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# REGRESSION AND RATIO TYPE ESTIMATORS FOR ESTIMATING THE FINITE POPULATION MEAN USING RANK AND EMPIRICAL DISTRIBUTION FUNCTION AS DUAL OF AUXILIARY VARIABLE

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## ABSTRACT

Accurately estimating the mean of a finite population is a fundamental objective in survey sampling methodologies. This study introduces an innovative framework for population mean estimation utilizing regression and ratio-type estimators. A key feature of the proposed approach is the incorporation of the empirical distribution function as an auxiliary variable in a dual capacity. Both the rank and empirical distribution function of the auxiliary variable demonstrate a strong correlation with the variable of interest, providing valuable supplementary information for improving estimation accuracy. By integrating this dual-use auxiliary variable into regression and ratio-type estimators, the proposed methodology achieves enhanced efficiency and precision. The performance of the estimators has been assessed through the analysis of their mean square errors (MSEs) and percentage relative efficiencies (PREs) across various populations with diverse characteristics. The results show that the proposed estimators, leveraging the rank and empirical distribution function, consistently outperform traditional methods in terms of accuracy and reliability. Notably, the proposed estimators achieve the lowest MSEs and the highest PREs when compared to existing counterparts, underscoring their superior precision.

## KEYWORDS

Finite population mean, regression estimator, ratio estimator, rank, empirical distribution function, auxiliary variable.

## 1. INTRODUCTION

The estimation of finite population parameters, particularly the mean, has long been a significant area of interest in survey sampling. Accurate estimation methods are critical for making informed decisions in various fields such as economics, health sciences, and social studies. To achieve more precise estimates, auxiliary information is often incorporated into the estimation process. Supplementary characteristics, which are interlinked with the variable of research, are employed in regression and ratio-type estimators to improve efficiency. These techniques have been widely studied and developed over the years, offering practical solutions to challenges in survey sampling (Muk and Kuk, 1983; Zaman and Kadilar, 2021).

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The role of auxiliary information is further enhanced by introducing advanced tools such as rank and empirical distribution functions (EDFs). These tools provide additional insights into the structure of the population, allowing for improved estimation accuracy. Incorporating ranks or EDFs as dual auxiliary variables combines the benefits of traditional auxiliary variables with the detailed information captured by rank-based measures or distributional properties. This dual approach has the potential to yield estimators with reduced bias and increased precision, even under simple random sampling (Pandey et al., 2021; Singh and Solanki, 2013).

Simple random sampling (SRS) remains a fundamental and widely used method in survey design due to its simplicity and unbiased nature. However, the precision of estimators under SRS can be limited, particularly when no stratification or complex designs are used. By leveraging dual auxiliary variables, such as ranks and EDFs, regression and ratio-type estimators can achieve better performance even within the straightforward framework of SRS. Recent studies, such as those by Cingi and Zaman (2006) and Yaqoob and Shabbir (2017), highlight the potential of integrating additional sources of auxiliary information for enhancing estimator efficiency.

This study focuses on the development and evaluation of regression and ratio-type estimators for the finite population mean $\bar{Y}$ under SRS, incorporating rank and EDFs as dual of auxiliary variable. The proposed estimators aim to addressed the key limitations in existing methods by combining the strengths of traditional and non-traditional auxiliary information. The theoretical properties of the estimators, including their bias and mean square error (MSE), are derived and analyzed. Additionally, numerical studies and simulations are conducted to demonstrate the efficiency and applicability of the proposed methods in real-world scenarios Haq et al. (2017). Khan et al. (2025) proposed a novel regression-type estimator using empirical distribution function, which achieved more efficiency compared to conventional estimators.

The outcomes of this research contributed to the literature by offering innovative estimation techniques that improve the accuracy of population mean estimation under SRS. These techniques have practical implications for researchers and practitioners seeking robust and efficient methods for survey data analysis.

Consider a finite population,

$Q = \{Q_1, Q_2, Q_3, \dots, Q_N\}$ of size $N$, where

$Y = \{Y_1, Y_2, \dots, Y_N\}$ represents the study variable, and

$Z = \{Z_1, Z_2, \dots, Z_N\}$ is the corresponding auxiliary variable.

Let $y_i$ and $z_i$ denote the $i^{th}$ observations of the study variable $Y$ and the supplementary variable $Z$, respectively. These variables exhibit a certain degree of correlation $(\rho_{yz})$ within the population $Q$. A random subset of size $n$ is selected from the population using SRSWOR.

Let the population mean and variance of the study characteristic $Y$ are given by:

$$\bar{Y} = \sum_{i=1}^{N} Y_i/N \quad \text{and} \quad S_y^2 = \sum_{i=1}^{N} (Y_i - \bar{Y})^2/N - 1,$$

respectively. In case of subset, the corresponding quantities for the study variable y are:

$$\bar{y} = \sum_{i=1}^{n} y_i/n \quad \text{and} \quad s_y^2 = \sum_{i=1}^{n} (y_i - \bar{y})^2/(n-1).$$

Similarly, the mean and variance of the auxiliary variable $Z$ for the population and subset are defined as follows:

$$\bar{Z} = \sum_{i=1}^{N} Z_i/N, \quad S_x^2 = \sum_{i=1}^{N} (Z_i - \bar{Z})^2/(N-1),$$

and for the sample:

$$\bar{z} = \sum_{i=1}^{n} z_i/n, \quad s_x^2 = \sum_{i=1}^{n} (z_i - \bar{z})^2/(n-1).$$

Next, we define the rank of the auxiliary variable as:

$R_z = \{r_{z1}, r_{z2}, ..., r_{zN}\}$, where $r_i$ represents the rank of $Z_i$, and the Empirical Distribution Function(EDF) of the auxiliary variable $Z = \{z_1, z_2, ..., z_N\}$ as:

$$F_z = \{f_{z1}, f_{z2}, ..., f_{zN}\},$$

where $f_{xi}$ is the EDF value for each $z_i$. The mean and variance of $R_z$ and $F_z$ in both the population and subset cases are defined as follows:

$$\bar{R}_z = \sum_{i=1}^{N} r_{zi}/N\,,\ S_{r_z}^2 = \sum_{i=1}^{N} (r_{zi} - \bar{R}_z)/(N-1), \quad \bar{r_z} = \sum_{i=1}^{n} r_{zi}/n \quad \text{and} \quad s_{r_z}^2 = \sum_{i=1}^{n} (r_{zi} - \bar{r_z})/(n-1).$$

Similarly, the mean and variance of $F_z$ for the population and sample are defined as:

$$\bar{F}_z = \sum_{i=1}^{N} f_{zi}/N, \quad S_{F_z}^2 = \sum_{i=1}^{N} (F_{zi} - \bar{F}_z)^2/(N-1),$$

for the population, and for the sample:

$$\bar{f}_z = \sum_{i=1}^{n} f_{zi}/n, \quad s_{\bar{f}_z}^2 = \sum_{i=1}^{n} (f_{zi} - \bar{f}_z)^2/(n-1).$$

