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AN EXTENDED SYMMETRICAL AND ASYMMETRICAL GENERATOR: PROPERTIES, INFERENCE, ACTUARIAL MEASURES, AND APPLICATIONS

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ABSTRACT

In this paper, we introduce the Burr-X Marshall–Olkin-F (BXMO-F) family, which proves to be highly effective for real-life data analysis. We establish several of its mathematical properties and demonstrate that the BXMO-F family can accommodate a variety of hazard rates and density functions. We derive six risk measures for the BXMO-Lomax distribution, which are crucial for portfolio optimization. The parameters of the BXMO-Lomax distribution are estimated using eight different estimation approaches, and these approaches are thoroughly evaluated through detailed numerical simulations. Finally, we explore the applicability of the BXMO-Lomax distribution by analyzing two real-life data sets from engineering and medicine. Our analysis shows that the BXMO-Lomax distribution offers a superior fit compared to several well-known extensions of the Lomax distribution.

KEYWORDS

Marshall–Olkin family; statistical model; value at risk; Cramér–von Mises estimation; Lomax distribution; simulations.

1. INTRODUCTION

Developing more flexible and reliable distributions by extending classical models or introducing new methods has become increasingly important for modeling diverse real-life data sets, including financial returns, medical data, and geological measurements. Recently, there has been significant interest among scholars in constructing new generalized families of distributions by incorporating additional shape parameters into classical baseline distributions. These extended models offer enhanced flexibility, making them better suited for a variety of real-world applications. Some notable families are the Marshall–Olkin-G (MO-G) [1], beta-G [2], gamma-G [3], Kumaraswamy-G [4], McDonald-G [5], transformer T-X [6], odd Lomax-G [7], exponentiated half-logistic [8], Burr X-G [9], alpha-power-G [10], Kumaraswamy alpha-power-G [11], Marshall–Olkin odd Burr III-G [12], Marshall–Olkin–Weibull-H [13], Kumaraswamy tan-G (KwT-G) [14], generalized Kavya–Manoharan-G [15], Lambert-G [16], and new exponential-H (NEx-H) [17] families.

The goal of this paper is to introduce a new and flexible family of distributions known as the Burr-X Marshall–Olkin-F (BXMO-F) family. This family is developed by extending the MO-G [1] and Burr X-G [9] families. The BXMO-F family boasts several advantageous properties, including the ability to model a wide range of hazard rate functions (HRFs) and accommodate skewed data. It is applicable to various fields such as finance, medicine, reliability, survival analysis, and economics.

The BXMO-F family includes special sub-models that can represent diverse HRFs, including bathtub, J-shape, reversed J-shape, unimodal, decreasing, upside-down bathtub, and increasing shapes. We validate the applicability of the BXMO-F family using real-life data, demonstrating that it offers greater flexibility compared to other existing families based on the same baseline distribution.

Additionally, we derived several key risk measures relevant to actuarial sciences, including Value at Risk (VaR), Tail-Value at Risk (TVaR), Tail-Variance Premium (TVP), Tail-Variance (TV), Expected Shortfall (ES), and Conditional Tail Expectation (CTE). We also estimated the parameters of a specific sub-model using eight different estimation techniques. These methods were evaluated through simulations for both small and large sample sizes. The results allow us to rank the estimation approaches, identifying the most effective methods, which will be valuable to engineers, applied statisticians, and practitioners.

This paper is outlined as follows. In Section 2, the BXMO-F family is defined. In Section 3, some statistical properties of the introduced family are presented. Three special sub-models of the family are defined in Section 4. Section 5 provides mathematical properties of one special sub-model. Additionally, six important risk measures for the proposed distribution are derived in Section 5. In Section 6, the parameters of a special sub-model are estimated using eight estimation methods. A simulation study is performed in Section 7. In Section 8, two real-life data applications are presented. Finally, the paper is concluded in Section 9.

2. THE BXMO-F FAMILY

In this section, we define the new BXMO-F family. We begin with the MO family which is proposed by [1]. Let $F(x;\boldsymbol{\gamma})$ be the baseline cumulative distribution function (cdf) with a vector of parameters $\boldsymbol{\gamma}$. Then, the cdf of the MO family takes the form

$$H(x;\alpha,\boldsymbol{\gamma}) = \frac{F(x;\boldsymbol{\gamma})}{1-\bar{\alpha}\bar{F}(x;\boldsymbol{\gamma})}, \quad x \in \mathcal{R}, \quad \alpha > 0, \tag{1}$$

where $\bar{F}(x;\boldsymbol{\gamma}) = 1 - F(x;\boldsymbol{\gamma})$ be the baseline survival function (sf) and $\bar{\alpha} = 1-\alpha$ is a shape parameter.

The corresponding probability density function (pdf) of (1) reduces to

$$h(x;\alpha,\boldsymbol{\gamma})=\frac{\alpha\, f(x;\boldsymbol{\gamma})}{[1-\bar{\alpha}\,\overline{F}(x;\boldsymbol{\gamma})]^2}\,. \tag{2}$$

where $f(x;\boldsymbol{\gamma})$ is the baseline pdf. Clearly, for $\alpha=1$, the baseline cdf follows directly, i.e., $H(x;\boldsymbol{\gamma})=F(x;\boldsymbol{\gamma})$.

The Burr X (BX) family [9] is specified by the cdf

$$G(x;\theta,\boldsymbol{\gamma})=\left\{1-\mathrm{e}^{\left[-\left(\frac{H(x;\boldsymbol{\gamma})}{1-H(x;\boldsymbol{\gamma})}\right)^2\right]}\right\}^{\theta},\ \ x\in\mathcal{R},\ \ \theta>0. \tag{3}$$

The pdf of the BX family takes the form

$$g(x;\theta,\boldsymbol{\gamma})=\frac{2\theta\, H(x;\boldsymbol{\gamma})\, h(x;\boldsymbol{\gamma})}{[1-H(x;\boldsymbol{\gamma})]^3}\,\mathrm{e}^{\left[-\left(\frac{H(x;\boldsymbol{\gamma})}{1-H(x;\boldsymbol{\gamma})}\right)^2\right]}\left\{1-\mathrm{e}^{\left[-\left(\frac{H(x;\boldsymbol{\gamma})}{1-H(x;\boldsymbol{\gamma})}\right)^2\right]}\right\}^{\theta-1}. \tag{4}$$

Now, we define the cdf of the BXMO-F family. By inserting (1) into Equation (3), the cdf of the BXMO-F family follows as

$$G(x;\alpha,\theta,\boldsymbol{\gamma})=\left\{1-\mathrm{e}^{\left[-\left(\frac{F(x;\boldsymbol{\gamma})}{\alpha\bar{F}(x;\boldsymbol{\gamma})}\right)^2\right]}\right\}^{\theta},\ \ x\in\mathcal{R},\ \ \alpha,\theta>0. \tag{5}$$

The pdf of the BXMO-F follows, by inserting (1) and (2) into Equation (4), as

$$g(x;\alpha,\theta,\boldsymbol{\gamma})=\frac{2\theta\alpha f(x;\boldsymbol{\gamma})\, F(x;\boldsymbol{\gamma})}{[\alpha\,\bar{F}(x;\boldsymbol{\gamma})]^3}\,\mathrm{e}^{\left[-\left(\frac{F(x;\boldsymbol{\gamma})}{\alpha\bar{F}(x;\boldsymbol{\gamma})}\right)^2\right]}\left\{1-\mathrm{e}^{\left[-\left(\frac{F(x;\boldsymbol{\gamma})}{\alpha\bar{F}(x;\boldsymbol{\gamma})}\right)^2\right]}\right\}^{\theta-1}. \tag{6}$$

The BX family [9] follows as a special case of (6) for $\alpha=1$.

The hrf, reversed-hrf (rhrf), and odds function are given by

$$\varphi(x;\alpha,\theta,\boldsymbol{\gamma})=\frac{2\theta\alpha f(x;\boldsymbol{\gamma})F(x;\boldsymbol{\gamma})}{[\alpha\,\bar{F}(x;\boldsymbol{\gamma})]^3}\,\mathrm{e}^{\left[-\left(\frac{F(x;\boldsymbol{\gamma})}{\alpha\bar{F}(x;\boldsymbol{\gamma})}\right)^2\right]}\left(1-\left\{1-\mathrm{e}^{\left[-\left(\frac{F(x;\boldsymbol{\gamma})}{\alpha\bar{F}(x;\boldsymbol{\gamma})}\right)^2\right]}\right\}^{\theta}\right)^{-1}$$
$$\times\left\{1-\mathrm{e}^{\left[-\left(\frac{F(x;\boldsymbol{\gamma})}{\alpha\bar{F}(x;\boldsymbol{\gamma})}\right)^2\right]}\right\}^{\theta-1},$$

$$r(x;\alpha,\theta,\boldsymbol{\gamma})=\frac{2\theta\alpha\, f(x;\boldsymbol{\gamma})\, F(x;\boldsymbol{\gamma})}{[\alpha\,\bar{F}(x;\boldsymbol{\gamma})]^3}\,\mathrm{e}^{\left[-\left(\frac{F(x;\boldsymbol{\gamma})}{\alpha\,\bar{F}(x;\boldsymbol{\gamma})}\right)^2\right]}\left\{1-\mathrm{e}^{\left[-\left(\frac{F(x;\boldsymbol{\gamma})}{\alpha\,\bar{F}(x;\boldsymbol{\gamma})}\right)^2\right]}\right\}^{-1},$$

and

$$O(x;\alpha,\theta,\boldsymbol{\gamma})=\left\{1-\mathrm{e}^{\left[-\left(\frac{F(x;\boldsymbol{\gamma})}{\alpha\,\bar{F}(x;\boldsymbol{\gamma})}\right)^2\right]}\right\}^{\theta}\left(1-\left\{1-\mathrm{e}^{\left[-\left(\frac{F(x;\boldsymbol{\gamma})}{\alpha\,\bar{F}(x;\boldsymbol{\gamma})}\right)^2\right]}\right\}^{\theta}\right)^{-1}.$$

3. GENERAL PROPERTIES

This section outlines several mathematical properties of the BXMO-F family.

3.1 Linear Representation

In this section, we provide a very useful linear representation for the BXMO-F density function.

$$g(x) = \frac{2\theta\alpha f(x)F(x)}{\left(\alpha\bar{F}(x)\right)^3} \mathrm{e}^{\left[-\left(\frac{F(x)}{\alpha\bar{F}(x)}\right)^2\right]} \left\{1 - \mathrm{e}^{\left[-\left(\frac{F(x)}{\alpha\bar{F}(x)}\right)^2\right]}\right\}^{\theta-1}.$$

The following power series

$$(1-z)^{b-1} = \sum_{w=0}^{\infty} \frac{(-1)^w \Gamma(b)}{w!\,\Gamma(b-w)} z^w, \tag{7}$$

holds for a positive real non-integer b and $|z| < 1$.

