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# WAVELET TRANSFORMATION TO ESTIMATE THE NONPARAMETRIC REGRESSION VIA SMOOTHING SPLINE

**Tahir R. Dikheel**[1] and **Tamheed Olewi Issa**[2]

[1] Department of Statistics, University of AL-Qadisiyah, Iraq
Email: tahir.dikheel@qu.edu.iq

[2] Department of Studies and Planning, University of Babylon, Iraq. Email: statistics.stp.24.5@qu.edu.iq

## ABSTRACT

Statistical methods focus on identifying key indicators of phenomena and studying the relationships between them using statistical estimates. While parametric methods rely on specific distributional assumptions that can be restrictive, nonparametric modelling does not require these assumptions, making it more flexible in handling complex and incomplete data. This thesis aims to employ smoothing spline and local polynomial regression combined with wavelet transforms to address the boundary issues arising when data transitions between the time and frequency domains. Various thresholding functions (visu and sure) will be tested, and the most suitable one will be selected using test functions such as Doppler, Heavisine, and Blocks. This evaluation will be conducted across different sample sizes (32, 64, 128, 256, 512, 1024) with a constant signal-to-noise ratio (SNR = 5). Finally, these methods will be applied to real data concerning monthly oil revenues and their impact on inflation.

## KEYWORDS

Wavelet transformation, nonparametric regression, threshold functions, threshold values, local polynomial, smoothing splines.

## INTRODUCTION

Regression analysis is a vital statistical tool for understanding and characterizing relationships between random variables, enhancing both prediction accuracy and the interpretation of causal relationships. Traditional methods predominantly rely on parametric models, which assume a known probability distribution for the data, such as the normal distribution. However, this assumption often fails to align with real-world data characteristics in many practical applications. In contrast, nonparametric models offer a more flexible alternative, as they do not impose strict assumptions about data distribution. This grants them greater capability in handling complex and nonlinear relationships. Over recent decades, nonparametric models have undergone significant advancements, becoming a cornerstone in statistical analysis and machine learning due to their efficiency in uncovering subtle and hidden data patterns. This study employs smoothing splines and wavelet transforms to estimate nonparametric regression models, leveraging advanced techniques such as thresholding functions and kernel methods. These approaches aim to address the challenges of processing complex-structured data, thereby improving the

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accuracy and reliability of statistical estimations across diverse fields, including finance, engineering, and medical sciences. introduced the concept of wavelet shrinkage to handle unknown smoothness in data. By improving the thresholding process in wavelet transformations, they successfully removed noise while preserving important details, leading to more accurate statistical estimations, this method has significantly contributed to enhancing statistical techniques, particularly in nonparametric regression and the analysis of noisy data Donoho & Johnstone,1995. investigated the critical aspect of selection of the threshold parameter in wavelet function estimation, an important aspect for denoising and accurate function estimation was introduced by Nason, he discussed the trade-off between bias and variance in thresholding methods, which improved wavelet-based statistical techniques, this work contributed to more efficient methods in signal processing, particularly in noise reduction applications (Nason, 1995). compared the techniques used for prediction in nonparametric regression models, these methods have been suggested to handle missing data in nonparametric regression by using wavelet to estimate nonparametric regression functions. Practical experiments showed that these methods are effective and can be applied successfully (Hamza, 2015). applied wavelet transform with AdaBoost modelling to improve wind speed forecasts in Yunnan, China. Their study showed significant accuracy improvements using advanced techniques and real- data**.** (Shao, 2017). The Wavelet transform increases the precision of medical diagnosis of degenerative changes in the spine, according to a study that used the Wavelet Gabor transform to analyse Magnetic Resonance Images (MRI)( Tao Yang,2020). Selecting the appropriate mother wavelet is crucial as it allows the majority of the signal's energy to be concentrated in a small number of wavelet coefficients. This enhances the ability to remove noise and obtain precise estimates for the regression function. Given the importance of analysing the price index in Iraq, various mother wavelets were evaluated using wavelet transform techniques. The aim was to identify the most suitable wavelet for this data, providing accurate estimates of the relationship between trading volume (independent variable) and the Iraq Stock Market Index (dependent variable). The results indicated that the Coif1, Coif5, and rbio1.3 The wavelets gave the best results, as they had the lowest Mean Squared Error (MSE) (Hamza & Ali, 2022). This research paper presents a two-step methodology for higher-accuracy estimation of nonparametric models. The methodology begins with model estimation using two distinct approaches: local polynomial regression and smoothing spline regression. Following estimation, the resulting residuals from each model are computed, representing the differences between actual and estimated values. In the second step, the Discrete Wavelet Transform (DWT) is applied exclusively to these residuals. This process aims to analyse them across multiple frequency levels and extract their noise components using thresholding techniques such as sure-shrink and visu-shrink. Subsequently, the denoised signal is reconstructed and combined with the initial estimation to refine the final model. The methodology's efficiency was evaluated using the Mean Squared Error (MSE) criterion, with results demonstrating clear superiority over conventional methods. This outcome highlights the proposed methodology's effectiveness in handling complex, nonlinear data across multiple domains.

Wavelet theory's advancements are an essential first step toward more thorough and precise data processing. Two primary filter types are employed while dealing with wavelet transform: high-pass filters are used to extract high-frequency information from the signal,

and low-pass filters are used to recover low-frequency information. There are two primary components to wavelets (Daubechies, 1992). In order to determine the form and behavior of the wavelets at various resolution levels, more wavelet functions are constructed using the father function (scaling function). It is obtained using the following formula:

$$\phi_{(x)} = \sum_{k=0}^{N} C(K)\sqrt{2}\phi(2x-k) \quad (1)$$

where C(k) are scalar coefficients that, when applied to a filter, alter the original signal. Each signal point's impact on its nearby points is determined by these weights. A mathematical function called the Mother function (Wavelet Function) is used to scale and shift a series of wave functions. The signals in the wavelet transform are examined using the functions that are produced.

$$\psi_{(x)} = \sum_{k=0}^{N} d(k)\sqrt{2}\psi(2x-k) \quad (2)$$

where d(k) is an integer that is used to extract fast changes and precise details from a signal. It can be compared to a "edge finder" that displays abrupt signal shifts. The Mother function's properties (Donald et al., 2004) Dimensions and displacement: All additional wavelet functions are created via transformation and scaling procedures, which allow the signal to be broken down into its constituent parts. It is acquired by applying the subsequent formula.:

$$\psi_{a,b}(t) = (1/\sqrt{|a|}\psi(t-\frac{b}{a}) \quad (3)$$

where,

$$a, b \in R, a \neq 0, \psi \in l^2(R), ||\psi|| = 1, \psi_{a,b}: R \to C.$$

where, $a$ is the scaling factor, $b$ is the displacement coefficient.

