ and $W_{\varsigma_4}^5$ exhibits a cardinality of 1. As the edge weight of $W_{\varsigma_4}^4$ has already been included in $W_{\varsigma_4}^6$, it is deemed superfluous and, therefore, it is excluded. Thus, $|W_{\varsigma_4}^3| = |W_{\varsigma_4}^5| = 1$ is considered. Consequently, the total number of unique edge weights is calculated as: $\frac{p}{2} + \frac{p}{2} + \frac{p}{2} + 2 = \frac{3p}{2} + 2$. In effect with this, $\varsigma_4 : V(H_p) \to \{1, 2, 3, \cdots, 2p+1\}$ induces a rainbow antimagic coloring with $\frac{3p}{2} + 2$ colors.

Case 2: For $p \equiv 1(mod 2)$

Now to establish the upper bound we define an antimagic labelling $\varsigma_5 : V(H_p) \to \{1, 2, 3, \cdots, 2p+1\}$ given by:

$$\varsigma_5(h) = 2p + 1 \qquad \forall p \in N$$

$$\varsigma_5(x_t) = \begin{cases} \frac{t+1}{2}, & for\ t = 1, 3, 5, \cdots, p \\ p + 1 - \left(\frac{t}{2}\right), & for\ t = 2, 4, 6, \cdots, p-1 \end{cases}$$

$$\varsigma_5(y_t) = \begin{cases} p + \left(\frac{t+1}{2}\right), & for\ t = 1, 3, 5, \cdots, p \\ 2p - \left(\frac{t-2}{2}\right), & for\ t = 2, 4, 6, \cdots, p-1 \end{cases}$$

Based on the above antimagic labelling, we can develop the following edge weight sets:

$W_{\varsigma_5}^1(hx_t) = 2p + \frac{t+3}{2}$ for $t = 1, 3, 5, \cdots, p$

$W_{\varsigma_5}^2(hx_t) = 3p + 2 - \frac{t}{2}$ for $t = 2, 4, 6, \cdots, p-1$

$W_{\varsigma_5}^3(x_t x_{t+1}) = p + 1$ for $t = 1, 3, 5, \cdots, p-2$

$W_{\varsigma_5}^4(x_t x_{t+1}) = p + 2$ for $t = 2, 4, 6, \cdots, p-1$

$W_{\varsigma_5}^5(x_p x_1) = \frac{p+3}{2}$ $\forall p \in N$

$W_{\varsigma_5}^6(x_t y_t) = p + t + 1$ for $t = 1, 3, 5, \cdots, p$

$W_{\varsigma_5}^7(x_t y_t) = 3p - t + 2$ for $t = 2, 4, 6, \cdots, p-1$

By clearly examining the above sets, we can determine the exact count of distinct edge weights. Furthermore, $W_{\varsigma_5}^1, W_{\varsigma_5}^2, W_{\varsigma_5}^6$ and $W_{\varsigma_5}^7$ compose an arithmetic sequence of the form:

$$W_{\varsigma_5}^1 = \left\{2p+2, 2p+3, 2p+4, \cdots, \frac{5p+3}{2}\right\}$$

$$W_{\varsigma_5}^2 = \left\{3p+1, 3p, 3p-1, \cdots, \frac{5p+5}{2}\right\}$$

$$W_{\varsigma_5}^6 = \{p+2, p+4, p+6, \cdots, 2p+1\}$$

$$W_{\varsigma_5}^7 = \{3p, 3p-2, 3p-4, \cdots, 2p+3\}$$

Utilizing the *N - term* formula for an arithmetic sequence, we can effortlessly calculate the following:

Weight Function	$\boldsymbol{U_N = a + (N-1)d}$	$\boldsymbol{N}$
$W_{\varsigma_5}^1$	$\frac{5p+3}{2} = 2p + 2 + (N-1)(1)$	$\frac{p+1}{2}$
$W_{\varsigma_5}^2$	$\frac{5p+5}{2} = 3p + 1 + (N-1)(-1)$	$\frac{p-1}{2}$
$W_{\varsigma_5}^6$	$2p + 1 = p + 2 + (N-1)(2)$	$\frac{p+1}{2}$
$W_{\varsigma_5}^7$	$2p + 3 = 3p + (N-1)(-2)$	$\frac{p-1}{2}$

To ensure consistency, we exclude $W_{\varsigma_5}^7$, since all the colors of $W_{\varsigma_5}^7$ are already included in $W_{\varsigma_5}^1$ and $W_{\varsigma_5}^2$. Notably, the sets $W_{\varsigma_5}^3$, $W_{\varsigma_5}^4$ and $W_{\varsigma_5}^5$ exhibits a cardinality of 1. As the edge weight of $W_{\varsigma_5}^4$ has already been included in $W_{\varsigma_5}^6$, it is deemed superfluous and, therefore, it is excluded. Thus, $\left|W_{\varsigma_5}^3\right| = \left|W_{\varsigma_5}^5\right| = 1$ is considered. Consequently, the total number of unique edge weights is calculated as: $\frac{p+1}{2} + \frac{p-1}{2} + \frac{p+1}{2} + 2 = \frac{3p+1}{2} + 2$. In effect with this, $\varsigma_5 : V(H_p) \to \{1, 2, 3, \cdots, 2p+1\}$ induces a rainbow antimagic coloring with $\frac{3p+1}{2} + 2$ colors.

On correlating the edge weights from Case 1 and Case 2, we get,

$$W_{\varsigma_4} = W_{\varsigma_5} \leq \begin{cases} \frac{3p}{2} + 2 = \lceil \frac{3p}{2} \rceil + 2 & for\ p \equiv 0 (mod 2) \\ \frac{3p+1}{2} + 2 = \lceil \frac{3p}{2} \rceil + 2 & for\ p \equiv 1 (mod 2) \end{cases}$$

As a result, $rac(H_p) \leq \lceil \frac{3p}{2} \rceil + 2$. After taking into account both the lower and upper bounds, it is clear that the rainbow antimagic connection number falls within $p + \lceil \frac{p}{2} \rceil + 2 \leq rac(H_p) \leq \lceil \frac{3p}{2} \rceil + 2$. However, the lower bound $p + \lceil \frac{p}{2} \rceil + 2 \simeq \lceil \frac{3p}{2} \rceil + 2$, which implies $rac(H_p) = \lceil \frac{3p}{2} \rceil + 2$.