The covariance terms between the study variable $Y$ and the auxiliary variables $Z$, $R_z$ and $F_z$ are defined as follows:

$$S_{yz} = \sum_{i=1}^{N}(Y_i - \bar{Y})(Z_i - \bar{Z})/(N-1), \quad S_{yr_z} = \sum_{i=1}^{N}(Y_i - \bar{Y})(R_{zi} - \bar{R}_z)/(N-1),$$

$$S_{zr_z} = \sum_{i=1}^{N}(Z_i - \bar{Z})(R_{zi} - \bar{R}_z)/(N-1), \quad S_{yf_z} = \sum_{i=1}^{N}(Y_i - \bar{Y})(F_{zi} - \bar{F}_z)/(N-1),$$

and

$$S_{zf_z} = \sum_{i=1}^{N}(Z_i - \bar{Z})(F_{zi} - \bar{F}_z)/(N-1).$$

Next, we define the correlation coefficients between $Z_i$ and $R_{zi}$, and between $Z_i$ and $F_{zi}$ as:

$$\rho_{zr_z} = \frac{S_{zr_z}}{S_z S_{r_z}} \quad \text{and} \quad \rho_{zf_z} = \frac{S_{zf_z}}{S_z S_{f_z}},$$

where $S_{zrz}$ and $S_{zfz}$ represent the population covariances between $Z_i$ and $R_{zi}$, and $Z_i$ and $F_{zi}$, respectively.

For the derivation of the biases and MSEs of the suggested estimators, we defined the error terms as:

Let

$$\xi_0 = (\bar{y} - \bar{Y})/\bar{Y}, \quad \xi_1 = (\bar{z} - \bar{Z})/\bar{Z}, \quad \xi_2 = (\bar{r}_z - \bar{R}_z)/\bar{R}_z, \quad \xi_3 = (\bar{f}_z - \bar{F}_z)/\bar{F}_z,$$

ensuring that

$$E(\xi_0) = E(\xi_1) = E(\xi_2) = E(\xi_3) = 0,$$
$$E(\xi_0^2) = \theta C_y^2, \quad E(\xi_1^2) = \theta C_z^2, \quad E(\xi_2^2) = \theta C_{r_z}^2, \quad E(\xi_3^2) = \theta C_{f_z}^2,$$
$$E(\xi_0\xi_1) = \theta \rho_{yz} C_y C_z, \quad E(\xi_0\xi_2) = \theta \rho_{yr_z} C_y C_{r_z},$$
$$E(\xi_0\xi_3) = \theta \rho_{yf_z} C_y C_{f_z}, \quad E(\xi_1\xi_2) = \theta \rho_{zr_z} C_z C_{r_z},$$
$$E(\xi_1\xi_3) = \theta \rho_{zf_z} C_z C_{f_z}.$$

and

$$\theta = (\frac{1}{n} - \frac{1}{N})$$

is the correction factor of finite population.

Some considered estimators that are currently available in the literature, along with their MSEs, are thoroughly explained in Section 2. In Section 3, we explored and derived the proposed efficient regression and ratio types estimators using rank and EDF as dual of supplementary variable and compared the estimators' outcomes under various scenarios. The practical usefulness of the suggested estimators is assessed by applying them to real-world data sets in Section 4, and the work is concluded and some future research topics are discussed in Section 5.

## 2. SOME CONSIDERED ESTIMATORS

This section reviews several related estimators currently used to calculate the finite population mean, providing their variances and MSEs up to the first-order approximation.

### 2.1 The Usual Mean Estimator

$\bar{y}$ is the mean of this unbiased estimator $\hat{y}_1$ and the variance is:

$$Var(\bar{y}_1) = \theta \bar{Y}^2 C_y^2 \tag{1}$$

### 2.2 The Foundational Ratio and Product Estimators

The foundational ratio proposed by Cochran (1940) and product suggested by Murthy (1964) estimators are given as:

$$\hat{\bar{Y}}_2 = \bar{y}\left(\frac{\bar{Z}}{\bar{z}}\right) \tag{2}$$

and

$$\hat{\bar{Y}}_3 = \bar{y}\left(\frac{\bar{z}}{\bar{Z}}\right) \tag{3}$$

The MSEs of the aforementioned ratio and product estimators are given as follows:

$$MSE(\hat{\bar{Y}}_2) \cong \theta \bar{Y}^2 (C_y^2 + C_z^2 - 2\rho_{yz} C_y C_z),$$

and

$$MSE(\hat{\bar{Y}}_3) \cong \theta \bar{Y}^2 (C_y^2 + C_z^2 + 2\rho_{yz} C_y C_z).$$

### 2.3 Conversion of Power Estimator

Respectively, Srivastava (1967) proposed the power transformation estimator as:

$$\hat{\bar{Y}}_4 = \bar{y}\left(\frac{\bar{z}}{\bar{Z}}\right)^{\pi} \tag{4}$$

The MSE of estimator $\hat{\bar{Y}}_4$ is

$$MSE(\hat{\bar{Y}}_4) \cong \theta \bar{Y}^2 [C_y^2 + \pi C_z^2 (\pi - 2\beta_{yz})],$$

where

$$\beta_{yz} = \frac{C_{yz}}{C_z^{\,2}}.$$

The minimum MSE of the $\hat{\bar{Y}}_4$ for the optimum level of $\upsilon = \beta_{yz}$ is

$$MSE_{(min.)}(\hat{\bar{Y}}_4) = \theta \bar{Y}^2 C_y^2 [1 - \rho_{yz}^2]$$

where

$$\rho_{yz}^2 = \frac{C_{yz}^2}{C_y^2 C_z^2}.$$

The MSE of $\hat{\bar{Y}}_4$ can be conveniently determined by substituting the value of $v = (1, -1)$.

**2.4 Standard Regression Estimator**

The standard regression estimator proposed by Cochran (1963) is described as:

$$\hat{\bar{Y}}_5 = \bar{y} + \hat{\beta}_{yz}(\bar{Z} - \bar{z}) \tag{5}$$

where

$$\hat{\beta}_{yz} = \frac{s_{yz}}{s_z^2}.$$

The MSE of the $\hat{\bar{Y}}_5$, up to the first approximation, is given as:

$$MSE(\hat{\bar{Y}}_5) = \theta \bar{Y}^2 [C_y^2 + P_{yz} C_z^2 (P_{yz} - 2\beta_{yz})],$$

where

$$P_{yz} = R\beta_{yz} \quad and \quad R = \frac{\bar{Z}}{\bar{Y}}.$$

so that

$$MSE_{(min.)}(\hat{\bar{Y}}_5) = min.\, MSE(\hat{\bar{Y}}_4)$$

**2.5 Traditional Difference-Type Estimator**

The Traditional difference-type estimator presented by Riaz et al. (2014) is

$$\hat{\bar{Y}}_6 = \bar{y} + w(\bar{Z} - \bar{z}) \tag{6}$$

The optimal value of $w$ is given by

$$w_{opt} = \frac{1}{R}\beta_{yz}.$$

An enhanced version of the estimator $\hat{\bar{Y}}_6$ as proposed by Riaz et al. (2014), is defined as:

$$\hat{\bar{Y}}_7 = w_1 \bar{y} + w_2(\bar{Z} - \bar{z}) \tag{7}$$

where the constants $w_1$ and $w_2$ are chosen at random.