Zero periodicity the mother function has the property that its integral over time is zero.

$$\int_{-\infty}^{\infty} \psi(x)dx = 0 \quad (4)$$

$$\int_{-\infty}^{\infty} |x|^{\propto}|\psi(x)|dx < \infty \ (5) \ , \propto > \frac{1}{2}$$

for some values.

Local in time and frequency the mother function is local in both time and frequency.

Mother function types We'll talk about a few popular wavelet functions, such as Wavelet of Haar the wavelet transform was invented by mathematician Alfred Haar in 1909 Haar, 1910. One of the most basic waveforms for wavelet processing and signal analysis is the Haar Wavelet. The Daubechies Wavelet, which is sometimes shortened to DbN, is a set of wavelets that were first presented by mathematician Ingrid Daubechies in 1988. The wavelet's order is denoted by $N$ (Cohen et al., 1993). Many mathematicians and

researchers found the wavelet transform in the late 20th century, which is a major improvement over earlier transforms. The wavelet transform uses a variable amplitude window, which allows for constant change of the window's size and location across the signal, in contrast to conventional techniques that use a fixed amplitude window. This makes it possible to efficiently extract information about frequency variations, giving signal analysis more flexibility. Therefore, even with noisy signals, wavelet transform may break down signals into components of various frequencies while maintaining the important information. Researchers across various disciplines agree that the wavelet transform is one of the most accurate and efficient methods for signal analysis, the most famous wavelet transforms are Continuous wavelet transform(CWT) (Zaidan, 2022). The continuous wavelet transform divides the signal into an infinite number of wavelets (the mother wavelet) that depend on location and scale which can be expressed mathematically as follows:

$$\Psi_{a,b(x)=\frac{1}{\sqrt{|a|}}}\psi\left(\frac{x-b}{a}\right) \tag{5}$$

where, $a$ is the Scale factor while $b$ is the time location of the wavelet function, as for the amount $\frac{1}{\sqrt{|a|}}$ represents the amount of energy of the signal. The continuous wavelet transform is defined as the internal multiplication of the signal $f(t)$ with the basis function of the wavelet (the mother wavelet) (Mallat, 1999):

$$\psi_{(a,b)} = \frac{1}{\sqrt{a}}\int_{-\infty}^{\infty}\psi_{a,b}\left(\frac{x-b}{a}\right)f(x)dx \tag{6}$$

The DWT is a discrete set of wavelet scales and translations. It is especially designed for the sampled value (Daubechies, 1992). This transformation decomposes the signal into a mutually orthogonal set of wavelets. This specificity is the main difference between DWT and CWT. In reality, the DWT's dyadic grid scales the mother wavelet by power two. $(a = 2^j)$ and converted using an integer $(b = k2^j)$ , where the location index, $k$, runs from 1 to $2^{-j}N$ (N is the J is the total number of scales, and $j$ ranges from 0 to J (number of observations). The following formula is used to express the DWT (Rhif et al., 2019):

$$\psi_{j,k}(x) = 2^{\frac{-j}{2}}\psi(2^{-j}x-k) \tag{7}$$

Wavelet Threshold Functions: A threshold term is used to eliminate noise from the signal in the wavelet function estimation process (Hedaoo & Godbole, 2011). A frequency cutoff that removes noise coefficients while maintaining the original signal coefficients is established using the discrete wavelet transform. According to (Donoho 1995), threshold pieces come in a variety of forms, such as hard, soft, and semi-soft. Controlling the magnitude of the interference or noise effect is the main objective of employing a threshold value; selecting the ideal threshold value is a crucial step in achieving the lowest mean square error (MSE). Sure Thresholding is one of the most popular methods for figuring out a threshold value in wavelet analysis, (Donoho & Johnstone, 1995).

$$\lambda_{j,\mathrm{SURE}} = arg\ min_{0\le\lambda\le\sqrt{2logN}}SURE((\lambda_{j,}\mathrm{d}_{jk}) \tag{8}$$

The value of $\lambda$ that minimizes the Stein Unbiased Risk (Sure) is determined. Transactions that represent noise and those that represent actual information are distinguished by this value. The comprehensive threshold approach, which serves as the fundamental building block for determining the threshold value, is improved upon by the Visu Threshold method. The following formula explains how this approach overcomes the drawbacks of the comprehensive threshold while maintaining acceptable performance even with larger sample sizes:

$$\lambda_{visushrink} = \sigma_n\sqrt{2log(n)} \tag{9}$$

$\sigma_n$ it is the standard deviation of the noise level, which can be found through the following relationship:

$$\sigma_n = \text{MAD(Y)}/0.6745 \tag{10}$$

where, MAD represents the median absolute deviation of the wave coefficients (Donoho, 1995).

**Spline Smoothing Wavelet Regression**

The method was introduced as an improvement to the approach proposed by researchers Lee, Naveen, and Oh (2001). This simplified method is called Polynomial Wavelet Regression (PWR), and its basic idea is to combine wavelet estimation with a smoothing function using splines instead of using polynomial functions. The mathematical formula for this method is as follows:

$$\hat{f}_{pw}(x) = \hat{f}(x) + \sum_{k=1}^{2^j-1} \hat{c}_{0,k}\,\phi_k(e_i) + \sum_{j=0}^{j-1}\sum_{k=1}^{2^j-1} \hat{d}^s{}_{j,k}\,\psi_{j,k}(e_i) \tag{11}$$

The first part $\hat{f}(x)$ represents the smoothing function, which is estimated using splines, while the second and third parts

$$\sum_{k=1}^{2^j-1} \hat{c}_{0,k}\,\phi_k(e_i), \sum_{j=0}^{j-1}\sum_{k=1}^{2^j-1} \hat{d}^s{}_{j,k}\,\psi_{j,k}(e_i)$$

represent the wavelet estimation. This can be written as:

$$\hat{f}_{pw}(x) = \hat{f}(x) + \hat{f}_w(e_i) \tag{12}$$

where

$\hat{\hat{f}}_{pw}(x)$ represent the final estimate of the function.