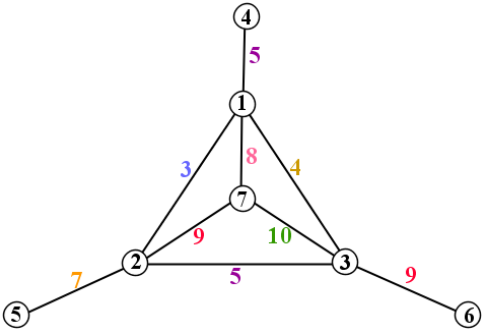


Figure 2: Illustration of rac Number of a Helm Graph $(H_3) = 7$ with Color Set $= \{3, 4, 5, 7, 8, 9, 10\}$

Theorem 6: *For any positive integer $p \geq 3$, the rainbow antimagic connection number of a closed helm graph is,*

$$rac(CH_p) = \begin{cases} \frac{3p}{2} + 1 & for\ p \equiv 0(mod2) \\ 2p + 1 & for\ p \equiv 1(mod2) \end{cases}$$

Proof: The closed helm graph CH_p is a graph obtained from a helm graph H_p by joining the pendant vertices to form a cycle. It's vertex set and edge set are defined by, $V(CH_p) = \{h\} \cup \{x_t y_t : 1 \leq t \leq p\}$ and $E(CH_p) = \{hx_t, x_t y_t : 1 \leq t \leq p\} \cup \{x_t x_{t+1}, y_t y_{t+1} : 1 \leq t \leq p-1\} \cup \{x_p x_1, y_p y_1\}$ respectively. The cardinality of vertices are, $|V(CH_p)| = 2p + 1$ vertices and edges are, $E|(CH_p)| = 4p$ edges. To prove the theorem, we have considered the following two cases.

Case 1: For $p \equiv 0(mod2)$

In order to establish the rainbow antimagic connection number, we must confirm both the lower and upper bounds. From Theorem-3 and 2, it is evident that for wheel graph W_p and cycle C_p, the rac number are p and $\lceil \frac{p}{2} \rceil$ respectively. Since there is one copy of wheel W_p and cycle C_p, we get $p + \lceil \frac{p}{2} \rceil$. However, since $p \equiv 0(mod2)$, we can obtain the lower bound by adding the edge cardinality of P_2, which is 1, to $p + \frac{p}{2}$, resulting in a lower bound of $\frac{3p}{2} + 1$. Thus $rac(CH_p) \geq \frac{3p}{2} + 1$. Now to establish the upper bound we define an antimagic labelling $\varsigma_6 : V(CH_p) \to \{1, 2, 3, \cdots, 2p + 1\}$ given by:

$$\varsigma_6(h) = 2p + 1 \quad \forall p \in N$$

$$\varsigma_6(x_t) = \begin{cases} 2t - 1, & for\ t = 1, 3, 5, \cdots, p - 1 \\ 2t, & for\ t = 2, 4, 6, \cdots, p \end{cases}$$

$$\varsigma_6(y_t) = \begin{cases} 2p - 2t + 1, & for\ t = 1, 3, 5, \cdots, p - 1 \\ 2p - 2t + 2, & for\ t = 2, 4, 6, \cdots, p \end{cases}$$

Based on the above antimagic labelling, we can develop the following edge weight sets:

$W^1_{\varsigma_6}(hx_t) = 2(p + t)$ for $t = 1, 3, 5, \cdots, p - 1$

$W_{\varsigma_6}^2(hx_t) = 2(p+t)+1$ for $t = 2,4,6,\cdots,p$

$W_{\varsigma_6}^3(x_t x_{t+1}) = 4t+1$ for $1 \le t \le p-1$

$W_{\varsigma_6}^4(y_t y_{t+1}) = 4p-4t+1$ for $1 \le t \le p-1$

$W_{\varsigma_6}^5(x_p x_1) = W_{\varsigma_6}^5(y_p y_1) = 2p+1 \; \forall p \in N$

$W_{\varsigma_6}^6(x_t y_t) = 2p$ for $t = 1,3,5,\cdots,p-1$

$W_{\varsigma_6}^7(x_t y_t) = 2p+2$ for $t = 2,4,6,\cdots,p$

By clearly examining the above sets, we can determine the exact count of distinct edge weights. Furthermore, $W_{\varsigma_6}^1, W_{\varsigma_6}^2, W_{\varsigma_6}^3$ and $W_{\varsigma_6}^4$ compose an arithmetic sequence of the form:

$W_{\varsigma_6}^1 = \{2p+2, 2p+6, 2p+10, \cdots, 4p-2\}$

$W_{\varsigma_6}^2 = \{2p+5, 2p+9, 2p+13, \cdots, 4p+1\}$

$W_{\varsigma_6}^3 = \{5, 9, 13, \cdots, 2p+1\}$

$W_{\varsigma_6}^4 = \{2p+1, 2p-3, 2p-7, \cdots, 5\}$

Utilizing the *N - term* formula for an arithmetic sequence, we can effortlessly calculate the following:

Weight Function	$\boldsymbol{U_N = a + (N-1)d}$	$\boldsymbol{N}$
$W_{\varsigma_6}^1$	$4p-2 = 2p+2+(N-1)(4)$	$\frac{p}{2}$
$W_{\varsigma_6}^2$	$4p+1 = 2p+5+(N-1)(4)$	$\frac{p}{2}$
$W_{\varsigma_6}^3$	$2p+1 = 5+(N-1)(4)$	$\frac{p}{2}$
$W_{\varsigma_6}^4$	$5 = 2p+1+(N-1)(-4)$	$\frac{p}{2}$