Up to the first order approximation, the minimal MSE of estimators $\hat{\bar{Y}}_6$ and $\hat{\bar{Y}}_7$ is

$$MSE(\hat{\bar{Y}}_6) \cong \theta \bar{Y}^2 [C_y^2 + \pi C_z^2 (\pi - 2\beta_{yz})],$$

the optimal level of the constants $w_1$ and $w_2$ are

$$w_{1opt} = \frac{1}{C_y^2(1-\rho_{yz}^2)} \quad \text{and} \quad w_{2opt} = (\frac{1}{R}\beta_{yz}) \left( \frac{1}{C_y^2(1-\rho_{yz}^2)} \right).$$

Following the entry of these values, the estimator $\hat{\bar{Y}}_6$ and $\hat{\bar{Y}}_7$'s minimal MSE is

$$MSE(\hat{\bar{Y}}_6) = MSE_{(min.)}(\hat{\bar{Y}}_4),$$

or

$$MSE_{(min.)}(\hat{\bar{Y}}_6) = MSE_{(min.)}(\hat{\bar{Y}}_4),$$

After simplification

$$MSE_{(min.)}(\hat{\bar{Y}}_7) = \frac{\bar{Y}^2 MSE_{(min.)}(\hat{\bar{Y}}_6)}{\bar{Y}^2 + MSE_{(min.)}(\hat{\bar{Y}}_6)}.$$

**2.6 The Ratio Type estimator proposed by Bhusan et al. (2018) is**

$$\hat{\bar{Y}}_8 = \delta\bar{y}\left(\frac{\bar{Z}}{\bar{z}}\right)^{\gamma} \tag{8}$$

The minimum MSE of $\hat{\bar{Y}}_8$ is

$$MSE_{(min.)}(\hat{\bar{Y}}_8) = \bar{Y}^2\left(1 - \frac{M^2}{L}\right) \tag{9}$$

where

$$L = 1 + C_y^2(1 - 2\rho_{yz}^2) + C_y C_z,$$

and

$$M = 1 + \frac{C_y^2}{2}\left[(1 - \rho_{yz}^2) - (1 - C_{yz})\right].$$

It is evident from all of these that the conversion of power, standard regression, and traditional difference estimators are equally effective.

**2.7 Difference Type Estimator of Yaqoob et al. (2017)**

They used rank as a dual use of auxiliary variable to define the difference type estimator as:

$$\hat{\bar{Y}}_9 = \bar{y} + a_1(\bar{Z} - \bar{z}) + a_2(\bar{R}_z - \bar{r}_z) \tag{10}$$

The optimal values of the arbitrary constants $a_1$ and $a_2$ are defined as follows:

$$a_1 = \frac{1}{T_1}\left(\frac{\beta_{yz} - \beta_{yr_z}\beta_{zr_z}}{1 - \rho_{zr_z}^2}\right), \quad a_2 = \frac{1}{T_2}\left(\frac{\beta_{yr_z} - \beta_{yr_z}\beta_{zr_z}}{1 - \rho_{zr_z}^2}\right),$$

$$\text{where} \quad T_1 = \frac{\bar{Z}}{\bar{Y}} \quad \text{and} \quad T_2 = \frac{\bar{R}_z}{\bar{Y}}.$$

Up to the first order approximation, the estimator $\hat{\bar{Y}}_9$'s minimal MSE is provided as:

$$MSE_{(min.)}(\hat{\bar{Y}}_9) = \theta\bar{Y}^2 C_y^2(1 - R_{yzr_z}^2) \tag{11}$$

where

$$R_{y,zr_z}^2 = \frac{\rho_{yz}^2 + \rho_{yr_z}^2 - 2\rho_{yz}\rho_{yr_z}\rho_{zr_z}}{1 - \rho_{zr_z}^2}, \quad \rho_{yz}^2 = \frac{C_{yz}^2}{C_y^2 C_z^2}, \; \rho_{yr_z}^2 = \frac{C_{yr_z}^2}{C_y^2 C_{r_z}^2} \text{ and } \rho_{zr_z}^2 = \frac{C_{zr_z}^2}{C_z^2 C_{r_z}^2}$$

**2.8 Ratio Type Estimator of Pandey et al. (2021)**

They proposed a ratio-type estimator that uses rank as a dual auxiliary variable use as:

$$\hat{\bar{Y}}_{10} = \bar{y}\left(\frac{\bar{Z}}{\bar{z}}\right)^{\gamma_1}\left(\frac{\bar{R}_z}{\bar{r}_z}\right)^{\gamma_2} \tag{12}$$

Up to the first order approximation, the estimator $\hat{\bar{Y}}_{10}$'s minimum MSE is:

$$MSE_{(min.)}(\hat{\bar{Y}}_{10}) = MSE_{(min.)}(\hat{\bar{Y}}_{9}); \tag{13}$$

It demonstrates that the regression type estimator $\hat{\bar{Y}}_{9}$ and the estimator $\hat{\bar{Y}}_{10}$ are equally efficient.

The optimum values of constants $\gamma_1$ and $\gamma_2$ are given as:

$$\gamma_{1opt} = \frac{\beta_{yz} - \beta_{yr_z}\beta_{zr_z}}{1-\rho^2_{zr_z}}, \quad \gamma_{2opt} = \frac{\beta_{yr_z} - \beta_{yr_z}\beta_{zr_z}}{1-\rho^2_{zr_z}},$$

and

$$\beta_{yz} = \frac{C_{yz}}{C_z^2}, \ \beta_{yr_z} = \frac{C_{yr_z}}{C_{r_z}^2} \quad \text{and} \quad \beta_{zr_z} = \frac{C_{zr_z}}{C_{r_z}^2}.$$

## 3. PROPOSED NEW ESTIMATORS

We utilized rank ( $R_z$ ) and Empirical Distribution Function ( $F_z$ ) as a dual of auxiliary variable ( $Z$ ) for estimation of $\bar{Y}$ of the study variable $Y$ Population mean, to improve the efficiency of the estimators under simple random sampling technique. For these four proposed estimators, we follow Yaqoob et al. (2017) for regression type and Pandey et al. (2021) for ratio type estimators. The following subsections provide the suggested regression and ratio-type estimators. The suggested estimators' bias and MSE are expanded to the first approximation.

### 3.1 Proposed Regression Type Estimator-I

The first proposed regression-type estimator, incorporating rank as a dual of auxiliary variable, is formulated as:

$$\hat{\bar{Y}}_{Prop1} = \delta_1 \, \bar{y} + \lambda_1(\bar{Z} - \bar{z}) + \lambda_2(\bar{R}_z - \bar{r}_z) \tag{14}$$

where the real constants that need to be appropriately selected are $\delta_1$, $\lambda_1$ and $\lambda_2$.