$\hat{f}(x)$ represent the smoothed model, which removes the problem of non-periodicity from the observations.

$\hat{f}_w(e_i)$ represent the wavelet estimate of the residual signal hidden in the residuals. This method provides a more effective way of dealing with non-periodic data while improving signal processing through wavelet transformations.

**Local Polynomial Wavelet Regression**

The local polynomial wavelet regression method was introduced by Oh & Lee (2005) as an improvement to address boundary issues in wavelet regression, instead of using an external polynomial as in Oh. They proposed using a suitable second-degree local polynomial model. The local polynomial wavelet regression estimator can be written as follows:

$$\hat{f}LPW(x) = \sum_{r=1}^{d} \hat{\alpha}_n x^r + \sum_{\mathrm{k}=1}^{2^{\mathrm{j}}-1} \hat{\mathrm{c}}_{0,\mathrm{k}}\, \phi_{\mathrm{k}}(\mathrm{e_i}) + \sum_{j=0}^{j-1} \sum_{k=1}^{2^j-1} \hat{d}^s{}_{j,k}\, \psi_{j,k}(e_i) \qquad (13)$$

The first part $\sum_{r=1}^{d} \hat{\alpha}_n x^r$ represents the polynomial, which is estimated using the least squares method, while the second and third parts

$$\sum_{\mathrm{K}=1}^{2^{\mathrm{j}}-1} \hat{\mathrm{c}}_{0,\mathrm{k}}\, \phi_{\mathrm{k}}(\mathrm{e_I}), \sum_{j=0}^{j-1} \sum_{k=1}^{2^j-1} \hat{d}^s{}_{j,k}\, \psi_{j,k}(e_i)$$

represent the wavelet estimate. This formula can be written as

$$\hat{f}LPW(x) = \hat{f}_{LP}(x) + \hat{f}_w(e_i) \qquad (14)$$

As explained in Oh & Lee (2005), the estimation is performed through an iterative algorithm inspired by the back-fitting algorithm of Hastie & Tibshirani (1990).

The summarized steps to find the local polynomial wavelet regression estimator are:

1. Initialization: Select initial values $f_0$ and $f_1$, and set $\hat{f}LPW(x) = \hat{f}_0^*$, where $\hat{f}_0^*$, represent $a$ an enhanced initial estimate.
2. Iteration: Repeat the following steps for $j = 1,2, \ldots, j$.
   a) Apply wavelet regression to the residuals.
   b) Estimate $\hat{f}_{LP}^{j}$ by fitting a local polynomial regression to $y_i - \hat{f}_W^j$.
3. Stop when the results are close enough $\hat{f}_{LPW} = \hat{f}_{LP}^{j} + \hat{f}_W^j$.

At the end of this theory chapter, we move to practical testing in the next chapter. We'll check how well the proposed method works by running simulations with different tests, measuring its efficiency and accuracy on real data.

**Simulation**

The wavelet transforms via local polynomial and smoothing spline was used to construct the nonparametric regression model, which was then implemented in multiple steps using R software. The specifics of the simulation's development and execution experiment are as follows:

**Step 1:** involves building the model and defining the variables. A nonparametric regression model with the equation $y_i = f(x_i) + e_i$, where the values of the independent variable $x_i$ is generated, such that $x_i = \frac{i}{n}$, where $i = 1,2, \ldots, n$. Calculating values $f(x_i)$ using the test functions (Doppler, heavisine, blocks). The errors $e_i$ is generated from a normal distribution $N(0,1)$ to ensure that the data contains random noise.

**Step 2:** Determine the sample size and signal-to-noise ratio (SNR=5) different sample sizes were used which are $n = 32, 64, 128, 256, 512, 1024$.One of the prerequisites for the wavelet transform is that the sample size must be dyadic $n = 2^j$, such $j$ is positive integr number.

**Step 3:** The wavelet transform was applied using the Daubechies wavelet function db10 as the mother wavelet. This high-order wavelet $(N\ 10)$ is characterized by:

- High analytical accuracy due to the presence of 10 vanishing moments, which enables the representation of complex signals.
- An optimal balance between time and frequency resolution.
- Superior capability in separating signal components from noise within the framework of multiresolution analysis (MRA)

**Step 4:** Three of threshold functions are used hard, soft, and semi soft with two threshold values, sure and visu, were used to adjust the conversion coefficients and improve the accuracy of the estimates.

**Step 5:** The nonparametric regression model is estimated using two distinct approaches:

The local polynomial regression employing the least squares method, and a smoothing spline technique. For the smoothing spline implementation, the smoothing parameter λ is optimally determined through cross-validation to achieve an optimal balance between model fit and smoothness characteristics.

**Step 6:** Comparing the results using the MSE, which is a measure of how close the estimates are to the true values.

**Step 7:** replicate each experiment 500 times to ensure the stability results.

The following diagram represents the steps that used in our simulation experiment:

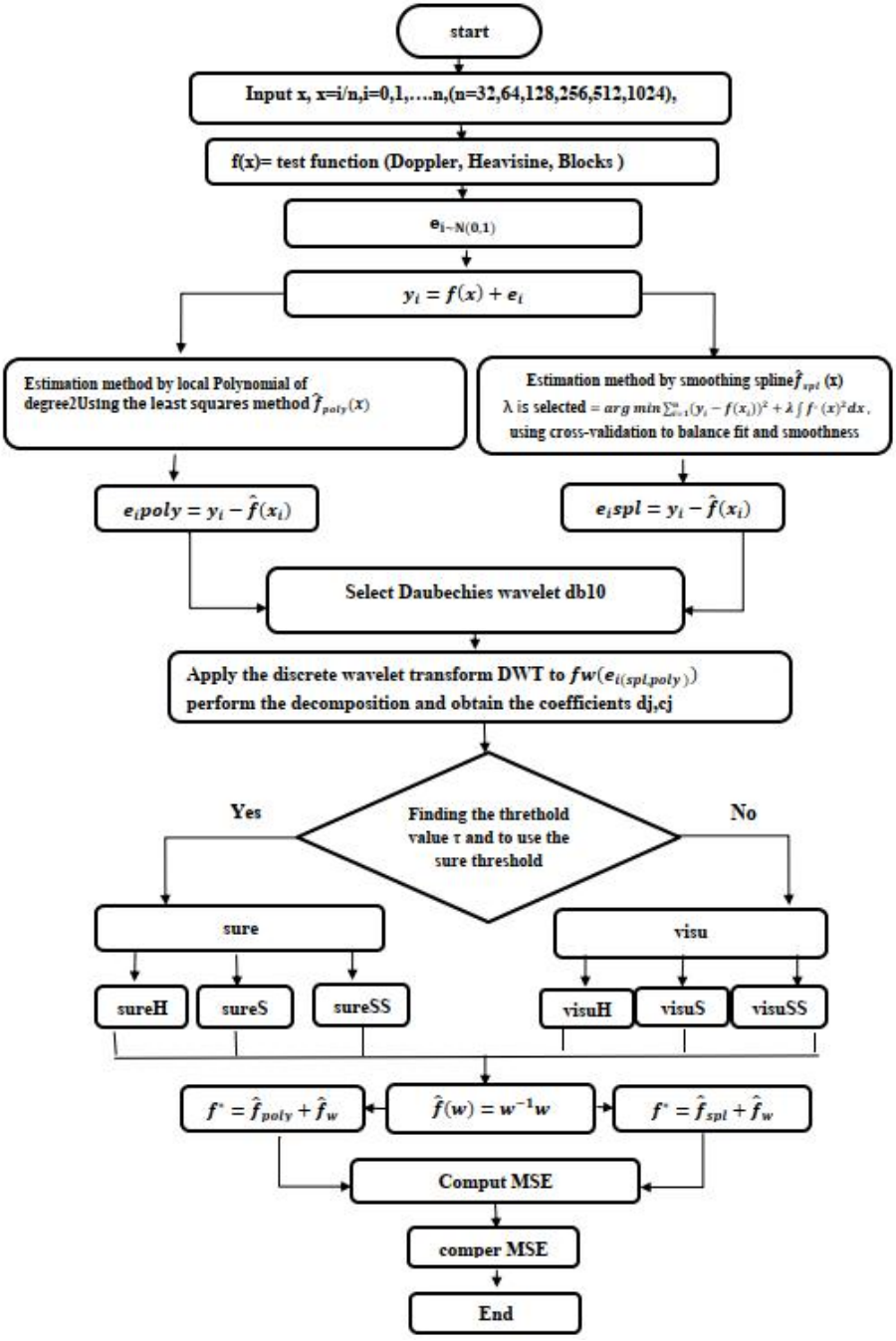


**Figure 1: Diagram of the Simulation**

The *R* program is run and the results are shown in the following table:

**Table 1**
**MSE of the Methods**

| Method used | Threshold function | Threshold value | Sample size | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | 32 | 64 | 128 | 256 | 513 | 1024 |
| Local Polynomial | Hard | Visu | 3.536224 | 2.228681 | 1.089453 | 0.496347 | 0.270797 | 0.201026 |
| | | Sure | 3.572651 | 2.281423 | 1.126279 | 0.553959 | 0.292342 | 0.189981 |
| | Soft | Visu | 1.111386 | 0.901128 | 0.7855 | 0.850221 | 0.971276 | 1.097325 |
| | | Sure | 2.722852 | 1.847898 | 0.8314 | 0.370555 | 0.130075 | 0.072852 |
| | Semi Soft | Visu | 1.852357 | 1.103191 | 0.56209 | 0.326227 | 0.281014 | 0.304074 |
| | | Sure | 3.10424 | 2.06683 | 0.964522 | 0.4625 | 0.204233 | 0.123032 |
| Smoothing Spline | Hard | Visu | 3.493136 | 2.174998 | 0.889128 | 0.407969 | 0.261878 | 0.165287 |
| | | Sure | 3.557541 | 2.287094 | 1.128985 | 0.558482 | 0.297199 | 0.192833 |
| | Soft | Visu | 1.790398 | 0.984841 | 0.336734 | 0.128065 | 0.135718 | 0.161124 |
| | | Sure | 3.295285 | 1.928657 | 0.726298 | 0.299119 | 0.100595 | 0.048486 |
| | Semi Soft | Visu | 2.550258 | 1.497262 | 0.53357 | 0.187646 | 0.105545 | 0.09743 |
| | | Sure | 3.424344 | 2.095942 | 0.899254 | 0.40301 | 0.166111 | 0.095391 |

The above table give us some findings are as follows:

- The soft threshold function combined with the Visu threshold value consistently yielded the best performance across most sample sizes.
- For small sample sizes ($n = 32$ and 64), local polynomial regression showed strong performance, particularly when using the soft threshold function.
- Beginning at sample size $n = 128$, the spline smoothing method outperformed local polynomial regression.
- At larger sample sizes ($n = 256$, 512, and 1024), spline smoothing with soft or semi-soft thresholding provided the lowest Mean Squared Error (MSE).
- In general, MSE decreases as sample size increases, indicating improved estimation accuracy.
- The soft threshold function outperformed both hard and semi-soft thresholding in most cases.
- An exception was observed at $n = 32$, where some fluctuations in performance were noted