Only colors upto $2p+1$ are considered in $W_{\varsigma_6}^3$ and $W_{\varsigma_6}^4$, whereas the remaining colors are already added in $W_{\varsigma_6}^1$ and $W_{\varsigma_6}^2$. To ensure consistency, we consider any one of $W_{\varsigma_6}^3$ and $W_{\varsigma_6}^4$, since they share the same colors, thus resulting in $U_N^{W_{\varsigma_6}^3} = U_N^{W_{\varsigma_6}^4}$. Notably, the sets $W_{\varsigma_6}^5$, $W_{\varsigma_6}^6$ and $W_{\varsigma_6}^7$ exhibits a cardinality of 1. As the edge weight of $W_{\varsigma_6}^5$ and $W_{\varsigma_6}^7$ has already been included in $W_{\varsigma_6}^3, W_{\varsigma_6}^4$ and $W_{\varsigma_6}^1$ respectively, it is deemed superfluous and, therefore, it is excluded. Therefore, $\left|W_{\varsigma_6}^6\right| = 1$ is considered. Consequently, the total number of unique edge weights is calculated as: $\frac{p}{2}+\frac{p}{2}+\frac{p}{2}+1 = \frac{3p}{4}+1$. In effect with this, $\varsigma_6 : V(CH_p) \to \{1,2,3,\cdots,2p+1\}$ induces a rainbow antimagic coloring with $\frac{3p}{2}+1$ colors. As a result, $rac(CH_p) \le \frac{3p}{2}+1$. After taking into account both the lower and upper bounds, it is clear that the rainbow antimagic connection number is, $rac(CH_p) = \frac{3p}{2}+1$.

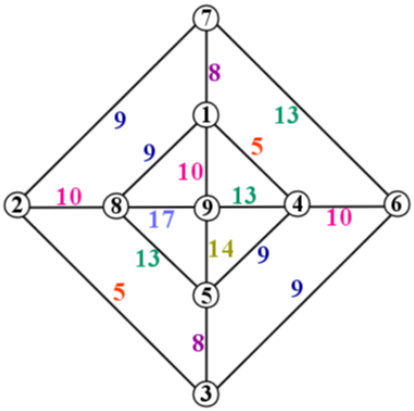


Figure 3: Illustration of rac Number of a Closed Helm Graph $(CH_4) = 7$ with Color Set $= \{5, 8, 9, 10, 13, 14, 17\}$

Case 2: For $p \equiv 1(mod2)$

In order to establish the rainbow antimagic connection number, we must confirm both the lower and upper bounds. From Theorem-2, it is evident that for both odd and even cycles C_p, the rac number varies between $\lceil \frac{p}{2} \rceil \leq rac(C_p) \leq \lceil \frac{p}{2} \rceil + 1$. Consider that the closed helm has two copies of cycle C_p and p distinct edges arising from the hub vertex with p unique colors, so we get $2\lceil \frac{p}{2} \rceil + p$. Thus $rac(CH_p) \leq 2\lceil \frac{p}{2} \rceil + p$. Now to establish the upper bound we define an antimagic labelling $\varsigma_7 : V(CH_p) \to \{1, 2, 3, \cdots, 2p + 1\}$ given by:

$$\varsigma_7(h) = 2p + 1, \quad \forall p \in N$$

$$\varsigma_7(x_t) = \begin{cases} 2t - 1, & for\ t = 1, 3, 5, \cdots, p \\ 2t, & for\ t = 2, 4, 6, \cdots, p - 1 \end{cases}$$

$$\varsigma_7(y_t) = \begin{cases} 2p - 2t + 2, & for\ t = 1, 3, 5, \cdots, p \\ 2p - 2t + 1, & for\ t = 2, 4, 6, \cdots, p - 1 \end{cases}$$

Based on the above antimagic labelling, we can develop the following edge weight sets:

$W^1_{\varsigma_7}(hx_t) = 2(p + t)$ for $t = 1, 3, 5, \cdots, p$

$W^2_{\varsigma_7}(hx_t) = 2(p + t) + 1$ for $t = 2, 4, 6, \cdots, p - 1$

$W^3_{\varsigma_7}(x_t x_{t+1}) = 4t + 1$ for $1 \leq t \leq p - 1$

$W^4_{\varsigma_7}(y_t y_{t+1}) = 4p - 4t + 1$ for $1 \leq t \leq p - 1$

$W^5_{\varsigma_7}(x_p x_1) = 2p \ \forall p \in N$

$W^6_{\varsigma_7}(y_p y_1) = 2p + 2 \ \forall p \in N$

$W^7_{\varsigma_7}(x_t y_t) = 2p + 1$ for $1 \leq t \leq p$

By clearly examining the above sets, we can determine the exact count of distinct edge weights. Furthermore, $W_{\varsigma_7}^1, W_{\varsigma_7}^2, W_{\varsigma_7}^3$ and $W_{\varsigma_7}^4$ compose an arithmetic sequence of the form:

$$W_{\varsigma_7}^1 = \{2p+2, 2p+6, 2p+10, \cdots, 4p\}$$

$$W_{\varsigma_7}^2 = \{2p+5, 2p+9, 2p+13, \cdots, 4p-1\}$$

$$W_{\varsigma_7}^3 = \{5, 9, 13, \cdots, 4p-3\}$$

$$W_{\varsigma_7}^4 = \{4p-3, 4p-7, 4p-11, \cdots, 5\}$$

Utilizing the *N - term* formula for an arithmetic sequence, we can effortlessly calculate the following:

Weight Function	$\boldsymbol{U_N = a + (N-1)d}$	$\boldsymbol{N}$
$W_{\varsigma_7}^1$	$4p = 2p + 2 + (N-1)(4)$	$\frac{p+1}{2}$
$W_{\varsigma_7}^2$	$4p - 1 = 2p + 5 + (N-1)(4)$	$\frac{p-1}{2}$
$W_{\varsigma_7}^3$	$4p - 3 = 5 + (N-1)(4)$	$p-1$
$W_{\varsigma_7}^4$	$5 = 4p - 3 + (N-1)(-4)$	$p-1$

To ensure consistency, we consider any one of $W_{\varsigma_7}^3$ or $W_{\varsigma_7}^4$, since they share the same colors. Notably, the sets $W_{\varsigma_7}^5, W_{\varsigma_7}^6$ and $W_{\varsigma_7}^7$ exhibits a cardinality of 1. As the edge weight of $W_{\varsigma_7}^6$ has already been included in $W_{\varsigma_7}^1$, it is deemed superfluous and, therefore, it is excluded. Thus, $\left|W_{\varsigma_7}^5\right| = \left|W_{\varsigma_7}^7\right| = 1$ is considered. Consequently, the total number of unique edge weights is calculated as: $\frac{p+1}{2} + \frac{p-1}{2} + (p-1) + 2 = 2p + 1$. In effect with this, $\varsigma_7 : V(CH_p) \to \{1, 2, 3, \cdots, 2p+1\}$ induces a rainbow antimagic coloring with $2p + 1$ colors. As a result, $rac(CH_p) \leq 2p + 1$.