Applying (7) to (6), we have

$$g(x) = \frac{2\theta\alpha f(x)F(x)}{\left(\alpha\bar{F}(x)\right)^3} \sum_{w=0}^{\infty} \frac{(-1)^w \Gamma(\theta)}{w!\,\Gamma(\theta-w)} \mathrm{e}^{\left[-(w+1)\left(\frac{F(x)}{\alpha\bar{F}(x)}\right)^2\right]}. \tag{8}$$

Applying the exponential series to the term

$$\mathrm{e}^{\left[-(w+1)\left(\frac{F(x)}{\alpha\bar{F}(x)}\right)^2\right]} = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} (w+1)^j \left(\frac{F(x)}{\alpha\bar{F}(x)}\right)^{2j}.$$

Hence, Equation (8) becomes

$$g(x) = 2\theta\alpha f(x) \sum_{w,j=0}^{\infty} \frac{(-1)^{w+j}\Gamma(\theta)(w+1)^j \left(F(x)\right)^{1+2j}}{w!\,j!\,\Gamma(\theta-w)\left(\alpha\bar{F}(x)\right)^{3+2j}}. \tag{9}$$

Consider the series expansion $(1-x)^{-s} = \sum_{k=0}^{\infty} \binom{s+k-1}{k} x^k, |x| < 1, b > 0.$

Then, we have

$$g(x) = 2\theta \sum_{w,j,k=0}^{\infty} \frac{(-1)^{w+j}\Gamma(\theta)(w+1)^j}{w!\,j!\,\Gamma(\theta-w)\alpha^{2(j+1)}} \binom{2+2j+k}{k} f(x)F(x)^{2j+k+1}. \tag{10}$$

Hence, the BXMO-F density can be written as

$$g(x) = \sum_{j,k=0}^{\infty} \delta_{j,k}\, h_{2j+k+2}(x), \tag{11}$$

where $h_{2j+k+2}(x) = (2j+k+2)f(x)F(x)^{2j+k+1}$ and

$$\delta_{j,k} = \sum_{w=0}^{\infty} \frac{2\theta(-1)^{w+j}\Gamma(\theta)(w+1)^j\binom{2+2j+k}{k}}{\alpha^{2(j+1)}w!\,j!\,\Gamma(\theta-w)(2j+k+2)},$$

Equation (11) demonstrates that the BXMO-F density can be represented as a double linear mixture of Exp-F densities. Consequently, many mathematical properties of the BXMO-F family can be derived from those of the Exp-F distribution.

3.2 Quantile Function and Moments

The quantile function (qf) of the BXMO-F family follows as

$$Q(u) = F^{-1}\left(\frac{\alpha\sqrt{\log\left[1/\left(1-\sqrt[\theta]{u}\right)\right]}}{\alpha\sqrt{\log\left[1/\left(1-\sqrt[\theta]{u}\right)\right]}+1}\right), \quad u \in (0,1).$$

The rth ordinary moments of the BXMO-F family is given by

$$\mu_r' = E(X^r) = \int_{-\infty}^{\infty} x^r g\,(x)dx.$$

Henceforth, let Y_{2j+k+2} denote the Exp-F random variable (rv) with power parameter $2j+k+2$. The rth moment of X, is obtained from Equation (11) as

$$\mu_r' = \sum_{j,k=0}^{\infty} \delta_{j,k} E\left(Y_{2j+k+2}^r\right) = \sum_{j,k=0}^{\infty} \delta_{j,k}\, \tau_{j,k,r}(x), \tag{12}$$

where $\tau_{j,k,r}(x) = (2j+k+1)\int_0^1 [Q_F(u)]^r u^{2j+k+1} du$, and $Q_F(u) = F^{-1}(u)$ is the baseline qf.

3.3 Probability Weighted Moments

The (s,r)th probability weighted moments (PWMs), say $\rho_{s,r}$, is defined by

$$\rho_{s,r} = E[X^s G(x)^r] = \int_{-\infty}^{\infty} x^s G(x)^r g(x) dx.$$

They can be defined for any rv whose ordinary moments exist.

Using the pdf and cdf of the BXMO-F family, we can write

$$G(x)^r g(x) = 2\theta \sum_{w,j,k=0}^{\infty} \frac{(-1)^{w+j}\Gamma[\theta(r+1)](w+1)^j}{w!\,j!\,\Gamma[\theta(r+1)-w]\alpha^{2(j+1)}} \binom{2j+k+2}{k} f(x)F(x)^{2j+k+1}.$$

The above equation can be expressed as

$$G(x)^r g(x) = \sum_{j,k=0}^{\infty} D_{j,k}\, h_{2j+k+2}(x),$$

where

$$D_{j,k} = \sum_{w=0}^{\infty} 2\theta \frac{(-1)^{w+j}\Gamma[\theta(r+1)](w+1)^{j}}{w!\,j!\,\Gamma[\theta(r+1)-w]\alpha^{2(j+1)}(2j+k+2)}\binom{2j+k+2}{k}.$$

Then, $\rho_{s,r}$ reduces to

$$\rho_{s,r} = \sum_{j,k=0}^{\infty} D_{j,k} \int_{-\infty}^{\infty} x^{s} h_{2j+k+2}(x)dx = \sum_{j,k=0}^{\infty} D_{j,k}\, E\left(Y_{2j+k+2}^{s}\right).$$

3.4 Order Statistics

Let $X_1, X_2, \dots, X_n$ be a random sample from the BXMO-F family and $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ be the corresponding order statistics. Then, the pdf of ith order statistic, say $X_{i:n}$, is defined by

$$g_{i:n}(x) = \frac{g(x)}{B(i,n-i+1)} \sum_{m=1}^{n-i} (-1)^{m} \binom{n-i}{m} G(x)^{m+i+1}, \tag{13}$$

where $B(.,.)$ is the beta function.

Therefore, we can write

$$G(x)^{m+i+1} g(x) = 2\theta \sum_{w,j,k=0}^{\infty} \frac{(-1)^{w+j}\Gamma[\theta(m+i+2)](w+1)^{j}}{w!\,j!\,\Gamma[\theta(m+i+2)-w]\alpha^{2(j+1)}} \binom{2j+k+2}{k} f(x)F(x)^{2j+k+1}.$$

Hence, the pdf of $X_{i:n}$ can be expressed as

$$g_{i:n}(x) = \sum_{j,k=0}^{\infty} M_{j,k}\, h_{2j+k+2}(x),$$

where

$$M_{j,k} = \sum_{m=1}^{n-i} \sum_{w=0}^{\infty} \frac{(-1)^{m}}{B(i,n-i+1)} \binom{n-i}{m} \frac{2\theta(-1)^{w+j}\Gamma[\theta(m+i+2)](w+1)^{j}}{w!\,j!\,\Gamma[\theta(m+i+2)-w]\alpha^{2(j+1)}(2j+k+2)} \binom{2j+k+2}{k}.$$

3.5 Entropies

The Rényi entropy of a rv X is defined by

$$U_{\eta}(x) = \frac{1}{1-\eta} \log\left[\int_{0}^{\infty} g(x)^{\eta}\, dx\right], \eta > 0 \text{ and } \eta \neq 1.$$

It is a measure of variation of uncertainty and has some applications is applied fields such as molecular imaging of tumors physics, and sparse kernel density estimation.

Using the pdf of the BXMO-F family, we can write

$$g(x)^{\eta} = \sum_{w,j,k=0}^{\infty} \frac{(-1)^{w+j}(2\theta)^{\eta}\alpha^{\eta}(w+\eta)^{j}\Gamma(\eta(\theta-1)+1)}{w!\,j!\,\Gamma[\eta(\theta-1)+1-w]\alpha^{3\eta+2j}} \binom{3\eta+2j+k-1}{k} f(x)^{\eta}F(x)^{\eta+2j+k}.$$

Therefore,

$$U_{\eta}(x) = \frac{1}{1-\eta}\log\left[\sum_{j,k=0}^{\infty} d_{j,k}\int_{0}^{\infty} f(x)^{\eta}F(x)^{\eta+2j+k}dx\right],$$

where

$$d_{j,k} = \sum_{w=0}^{\infty} \frac{(-1)^{w+j}(2\theta)^{\eta}\alpha^{\eta}(w+\eta)^{j}\Gamma(\eta(\theta-1)+1)}{w!\,j!\,\Gamma[\eta(\theta-1)+1-w]\alpha^{3\eta+2j}}\binom{3\eta+2j+k-1}{k}.$$

The η-entropy, say $H_{\eta}(X)$, can be obtained as

$$H_{\eta}(X) = \frac{1}{1-\eta}\log\left\{1-\left[\sum_{j,k=0}^{\infty} d_{j,k}\int_{0}^{\infty} f(x)^{\eta}F(x)^{\eta+2j+k}dx\right]\right\}.$$

The Shannon entropy of X, say $K_X(1)$, is defined by

$$K_X(1) = \lim_{\eta\to 1} U_{\eta}(x).$$

4. SPECIAL BXMO-F MODELS

In this section, we provide three special cases of the BXMO-F family. We consider the exponential (E), Weibull (W) and Lomax (L) distributions as baseline models in the BXMO-F family to generate the BXMO-exponential (BXMO-E), BXMO-Weibull (BXMO-W) and BXMO-Lomax (BXMO-L) distributions which have many desirable properties. These special sub-models generalize some well-known distributions in literature. Plots of the pdf and hrf of BXMO-E, BXMO-W, and BXMO-L distributions are shown in Figures 1-3, respectively. These plots reveal that the sub-models of the BXMO-F family accommodate bathtub, upside-down bathtub, J-shape, reversed J-shape, unimodal, decreasing and increasing failure rates as well as symmetrical, left-skewed, right-skewed, and reversed-J shaped densities.

4.1 The BXMO-E Distribution

Consider the pdf of the E distribution (for $x > 0$) $f(x;\lambda) = \lambda e^{-\lambda x}$, with positive parameter λ. Then, the pdf of the BXMO-E model is

$$\mathrm{g}(x;\alpha,\theta,\lambda) = \frac{2\theta\lambda}{\alpha^2} e^{2\lambda x}\left(1-e^{-\lambda x}\right)e^{-\frac{1}{\alpha^2}\left(e^{\lambda x}-1\right)^2}\left[1-e^{-\frac{1}{\alpha^2}\left(e^{\lambda x}-1\right)^2}\right]^{\theta-1}.$$

For $\alpha = 1$, the BXMO-E model reduces to BX-E model.