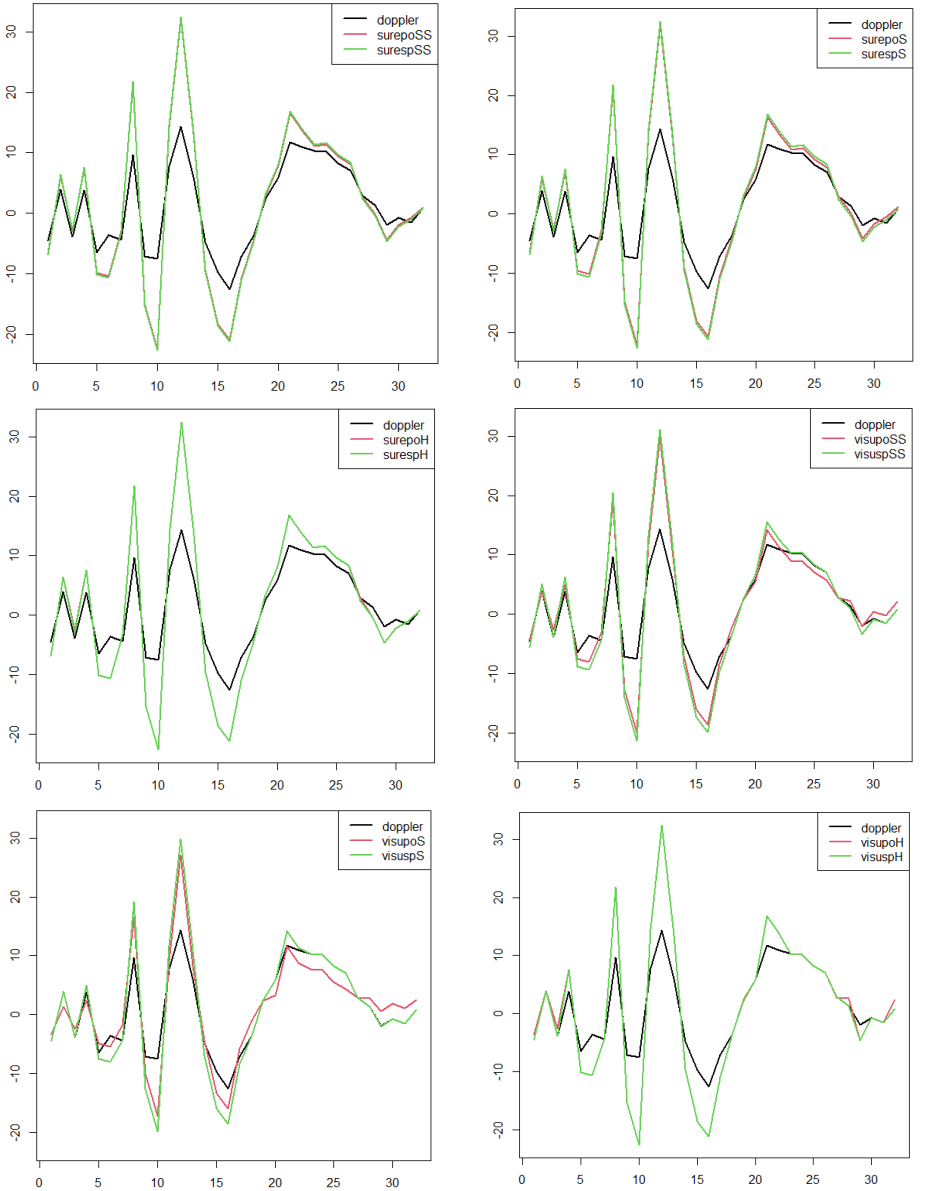


**Figure 2: Results of the Doppler Function Analysis for a Sample Size (32)**

The original signal is shown by the black line in Figure 2, the estimated signal using the local polynomial approach is shown by the red line, and the estimated signal using the smoothing spline method is shown by the green line.

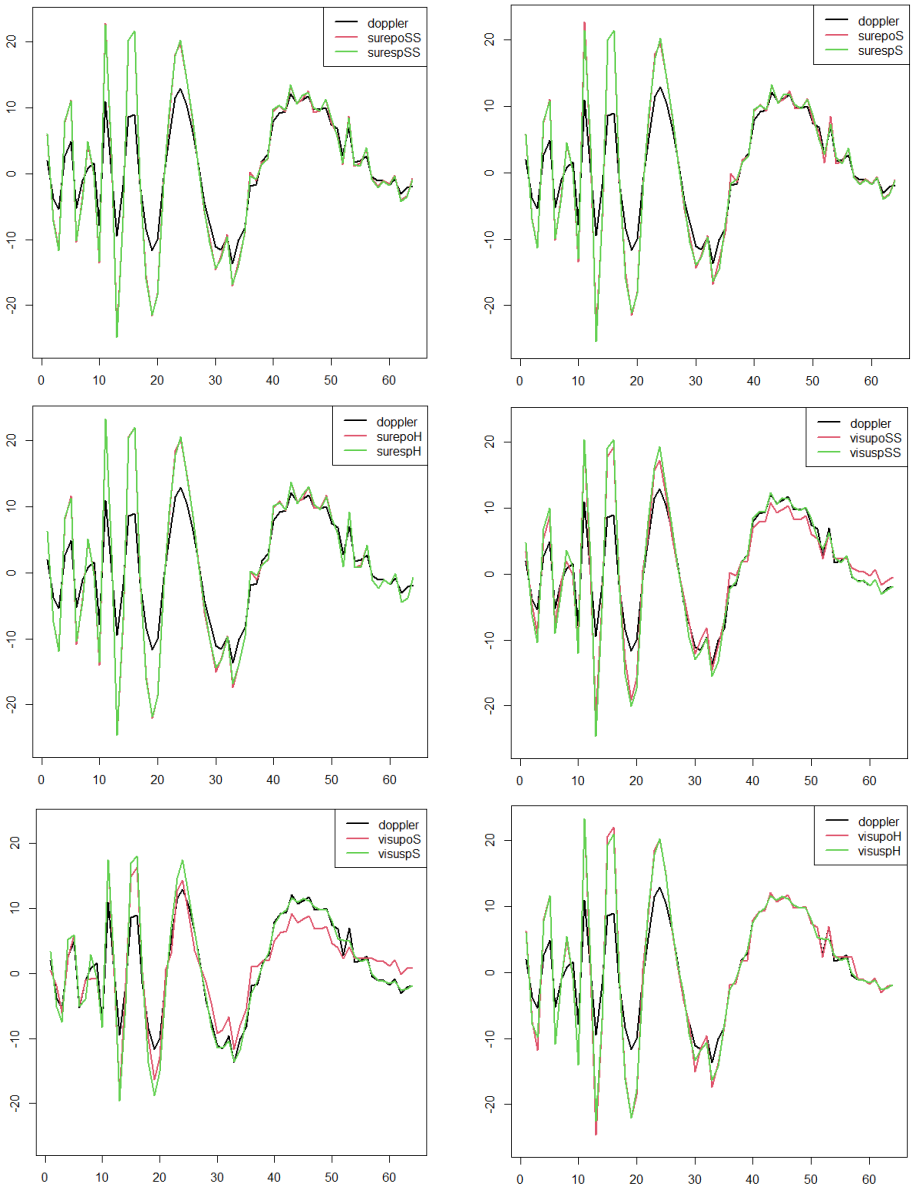


**Figure 3: Results of the Doppler Function Analysis for a Sample Size (64)**

The actual signal is represented by the black line in Figure 3, the estimated signal using the local polynomial approach is represented by the red line, and the estimated signal using the smoothing spline method is represented by the green line.