We obtain after assuming the values of $\bar{y}, \bar{z}$ and $\bar{r}_z$:

$$\hat{\bar{Y}}_{Prop1} = \delta_1 \, \bar{Y}(1+\xi_0) + \lambda_1\{\bar{Z} - \bar{Z}(1+\xi_1)\} + \lambda_2\{\bar{R}_z - \bar{R}_z(1+\xi_2)\},$$

or

$$\hat{\bar{Y}}_{Prop1} = \delta_1 \, \bar{Y} + \delta_1 \, \bar{Y}\xi_0 - \lambda_1\bar{Z}\xi_1 - \lambda_2\bar{R}_z\xi_2,$$

Subtracting $\bar{Y}$ from both sides for bias:

$$\hat{\bar{Y}}_{Prop1} - \bar{Y} = \bar{Y}(\delta_1 - 1) + \delta_1 \, \bar{Y}\xi_0 - \lambda_1\bar{Z}\xi_1 - \lambda_2\bar{R}_z\xi_2, \tag{15}$$

Considering expectations and simplifying, the suggested estimator's bias is:

$$Bias(\hat{\bar{Y}}_{Prop1}) = (\delta_1 - 1) \, \bar{Y} \tag{16}$$

For (MSE) we square on both sides of the (15) and after taking expectation

$$E(\hat{\bar{Y}}_{Prop1}-\bar{Y})^2=\bar{Y}^2(\delta_1-1)^2+\delta_1^2\,\bar{Y}^2E(\xi_0^2)+\lambda_1^2\bar{Z}^2E(\xi_1^2)+\lambda_2^2\bar{R}_z^2E(\xi_2^2)$$
$$-2\delta_1\lambda_1\bar{Y}\bar{Z}E(\xi_0\xi_1)-2\delta_1\lambda_2\bar{Y}\bar{R}_zE(\xi_0\xi_2)+2\lambda_1\lambda_2\bar{Z}\bar{R}_zE(\xi_1\xi_2)$$
$$+2\bar{Y}(\delta_1-1)(\delta_1\bar{Y}E(\xi_0)-\lambda_1\bar{Z}E(\xi_1)-\lambda_2\bar{R}_zE(\xi_2))\,,$$

By simplifying, the MSE of the proposed estimator is given as:

$$MSE(\hat{\bar{Y}}_{Prop1})=\bar{Y}^2(\delta_1-1)^2+\theta(\delta_1^2\,\bar{Y}^2C_y^2+\lambda_1^2\bar{Z}^2C_z^2+\lambda_2^2\bar{R}_z^2C_{R_z}^2$$
$$-2\delta_1\lambda_1\bar{Y}\bar{Z}\rho_{yz}C_yC_z-2\delta_1\lambda_2\bar{Y}\bar{R}_z\rho_{yR_z}C_yC_{r_z}+2\lambda_1\lambda_2\bar{Z}\bar{R}_z\rho_{zr_z}C_zC_{r_z})+2\bar{Y}(\delta_1-1)(0)\,,$$

or equivalently,

$$MSE(\hat{\bar{Y}}_{Prop1})=\bar{Y}^2(\delta_1-1)^2+\theta(\delta_1^2\,\bar{Y}^2C_y^2+\lambda_1^2\bar{Z}^2C_z^2+\lambda_2^2\bar{R}_z^2C_{R_z}^2$$
$$-2\delta_1\lambda_1\bar{Y}\bar{Z}\rho_{yz}C_yC_z-2\delta_1\lambda_2\bar{Y}\bar{R}_z\rho_{yR_z}C_yC_{r_z}+2\lambda_1\lambda_2\bar{Z}\bar{R}_z\rho_{zr_z}C_zC_{r_z})\quad(17)$$

The optimal values of the constants $\delta_1$, $\lambda_1$ and $\lambda_2$ are given as follows:

$$\delta_{1opt}=\frac{1}{1+\theta C_y^2(1-R_{yzr_z}^2)}\,,$$
$$\lambda_{1opt}=\frac{1}{R_1}\left(\frac{1}{1+C_y^2(1-R_{yzr_z}^2)}\right)\left(\frac{\beta_{yz}-\beta_{yr_z}\beta_{zr_z}}{1-\rho_{zr_z}^2}\right),$$
$$\lambda_{2opt}=\frac{1}{R_2}\left(\frac{1}{1+C_y^2(1-R_{yzr_z}^2)}\right)\left(\frac{\beta_{yr_z}-\beta_{yz}\beta_{zr_z}}{1-\rho_{zr_z}^2}\right),$$

where

$$R_1=\frac{\bar{Z}}{\bar{Y}}\quad\text{and}\quad R_2=\frac{\bar{R}_z}{\bar{Y}}\,.$$

By substituting the values of $\delta_1$, $\lambda_1$ and $\lambda_2$ in Equation (17) and simplifying we obtain the following:

$$MSE_{(min.)}(\hat{\bar{Y}}_{Prop1})=\frac{\bar{Y}^2MSE_{(min.)}(\hat{\bar{Y}}_9)}{\bar{Y}^2+MSE_{(min.)}(\hat{\bar{Y}}_9)}\qquad(18)$$

where

$$MSE_{(min.)}(\hat{\bar{Y}}_9)=\bar{Y}^2C_y^2(1-R_{yzr_z}^2)\,,$$
$$R_{y,zr_z}^2=\frac{\rho_{yz}^2+\rho_{yr_z}^2-2\rho_{yz}\rho_{yr_z}\rho_{zr_z}}{1-\rho_{zr_z}^2}\,,$$
$$\rho_{yz}^2=\frac{C_{yz}^2}{C_y^2C_z^2},\quad\rho_{yr_z}^2=\frac{C_{yr_z}^2}{C_y^2C_{r_z}^2}\quad and\quad\rho_{zr_z}^2=\frac{C_{zr_z}^2}{C_z^2C_{r_z}^2}\,,$$
$$C_y=\frac{S_y}{\bar{Y}},\ C_z=\frac{S_z}{\bar{Z}},\ C_{r_z}=\frac{S_{r_z}}{\bar{Z}},\ C_{yz}=\frac{S_{yz}}{\bar{Y}\bar{Z}}=\rho_{yz}C_yC_z\,,$$
$$C_{yr_z}=\frac{S_{yr_z}}{\bar{Y}\bar{R}_z}=\rho_{yr_z}C_yC_{r_z},\quad and\quad C_{zr_z}=\frac{S_{zr_z}}{\bar{Z}\bar{R}_z}=\rho_{zr_z}C_zC_{r_z}.$$

The minimum MSE of the suggested regression type estimator after entering all of these data in (17) is:

$$MSE_{(min.)}(\hat{\bar{Y}}_{Prop1}) = \bar{Y}^2\,(1-\delta_{1opt}) \tag{19}$$

**3.2 Proposed Ratio Type Estimator-II**

The second suggested a ratio-type estimator that uses rank as a dual of auxiliary variable is

$$\hat{\bar{Y}}_{Prop2} = \eta_1\,\bar{Y}\left(\frac{\bar{Z}}{\bar{z}}\right)^{\kappa_1}\left(\frac{\bar{R}_z}{\bar{r}_z}\right)^{\kappa_2} \tag{20}$$

where $\eta_1, \kappa_1$ and $\kappa_2$ are real constants that can be appropriately selected.