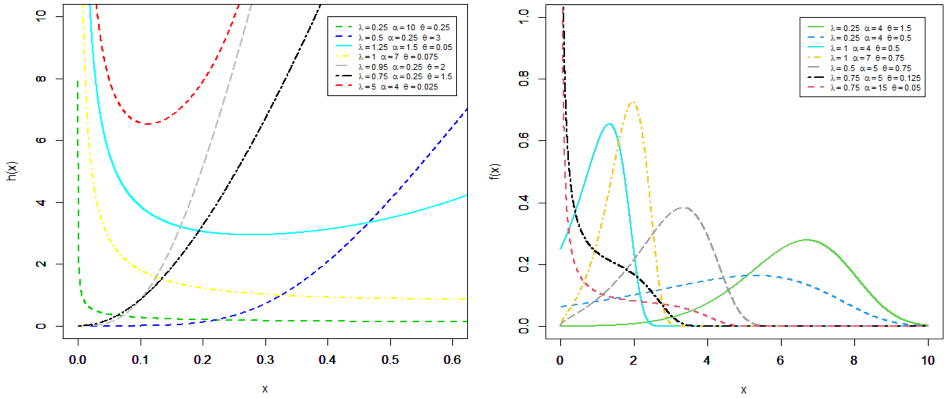

Figure 1: Plots of the hrf (left panel) and pdf (right panel) of the BXMO-E Distribution

4.2 The BXMO-W Distribution

Consider the pdf of the W distribution $f(x;\beta,\lambda)=\beta\lambda^{\beta}x^{\beta-1}\mathrm{e}^{-(\lambda x)^{\beta}}, x>0$. Then, the pdf of the BXMO-W distribution has the form

$$g(x;\alpha,\theta,\beta,\lambda)=\frac{2\theta\beta\lambda^{\beta}x^{\beta-1}e^{2(\lambda x)^{\beta}}\left[1-e^{-(\lambda x)^{\beta}}\right]}{\alpha^{2}}e^{-\frac{1}{\alpha^{2}}\left[e^{(\lambda x)^{\beta}}-1\right]^{2}} \times\left\{1-e^{-\frac{1}{\alpha^{2}}\left[e^{(\lambda x)^{\beta}}-1\right]^{2}}\right\}^{\theta-1}.$$

The BXMO-W model has five special cases including the BX-W when $\alpha=1$, the BXMO-E for $\beta=1$, the BXMO-Rayleigh for $\beta=2$, the BX-E for $\alpha=\beta=1$, and the BX-Rayleigh distribution for $\alpha=1$ and $\beta=2$

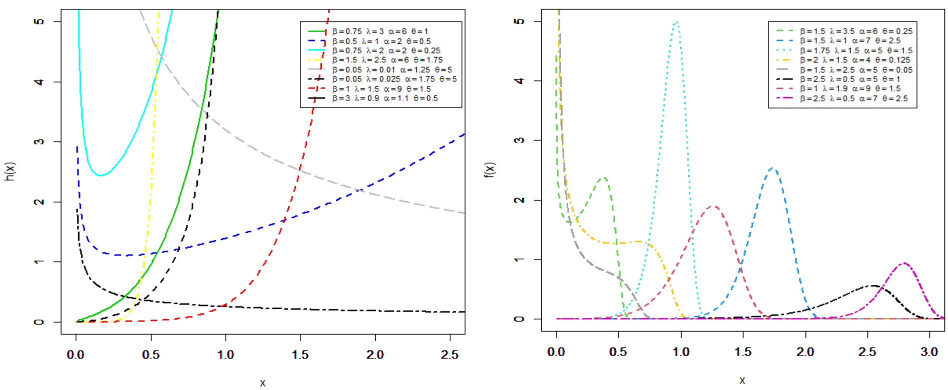

Figure 2: Plots of the hrf (left panel) and pdf (right panel) of the BXMO-W Distribution

4.3 The BXMO-L Distribution

Consider the pdf of the Lomax distribution with positive parameters β and λ $f(x;\beta,\lambda)=\frac{\beta}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-\beta-1}$, $x>0$. Then, the pdf of the BXMO-L distribution takes the form

$$\mathrm{g}(x;\alpha,\theta,\beta,\lambda)=\frac{2\theta\beta}{\lambda\alpha^2}\left(1+\frac{x}{\lambda}\right)^{2\beta-1}\left[1-\left(1+\frac{x}{\lambda}\right)^{-\beta}\right]e^{-\frac{1}{\alpha^2}\left[\left(1+\frac{x}{\lambda}\right)^{\beta}-1\right]^2}.$$

For $\alpha=1$, the BXMO-L model reduces to the BX-L model.

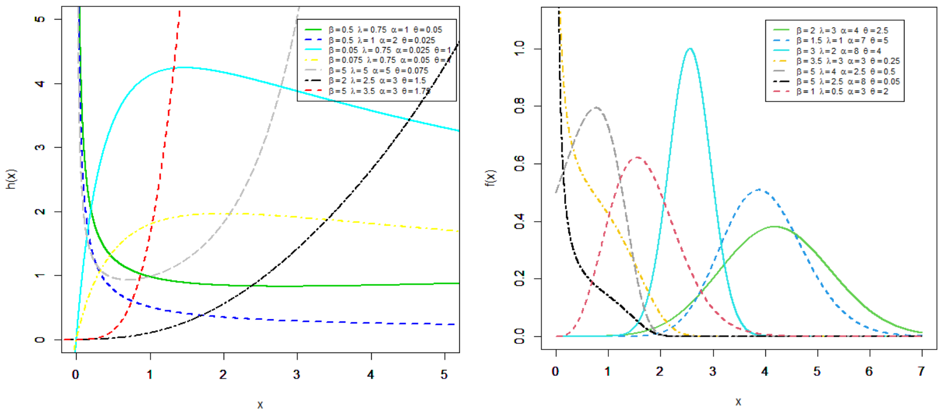

Figure 3: Plots of the hrf (left panel) and pdf (right panel) of the BXMO-L Distribution

5. PROPERTIES OF THE BXMO-L DISTRIBUTION

5.1 Quantile and Moments

The quantile function (qf) of the BXMO-L distribution follows as

$$Q(u)=\lambda\left(\left\{\alpha\sqrt{\log[1/(1-\sqrt[\theta]{u})]}+1\right\}^{1/\beta}-1\right),\quad u\in(0,1).$$

The Bowley's skewness (SK_B) [18] of X is given by

$$SK_B=\frac{Q(1/4)+Q(3/4)-2Q(1/2)}{Q(3/4)-Q(1/4)},$$

The moor's kurtosis (KU_M) [19] of X is given by

$$KU_M=\frac{Q(7/8)-Q(5/8)+Q(3/8)-Q(1/8)}{Q(6/8)-Q(2/8)}.$$

Figure 4 shows the SK_B and KU_M plots of the BXMO-L distribution according to [18] and [19]. The values of SK_B and KU_M are decreasing when the value parameter θ increasing for various choices of α, λ, β. The value of skewness falls between 0 and 1.

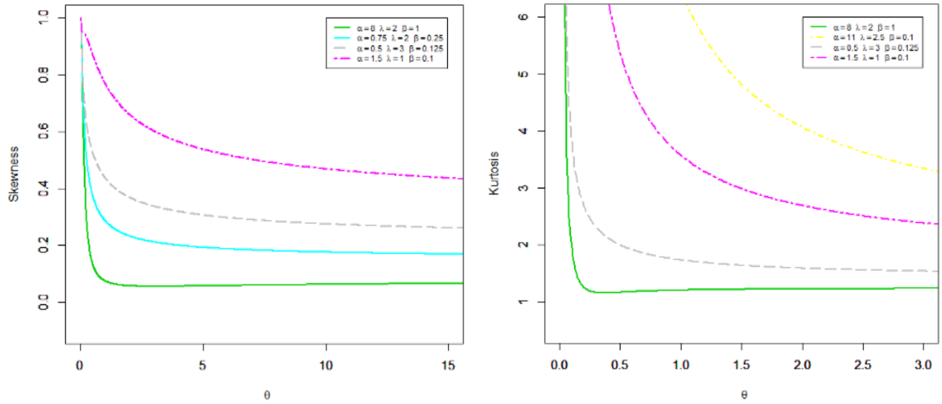

Figure 4: Plots of the SK_B hrf (left panel) and KU_M (right panel) of the BXMO-L Distribution

Theorem 1:

The rth ordinary moments of the BXMO-L distribution follows as

$$\mu_r' = \sum_{s=0}^{\infty} \vartheta_s \, B(\beta(1+s) - r, r+1), \tag{14}$$

where

$$\vartheta_s = \sum_{j,k=0}^{\infty} \delta_{j,k} \frac{(-1)^s \Gamma(2j+k+2)\beta\lambda^r}{s!\,\Gamma(2j+k+2-s)}.$$

Proof:

By definition and using Equation (11), we can write

$$\mu_r' = E(X^r) = \sum_{j,k=0}^{\infty} \delta_{j,k} \int_{-\infty}^{\infty} x^r \, h_{2j+k+2}(x) dx,$$

Using the pdf and cdf of the L distribution, μ_r' follows as

$$\mu_r' = \sum_{j,k=0}^{\infty} \delta_{j,k} \int_0^{\infty} x^r \frac{\beta}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-\beta-1} \left[1-\left(1+\frac{x}{\lambda}\right)^{-\beta}\right]^{2j+k+1} dx,$$

Applying Equation (7), we have

$$\mu_r' = \sum_{j,k,s=0}^{\infty} \delta_{j,k} \frac{(-1)^s \Gamma(2j+k+2)}{s!\,\Gamma(2j+k+2-s)} \beta\lambda^{r-1} \int_0^{\infty} \left(\frac{x}{\lambda}\right)^r \left(1+\frac{x}{\lambda}\right)^{-\beta(1+s)-1} dx.$$

Simplifying the integral, we obtain

$$\int_0^\infty \left(\frac{x}{\lambda}\right)^r \left(1+\frac{x}{\lambda}\right)^{-\beta(1+s)-1} dx = \lambda B(r+1, \beta(1+s)-r).$$

Hence, combining the last two equation completes the proof.

Table 1 lists some numerical values for $\mu_1', \mu_2', \mu_3', \mu_4'$ (first four moments), variance (ε^2), kurtosis (τ_1), and skewness (τ_2) of the BXMO-L distribution for some choices of α, θ, β and λ. The results are obtained for ($\alpha = 3, \theta = 0.125, \beta = 0.5,1,1.5,2,3,5$), ($\alpha = 5, \beta = 5, \theta = 0.5,1,1.5,2,3,5$), ($\theta = 2, \beta = 2, \alpha = 0.5,1,1.5,2,3,5$). The results illustrate that, for fixed α and θ, the $\mu_1', \mu_2', \mu_3', \mu_4', \varepsilon^2$, and τ_2 are decreasing functions of β. Also, for fixed α and β, the $\mu_1', \mu_2', \mu_3', \mu_4'$, and τ_2 are increasing functions of θ, while ε^2 is decreasing function of θ. Additionally, for fixed θ and β, the $\mu_1', \mu_2', \mu_3', \mu_4'$ and ε^2 are increasing functions of α. Furthermore, Table 1 illustrates that the BXMO-L distribution can be right-skewed, left-skewed, platykurtic (τ_1<3), and leptokurtic (τ_1>3). Hence, the BXMO-L distribution is a flexible distribution which can be adapted to model skewed data.