However, the lower bound $2\lceil \frac{p}{2} \rceil + p$, evaluates to $2p + 1$, resulting in $rac(CH_p) \geq 2p + 1$. After taking into account both the lower and upper bounds, it is clear that the rainbow antimagic connection number is, $rac(CH_p) = 2p + 1$.

Theorem 7: *For any positive integer $p \geq 3$, the rainbow antimagic connection number of a web graph is,*

$$rac(WB_p) = \begin{cases} \left\lceil \frac{9p-1}{4} \right\rceil + 2 & for\ p \equiv 0,1 (mod\ 4) \\ \left\lceil \frac{5p}{2} \right\rceil & for\ p \equiv 2,3 (mod\ 4) \end{cases}$$

***Proof*:** The web graph WB_p is a graph obtained from a closed helm graph CH_p by attaching pendant vertices to each of its node whose vertex set and edge set are defined by, $V(WBp) = \{h\} \cup \{x_t, y_t, z_t : 1 \leq t \leq p\}$ and $E(W\ Bp) = \{hx_t, x_t y_t,\ y_t z_t : 1 \leq t \leq p\} \cup \{x_t x_t + 1, y_t y_t + 1 : 1 \leq t \leq p-1\} \cup \{x_p x_1, y_p y_1\}$ respectively. The

cardinality of vertices are, $|V(WBp)| = 3p + 1$ vertices and edges are, $|E(WBp)| = 5p$ edges. To prove the theorem, we have considered the following two cases.

Case 1: *For* $p \equiv 0,1(mod\ 4)$

Subcase 1.1 $p \equiv 0(mod\ 4)$

In order to establish the rainbow antimagic connection number, we must confirm both the lower and upper bounds. From Theorem-6, it is evident that for $p \equiv 0(mod\ 4), rac(CH_p) = \frac{3p}{2} + 1$ and pendant edges has $\frac{3p}{2} + 1$ new colors, so we get, $\frac{3p}{2} + 1 + \frac{3p}{4} + 1 = \frac{9p}{4} + 2$ as the bound. Thus $rac(WB_p) \geq \frac{9p}{4} + 2$. Now to establish the upper bound we define an antimagic labelling $\varsigma_8: (WB_p) \rightarrow \{1,2,3, \dots, 3p + 1\}$ given by:

$$\varsigma_8(h) = 3p + 1 \quad \forall p \in N$$

$$\varsigma_8(x_t) = \begin{cases} 2t - 1, & for\ t = 1,3,5, \dots, p - 1 \\ 2t, & for\ t = 2,4,6, \dots, p \end{cases}$$

$$\varsigma_8(y_t) = \begin{cases} 2p - 2t + 1, & for\ t = 1,3,5, \dots, p - 1 \\ 2p - 2t + 2, & for\ t = 2,4,6, \dots, p \end{cases}$$

$$\varsigma_8(z_t) = 3p - t + 1\ for\ 1 \leq t \leq p$$

Based on the above antimagic labelling, we can develop the following edge weight sets:

$$W_{\varsigma 8}^1(hx_t) = 3p + 2t\ for\ t = 1,3,5, \dots, p - 1$$

$$W_{\varsigma 8}^2(hx_t) = 3p + 2t + 1\ for\ t = 2,4,6, \dots, p$$

$$W_{\varsigma 8}^3(x_t x_t + 1) = 4t + 1\ for\ 1 \leq t \leq p - 1$$

$$W_{\varsigma 8}^4(y_t y_t + 1) = 4p - 4t + 1\ for\ 1 \leq t \leq p - 1$$

$$W_{\varsigma 8}^5(x_p x_1) = W_{\varsigma 8}^5(y_p y_1) = 2p + 1 \forall p \in N$$

$$W_{\varsigma 8}^6(x_t y_t) = 2p\ for\ t = 1,3,5, \dots, p - 1$$

$$W_{\varsigma 8}^7(x_t y_t) = 2p + 2\ for\ t = 2,4,6, \dots, p$$

$$W_{\varsigma 8}^8(y_t z_t) = 5p - 3t + 2\ for\ t = 1,3,5, \dots, p - 1$$

$$W_{\varsigma 8}^9(y_t z_t) = 5p - 3t + 3\ for\ t = 2,4,6, \dots, p$$

By clearly examining the above sets, we can determine the exact count of distinct edge weights.

Furthermore, $W_{\varsigma 8}^1$, $W_{\varsigma 8}^2$, $W_{\varsigma 8}^3$, $W_{\varsigma 8}^3$, $W_{\varsigma 8}^4$, $W_{\varsigma 8}^5$, $W_{\varsigma 8}^6$, $W_{\varsigma 8}^7$, $W_{\varsigma 8}^8$ and $W_{\varsigma 8}^9$ compose an arithmetic sequence of the form:

$$W_{\varsigma 8}^1 = \{3p + 2, 3p + 6, 3p + 10, \dots, 5p - 2\}$$

$$W_{\varsigma 8}^2 = \{3p + 5, 3p + 9, 3p + 13, \dots, 5p + 1\}$$

$$W_{\varsigma 8}^3 = \{5, 9, 13, \dots, 3p + 1\}$$

$W_{\varsigma 8}^{4} = \{3p+1, 3p-3, 3p-7, \ldots, 5\}$

$W_{\varsigma 8}^{8} = \{5p-1, 5p-13, 5p-25, \ldots, 2p+11\}$

$W_{\varsigma 8}^{9} = \{5p-9, 5p-21, 5p-33, \ldots, 2p+3\}$

Utilizing the $N-term$ formula for an arithmetic sequence, we can effortlessly calculate the
following:

Weight Function	$\boldsymbol{U_N = a + (N-1)d}$	$\boldsymbol{N}$
$W_{\varsigma 8}^{1}$	$5p-2 = 3p+2+(N-1)(4)$	$\frac{p}{2}$
$W_{\varsigma 8}^{2}$	$5p+1 = 3p+5+(N-1)(4)$	$\frac{p}{2}$
$W_{\varsigma 8}^{3}$	$3p+1 = 5+(N-1)(4)$	$\frac{3p}{4}$
$W_{\varsigma 8}^{4}$	$5 = 3p+1+(N-1)(-4)$	$\frac{3p}{4}$
$W_{\varsigma 8}^{8}$	$2p+11 = 5p-1+(N-1)(-12)$	$\frac{p}{4}$
$W_{\varsigma 8}^{9}$	$2p+3 = 5p-9+(N-1)(-12)$	$\frac{p}{4}$

Only colors up to $3p+1$ are considered in $W_{\varsigma 8}^{3}$ and $W_{\varsigma 8}^{4}$, whereas the remaining colors are already included in $W_{\varsigma 8}^{2}$. To ensure consistency, we consider any one of $W_{\varsigma 8}^{3}$ and $W_{\varsigma 8}^{4}$, since they share the same colors, thus resulting in $U_N^{W_{\varsigma 8}^{3}} = U_N^{W_{\varsigma 8}^{4}}$. Also, $W_{\varsigma 8}^{8} = 5p-3t+2$ for $t = 3, 7, 11, \ldots, p-1$ and $W_{\varsigma 8}^{9} = 5p-3t+3$ for $t = 2, 6, 10, \ldots, p-2$ has already been included in $W_{\varsigma 8}^{2}$, $W_{\varsigma 8}^{3}$, and $W_{\varsigma 8}^{4}$. Thus, $W_{\varsigma 8}^{8} = 5p-3t+2$ for $t = 1,5,9, \ldots, p-3$ and $W_{\varsigma 8}^{9} = 5p-3t+3$ for $t = 4, 8, 12, \ldots, p$ are considered.

Notably, the sets $W_{\varsigma 8}^{5}$, $W_{\varsigma 8}^{6}$, and $W_{\varsigma 8}^{7}$ exhibits a cardinality of 1. As the edge weight of $W_{\varsigma 8}^{5}$ has already been included in $W_{\varsigma 8}^{3}$ and $W_{\varsigma 8}^{4}$, it is deemed superfluous and, therefore, it is excluded. Thus, $\left|W_{\varsigma 8}^{6}\right| = \left|W_{\varsigma 8}^{7}\right| = 1$ s considered.Consequently, the total number of unique edge weights is calculated as: $\frac{p}{2} + \frac{p}{2} + \frac{3p}{4} + \frac{p}{4} + \frac{p}{4} + 1 + 1 = \frac{9p}{4} + 2$.

In effect with this, $\varsigma_8 : V(W\ B_p) \rightarrow \{1, 2, 3, \ldots, 3p+1\}$ induces a rainbow antimagic coloring with $\frac{9p}{4} + 2$ colors. As a result, $rac(W\ B_p) \leq \frac{9p}{4} + 2$.

Subcase 1.2 $p \equiv 1(mod\ 4)$

In order to establish the rainbow antimagic connection number, we must confirm both the lower and upper bounds. From Theorem – 6, it is evident that for $p \equiv 1(mod\ 4), rac(CH_p) = 2p+1$ and pendant edges has $\frac{p+3}{4}$ new colors, so we get, $2p + 1 + \frac{p+3}{4} = \frac{9p+7}{4} \simeq \frac{9p-1}{4} + 2$ as the lower bound. Thus $rac(W\ B_p) \geq \frac{9p-1}{4} + 2$. Now to establish the upper bound we define an antimagic labelling $\varsigma_9 : V(W\ B_p) \rightarrow \{1, 2, 3, \ldots, 3p+1\}$ given by:

$$\varsigma_9(h) = 3p + 1 \quad \forall p \in N$$

$$\varsigma_9(x_t) = \begin{cases} 2t - 1, & for\ t = 1,3,5, \dots, p \\ 2t, & for\ t = 2,4,6, \dots, p-1 \end{cases}$$

$$\varsigma_9(y_t) = \begin{cases} 2p - 2t + 2, & for\ t = 1,3,5, \dots, p \\ 2p - 2t + 1, & for\ t = 2,4,6, \dots, p-1 \end{cases}$$

$$\varsigma_9(z_t) = 3p - t + 1\ for\ 1 \le t \le p$$

Based on the above antimagic labelling, we can develop the following edge weight sets:

$$W^1_{\varsigma 9}(hx_t) = 3p + 2t\ for\ t = 1,3,5, \dots, p$$

$$W^2_{\varsigma 9}(hx_t) = 3p + 2t + 1\ for\ t = 2,4,6, \dots, p-1$$

$$W^3_{\varsigma 9}(x_t x_t + 1) = 4t + 1\ for\ 1 \le t \le p-1$$

$$W^4_{\varsigma 9}(y_t y_t + 1) = 4p - 4t + 1\ for\ 1 \le t \le p-1$$

$$W^5_{\varsigma 9}(x_p x_1) = 2p\ \forall p \in N$$

$$W^6_{\varsigma 9}(y_p y_1) = 2p + 2\ \forall p \in N$$

$$W^7_{\varsigma 9}(x_t y_t) = 2p + 1\ \forall p \in N$$

$$W^8_{\varsigma 9}(y_t z_t) = 5p - 3t + 3\ for\ t = 1,3,5, \dots, p$$

$$W^9_{\varsigma 9}(y_t z_t) = 5p - 3t + 2\ for\ t = 2,4,6, \dots, p-1$$

By clearly examining the above sets, we can determine the exact count of distinct edge weights.