By substituting the values of $\bar{Y}, \bar{Z}$ and $\bar{R}_z$ into (20), we obtain:

$$\hat{\bar{Y}}_{Prop2} = \eta_1\,\bar{Y}(1+\xi 0)\left(\frac{\bar{Z}}{\bar{Z}(1+\xi_1)}\right)^{\kappa_1}\left(\frac{\bar{R}_z}{\bar{R}_z(1+\xi_2)}\right)^{\kappa_2},$$

or

$$\hat{\bar{Y}}_{Prop2} = \eta_1\,\bar{Y}(1+\xi_0)\,(1-\kappa_1\xi_1)(1-\kappa_2\xi_2),$$

or

$$\hat{\bar{Y}}_{Prop2} = \eta_1\,\bar{Y}(1+\xi_0)\,(1-\kappa_1\xi_1-\kappa_2\xi_2+\kappa_1\kappa_2\xi_1\xi_2),$$

or

$$\hat{\bar{Y}}_{Prop2} = \eta_1\,\bar{Y}(1+\xi_0-\kappa_1\xi 1-\kappa_2\xi_2-\kappa_1\xi_0\xi_1-\kappa_2\xi_0\xi_2+\kappa_1\kappa_2\xi_1\xi_2),$$

or

$$\hat{\bar{Y}}_{Prop2} = \eta_1\,\bar{Y}+\eta_1\,\bar{Y}(\xi_0-\kappa_1\xi_1-\kappa_2\xi_2-\kappa_1\xi_0\xi_1-\kappa_2\xi_0\xi_2+\kappa_1\kappa_2\xi 1\xi_2) \tag{21}$$

$\bar{Y}$ is subtracted from both sides of (21), we obtain

$$\hat{\bar{Y}}_{Prop2}-\bar{Y} = \bar{Y}(\eta_1-1)+\eta_1\,\bar{Y}(\xi_0-\kappa_1\xi_1-\kappa_2\xi_2-\kappa_1\xi_0\xi_1-\kappa_2\xi_0\xi_2+\kappa_1\kappa_2\xi_1\xi_2) \tag{22}$$

Taking expectation of (22)

$$E(\hat{\bar{Y}}_{Prop2}-\bar{Y}) = \bar{Y}(\eta_1-1)+\eta_1\,\bar{Y}E(\xi_0-\kappa_1\xi_1-\kappa_2\xi_2-\kappa_1\xi_0\xi_1-\kappa_2\xi_0\xi 2+\kappa_1\kappa_2\xi_1\xi_2).$$

By further simplification we have

$$Bias(\hat{\bar{Y}}_{Prop2}) = \bar{Y}(\eta_1-1)+\eta_1\,B(\hat{\bar{Y}}_{10}) \tag{23}$$

By squaring both sides of equation (22), using the expectation operator, and then simplifying the resultant expression, the MSE of this estimator is obtained.

$$\begin{aligned}E(\hat{\bar{Y}}_{Prop2}-\bar{Y})^2 &= (\eta_1-1)^2\bar{Y}^2\\ &+\eta_1^2\,\bar{Y}^2E(\xi_0^2+\kappa_1\xi_1^2+\kappa_2\xi_2^2-2\kappa_1\xi_0\xi_1-2\kappa_2\xi_0\xi_2+2\kappa_1\kappa_2\xi_1\xi_2)\\ &+2\bar{Y}(\eta_1-1)\eta_1\bar{Y}E(\xi_0-\kappa_1\xi 1-\kappa_2\xi_2-\kappa_1\xi_0\xi_1-\kappa_2\xi_0\xi_2+\kappa_1\kappa_2\xi_1\xi_2),\end{aligned}$$

After further simplification, the minimum MSE of the estimator $(\hat{\bar{Y}}_{Prop2})$ is given as:

$$MSE(min.)(\hat{\bar{Y}}_{Prop2}) = \bar{Y}^2 \left(1 - \frac{K_1^2}{J_1}\right) \tag{24}$$

where

$$J_1 = 1 + C_y^2 + C_z^2\{\kappa_{1opt}(1+\kappa_{1opt}) - 4\beta_{yz}\} + \kappa_{2opt}\, C_{r_z}^2\{1 + 2\kappa_{2opt} - 4(\beta_{yr_z} - \kappa_{1opt}\beta_{xr_z})\},$$

$$K_1 = 1 + C_z^2\, \frac{\kappa_{1opt}}{2}(1 + \kappa_{1opt} - 2\beta_{yz}) + C_{r_z}^2\, \frac{\kappa_{2opt}}{2}\{1 + \kappa_{2opt} + 2(\kappa_{1opt} - 1)\beta_{yr_z}\}\,,$$

$$\kappa_{1opt} = \frac{\beta_{yz} - \beta_{yr_z}\beta_{xr_z}}{1 - \rho_{xr_z}^2}, \quad \kappa_{2opt} = \frac{\beta_{yr_z} - \beta_{yz}\beta_{xr_z}}{1 - \rho_{xr_z}^2}\,.$$

$$\beta_{yz} = \frac{C_{yz}}{C_y^2}, \beta_{yr_z} = \frac{C_{yr_z}}{C_{r_z}^2} \quad and \quad \beta_{xr_z} = \frac{C_{xr_z}}{C_{r_z}^2}\,.$$

**3.3 Proposed Regression Type Estimator-III**

The third suggested regression-type estimator uses the Empirical Distribution Function as a dual application of the auxiliary variable is

$$\hat{\bar{Y}}_{Prop3} = \delta_2\, \bar{y} + \lambda_3(\bar{Z} - \bar{z}) + \lambda_4(\bar{F}_z - \bar{f}_z) \tag{26}$$

where $\delta_2, \lambda_3$ and $\lambda_4$ are constants to be chosen appropriately.