Table 1
Some Numerical Results of Moments, ε^2, τ_1 and τ_2 of the BXMO-L Model for $\lambda = 1$ and Different Values of α, θ and β

α	θ	β	μ_1'	μ_2'	μ_3'	μ_4'	ε^2	τ_1	τ_2
3	0.125	0.5	3.121832	53.46793	1586.434	64953.08	43.72210	0.52289	5.114630
		1	0.711662	1.698507	5.747032	23.68578	1.192044	8.24639	2.183359
		1.5	0.383301	0.435610	0.665098	1.195555	0.288691	5.94016	1.784615
		2	0.260514	0.190634	0.183037	0.203824	0.122767	5.10224	1.613592
		3	0.158195	0.066910	0.036345	0.022588	0.041885	4.42178	1.459237
		5	0.088396	0.020147	0.005806	0.001896	0.012333	3.97018	1.346687
5	0.5	5	0.294840	0.109674	0.045846	0.020687	0.022744	1932338	-31410.1
	1		0.381391	0.160822	0.072618	0.034533	0.015364	2.68595	-0.22981
	1.5		0.424662	0.191813	0.090889	0.044788	0.011476	2.89616	-0.25457
	2		0.451714	0.213320	0.104534	0.052875	0.009275	2.97240	-0.22908
	3		0.485177	0.242324	0.124201	0.065163	0.006928	3.00647	-0.15841
	5		0.520839	0.276207	0.148982	0.081658	0.004934	2.99990	-0.04748
0.5	2	2	0.251240	0.070418	0.021556	0.007104	0.007297	18773987	-172890.7
1			0.457522	0.230753	0.125903	0.073378	0.021426	2.892071	0.2303341
1.5			0.637221	0.444253	0.333111	0.265542	0.038203	2.867421	0.1790009
2			0.798687	0.694219	0.646355	0.637702	0.056318	2.858204	0.1444366
3			1.083935	1.269521	1.584090	2.085559	0.094606	2.855315	0.1003196
5			1.560126	2.608731	4.619156	8.588452	0.174738	2.863351	0.0544883

5.2 Actuarial Measures

In this section, we derive six important risk measures for the BXMO-L distribution.

5.2.1 VaR

The VaR is also known as quantile-premium principle. The VaR of a rv is the qf of its cdf [20]. Hence, it is defined as

$$VaR_q(x) = \inf\{x: F_X(x) \geq q\} = F_X^{-1}(q).$$

Then, $VaR_q(x`)$ of the BXMO-L model has the form

$$VaR_q(x) = \lambda\left(\sqrt[\beta]{\left\{1 + \alpha\sqrt{\log\left[1/\left(1 - \sqrt[\theta]{q}\right)\right]}\right\}} - 1\right).$$

5.2.2 TVaR

The TVaR is a measure for quantifying the expected value of losses. The TVaR of X is defined as

$$TVaR_q = \frac{1}{1-q}\int_{VaR_q}^{\infty} x\, g(x;\alpha,\theta,\boldsymbol{\gamma})dx. \tag{15}$$

Substituting Equation (10) in (15)

$$TVaR_q = \frac{1}{1-q}\sum_{w,j,k=0}^{\infty}\frac{2\theta(-1)^{w+j}\Gamma(\theta)(w+1)^j}{w!\,j!\,\Gamma(\theta-w)\alpha^{2(j+1)}}\binom{2+2j+k}{k} \times \int_{VaR_q}^{\infty} xf(x;\boldsymbol{\gamma})F(x;\boldsymbol{\gamma})^{2j+k+1}dx.$$

Then by inserting for the cdf and pdf for Lomax distribution, we get

$$TVaR_q = \frac{1}{1-q}\sum_{w,j,k=0}^{\infty}\frac{2\theta(-1)^{w+j}\Gamma(\theta)(w+1)^j}{w!\,j!\,\Gamma(\theta-w)\alpha^{2(j+1)}}\binom{2+2j+k}{k} \times \int_{VaR_q}^{\infty} x\frac{\beta}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-\beta-1}\left[1-\left(1+\frac{x}{\lambda}\right)^{-\beta}\right]^{2j+k+1} dx.$$

Appling the power series (7) to the last term and after some simplifications, we have

$$TVaR_q = \frac{1}{1-q}\sum_{w,j,k,r=0}^{\infty}\frac{\begin{array}{c}2\theta(-1)^{w+j+r}\Gamma(\theta)(w+1)^j\lambda^{\beta(1+r)-1}\\ \left[VaR_q(1+r)\beta+\lambda\right]\end{array}}{\begin{array}{c}w!\,j!\,\Gamma(\theta-w)\alpha^{2(j+1)}(1+r)\\ \left[-1+\beta(1+r)\right]\left(VaR_q+\lambda\right)^{\beta(1+r)}\end{array}} \binom{2j+k+2}{k}\binom{2j+k+1}{r}. \tag{16}$$

5.2.3 TV Measure

The TV measure is the variance of the loss distribution beyond some critical value Landsman [21]. The TV is given by

$$TV_q(X) = E\left(X^2|X > x_q\right) - \left(TVaR_q\right)^2. \tag{17}$$

First, we derived (for $x_q > 0,\ \beta(1+r) > 2$)

$$E\left(X^2|X>x_q\right) = \sum_{w,j,k,r=0}^{\infty} \frac{\begin{array}{c}2\theta(-1)^{w+j+r}\Gamma(\theta)(w+1)^j\lambda^{\beta(1+r)} \\ \left\{{x_q}^2\beta(1+r)[-1+\beta(1+r)]+2x_q(1+r)\beta\lambda+2\lambda^2\right\}\end{array}}{\begin{array}{c}w!\,j!\,\Gamma(\theta-w)\alpha^{2(j+1)}(1+r)[-1+\beta(1+r)] \\ [-2+\beta(1+r)]\left(x_q+\lambda\right)^{\beta(1+r)}\end{array}}\binom{2j+k+2}{k} \quad (18)$$

Then, by replacing Equations (18) and (16) in Equation (17), we obtain the TV.

5.2.4 TVP

The TVP measure is defined by

$$TVP_q(X) = TVaR_q + \sigma\, TV_q(X), 0 < \sigma < 1.$$

5.2.5 ES Measure

The ES measure of financial risk is introduced by Artzner et al. [22]. It is given by

$$ES_q(X) = \frac{1}{q}\int_0^q VaR_t dt = \frac{1}{q}\int_0^q \lambda\left(\sqrt[\beta]{\left\{1+\alpha\sqrt{\log\left[1/\left(1-\sqrt[\theta]{t}\right)\right]}\right\}}-1\right)dt.$$

5.2.6 CTE

The CTE [23] risk measure has the form

$$CTE_q(X) = \frac{1}{1-q}\int_q^1 VaR_t dt = \frac{1}{1-q}\int_q^1 \lambda\left(\sqrt[\beta]{\left\{1+\alpha\sqrt{log\left[1/\left(1-\sqrt[\theta]{t}\right)\right]}\right\}}-1\right)dt.$$

5.3 Simulations of Risk Measures

This section provides some computational results for the VaR, TVaR, TVP, TV, ES and CTE measures of the BXMO-L distribution for some parametric values. The results in Table 2 are obtained as follows:

1. Random sample of size $n = 100$ is generated from the BXMO-L distribution, and parameters are estimated by using the maximum likelihood (ML) method.
2. 1000 repetitions are conducted to calculate all risk measures of the BXMO-L distribution.

Simulation results for the VaR, TVaR, TVP, TV, ES and CTE of the BXMO-L distribution are obtained for some choices of $\alpha = (0.5, 0.9, 1.5, 2, 3), \theta = (0.5, 1, 1.5, 2, 3, 5)$, $\lambda = (5, 1.5, 0.5, 3)$ and $\beta = (1.5, 0.5, 3, 2)$.

The first part of Table 2 is presented visually in Figure 5. The plots of the VaR, TVaR, TVP, TV, ES and CTE of the BXMO-L distribution show that, for fixed λ and β, the risk measures are increasing functions of α.