Furthermore, $W^1_{\varsigma 9}$, $W^2_{\varsigma 9}$, $W^3_{\varsigma 9}$, $W^4_{\varsigma 9}$, $W^8_{\varsigma 9}$, and $W^9_{\varsigma 9}$ compose an arithmetic sequence of the form:

$$W^1_{\varsigma 9} = \{3p + 2, 3p + 6, 3p + 10, \dots, 5p\}$$

$$W^2_{\varsigma 9} = \{3p + 5, 3p + 9, 3p + 13, \dots, 5p - 1\}$$

$$W^3_{\varsigma 9} = \{5, 9, 13, \dots, 3p - 2\}$$

$$W^4_{\varsigma 9} = \{3p - 2, 3p - 6, 3p - 10, \dots, 5\}$$

$$W^8_{\varsigma 9} = \{5p - 6, 5p - 18, 5p - 30, \dots, 2p + 9\}$$

$$W^9_{\varsigma 9} = \{5p - 10, 5p - 22, 5p - 34, \dots, 2p + 5\}$$

Utilizing the $N - term$ formula for an arithmetic sequence, we can effortlessly calculate the following:

Weight Function	$U_N = a + (N-1)d$	N
$W_{\varsigma 9}^1$	$5p = 3p + 2 + (N-1)(4)$	$\frac{p+1}{2}$
$W_{\varsigma 9}^2$	$5p - 1 = 3p + 5 + (N-1)(4)$	$\frac{p-1}{2}$
$W_{\varsigma 9}^3$	$3p - 2 = 5 + (N-1)(4)$	$\frac{3(p-1)}{4}$
$W_{\varsigma 9}^4$	$5 = 3p - 2 + (N-1)(-4)$	$\frac{3(p-1)}{4}$
$W_{\varsigma 9}^8$	$2p + 9 = 5p - 6 + (N-1)(-12)$	$\frac{p-1}{4}$
$W_{\varsigma 9}^9$	$2p + 5 = 5p - 10 + (N-1)(-12)$	$\frac{p-1}{4}$

Only colors up to $3p-2$ are considered in $W_{\varsigma 9}^3$ and $W_{\varsigma 9}^4$, whereas the remaining colors are already included in $W_{\varsigma 9}^1$. To ensure consistency, we consider any one of $W_{\varsigma 9}^3$ and $W_{\varsigma 9}^4$, since they share the same colors, thus resulting in $U_N^{W_{\varsigma 9}^3} = U_N^{W_{\varsigma 9}^4}$. Also, $W_{\varsigma 9}^8 = 5p - 3t + 3$, for $t = 1, 5, 9, \dots, p$ and , $W_{\varsigma 9}^9 = 5p - 3t + 2$, for $t = 2, 6, 10, \dots, p-3$ has already been included in $W_{\varsigma 9}^1$, $W_{\varsigma 9}^7$ and $W_{\varsigma 9}^8$. Thus, $W_{\varsigma 9}^8 = 5p - 3t + 3, for\ t = 3, 7, 11, \dots, p-2$ and $W_{\varsigma 9}^9 = 5p - 3t + 3, for\ t = 4, 8, 12, \dots, p-1$ are considered.

Notably, the sets $W_{\varsigma 9}^5$, $W_{\varsigma 9}^6$ and $W_{\varsigma 9}^7$ exhibits a cardinality of 1. So, $\left|W_{\varsigma 9}^5\right| = \left|W_{\varsigma 9}^6\right| = \left|W_{\varsigma 9}^7\right| = 1$. Consequently, the total number of unique edge weights is calculated as: $\frac{p+1}{2} + \frac{p-1}{2} + \frac{3(p-1)}{4} + \frac{p-1}{4} + \frac{p-1}{4} + 1 + 1 + 1 = \frac{9p-1}{4} + 2$ colors. As a result $rac(W\ B_p) \leq \frac{9p-1}{4} + 2$. On correlating the edge weights from Subcase 1.1 and Subcase 1.2, we get the upper bound as,

$$W_{\varsigma 8} = W_{\varsigma 9} \leq \begin{cases} \frac{9p}{4} + 2 = \left[\frac{9p-1}{4}\right] + 2 & for\ p \equiv 0 (mod 4) \\ \frac{9p-1}{4} + 2 = \left[\frac{9p-1}{4}\right] + 2 & for\ p \equiv 0 (mod 4) \end{cases}$$

After taking into account both the lower and upper bounds, it is clear that the rainbow antimagic connection number is, $rac(W\ B_p) = \left[\frac{9p-1}{4}\right] + 2$.

Case 2: For $p \equiv 2,3 (mod\ 4)$

Subcase 2.1 $p \equiv 2 (mod\ 4)$

In order to establish the rainbow antimagic connection number, we must confirm both the lower and upper bounds. From Theorem-6, it is evident that for $p \equiv 2 (mod\ 4), rac(CH_p) = \frac{3p}{2} + 1$ and pendant edges has $p-1$ new colors, so we get, $\frac{3p}{2} + 1 + p - 1 = \frac{5p}{2}$ as the lower bound. Thus $rac(W\ B_p) \geq \frac{5p}{2}$. Now to establish the upper bound we define an antimagic labelling $\varsigma_{10} : V(W\ B_p) \to \{1, 2, 3, \dots, 3p+1\}$ given by:

$$\varsigma_{10}(h) = 3p + 1 \quad \forall p \in N$$

$$\varsigma_{10}(x_t) = \begin{cases} 2t - 1, & for\ t = 1, 3, 5, \dots, p - 1 \\ 2t, & for\ t = 2, 4, 6, \dots, p \end{cases}$$

$$\varsigma_{10}(y_t) = \begin{cases} 2p - 2t + 1, & for\ t = 1, 3, 5, \dots, p - 1 \\ 2p - 2t + 2, & for\ t = 2, 4, 6, \dots, p \end{cases}$$