After substituting the values of $\bar{y}, \bar{z}$ and $\bar{f}_z$ we obtain:

$$\hat{\bar{Y}}_{Prop3} = \delta_2\, \bar{Y}(1+\xi_0) + \lambda_3\{\bar{Z} - \bar{Z}(1+\xi_1)\} + \lambda_4\{\bar{F}_z - \bar{F}_z(1+\xi_3)\}\,,$$

or

$$\hat{\bar{Y}}_{Prop3} = \delta_2\, \bar{Y}(1+\xi_0) + \lambda_3\{\bar{Z} - \bar{Z}(1+\xi_1)\} + \lambda_4\{\bar{F}_z - \bar{F}_z(1+\xi_3)\}\,,$$

Subtracting $\bar{Y}$ from (3.26), we have:

$$\hat{\bar{Y}}_{Prop3} - \bar{Y} = \bar{Y}(\delta_2 - 1) + \delta_2\, \bar{Y}\xi_0 - \lambda_3\bar{Z}\xi_1 - \lambda_4\bar{F}_z\xi_3 \tag{26}$$

By taking expectation, we get:

$$E(\hat{\bar{Y}}_{Prop3} - \bar{Y}) = \delta_2\, \bar{Y} - \bar{Y} + \delta_2\, \bar{Y}E(\xi_0) - \lambda_3\bar{Z}E(\xi_1) - \lambda_4\bar{F}_zE(\xi_3)\,.$$

The bias of the proposed estimator is given by:

$$Bias(\hat{\bar{Y}}_{Prop3}) = (\delta_2 - 1)\, \bar{Y} \tag{27}$$

In order to get the MSE of the suggested $\hat{\bar{Y}}_{Prop3}$ estimator, we square (27), and after taking expectation we have

$$\begin{aligned} E(\hat{\bar{Y}}_{Prop3} - \bar{Y})^2 = {} & \bar{Y}^2(\delta_2 - 1)^2 + \delta_2^2\bar{Y}^2E(\xi_0^2) + \lambda_3^2\bar{Z}^2E(\xi_1^2) + \lambda_4^2\bar{F}_z^2E(\xi_3^2) \\ & -2\delta_2\lambda_3\bar{Y}\bar{Z}E(\xi_0\xi_1) - 2\delta_2\lambda_4\bar{Y}\bar{F}_zE(\xi_0\xi_3) + 2\lambda_3\lambda_4\bar{Z}\bar{F}_zE(\xi_1\xi_3) \\ & +2\bar{Y}(\delta_2 - 1)(\delta_2\bar{Y}E(\xi_0) - \lambda_3\bar{Z}E(\xi_1) - \lambda_4\bar{F}_zE(\xi_3))\,, \end{aligned}$$

After simplification, the (MSE) of the estimator $\hat{\bar{Y}}_{Prop3}$ is:

$$MSE(\hat{\bar{Y}}_{Prop3}) = \bar{Y}^2(\delta_2 - 1)^2 + \theta\Big(\delta_2^2\bar{Y}^2C_y^2 + \lambda_3^2\bar{Z}^2C_z^2 + \lambda_4^2\bar{F}_z^2C_{f_z}^2 - 2\delta_2\lambda_3\bar{Y}\bar{Z}\rho_{yz}C_yC_z - 2\delta_2\lambda_4\bar{Y}\bar{F}_z\rho_{yf_z}C_yC_{f_z} - 2\lambda_3\lambda_4\bar{Z}\bar{F}_z\rho_{zf_z}C_zC_{f_z}\Big) + 2\bar{Y}(\delta_2 - 1)(0),$$

or equivalently,

$$MSE(\hat{\bar{Y}}_{Prop3}) = \bar{Y}^2(\delta_2 - 1)^2 + \theta\Big(\delta_2^2\bar{Y}^2C_y^2 + \lambda_3^2\bar{Z}^2C_z^2 + \lambda_4^2\bar{F}_z^2C_{f_z}^2 - 2\delta_2\lambda_3\bar{Y}\bar{Z}\rho_{yz}C_yC_z - 2\delta_2\lambda_4\bar{Y}\bar{F}_z\rho_{yf_z}C_yC_{f_z} + 2\lambda_3\lambda_4\bar{Z}\bar{F}_z\rho_{zf_z}C_zC_{f_z}\Big) \quad (28)$$

The optimum values of constants $\delta_2, \lambda_3$ and $\lambda_4$ are

$$\delta_{2opt} = \frac{1}{1 + C_y^2(1 - R_{yzf_z}^2)},$$

$$\lambda_{3opt} = \frac{1}{Z_1}\left(\frac{1}{1 + C_y^2(1 - R_{yzf_z}^2)}\right)\left(\frac{\beta_{yz} - \beta_{yzf_z}\beta_{zf_z}}{1 - \rho_{zf_z}^2}\right),$$

and

$$\lambda_{4opt} = \frac{1}{Z_3}\left(\frac{1}{1 + C_y^2(1 - R_{yzf_z}^2)}\right)\left(\frac{\beta_{yzf_z} - \beta_{yzf_z}\beta_{zf_z}}{1 - \rho_{zf_z}^2}\right), \quad (29)$$

where

$$R_1 = \frac{\bar{Z}}{\bar{Y}} \quad and \quad R_3 = \frac{\bar{F}_z}{\bar{Y}},$$

Putting the values of $\delta_2, \lambda_3$ and $\lambda_4$ in (29) and after simplification we get

$$MSE_{(min.)}(\hat{\bar{Y}}_{Prop3}) = \frac{\bar{Y}^2 MSE_{(min.)}(\hat{\bar{Y}}_9)}{\bar{Y}^2 + MSE_{(min.)}(\hat{\bar{Y}}_9)} \quad (30)$$

where

$$min.\,MSE(\hat{\bar{Y}}_9) = \bar{Y}^2C_y^2(1 - R_{yzr_z}^2),$$

$$R_{y,zf_z}^2 = \frac{\rho_{yz}^2 + \rho_{yf_z}^2 - 2\rho_{yz}\rho_{yf_z}\rho_{zf_z}}{1 - \rho_{zf_z}^2},$$

$$\rho_{yz}^2 = \frac{C_{yz}^2}{C_y^2C_z^2}, \quad \rho_{yf_z}^2 = \frac{C_{yf_z}^2}{C_y^2C_{f_z}^2} \quad and \quad \rho_{zf_z}^2 = \frac{C_{zf_z}^2}{C_z^2C_{f_z}^2},$$

$$C_y = \frac{S_y}{\bar{Y}},\ C_z = \frac{S_z}{\bar{Z}},\ C_{f_z} = \frac{S_{f_z}}{\bar{Z}},\ C_{yz} = \frac{S_{yz}}{\bar{Y}\bar{Z}} = \rho_{yz}C_yC_z,$$

$$C_{yf_z} = \frac{S_{yf_z}}{\bar{Y}\bar{F}_z} = \rho_{yf_z}C_yC_{f_z}, \quad and \quad C_{zf_z} = \frac{S_{zf_z}}{\bar{Z}\bar{F}_z} = \rho_{zf_z}C_zC_{f_z}.$$

The minimum MSE of the suggested regression type estimator after entering all of these values in (30) is

$$MSE_{(min.)}(\hat{\bar{Y}}_{Prop3}) = \bar{Y}^2\,(1 - \delta_{2opt}) \quad (31)$$

**3.4 Proposed Ratio Type Estimator-IV**

The fourth proposed estimator is a ratio type estimator that utilizes the EDF in a dual capacity is

$$\hat{\bar{Y}}_{Prop4} = \eta_2 \, \bar{Y} \left( \frac{\bar{Z}}{\bar{z}} \right)^{\kappa_3} \left( \frac{\bar{F}_x}{\bar{f}_x} \right)^{\kappa_4} \tag{32}$$

where $\eta_2, \kappa_3$ and $\kappa_4$ are real constants to be chosen suitably. After assuming the values of