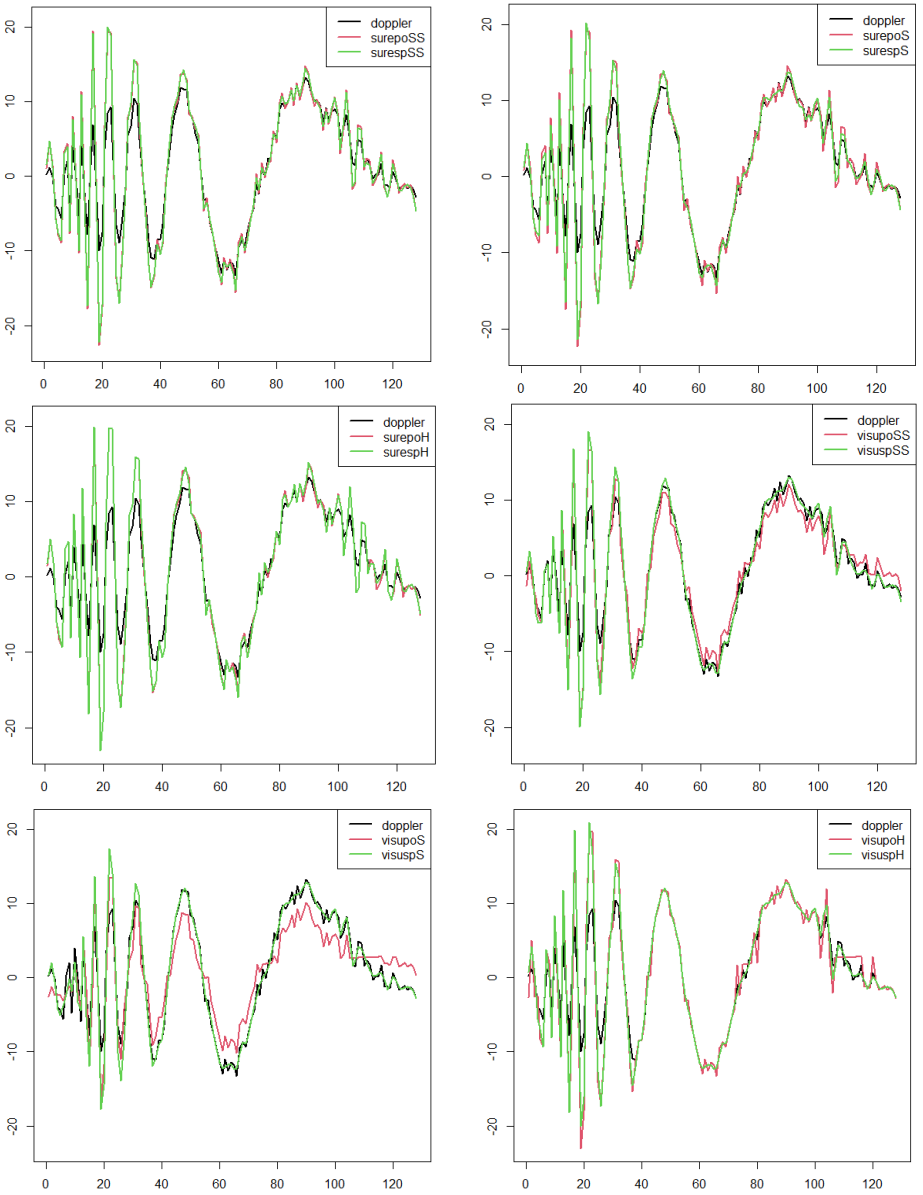


**Figure 4: Results of the Doppler Function Analysis for a Sample Size (128)**

The actual signal is shown by the black line in Figure 4, the estimated signal using the local polynomial approach is shown by the red line, and the estimated signal using the smoothing spline method is shown by the green line.