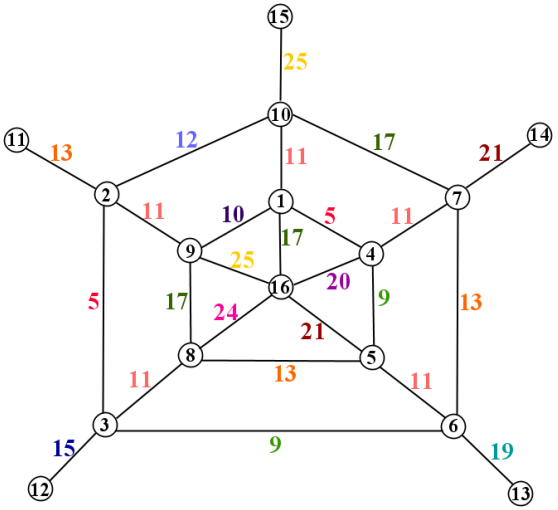


Figure 4: Illustration of RAC Number of a Web Graph $(WB_5) = 13$ with Color Set = $\{5, 9, 10, 11, 12, 13, 15, 17, 19, 20, 21, 24, 25\}$

$$\varsigma_{10}(z_t) = 3p - t + 1\ for\ 1 \le t \le p$$

Based on the above antimagic labelling, we can develop the following edge weight sets:

$$W^1_{\varsigma 10}(hx_t) = 3p + 2t\ for\ t = 1,3,5, \dots, p - 1$$

$$W^2_{\varsigma 10}(hx_t) = 3p + 2t + 1\ for\ t = 2,4,6, \dots, p$$

$$W^3_{\varsigma 10}(x_t x_t + 1) = 4t + 1\ for\ 1 \le t \le p - 1$$

$$W^4_{\varsigma 10}(y_t y_t + 1) = 4p - 4t + 1\ for\ 1 \le t \le p - 1$$

$$W^5_{\varsigma 10}(x_p x_1) = W^5_{\varsigma 10}(y_p y_1) = 2p + 1\ \forall p \in N$$

$$W^6_{\varsigma 10}(x_t y_t) = 2p\ for\ t = 1, 3, 5, \dots, p - 1$$

$$W^7_{\varsigma 10}(x_t y_t) = 2p + 2\ for\ t = 2, 4, 6, \dots, p$$

$$W^8_{\varsigma 10}(y_t z_t) = 5p - 3t + 2\ for\ t = 1, 3, 5, \dots, p - 1$$

$$W^9_{\varsigma 10}(y_t z_t) = 5p - 3t + 3\ for\ t = 2, 4, 6, \dots, p$$

By clearly examining the above sets, we can determine the exact count of distinct edge weights.

Furthermore, $W^1_{\varsigma 10}$, $W^2_{\varsigma 10}$, $W^3_{\varsigma 10}$, and $W^4_{\varsigma 10}$ compose an arithmetic sequence of the form:

$$W^1_{\varsigma 10} = \{3p+2, 3p+6, 3p+10, \dots, 5p-2\}$$

$$W^2_{\varsigma 10} = \{3p+5, 3p+9, 3p+13, \dots, 5p+1\}$$

$$W^3_{\varsigma 10} = \{5, 9, 13, \dots, 4p-3\}$$

$$W^4_{\varsigma 10} = \{4p-3, 4p-7, 4p-11, \dots, 5\}$$

Utilizing the $N-term$ formula for an arithmetic sequence, we can effortlessly calculate the following:

Weight Function	$\boldsymbol{U_N = a + (N-1)d}$	$\boldsymbol{N}$
$W^1_{\varsigma 10}$	$5p-2 = 3p+2+(N-1)(4)$	$\frac{p}{2}$
$W^2_{\varsigma 10}$	$5p+1 = 3p+5+(N-1)(4)$	$\frac{p}{2}$
$W^3_{\varsigma 10}$	$4p-3 = 5+(N-1)(4)$	$p-1$
$W^4_{\varsigma 10}$	$5 = 4p-3+(N-1)(-4)$	$p-1$

In $W^8_{\varsigma 10}$ and $W^9_{\varsigma 10}$, the following edge weights in:

$$t = 1, p \; for \; p = 6$$

$$t = \begin{cases} 1, 4, 5, 6, \dots, \frac{p+6}{4} \\ \frac{3p-2}{4}, \frac{3p+2}{4}, \frac{3p+6}{4}, \dots, p-3, p \end{cases} \quad for \; p \equiv 2(mod 4); \; p \geq 10$$

are already included in $W^1_{\varsigma 10}$, $W^2_{\varsigma 10}$, $W^3_{\varsigma 10}$, and $W^4_{\varsigma 10}$. So the total number of colors produced by $W^8_{\varsigma 10}$ and $W^9_{\varsigma 10}$ are $\frac{p-2}{2}$ colors.

To ensure consistency, we consider any one of $W^3_{\varsigma 10}$ and $W^4_{\varsigma 10}$, since they share the same colors, thus resulting in $U_N^{W^3_{\varsigma 10}} = U_N^{W^4_{\varsigma 10}}$. Notably, the sets $W^5_{\varsigma 10}, W^6_{\varsigma 10}$, and $W^7_{\varsigma 10}$ exhibits a cardinality of 1. As the edge weight of $W^5_{\varsigma 10}$ has already been included in $W^3_{\varsigma 10}$ and $W^4_{\varsigma 10}$, t is deemed super-fluous and, therefore, it is excluded. Thus, $\left[W^6_{\varsigma 10}\right] = \left[W^7_{\varsigma 10}\right] = 1$ is considered. Consequently, the total number of unique edge weights is calculated as: $\frac{p}{2} + \frac{p}{2} + p - 1 + \frac{p-2}{2} + 1 + 1 = \frac{5p}{2}$.

In effect with this, $\varsigma_{10}: V(W\,B_p) \rightarrow \{1, 2, 3, \dots, 3p+1\}$ induces a rainbow antimagic coloring with $\frac{5p}{4}$ colors. As a result, $rac(W\,B_p) \leq \frac{5p}{4}$.

Subcase 2.2 $p \equiv 3(mod\ 4)$

In order to establish the rainbow antimagic connection number, we must confirm both the lower and upper bounds. From Theorem-6, it is evident that for