Table 2
Simulated Results of Risk Measures for the BXMOL Distribution

	$(\lambda, \beta, \alpha, \theta)$	0.70	0.75	0.80	0.85	0.90	0.95	0.99
VaR	(1.5,0.5,0.5,1.5)	2.44369	2.62558	2.83786	3.09876	3.44865	4.01474	5.24937
	(1.5,0.5,2,5)	25.68241	27.31795	29.26753	31.72729	35.14223	40.96367	54.98687
	(1.5,0.5,3,3)	42.55435	45.92123	49.95813	55.08349	62.25054	74.585	104.7901
	(1.5,0.5,0.9,2)	5.76659	6.21529	6.75182	7.43087	8.37673	9.9957	13.92134
TVaR	(1.5,0.5,0.5,1.5)	3.30316	3.46028	3.64751	3.88232	4.20369	4.73517	5.9245
	(1.5,0.5,2,5)	33.8956	35.43628	37.31012	39.71942	43.12554	49.0385	63.54472
	(1.5,0.5,3,3)	59.99745	63.24247	67.2052	72.32388	79.6012	92.33543	124.0438
	(1.5,0.5,0.9,2)	8.09825	8.52840	9.05257	9.72792	10.68499	12.35187	16.46517
TV	(1.5,0.5,0.5,1.5)	0.77073	0.76524	0.76385	0.76897	0.78666	0.83888	1.04258
	(1.5,0.5,2,5)	94.75335	97.41152	101.1646	106.7288	115.8279	134.5858	194.2794
	(1.5,0.5,3,3)	411.7366	424.6532	442.7903	469.6581	513.8637	606.6478	918.1005
	(1.5,0.5,0.9,2)	6.80244	6.96465	7.19846	7.55067	8.13488	9.35701	13.32517
TVP	(1.5,0.5,0.5,1.5)	3.84267	4.03421	4.25859	4.53595	4.91168	5.53211	6.956660
	(1.5,0.5,2,5)	100.2229	108.4949	118.2418	130.4389	147.3707	176.895	255.8814
	(1.5,0.5,3,3)	348.2131	381.7324	421.4374	471.5332	542.0785	668.6508	1032.963
	(1.5,0.5,0.9,2)	12.85996	13.75189	14.81133	16.14599	18.00638	21.24102	29.65708
ES	(1.5,0.5,0.5,1.5)	1.48473	1.55458	1.62796	1.70650	1.79299	1.89346	1.99795
	(1.5,0.5,2,5)	17.48706	18.08654	18.72241	19.41182	20.18465	21.10735	22.11687
	(1.5,0.5,3,3)	26.28326	27.47724	28.75184	30.1431	31.71445	33.60805	35.70798
	(1.5,0.5,0.9,2)	3.57495	3.73563	3.90681	4.09323	4.30319	4.55523	4.83295
CTE	(1.5,0.5,0.5,1.5)	3.32859	3.48779	3.67759	3.91575	4.24193	4.78190	5.99228
	(1.5,0.5,2,5)	34.33732	35.90894	37.82106	40.28063	43.75957	49.80301	64.64572
	(1.5,0.5,3,3)	60.71115	64.01477	68.05076	73.26658	80.68622	93.67964	126.0757
	(1.5,0.5,0.9,2)	8.16216	8.59757	9.12833	9.81245	10.78243	12.47293	16.64954
VaR	(5,1.5,0.5,1)	1.66834	1.78588	1.92011	2.08078	2.28897	2.60905	3.24278
	(1.5,0.5,1.5,1.5)	10.75723	11.74626	12.93287	14.43934	16.54383	20.15626	28.94681
	(0.5,3,1.5,0.5)	0.14509	0.15866	0.17415	0.19266	0.21650	0.25271	0.32252
	(3,2,0.5,5)	1.03846	1.07451	1.11589	1.16584	1.23137	1.33407	1.54450
TVaR	(5,1.5,0.5,1)	2.18295	2.27612	2.38488	2.51792	2.69420	2.97202	3.53910
	(1.5,0.5,1.5,1.5)	15.93894	16.89415	18.05992	19.56419	21.69926	25.42444	34.64636
	(0.5,3,1.5,0.5)	0.20442	0.21507	0.22747	0.24255	0.26238	0.29324	0.35464
	(3,2,0.5,5)	1.19446	1.22370	1.25808	1.30051	1.35740	1.44860	1.64010
TV	(5,1.5,0.5,1)	0.23555	0.22691	0.21896	0.21197	0.20674	0.20651	0.23134
	(1.5,0.5,1.5,1.5)	33.60338	34.47477	35.71476	37.57533	40.65048	47.11343	68.36821
	(0.5,3,1.5,0.5)	0.00280	0.00266	0.00251	0.00238	0.00224	0.00214	0.00224
	(3,2,0.5,5)	0.03329	0.03312	0.03313	0.03340	0.03418	0.03623	0.04325
TVP	(5,1.5,0.5,1)	2.34784	2.44631	2.56005	2.69809	2.88027	3.16821	3.76813
	(1.5,0.5,1.5,1.5)	39.46131	42.75023	46.63173	51.50322	58.28469	70.1822	102.3309
	(0.5,3,1.5,0.5)	0.20638	0.21707	0.22948	0.24457	0.26440	0.29527	0.35686
	(3,2,0.5,5)	1.21777	1.24854	1.28458	1.32890	1.38816	1.48301	1.68291
ES	(5,1.5,0.5,1)	1.00421	1.05233	1.10227	1.15493	1.21185	1.27617	1.33993
	(1.5,0.5,1.5,1.5)	6.07684	6.42095	6.7896	7.19325	7.65037	8.20239	8.81492
	(0.5,3,1.5,0.5)	0.07352	0.07874	0.08421	0.09002	0.09635	0.10354	0.11067
	(3,2,0.5,5)	0.82748	0.84272	0.85847	0.87502	0.89290	0.91315	0.93342
CTE	(5,1.5,0.5,1)	2.19780	2.29216	2.40238	2.53730	2.71623	2.99861	3.57632
	(1.5,0.5,1.5,1.5)	16.07952	17.04772	18.22983	19.75589	21.9231	25.7074	35.08765
	(0.5,3,1.5,0.5)	0.20558	0.21635	0.22888	0.24414	0.26423	0.29553	0.35796
	(3,2,0.5,5)	1.20480	1.23454	1.26951	1.31270	1.37065	1.46364	1.65926

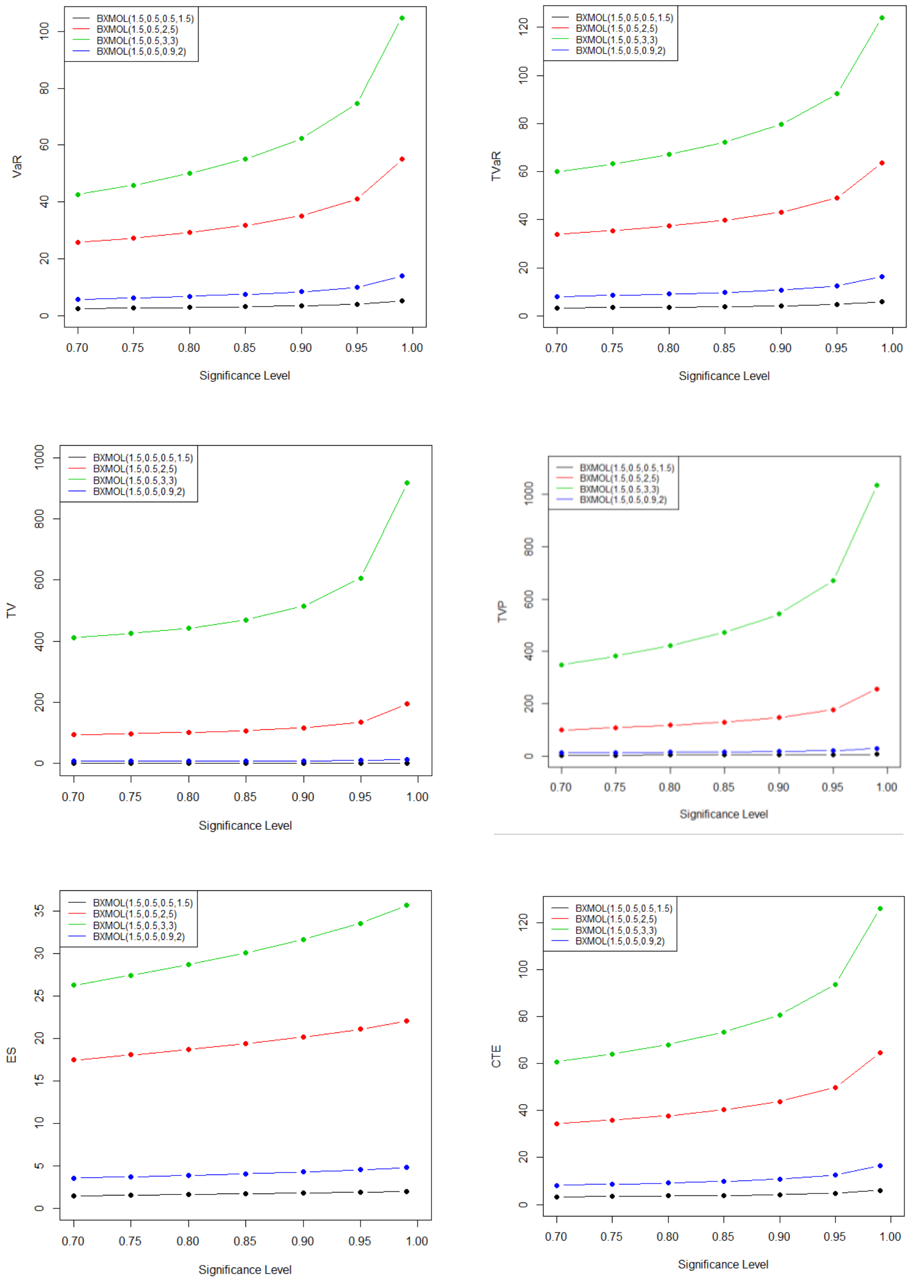

Figure 5: The VaR, TVaR, TV, TVP, ES and CTE for the BXMO-L $(\lambda, \beta, \alpha, \theta)$

6. ESTIMATION METHODS

In this section, the parameters of the BXMO-L distribution are estimated using eight different methods, namely: the ML estimators (MLEs), least-squares estimators (LSEs), Anderson–Darling estimators (ADEs), weighted least-squares estimators (WLSEs), maximum product of spacing estimators (MPSEs), percentiles estimators (PCEs), Cramér–von Mises estimators (CVMEs), and right-tail Anderson–Darling estimators (RADEs).

Let $x_1, x_2, \dots, x_n$ be a random sample from the BXMO-L distribution and let $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ be the corresponding order statistic.

The MLEs of the BXMO-L parameters are obtained by maximizing the following log-likelihood function

$$\ell = n\log(2) + n\log(\theta) + n\log(\beta) - n\log(\lambda) - 2n\log(\alpha) + (2\beta-1)\sum_{i=1}^{n}\log(s_i) + \sum_{i=1}^{n}\log\left(1 - s_i{}^{-\beta}\right) - \frac{1}{\alpha^2}\sum_{i=1}^{n}\left(s_i{}^{\beta}-1\right)^2 + (\theta-1)\sum_{i=1}^{n}\log(1-d_i),$$

where

$$s_i = \left(1 + \frac{x_i}{\lambda}\right) \text{ and } d_i = e^{-\frac{1}{\alpha^2}\left[(s_i)^{\beta}-1\right]^2}$$

The LSEs [23] of the BXMO-L parameters can be obtained by minimizing:

$$Q(\alpha,\theta,\beta,\lambda) = \sum_{i=1}^{n}\left[G(x_{i:n};\alpha,\theta,\beta,\lambda) - \frac{i}{n+1}\right]^2.$$

The LSEs are also follow by solving the nonlinear equations

$$\sum_{i=1}^{n}\left[(1-b_{i:n})^{\theta} - \frac{i}{n+1}\right]\Delta_k(x_{i:n};\alpha,\theta,\beta,\lambda) = 0\,, k = 1,2,3,4,$$

where

$$a_{i:n} = \left(1 + \frac{x_{i:n}}{\lambda}\right) \text{ and } b_{i:n} = e^{\frac{-1}{\alpha^2}\left[(a_{i:n})^{\beta}-1\right]^2}$$

$$\Delta_1(x_{i:n};\alpha,\theta,\beta,\lambda) = \frac{\partial}{\partial\alpha}G(x_{i:n};\alpha,\theta,\beta,\lambda) = \frac{-\theta}{\alpha^3}\left(a_{i:n}{}^{\beta}-1\right)^2 b_{i:n}(1-b_{i:n})^{\theta-1},$$

$$\Delta_2(x_{i:n};\alpha,\theta,\beta,\lambda) = \frac{\partial}{\partial\theta}G(x_{i:n};\alpha,\theta,\beta,\lambda) = (1-b_{i:n})^{\theta}\log(1-b_{i:n}),$$

$$\Delta_3(x_{i:n};\alpha,\theta,\beta,\lambda) = \frac{\partial}{\partial\beta}G(x_{i:n};\alpha,\theta,\beta,\lambda) = \frac{2\theta}{\alpha^2}a_{i:n}{}^{\beta}\log(a_{i:n})\left(a_{i:n}{}^{\beta}-1\right)b_{i:n}(1-b_{i:n})^{\theta-1},$$

$$\Delta_4(x_{i:n};\alpha,\theta,\beta,\lambda)=\frac{\partial}{\partial\lambda}G(x_{i:n};\alpha,\theta,\beta,\lambda)$$
$$=\frac{-2\theta\beta x}{\alpha^2\lambda^2}a_{i:n}{}^{\beta-1}\left(a_{i:n}{}^{\beta}-1\right)b_{i:n}(1-b_{i:n})^{\theta-1}. \quad (19)$$

The WLSEs of the BXMO-L parameters are derived by minimizing the following equation:

$$W(\alpha,\theta,\beta,\lambda)=\sum_{i=1}^{n}\omega_i\left[(1-b_{i:n})^{\theta}-\frac{i}{n+1}\right]^2,$$

Additionally, the WLSEs are determined by solving the nonlinear equations:

$$\sum_{i=1}^{n}\omega_i\left[(1-b_{i:n})^{\theta}-\frac{i}{n+1}\right]\Delta_k(x_{i:n};\alpha,\theta,\beta,\lambda)=0,$$

where $\omega_i=[(n+1)^2(n+2)]/[i(n-i+1)]$, and $\Delta_k(x_{i:n};\alpha,\theta,\beta,\lambda)$ is defined in (19).