$\bar{Y}, \bar{Z}$ and $\bar{F}_x$ we have:

$$\hat{\bar{Y}}_{Prop4} = \kappa_2 \, \bar{Y}(1+\xi_0) \left( \frac{\bar{Z}}{\bar{Z}(1+\xi_1)} \right)^{\kappa_3} \left( \frac{\bar{F}_x}{\bar{F}_x(1+\xi_3)} \right)^{\kappa_4}$$

or

$$\hat{\bar{Y}}_{Prop4} = \kappa_2 \, \bar{Y}(1+\xi_0)(1-\kappa_3\xi_1)(1-\kappa_4\xi_3)$$

or

$$\hat{\bar{Y}}_{Prop4} = \kappa_2 \, \bar{Y}(1+\epsilon_0)(1-\kappa_3\epsilon_1-\kappa_4\epsilon_2+\kappa_3\kappa_4\epsilon_1\epsilon_2)$$

or

$$\hat{\bar{Y}}_{Prop4} = \kappa_2 \, \bar{Y}(1+\epsilon_0-\kappa_3\epsilon_1-\kappa_4\epsilon_2-\kappa_3\epsilon_0\epsilon_1-\kappa_4\epsilon_0\epsilon_2+\kappa_3\kappa_4\epsilon_1\epsilon_2)$$

or

$$\hat{\bar{Y}}_{Prop4} = \kappa_2 \, \bar{Y} + \kappa_2 \, \bar{Y}(\epsilon_0-\kappa_3\epsilon_1-\kappa_4\epsilon_2-\kappa_3\epsilon_0\epsilon_1-\kappa_4\epsilon_0\epsilon_2+\kappa_3\kappa_4\epsilon_1\epsilon_2) \tag{33}$$

Subtracting $\bar{Y}$ from both sides of (33), we get

$$\hat{\bar{Y}}_{Prop4} - \bar{Y} = \bar{Y}(\kappa_2-1) + \kappa_2 \, \bar{Y}(\epsilon_0-\kappa_3\epsilon_1-\kappa_4\epsilon_2-\kappa_3\epsilon_0\epsilon_1-\kappa_4\epsilon_0\epsilon_2+\kappa_3\kappa_4\epsilon_1\epsilon_2) \tag{34}$$

Taking expectation of (34), we have

$$E(\hat{\bar{Y}}_{Prop4} - \bar{Y}) = \bar{Y}(\kappa_2-1) + \kappa_2 \, \bar{Y}E(\epsilon_0-\kappa_3\epsilon_1-\kappa_4\epsilon_2-\kappa_3\epsilon_0\epsilon_1-\kappa_4\epsilon_0\epsilon_2+\kappa_3\kappa_4\epsilon_1\epsilon_2)$$

After simplification, we get

$$Bias(\hat{\bar{Y}}_{Prop4}) = \bar{Y}(\kappa_2-1) + \kappa_2 \, B(\hat{\bar{Y}}_{10}) \tag{35}$$

For the expression of the MSE of this estimator, we square both sides of (34), take expectation, and after simplification, we get

$$\begin{aligned} E(\hat{\bar{Y}}_{Prop4} - \bar{Y})^2 &= (\eta_2-1)^2\bar{Y}^2 \\ &+ \eta_2^2 \, \bar{Y}^2 E(\epsilon_0^2+\kappa_3\epsilon_1^2+\kappa_4\epsilon_2^2-2\kappa_3\epsilon_0\epsilon_1-2\kappa_4\epsilon_0\epsilon_2+2\kappa_3\kappa_4\epsilon_1\epsilon_2) \\ &+2\bar{Y}(\eta_2-1)\eta_2\bar{Y}E(\epsilon_0-\kappa_3\epsilon_1-\kappa_4\epsilon_2-\kappa_3\epsilon_0\epsilon_1-\kappa_4\epsilon_0\epsilon_2+\kappa_3\kappa_4\epsilon_1\epsilon_2) \end{aligned}$$

or

$$\begin{aligned} MSE(\hat{\bar{Y}}_{Prop4}) &= (\eta_2-1)^2\bar{Y}^2 + \eta_2^2 \, \bar{Y}^2\tau(C_y^2+\kappa_3^2C_z^2+\kappa_4^2C_{F_z}^2 \\ &- 2\kappa_3\rho_{yz}C_yC_z - 2\kappa_4\rho_{yf_z}C_yC_{f_z}+2\kappa_3\kappa_4\rho_{zf_z}C_zC_{f_z}) \\ &+ \eta_2(\eta_2-1)\bar{Y}^2\tau(-2\kappa_3\rho_{yz}C_yC_z - 2\kappa_4\rho_{yf_z}C_yC_{f_z} + 2\kappa_3\kappa_4\rho_{zf_z}C_zC_{f_z}) \end{aligned}$$

After simplification, the minimum MSE of $(\hat{\bar{Y}}_{Prop4})$ is given as:

$$MSE_{(min.)}(\hat{\bar{Y}}_{Prop4}) = \bar{Y}^2 \left(1 - \frac{K_2^2}{J_2}\right) \tag{36}$$

where

$$J_2 = 1 + C_y^2 + C_z^2 \{\kappa_{3opt}(1 + \kappa_{3opt}) - 4\beta_{yz}\} + \kappa_{4opt}\, C_{f_z}^2 \{1 + 2\kappa_{4opt} - 4(\beta_{yf_z} - \kappa_{3opt}\beta_{zf_z})\},$$

$$K_2 = 1 + C_z^2 \frac{\kappa_{3opt}}{2}(1 + \kappa_{3opt} - 2\beta_{yz}) + C_{f_z}^2 \frac{\kappa_{4opt}}{2}\{1 + \kappa_{4opt} + 2(\kappa_{3opt} - 1)\beta_{yf_z}\},$$

$$\kappa_{3opt} = \frac{\beta_{yz} - \beta_{yf_z}\beta_{zf_z}}{1 - \rho_{zf_z}^2},\ \kappa_{4opt} = \frac{\beta_{yf_z} - \beta_{yz}\beta_{zf_z}}{1 - \rho_{zf_z}^2},$$

$$\beta_{yz} = \frac{C_{yz}}{C_y^2}, \beta_{yf_z} = \frac{C_{yf_z}}{C_{f_z}^2} \quad and \quad \beta_{zf_z} = \frac{C_{zf_z}}{C_{f_z}^2}.$$

## 4. COMPARATIVE STUDY

In this section, we have assessed and compared multiple population datasets to examined the performance of the newly proposed estimators in relation to existing similar estimators. The evaluation was carried out by assessing their MSEs and PREs, providing a comprehensive comparison of their effectiveness

**i. Population-I** (Source of data: Gujarati, 2009)

$Y =$ Eggs produced in Millions in 1991,
$X =$ Per dozen price of eggs in 1991.