$p \equiv 3(mod\ 4), rac(CH_p) = 2p + 1$ and pendant edges has $\frac{p-1}{2}$ new colors, so we get, $2p + 1 + \frac{p-1}{2} = \frac{5p+1}{2}$ as the lower bound. Thus $rac(W\ B_p) \geq \frac{5p+1}{2}$. Now to establish the upper bound we define an antimagic labelling $\varsigma_{11} : V(W\ B_p) \rightarrow \{1, 2, 3, \dots, 3p + 1\}$ given by:

$$\varsigma_{11}(h) = 3p + 1\ \forall p \in N$$

$$\varsigma_{11}(x_t) = \begin{cases} 2t - 1, & for\ t = 1, 3, 5, \dots, p \\ 2t, & for\ t = 2, 4, 6, \dots, p - 1 \end{cases}$$

$$\varsigma_{11}(y_t) = \begin{cases} 2p - 2t + 2, & for\ t = 1, 3, 5, \dots, p \\ 2p - 2t + 1, & for\ t = 2, 4, 6, \dots, p - 1 \end{cases}$$

$$\varsigma_{11}(z_t) = 3p - t + 1\ for\ 1 \leq t \leq p$$

Based on the above antimagic labelling, we can develop the following edge weight sets:

$$W^1_{\varsigma 11}(hx_t) = 3p + 2t\ for\ t = 1,3,5, \dots, p$$

$$W^2_{\varsigma 11}(hx_t) = 3p + 2t + 1\ for\ t = 2,4,6, \dots, p - 1$$

$$W^3_{\varsigma 11}(x_t x_t + 1) = 4t + 1\ for\ 1 \leq t \leq p - 1$$

$$W^4_{\varsigma 11}(y_t y_t + 1) = 4p - 4t + 1\ for\ 1 \leq t \leq p - 1$$

$$W^5_{\varsigma 11}(x_p x_1) = 2p\ \forall p \in N$$

$$W^6_{\varsigma 11}(y_p y_1) = 2p + 2\ \forall p \in N$$

$$W^7_{\varsigma 11}(x_t y_t) = 2p + 1\ \forall p \in N$$

$$W^8_{\varsigma 11}(y_t z_t) = 5p - 3t + 3\ for\ t = 1, 3, 5, \dots, p$$

$$W^9_{\varsigma 11}(y_t z_t) = 5p - 3t + 2\ for\ t = 2, 4, 6, \dots, p - 1$$

By clearly examining the above sets, we can determine the exact count of distinct edge weights.

Furthermore, $W^1_{\varsigma 11}$, $W^2_{\varsigma 11}$, $W^3_{\varsigma 11}$ and $W^4_{\varsigma 11}$ compose an arithmetic sequence of the form:

$$W^1_{\varsigma 11} = \{3p + 2, 3p + 6, 3p + 10, \dots, 5p\}$$

$$W^2_{\varsigma 11} = \{3p + 5, 3p + 9, 3p + 13, \dots, 5p - 1\}$$

$$W^3_{\varsigma 11} = \{5, 9, 13, \dots, 4p - 3\}$$

$$W^4_{\varsigma 10} = \{4p - 3, 4p - 7, 4p - 11, \dots, 5\}$$

Utilizing the $N - term$ formula for an arithmetic sequence, we can effortlessly calculate the following:

Weight Function	$U_N = a + (N-1)d$	N
$W_{\varsigma 11}^1$	$5p = 3p + 2 + (N-1)(4)$	$\frac{p+1}{2}$
$W_{\varsigma 11}^2$	$5p - 1 = 3p + 5 + (N-1)(4)$	$\frac{p-1}{2}$
$W_{\varsigma 11}^3$	$4p - 3 = 5 + (N-1)(4)$	$p-1$
$W_{\varsigma 11}^4$	$5 = 4p - 3 + (N-1)(-4)$	$p-1$

In $W_{\varsigma 11}^8$ and $W_{\varsigma 11}^9$, the following edge weights in:

$$t = \begin{cases} 3, 4, 5, \ \ldots, \frac{p+5}{4} \\ p-1, p-2, p-3, \ldots, \frac{3p+3}{4} \end{cases} \quad for\ p \equiv 2(mod 4); p \geq 7$$

are considered, whereas the remaining colors are included in $W_{\varsigma 11}^1$ and $W_{\varsigma 11}^4$. So the total number of colors produced by $W_{\varsigma 11}^8$ and $W_{\varsigma 11}^9$ are $\frac{p-3}{2}$ colors.

To ensure consistency, we consider any one of $W_{\varsigma 11}^3$ and $W_{\varsigma 11}^4$, since they share the same colors, thus resulting in $U_N^{W_{\varsigma 11}^3} = U_N^{W_{\varsigma 11}^4}$. Notably, the sets $W_{\varsigma 11}^5, W_{\varsigma 11}^6,$ and $W_{\varsigma 11}^7$ exhibits a cardinality of 1. So, $\left[W_{\varsigma 11}^5\right] = \left[W_{\varsigma 11}^6\right] = \left[W_{\varsigma 11}^7\right] = 1$. Consequently, the total number of unique edge weights is calculated as: $\frac{p+1}{2} + \frac{p-1}{2} + p - 1 + \frac{p-3}{2} + 1 + 1 + 1 = \frac{5p+1}{2}$. In effect with this, $\varsigma_{11}: V(WB_p) \rightarrow \{1, 2, 3, \ldots, 3p+1\}$ induces a rainbow antimagic coloring with $\frac{5p}{2}$ colors. As a result, $rac(WB_p) \leq \frac{5p+1}{2}$.

On correlating the edges weights from Subcase 2.1 and Subcase 2.2, we get the upper bound as,

$$W_{\varsigma 10} = W_{\varsigma 11} \leq \begin{cases} \frac{5p}{2} = \left[\frac{5p}{2}\right] & for\ p \equiv 2(mod 4) \\ \frac{5p+1}{2} = \left[\frac{5p}{2}\right] & for\ p \equiv 3(mod 4) \end{cases}$$

After taking into account both the lower and upper bounds, it is clear that the rainbow antimagic connection number is, $rac(W\ B_p) \leq \left[\frac{5p}{2}\right]$.

4. CONCLUSIONS

The proofs provided in this paper establishes an optimal solution for the rainbow antimagic coloring of Wheel-related graphs such as, Sunlet graph SL_p, Helm graph SL_p, Closed helm graphs CH_p and Web graph WB_p. The rainbow antimagic connection number (rac) of a related wheel graph is higher than rac of wheel graph due to modifications, such as the addition of pendants to the graph. Moreover, the rac of the related wheel graph also contains elements from the wheel graph, where the value is influenced by the rim of the wheel. However, finding the exact value of rac number is not easy task, even it is

considered to be NP-hard problems. Thus, we propose the following open problems: Characterize the existence of the rainbow antimagic coloring for any graph satisfied some specific properties, i.e. regular graph, graph with pendants, cyclic graph, planar graph or other family of graph with specific properties. Also characterize their rac number.

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