The PCEs of the BXMO-L parameters are obtained by minimizing:

$$P(\alpha,\theta,\beta,\lambda)=\sum_{i=1}^{n}\left[x_{i:n}-\lambda\left(\sqrt[\beta]{\left\{1+\alpha\sqrt{\log\left[1/\left(1-\sqrt[\theta]{p_i}\right)\right]}\right\}}-1\right)\right]^2,$$

where $p_i=i/(n+1)$, then p_i is the unbiased estimator of $G(x_{i:n};\alpha,\theta,\beta,\lambda)$.

Suppose we have a random sample $X_1,X_2,\dots,X_n$ of size n from a cdf $G(x_{i:n};\alpha,\theta,\beta,\lambda)$ with corresponding pdf $g(x_{i:n};\alpha,\theta,\beta,\lambda)$ and we estimate and $x_{i:n}$ denote the i^{th} order statistics.

The uniform spacings of a random sample of size n from the BXMO-L distribution are defined by $D_i(\alpha,\theta,\beta,\lambda)=G(x_{i:n};\alpha,\theta,\beta,\lambda)-G(x_{i-1:n};\alpha,\theta,\beta,\lambda)$, where $G(x_{0:n};\alpha,\theta,\beta,\lambda)=0$, and $G(x_{n+1:n};\alpha,\theta,\beta,\lambda)=1$. Then, The MPSEs of the BXMO-L parameters can obtained by maximizing:

$$\log[M_i(\alpha,\theta,\beta,\lambda)]=\frac{1}{n+1}\sum_{i=1}^{n+1}\log\left[(1-b_{i:n})^{\theta}-(1-b_{i-1:n})^{\theta}\right],$$

where $M_i(\alpha,\theta,\beta,\lambda)=[\prod_{i=1}^{n+1}D_i(\alpha,\theta,\beta,\lambda)]^{\frac{1}{n+1}}$.

The CVMEs of the BXMO-L parameters are also calculated by minimizing the following function:

$$\delta(\alpha,\theta,\beta,\lambda)=\frac{1}{12n}+\sum_{i=1}^{n}\left[(1-b_{i:n})^{\theta}-\frac{2i-1}{2n}\right]^2.$$

Similarly, the CVMEs follow by solving the nonlinear equations

$$\sum_{i=1}^{n}\left[(1-b_{i:n})^{\theta}-\frac{2i-1}{2n}\right]\Delta_k(x_{i:n};\alpha,\theta,\beta,\lambda)=0,$$

where $\Delta_k(x_{i:n};\alpha,\theta,\beta,\lambda)$ is defined in Equation (19).

The ADEs the BXMO-L parameters are obtained by minimizing

$$A(\alpha,\theta,\beta,\lambda)=-n-\frac{1}{n}\sum_{i=1}^{n}(2i-1)$$
$$\{\log[G(x_{i:n};\alpha,\theta,\beta,\lambda)]+\log[\bar{G}(x_{n+1-i:n};\alpha,\theta,\beta,\lambda)]\},$$

with respect to θ,α,β and λ. The ADEs are also determined by solving the nonlinear equations:

$$\sum_{i=1}^{n}(2i-1)\left\{\frac{\Delta_k(x_{i:n};\alpha,\theta,\beta,\lambda)}{G(x_{i:n};\alpha,\theta,\beta,\lambda)}-\frac{\Delta_k(x_{i:n};\alpha,\theta,\beta,\lambda)}{\bar{G}(x_{n+1-i:n};\alpha,\theta,\beta,\lambda)}\right\}=0,$$

where $\Delta_k(x_{i:n};\alpha,\theta,\beta,\lambda)$ is defined in (19).

The RADEs of the BXMO-L parameters are obtained by minimizing the following function:

$$R(\alpha,\theta,\beta,\lambda)=\frac{n}{2}-2\sum_{i=1}^{n}G(x_{i:n};\alpha,\theta,\beta,\lambda)$$
$$-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\{\log[\bar{G}(x_{n+1-i:n};\alpha,\theta,\beta,\lambda)]\},$$

with respect to θ,α,β and λ.

The RADEs can also be obtained by solving the nonlinear equations:

$$\sum_{i=1}^{n}\Delta_k(x_{i:n};\alpha,\theta,\beta,\lambda)+\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\frac{\Delta_k(x_{i:n};\alpha,\theta,\beta,\lambda)}{\bar{G}(x_{n+1-i:n};\alpha,\theta,\beta,\lambda)}\right]=0,$$

where $\Delta_k(x_{i:n};\alpha,\theta,\beta,\lambda)$ is defined in (19).

7. SIMULATION RESULTS

In this section, the performance of the proposed estimators of the BXMO-L parameters is explored based on numerical simulation results. Some sample sizes $n=\{20,50,150\}$ and different parametric values $\theta=(2.5,5,4,1,.75,1.2,2,1.1)$, $\alpha=(4,2,0.5,2,2.75,5,3,0.9)$, $\beta=(2,1.5,3,4,6,5,1,0.5)$ and $\lambda=(3,1,2,7,3,2.5,0.5,0.3)$ are considered. We generate $N=2000$ random samples from the BXMO-L distribution. The average absolute biases (|BIAS|), mean square errors (MSE), and mean relative estimates (MRE) are obtained for all parameter combinations and sample sizes using the R software©.

Four of the eight simulated outcomes are presented in Tables 3–6. These tables display the BIAS|, MSE, and MRE for each estimator: MLEs, LSEs, WLSEs, MPSEs, PCEs, CVMEs, ADEs, and RTADEs. Additionally, the tables provide the rank of each estimator in every row, with superscripts indicating rank positions, and $\sum R$ representing the partial sum of ranks for each column and sample size. Table 7 summarizes both the partial and overall ranks of the studied estimators.

The results indicate that the proposed estimators perform very well, with the following order from best to worst based on overall ranks: WLSEs, LSEs, CVMEs, ADEs, PCEs, RTADEs, MPSEs, and MLEs. All estimators demonstrate consistency, as evidenced by the decrease in MSE, |BIAS|, and MRE with increasing sample size for all parameter values. In conclusion, various estimation methods effectively estimate the BXMO-L parameters.

8. REAL-LIFE DATA APPLICATIONS

In this section, we present two real-life data applications to show the importance and applicability of the BXMO-L distribution. The first data represents time-to-failure of turbocharger of one type of engine which consists of 40 observations [24]: 2.6, 7.3, 1.6, 2, 3, 3.9, 4.5, 4.6, 3.5, 4.8, 5.1, 5.3, 5, 5.4, 5.8, 5.6, 6, 6.1, 6.3, 6.5, 6, 6.5, 7.7, 7.8, 7.7, 7.3, 7.1, 7.3, 7, 6.7, 8.7, 8.5, 8.8, 8.4, 8.3, 8.4, 8.1, 7.9, 8, 9.

The second set of data represents survival times of 40 patients (in weeks) who are suffering from acute myelogenous leukemia [25]. The data are: 255, 115, 181, 418, 441, 461, 516, 739, 743, 789, 807, 865, 924, 983, 1024, 1062, 1063, 1165, 1191, 1222, 1251, 1277, 1290, 1357, 1815, 1369, 1455, 1408, 1478, 1222, 1549, 1578, 1603, 1578, 1605, 1696, 1599, 1735, 1799, 1852.

The BXMO-L distribution is compared with several competing Lomax distributions, namely: the odd exponentiated half-logistic Lomax (OEHL-L) [26], exponential Lomax (E-L) [27], Marshal–Olkin Burr-X Lomax (MOBX-L) [28], Weibull–Lomax (W-L) [29], Kumaraswamy–Lomax (KW-L) and McDonald–Lomax (Mc-L) [30], transmuted Weibull–Lomax (TW-L) [31], odd log- logistic Lomax (OLL-L) [32], Burr-X Lomax (BX-L) [33], Fréchet Topp–Leone Lomax (FTL-L) [34], Poisson–Lomax (PO-L) [35], and Lomax (L) distributions. The fitted distributions are compared based on the following criteria namely: the Akaike-information criterion (AIC), consistent AIC (CAIC), Hannan–Quinn information criterion (HQIC), Bayesian-information criterion (BIC), the maximized log-likelihood ($-\ell$), Cramér–Von Mises (CVM), Anderson–Darling (AD), Kolmogorov–Smirnov (KS) and the KS p-value.

The R program is used to obtain the numerical results in this section. The goodness-of-fit measures for both data sets are presented in Tables 8 and 10, respectively. Tables 9 and 11report the MLEs of the parameters of the fitted models and their standard errors (SEs) for both data sets, respectively.