**ii. Population-II** (Source of data: Koyuncu and Kadilar, 2009)

$Y =$ Number of teachers,
$X =$ Number of students.

**iii. Population-III** (Source of data: Punjab Bureau of Statistics, 2021-2022). Book Published by Bureau of Statistics Pakistan.

$Y =$ Chance of dying between one year of age and
$X =$ Chance of dying at the age of between first and fifth birth day.

**Table 1**
**Overview of the Comparative Study Data**

| Statistic↓ | Population-I | Population-II | Population-III |
|---|---|---|---|
| $N$ | 50 | 923 | 36 |
| $n$ | 5 | 180 | 5 |
| $\tau$ | 0.18 | 0.004472132 | 0.004472132 |
| $\bar{Y}$ | 1357.622 | 436.4345 | 660.1389 |
| $\bar{X}$ | 78.29 | 11440.5 | 215.6389 |
| $\bar{R}_x$ | 25.5 | 462 | 18.5 |
| $\bar{F}_x$ | 0.512 | 0.5005804 | 0.5146605 |
| $C_y$ | 1.223641 | 1.718333 | 1.471357 |
| $C_x$ | 0.2722885 | 1.864528 | 1.504528 |
| $C_{rx}$ | 0.5715933 | 0.5770377 | 0.5694581 |
| $C_{fx}$ | 0.5717075 | 0.5769416 | 0.5664392 |
| $\rho_{yx}$ | -0.2888328 | 0.9543029 | 0.9696729 |
| $\rho_{yrx}$ | -0.2460575 | 0.6444158 | 0.6034372 |
| $\rho_{yfx}$ | -0.2469467 | 0.6444185 | 0.6044836 |
| $\rho_{xrx}$ | 0.9460146 | 0.6306615 | 0.6071001 |
| $\rho_{xfx}$ | 0.9467713 | 0.6306635 | 0.6080477 |
| $\beta_{yx}$ | -2.871764 | 1.6968120 | 1.0278240 |
| $\beta_{yrx}$ | -0.001162381 | 9.0672150 | 0.005437071 |
| $\beta_{yfx}$ | -2.871764 | 7.726033 | 7.11273 |
| $\beta_{xrx}$ | 0.004456465 | 8.8736840 | 0.005470075 |
| $\beta_{xfx}$ | 11.04104 | 7.561122 | 7.154667 |

**Table 2**
**Variance & MSEs of the Proposed and Considered Estimators**

| Estimators | Population-I | Population-II | Population-III |
|---|---|---|---|
| $\bar{y}$ | 496750.7 | 2515.169 | 4219.113 |
| $\hat{\bar{Y}}_1$ | 585202.3 | 267.6354 | 263.8202 |
| $\hat{\bar{Y}}_2$ | 457493.7 | 10685.41 | 16997.39 |
| $\hat{\bar{Y}}_3 = \hat{\bar{Y}}_5 = \hat{\bar{Y}}_6$ | 455309.5 | 224.6194 | 252.0261 |
| $\hat{\bar{Y}}_7$ | 365115.3 | 224.3549 | 251.8804 |
| $\hat{\bar{Y}}_8$ | 412152.9 | 239.9453 | 255.4066 |
| $\hat{\bar{Y}}_9 = \hat{\bar{Y}}_{10}$ | 451940.1 | 217.05 | 250.5726 |
| $\hat{\bar{Y}}_{\Pr op1}$ | **70470.02** | **11.10148** | **18.68715** |
| $\hat{\bar{Y}}_{\Pr op2}$ | **69895.18** | **11.1012** | **18.35451** |
| $\hat{\bar{Y}}_{\Pr op3}$ | **65330.16** | **0.9695714** | **1.11995** |
| $\hat{\bar{Y}}_{\Pr op4}$ | **65315.72** | **0.9695701** | **1.119825** |

**Table 3**
**PREs of the Proposed & Considered Estimators**

| Estimators | Population-I | Population-II | Population-III |
|---|---|---|---|
| $\bar{y}$ | 100 | 100 | 100 |
| $\hat{\bar{Y}}_1$ | 84.88528 | 939.7742 | 1599.238 |
| $\hat{\bar{Y}}_2$ | 108.5809 | 23.53834 | 24.82212 |
| $\hat{\bar{Y}}_3 = \hat{\bar{Y}}_5 = \hat{\bar{Y}}_6$ | 109.1017 | 1119.747 | 1674.078 |
| $\hat{\bar{Y}}_7$ | 136.0531 | 1121.067 | 1675.046 |
| $\hat{\bar{Y}}_8$ | 120.5258 | 1048.2260 | 1651.9201 |
| $\hat{\bar{Y}}_9 = \hat{\bar{Y}}_{10}$ | 136.0531 | 1158.797 | 1683.788 |
| $\hat{\bar{Y}}_{\Pr op1}$ | **704.9107** | **22656.15** | **22577.62** |
| $\hat{\bar{Y}}_{\Pr op2}$ | **710.708** | **22656.73** | **22986.79** |
| $\hat{\bar{Y}}_{\Pr op3}$ | **760.3696** | **259410.3** | **376723.4** |
| $\hat{\bar{Y}}_{\Pr op4}$ | **760.5376** | **259410.7** | **376765.4** |

After analyzing the datasets, the Percentage Relative Efficiencies (PREs) were computed relative to estimator $\bar{y}$ for all proposed and existing estimators considered in the study. These results are summarized in Table 3.3. The data in Table 3.3 clearly illustrates that the proposed estimators outperform all the existing estimators included in this comparison. In the three Populations the PREs associated with the proposed estimators consistently exhibit substantially higher values than their counterparts. Furthermore, it is evident that among the proposed estimators, the use of the EDF as a dual auxiliary variable provides superior efficiency compared to the rank-based approach.

## 5. DISCUSSIONS AND CONCLUSION

This study makes a significant contribution to the enhancement of estimation methods aimed at determining the mean of finite populations. The primary objective of this research was to develop a novel class of estimators that utilize auxiliary information by incorporating both rank and EDF. The performance of the proposed estimators was assessed using three widely recognized datasets from various disciplines.

Additionally, a thorough comparative analysis was conducted, evaluating the proposed estimators against existing frameworks, including those introduced by Yaqoob et al., Pandey et al., and other related methodologies. The concise and structured approach of rank and empirical distribution functions was anticipated to enable the efficient integration of auxiliary information. The outcomes strongly validate the potential of rank and EDF as an effective tools for incorporating extra information into the estimation process. The proposed estimators consistently demonstrated superior performance over existing counterparts across all datasets.

Future research could explore the adaptability of these estimators in more complex designs, particularly those involving confounding variables within the target population. Moreover, investigating broader frameworks that incorporate cumulative functions in various forms offers a promising avenue for further development.

The analysis also underscores that the EDF yields consistently lower Mean Squared Errors (MSEs) and higher PREs compared to the rank-based approach across all datasets. Introducing the EDF as a dual-use auxiliary variable significantly enhances the efficiency of the estimation process.

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