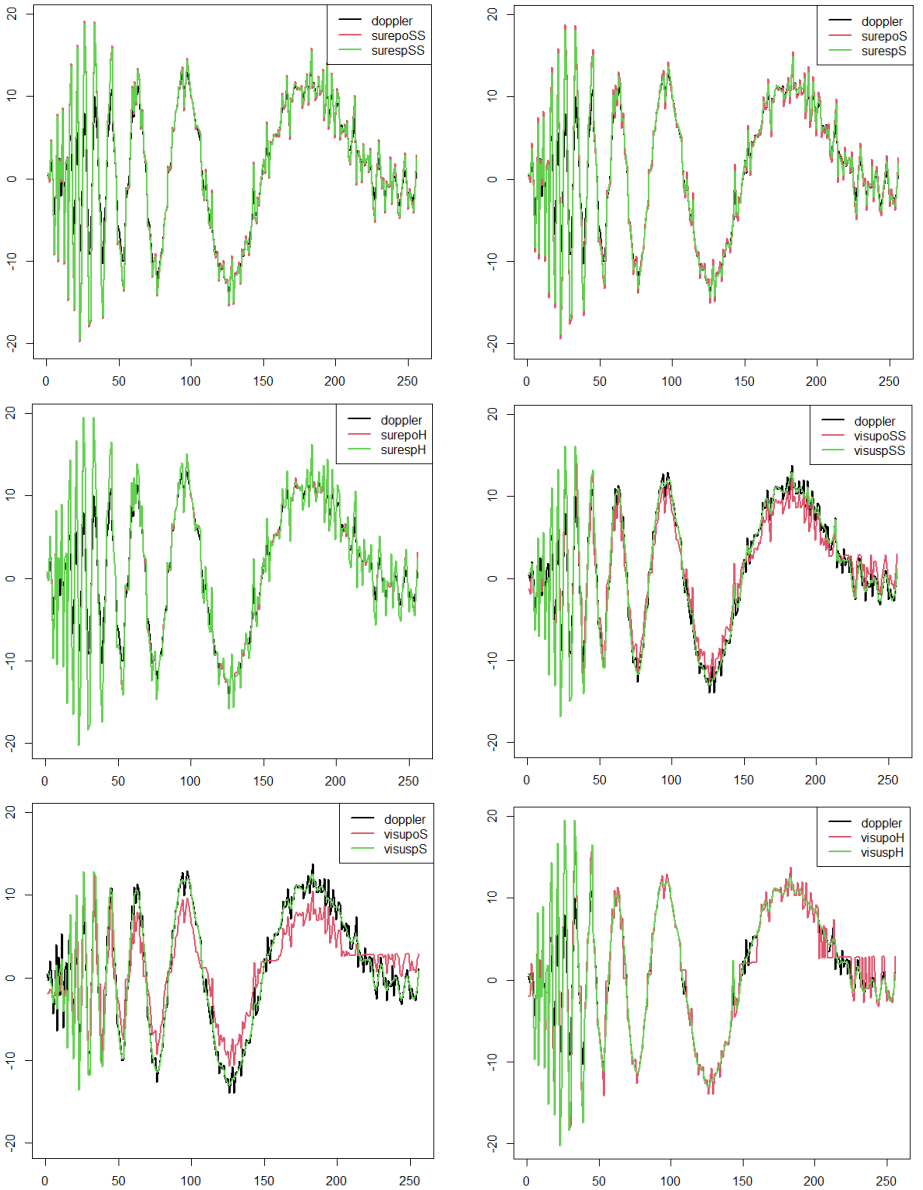


**Figure 5: Results of the Doppler Function Analysis for a Sample Size (256)**

The actual signal is shown by the black line in Figure 5, the estimated signal using the local polynomial approach is shown by the red line, and the estimated signal using the smoothing spline method is shown by the green line.

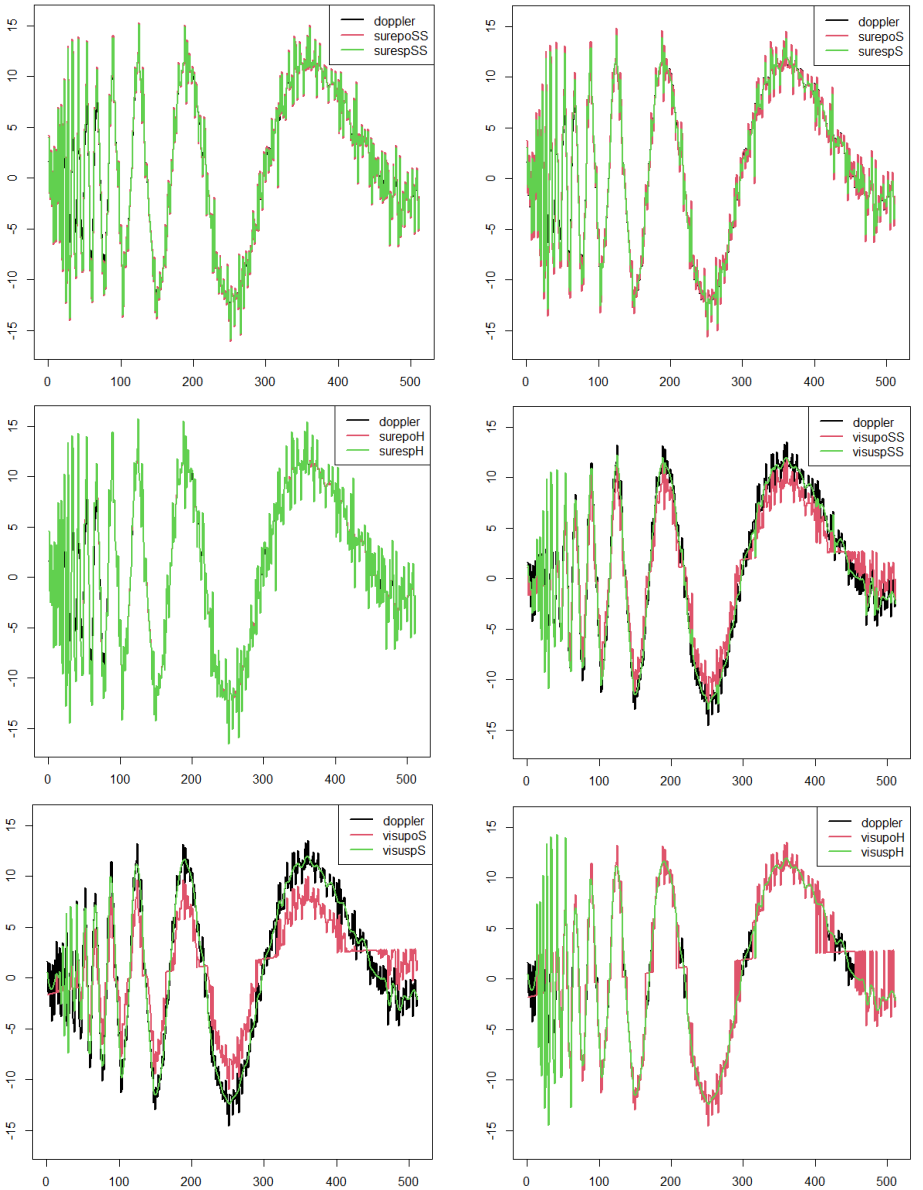


**Figure 6: Results of the Doppler Function Analysis for a Sample Size (512)**

The actual signal is shown by the black line in Figure 6, the estimated signal using the local polynomial approach is shown by the red line, and the estimated signal using the smoothing spline method is shown by the green line.