Table 3
Simulation Results of Several Estimation Methods for θ=2.5, α=4, β=2, λ=3

n	Est.	Param.	MLEs	LSEs	WLSEs	CVMEs	MPSEs	PCEs	ADEs	RADEs
20	|BIAS|	θ	0.82136{8}	0.59099{1}	0.61773{2}	0.66146{5}	0.72296{7}	0.61942{3}	0.65497{4}	0.71021{6}
		α	0.45885{8}	0.29502{3}	0.29815{5}	0.28161{2}	0.24836{1}	0.30057{6}	0.29718{4}	0.31507{7}
		β	0.13402{8}	0.07340{3}	0.07283{1}	0.07981{5}	0.11628{7}	0.07382{4}	0.07339{2}	0.08598{6}
		λ	0.26135{7}	0.17505{2}	0.17569{3}	0.17264{1}	0.27047{8}	0.18626{5}	0.18473{4}	0.19864{6}
	MSE	θ	1.41311{8}	0.61553{1}	0.70846{3}	0.85620{5}	0.95685{6}	0.63167{2}	0.79930{4}	0.97859{7}
		α	0.40262{8}	0.16185{2}	0.16835{5}	0.15082{1}	0.34670{7}	0.16625{4}	0.16479{3}	0.19408{6}
		β	0.04054{8}	0.01105{2}	0.01076{1}	0.01196{5}	0.03397{7}	0.01133{4}	0.01117{3}	0.01653{6}
		λ	0.15291{7}	0.07107{2}	0.07125{3}	0.06536{1}	0.15830{8}	0.07973{5}	0.07886{4}	0.08712{6}
	MRE	θ	0.32854{8}	0.23640{1}	0.24709{2}	0.26458{5}	0.28918{7}	0.24777{3}	0.26199{4}	0.28409{6}
		α	0.11471{8}	0.07376{2}	0.07454{4}	0.07040{1}	0.11010{7}	0.07514{5}	0.07430{3}	0.07877{6}
		β	0.06701{8}	0.03670{2.5}	0.03641{1}	0.03990{5}	0.05814{7}	0.03691{4}	0.03670{2.5}	0.04299{6}
		λ	0.08712{7}	0.05835{2}	0.05856{3}	0.05755{1}	0.09016{8}	0.06209{5}	0.06158{4}	0.06621{6}
		$\sum R$	93{8}	23.5{1}	33{2}	37{3}	80{7}	50{5}	41.5{4}	74{6}
50	|BIAS|	θ	0.47357{5}	0.44290{4}	0.42703{3}	0.47909{7}	0.47742{6}	0.40630{1}	0.42475{2}	0.48709{8}
		α	0.31802{7}	0.21581{5}	0.19905{1}	0.20648{3}	0.34458{8}	0.21404{4}	0.20248{2}	0.22915{6}
		β	0.10065{8}	0.05099{3}	0.05079{2}	0.05578{5}	0.09386{7}	0.04802{1}	0.05284{4}	0.06118{6}
		λ	0.16465{7}	0.10237{2}	0.10359{3}	0.10735{5}	0.17192{8}	0.10688{4}	0.10044{1}	0.12691{6}
	MSE	θ	0.43199{7}	0.35315{5}	0.33871{3}	0.43675{8}	0.35109{4}	0.26582{1}	0.32032{2}	0.42120{6}
		α	0.23275{8}	0.08776{5}	0.07556{1}	0.08254{3}	0.23257{7}	0.08743{4}	0.07981{2}	0.10440{6}
		β	0.02686{8}	0.00497{3}	0.00486{2}	0.00613{5}	0.02205{7}	0.00481{1}	0.00510{4}	0.00816{6}
		λ	0.07634{8}	0.02331{2}	0.02372{3}	0.02625{4}	0.06773{7}	0.02781{5}	0.02189{1}	0.03972{6}
	MRE	θ	0.18943{5}	0.17716{4}	0.17081{3}	0.19164{7}	0.19097{6}	0.16252{1}	0.16990{2}	0.19484{8}
		α	0.07951{7}	0.05395{5}	0.04976{1}	0.05162{3}	0.08615{8}	0.05351{4}	0.05062{2}	0.05729{6}
		β	0.05033{8}	0.02549{3}	0.02540{2}	0.02789{5}	0.04693{7}	0.02401{1}	0.02642{4}	0.03059{6}
		λ	0.05488{7}	0.03412{2}	0.03453{3}	0.03578{5}	0.05731{8}	0.03563{4}	0.03348{1}	0.04230{6}
		$\sum R$	85{8}	43{4}	27{1.5}	60{5}	83{7}	31{3}	27{1.5}	76{6}
150	|BIAS|	θ	0.26770{4}	0.27702{6}	0.23469{2}	0.27261{5}	0.28004{7}	0.23192{1}	0.24025{3}	0.28344{8}
		α	0.21884{7}	0.13277{5}	0.12037{2}	0.12424{3}	0.24836{8}	0.13377{6}	0.11890{1}	0.13159{4}
		β	0.06927{7}	0.03913{4}	0.03491{2}	0.03962{5}	0.08060{8}	0.03281{1}	0.03616{3}	0.04277{6}
		λ	0.08937{7}	0.06420{5}	0.05807{2}	0.06179{4}	0.11724{8}	0.06142{3}	0.05635{1}	0.07493{6}
	MSE	θ	0.12406{5}	0.12908{7}	0.09200{2}	0.12808{6}	0.12275{4}	0.08261{1}	0.09492{3}	0.13422{8}
		α	0.11151{7}	0.03276{4}	0.02641{2}	0.02808{3}	0.14540{8}	0.03416{6}	0.02476{1}	0.03331{5}
		β	0.01143{7}	0.00265{4}	0.00209{1}	0.00267{5}	0.01828{8}	0.00221{3}	0.00220{2}	0.00397{6}
		λ	0.02230{7}	0.00809{4}	0.00673{1}	0.00751{3}	0.0425{8}1	0.00848{5}	0.00696{2}	0.01769{6}
	MRE	θ	0.10708{4}	0.11081{6}	0.09388{2}	0.10904{5}	0.11202{7}	0.09277{1}	0.09610{3}	0.11338{8}
		α	0.05471{7}	0.03319{5}	0.03009{2}	0.03106{3}	0.06209{8}	0.03344{6}	0.02972{1}	0.03290{4}
		β	0.03464{7}	0.01956{4}	0.01745{2}	0.01981{5}	0.04030{8}	0.01641{1}	0.01808{3}	0.02139{6}
		λ	0.02979{7}	0.02140{5}	0.01936{2}	0.02060{4}	0.03908{8}	0.02047{3}	0.01878{1}	0.02498{6}
		$\sum R$	76{7}	59{5}	22{1}	51{4}	90{8}	37{3}	24{2}	73{6}

Table 4
Simulation Results of Several Estimation Methods for $\theta = 5, \alpha = 2, \beta = 1.5, \lambda = 1$

n	Est.	Param.	MLEs	LSEs	WLSEs	CVMEs	MPSEs	PCEs	ADEs	RADEs
20	|BIAS|	θ	1.06341{7}	0.63954{3}	0.69071{5}	0.62023{1}	1.17252{8}	0.78580{6}	0.68986{4}	0.62623{2}
		α	0.24884{8}	0.14649{3}	0.15630{5}	0.14295{2}	0.11656{1}	0.17920{6}	0.15063{4}	0.19214{7}
		β	0.18707{7}	0.08405{1}	0.08813{3}	0.08646{2}	0.17170{6}	0.09215{4}	0.09558{5}	0.20311{8}
		λ	0.11851{6}	0.05183{2}	0.05694{4}	0.04756{1}	0.11869{7}	0.05652{3}	0.05970{5}	0.15643{8}
	MSE	θ	2.28203{7}	0.94168{2}	1.13122{5}	0.93010{1}	2.37276{8}	1.24591{6}	1.06721{4}	0.95110{3}
		α	0.13210{8}	0.04009{2}	0.04607{4}	0.03770{1}	0.12740{7}	0.06236{5}	0.04110{3}	0.07680{6}
		β	0.08612{7}	0.01576{2}	0.01631{3}	0.01561{1}	0.05937{6}	0.01861{4}	0.01948{5}	0.10378{8}
		λ	0.03410{7}	0.00660{2}	0.00763{4}	0.00539{1}	0.02834{6}	0.00713{3}	0.00937{5}	0.05918{8}
	MRE	θ	0.21268{7}	0.12791{3}	0.13814{5}	0.12405{1}	0.23450{8}	0.15716{6}	0.13797{4}	0.12525{2}
		α	0.12442{7}	0.07324{2}	0.07815{4}	0.07148{1}	0.12771{8}	0.08960{5}	0.07532{3}	0.09607{6}
		β	0.12471{7}	0.05603{1}	0.05875{3}	0.05764{2}	0.11447{6}	0.06144{4}	0.06372{5}	0.13541{8}
		λ	0.11851{6}	0.05183{2}	0.05694{4}	0.04756{1}	0.11869{7}	0.05652{3}	0.05970{5}	0.15643{8}
		$\sum R$	84{8}	25{2}	49{3}	15{1}	78{7}	55{5}	52{4}	74{6}
50	|BIAS|	θ	0.79183{7}	0.54345{3}	0.57409{4}	0.54189{2}	0.82593{8}	0.65585{6}	0.58840{5}	0.48689{1}
		α	0.17621{7}	0.11112{2}	0.12324{4}	0.10736{1}	0.17825{8}	0.13598{5}	0.11192{3}	0.13906{6}
		β	0.12104{6}	0.06584{1}	0.06927{3}	0.06615{2}	0.12264{7}	0.06947{4}	0.07372{5}	0.16535{8}
		λ	0.06638{6}	0.03361{2}	0.03700{3}	0.03236{1}	0.07631{7}	0.03991{5}	0.03731{4}	0.12952{8}
	MSE	θ	1.21064{8}	0.62771{2}	0.63663{4}	0.63016{3}	1.10638{7}	0.77527{6}	0.69245{5}	0.48035{1}
		α	0.06350{8}	0.02155{2}	0.02621{4}	0.02012{1}	0.06303{7}	0.03230{5}	0.02268{3}	0.03760{6}
		β	0.03202{6}	0.00801{1}	0.00974{4}	0.00813{2}	0.03227{7}	0.00922{3}	0.01058{5}	0.06058{8}
		λ	0.01080{6}	0.00249{2}	0.00360{4}	0.00244{1}	0.01380{7}	0.00367{5}	0.00351{3}	0.03594{8}
	MRE	θ	0.15837{7}	0.10869{3}	0.11482{4}	0.10838{2}	0.16519{8}	0.13117{6}	0.11768{5}	0.09738{1}
		α	0.08811{7}	0.05556{2}	0.06162{4}	0.05368{1}	0.08912{8}	0.06799{5}	0.05596{3}	0.06953{6}
		β	0.08069{6}	0.04389{1}	0.04618{3}	0.04410{2}	0.08176{7}	0.04631{4}	0.04915{5}	0.11023{8}
		λ	0.06638{6}	0.03361{2}	0.03700{3}	0.03236{1}	0.07631{7}	0.03991{5}	0.03731{4}	0.12952{8}
		$\sum R$	80{7}	23{2}	44{3}	19{1}	88{8}	59{5}	50{4}	69{6}
150	|BIAS|	θ	0.48683{7}	0.40656{3}	0.41533{5}	0.39676{2}	0.52843{8}	0.44275{6}	0.41082{4}	0.33545{1}
		α	0.11238{7}	0.07687{2}	0.08285{4}	0.07559{1}	0.11656{8}	0.08993{5}	0.08231{3}	0.09918{6}
		β	0.07356{6}	0.04599{1}	0.04790{4}	0.04717{3}	0.08168{7}	0.04660{2}	0.05785{5}	0.12726{8}
		λ	0.03499{6}	0.01984{2}	0.02207{3}	0.01954{1}	0.04817{7}	0.02388{4}	0.02778{5}	0.10000{8}
	MSE	θ	0.41841{7}	0.30625{4}	0.31626{5}	0.29210{2}	0.46518{8}	0.32797{6}	0.30506{3}	0.21378{1}
		α	0.02886{8}	0.01050{2}	0.01211{3}	0.01015{1}	0.02742{7}	0.01442{5}	0.01314{4}	0.01955{6}
		β	0.01279{6}	0.00358{1}	0.00409{4}	0.00380{3}	0.01504{7}	0.00377{2}	0.00660{5}	0.03191{8}
		λ	0.00371{6}	0.00097{2}	0.00136{4}	0.00095{1}	0.00731{7}	0.00129{3}	0.00243{5}	0.01944{8}
	MRE	θ	0.09737{7}	0.08131{3}	0.08307{5}	0.07935{2}	0.10569{8}	0.08855{6}	0.08216{4}	0.06709{1}
		α	0.05619{7}	0.03843{2}	0.04143{4}	0.03780{1}	0.05828{8}	0.04497{5}	0.04115{3}	0.04959{6}
		β	0.04904{6}	0.03066{1}	0.03193{4}	0.03145{3}	0.05445{7}	0.03107{2}	0.03857{5}	0.08484{8}
		λ	0.03499{6}	0.01984{2}	0.02207{3}	0.01954{1}	0.04817{7}	0.02388{4}	0.02778{5}	0.10000{8}
		$\sum R$	79{7}	25{2}	48{3}	21{1}	89{8}	50{4}	51{5}	69{6}