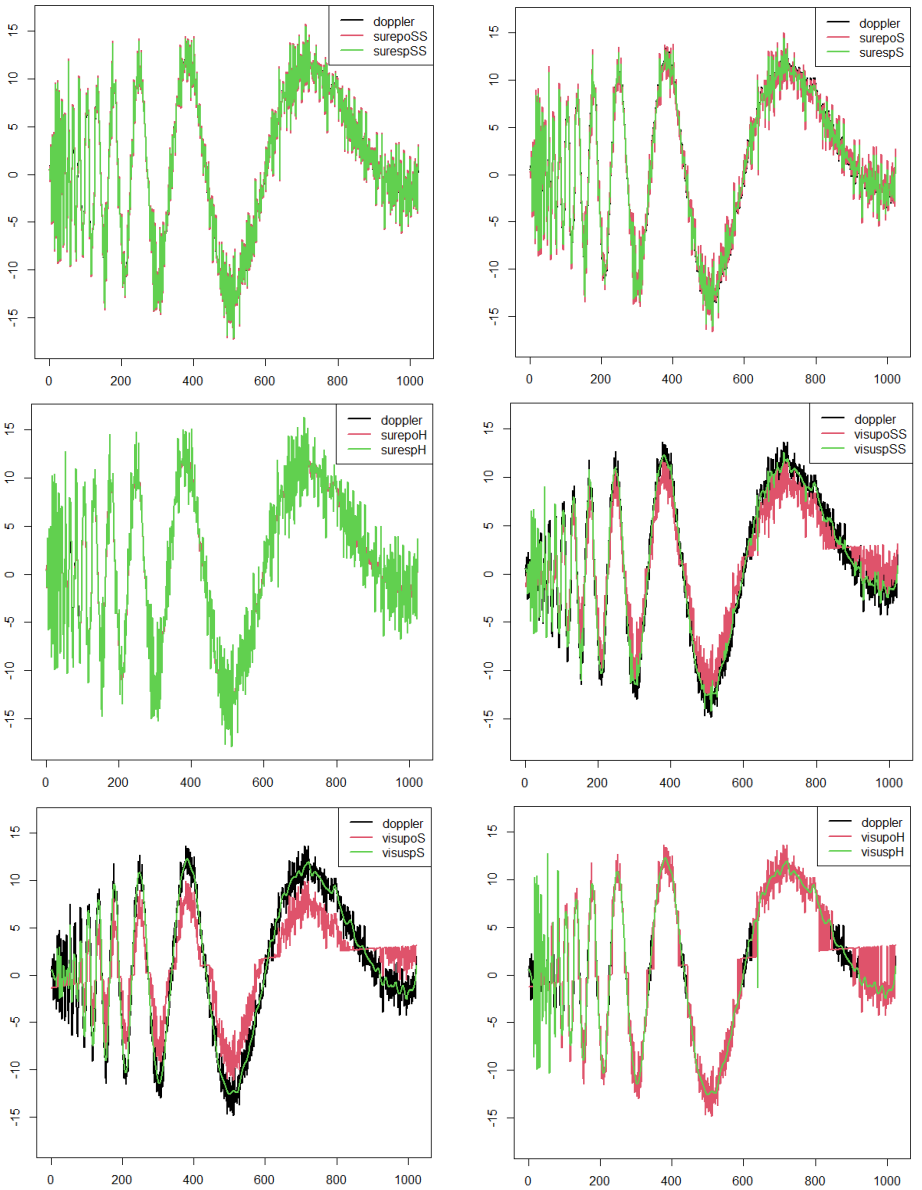


**Figure 7: Results of the Doppler Function Analysis for a Sample Size (1024)**

In Figure 7, the original signal is shown by the black line, the estimated signal using the local polynomial approach is shown by the red line, and the estimated signal using the smoothing spline method is shown by the green line.

**Iraqi Oil Revenue and Inflation Data**

Two variables oil revenues and inflation are included in the data, which spans the period from May 31, 2007, to December 31, 2017, and has a sample size of 128. The explanatory variable, which represents the entire amount of money the nation makes from the production and sale of oil, is oil revenue. Numerous intricate elements, such as OPEC's production decisions, political upheavals, and shifts in the economy, such as expansion or recession, have an impact on oil profits. Predicting oil prices is very difficult due to international difficulties, market fluctuations, and worldwide supply and demand. Traditional models are not appropriate for predicting oil income because of the unpredictable nature of these components. We chose non-parametric models as a result, particularly the Smoothing Spline method combined with Wavelet Transform, to estimate the relationship. This approach allows for better data cleaning and analysis while adapting to the ongoing changes in the relationship between oil revenues and inflation, improving accuracy and effectively managing data fluctuations.the data is available at the OAPEC data base link http://oapecdbsys.oapecorg.org:8081/ords/f?p=100:23 and Central Bank database link https://cbiraq.org/SeriesChart.aspx?TseriesID=423

**Table 2**
**MSE Values for the Inflation Rate with the Best Method**

| Estimation Method | Threshold Function | Threshold Value | MSE |
|---|---|---|---|
| Smoothing Spline | Soft | Visu | 12.98345 |
| | Semi Soft | | 14.67383 |

Table 2 shows that the smoothing spline with soft threshold function and visu threshold value is the best method due to MSE, followed by the estimation method using smoothing spline with semi-soft threshold function and visu threshold value.

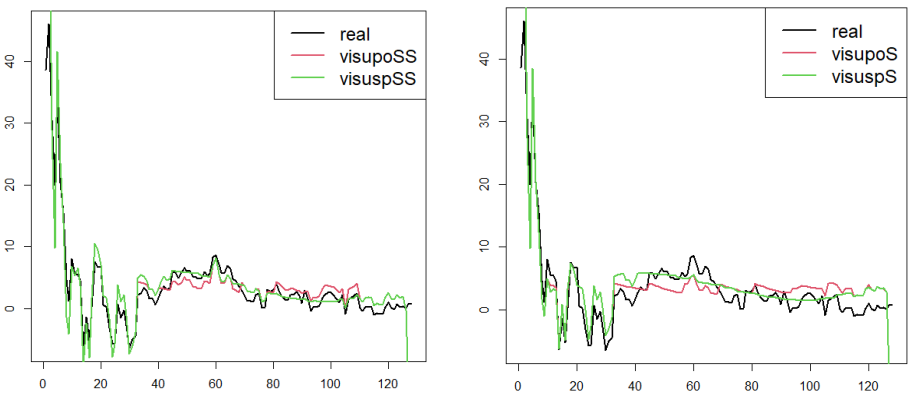


**Figure 8: Real and Estimated Data using Local Polynomial Estimation and Smoothing Spline (Left: Semi Soft, Right: Soft)**

In the figure (8) the black line represents the original data, while the red line represents the results of estimating the function using the local polynomial method, while the green line indicates the results of estimating the function using the smoothing spline method with the threshold function soft and semi soft the threshold value visu.

## CONCLUSIONS

The proposed model demonstrated high accuracy in analyzing the relationship between oil revenues and inflation in Iraq, achieving an MSE of 12.98 when employing smoothing spline regression with soft thresholding. This performance metric significantly outperformed conventional estimation methods in handling economic time-series data. When applying the Doppler function, the results demonstrated that combining the soft thresholding function with the Visu threshold value provided the best performance in terms of minimizing the Mean Squared Error (MSE), particularly when using spline smoothing for larger sample sizes. This highlights the effectiveness of this approach in handling signals characterized by sharp changes and sudden fluctuations, which are typical of the Doppler function

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