Table 5
Simulation Results of Several Estimation Methods for $\theta = 4, \alpha = 0.5, \beta = 3, \lambda = 2$

n	Est.	Param.	MLEs	LSEs	WLSEs	CVMEs	MPSEs	PCEs	ADEs	RADEs
20	|BIAS|	θ	1.09866{8}	0.61498{1.5}	0.66531{3}	0.61498{1.5}	0.96026{7}	0.77727{6}	0.69586{5}	0.67269{4}
		α	0.07024{8}	0.03837{4}	0.04153{5}	0.03468{3}	0.02241{1}	0.04743{7}	0.04171{6}	0.03436{2}
		β	0.25547{8}	0.17495{3}	0.18400{5}	0.17031{1}	0.23667{7}	0.21081{6}	0.17640{4}	0.17081{2}
		λ	0.19654{7}	0.11597{2}	0.12922{3}	0.10050{1}	0.25877{8}	0.14911{6}	0.13780{5}	0.13503{4}
	MSE	θ	2.81754{8}	0.90327{1}	0.97631{3}	0.92382{2}	1.47171{7}	1.16955{6}	1.09441{5}	1.02077{4}
		α	0.01051{8}	0.00295{3}	0.00340{4}	0.00230{1}	0.00730{7}	0.00438{6}	0.00349{5}	0.00248{2}
		β	0.16011{8}	0.06498{3}	0.06913{4}	0.05862{1}	0.11019{7}	0.09302{6}	0.06371{2}	0.07086{5}
		λ	0.07927{7}	0.03007{2}	0.03632{3}	0.02151{1}	0.13779{8}	0.04626{6}	0.04055{5}	0.04044{4}
	MRE	θ	0.27467{8}	).15375{1.5}	0.16633{3}	).15375{1.5}	0.24006{7}	0.19432{6}	0.17397{5}	0.16817{4}
		α	0.14048{8}	0.07674{3}	0.08306{4}	0.06936{2}	0.12738{7}	0.09486{6}	0.08343{5}	0.06871{1}
		β	0.08516{8}	0.05832{3}	0.06133{5}	0.05677{1}	0.07889{7}	0.07027{6}	0.05880{4}	0.05694{2}
		λ	0.09827{7}	0.05798{2}	0.06461{3}	0.05025{1}	0.12938{8}	0.07455{6}	0.06890{5}	0.06751{4}
		$\sum R$	93{8}	29{2}	45{4}	17{1}	81{7}	73{6}	56{5}	38{3}
50	|BIAS|	θ	0.68572{8}	0.48865{1}	0.53616{3}	0.49699{2}	0.68054{7}	0.57767{6}	0.56504{5}	0.55790{4}
		α	0.03732{7}	0.02745{3}	0.02838{5}	0.02584{2}	0.04016{8}	0.03472{6}	0.02773{4}	0.02498{1}
		β	0.17202{7}	0.14332{2}	0.14447{3}	0.13789{1}	0.17538{8}	0.15126{6}	0.14618{5}	0.14593{4}
		λ	0.11074{7}	0.07736{2}	0.07945{3}	0.06793{1}	0.14291{8}	0.10049{6}	0.08541{4}	0.08636{5}
	MSE	θ	0.87958{8}	0.49659{1}	0.59049{3}	0.53767{2}	0.72972{7}	0.59367{4}	0.66253{6}	0.65040{5}
		α	0.00280{7}	0.00142{3}	0.00154{5}	0.00118{1}	0.00310{8}	0.00223{6}	0.00150{4}	0.00127{2}
		β	0.05513{7}	0.03673{2}	0.03714{3}	0.03330{1}	0.05728{8}	0.04345{5}	0.03990{4}	0.04435{6}
		λ	0.02656{7}	0.01282{2}	0.01302{3}	0.00967{1}	0.04304{8}	0.02051{6}	0.01770{5}	0.01634{4}
	MRE	θ	0.17143{8}	0.12216{1}	0.13404{3}	0.12425{2}	0.17014{7}	0.14442{6}	0.14126{5}	0.13948{4}
		α	0.07463{7}	0.05490{3}	0.05676{5}	0.05169{2}	0.08032{8}	0.06944{6}	0.05546{4}	0.04996{1}
		β	0.05734{7}	0.04777{2}	0.04816{3}	0.04596{1}	0.05846{8}	0.05042{6}	0.04873{5}	0.04864{4}
		λ	0.05537{7}	0.03868{2}	0.03972{3}	0.03397{1}	0.07145{8}	0.05024{6}	0.04271{4}	0.04318{5}
		$\sum R$	87{7}	24{2}	42{3}	17{1}	93{8}	69{6}	55{5}	45{4}
150	|BIAS|	θ	0.40841{6}	0.34970{1}	0.37495{3}	0.37294{2}	0.42256{8}	0.39190{5}	0.38625{4}	0.40930{7}
		α	0.02029{6}	0.01889{4}	0.01920{5}	0.01849{3}	0.02241{7}	0.02254{8}	0.01769{2}	0.01763{1}
		β	0.11424{7}	0.10438{4}	0.10123{2}	0.10461{5}	0.12068{8}	0.09815{1}	0.10426{3}	0.11087{6}
		λ	0.05701{6}	0.04365{2}	0.04594{3}	0.04276{1}	0.07039{8}	0.05940{7}	0.04616{4}	0.05418{5}
	MSE	θ	0.27943{7}	0.24033{1}	0.26921{3}	0.27483{5}	0.27771{6}	0.25594{2}	0.27479{4}	0.32483{8}
		α	0.00080{6}	0.00065{4}	0.00067{5}	0.00059{2}	0.00094{7}	0.00097{8}	0.00055{1}	0.00061{3}
		β	0.02311{6}	0.01860{5}	0.01752{1}	0.01850{3}	0.02649{8}	0.01754{2}	0.01858{4}	0.02355{7}
		λ	0.00685{6}	0.00446{2}	0.00468{3}	0.00410{1}	0.00971{8}	0.00843{7}	0.00519{4}	0.00676{5}
	MRE	θ	0.10210{6}	0.08742{1}	0.09374{3}	0.09323{2}	0.10564{8}	0.09797{5}	0.09656{4}	0.10232{7}
		α	0.04059{6}	0.03778{4}	0.03840{5}	0.03699{3}	0.04481{7}	0.04508{8}	0.03537{2}	0.03526{1}
		β	0.03808{7}	0.03479{4}	0.03374{2}	0.03487{5}	0.04023{8}	0.03272{1}	0.03475{3}	0.03696{6}
		λ	0.02850{6}	0.02182{2}	0.02297{3}	0.02138{1}	0.03519{8}	0.02970{7}	0.02308{4}	0.02709{5}
		$\sum R$	75{7}	34{2}	38{3}	33{1}	91{8}	61{5.5}	39{4}	61{5.5}

Table 6

Simulation Results of Several Estimation Methods for $\theta = 1, \alpha = 2, \beta = 4, \lambda = 7$

n	Est.	Param.	MLEs	LSEs	WLSEs	CVMEs	MPSEs	PCEs	ADEs	RADEs
20	|BIAS|	θ	0.28620{6}	0.26764{4}	0.27354{5}	0.29500{7}	0.24867{3}	0.09221{1}	0.24256{2}	0.29878{8}
		α	0.43678{8}	0.17120{3}	0.17233{4}	0.18044{5}	0.20076{7}	0.05965{1}	0.18527{6}	0.16743{2}
		β	0.38837{8}	0.18987{4}	0.16273{2}	0.20921{6}	0.31489{7}	0.04209{1}	0.20323{5}	0.17982{3}
		λ	0.48248{7}	0.39686{5}	0.35803{3}	0.37412{4}	0.53106{8}	0.21736{1}	0.34235{2}	0.39854{6}
	MSE	θ	0.16613{6}	0.13671{4}	0.14552{5}	0.18495{7}	0.10208{2}	0.09221{1}	0.11947{3}	0.19286{8}
		α	0.39301{7}	0.06431{5}	0.06036{2}	0.06428{4}	0.44432{8}	0.05965{1}	0.06768{6}	0.06217{3}
		β	0.43653{8}	0.07672{4}	0.05354{2}	0.10087{6}	0.33052{7}	0.04209{1}	0.08194{5}	0.07261{3}
		λ	0.50382{7}	0.27061{5}	0.22430{3}	0.25969{4}	0.55914{8}	0.21736{2}	0.19293{1}	0.27442{6}
	MRE	θ	0.28620{6}	0.26764{4}	0.27354{5}	0.29500{7}	0.24867{3}	0.23002{1}	0.24256{2}	0.29878{8}
		α	0.21839{7}	0.08560{2}	0.08616{4}	0.09022{5}	0.22897{8}	0.08590{3}	0.09264{6}	0.08371{1}
		β	0.09709{8}	0.04747{4}	0.04068{2}	0.05230{6}	0.07872{7}	0.03647{1}	0.05081{5}	0.04496{3}
		λ	0.06893{7}	0.05669{5}	0.05115{3}	0.05345{4}	0.07587{8}	0.04943{2}	0.04891{1}	0.05693{6}
		$\sum R$	85{8}	49{4}	40{2}	65{6}	76{7}	16{1}	44{3}	57{5}
50	|BIAS|	θ	0.17048{6}	0.16021{5}	0.15213{2}	0.17991{8}	0.15356{3}	0.03171{1}	0.15442{4}	0.17211{7}
		α	0.29199{7}	0.10586{4}	0.10480{3}	0.11352{5}	0.32100{8}	0.02338{1}	0.12066{6}	0.09576{2}
		β	0.29888{8}	0.14982{4}	0.12683{2}	0.16363{5}	0.27626{7}	0.02505{1}	0.16839{6}	0.13942{3}
		λ	0.36623{7}	0.30091{6}	0.24365{2}	0.28310{4}	0.41927{8}	0.12192{1}	0.28057{3}	0.30032{5}
	MSE	θ	0.05172{6}	0.04389{5}	0.03886{3}	0.05985{8}	0.03537{2}	0.03171{1}	0.03965{4}	0.05258{7}
		α	0.18989{7}	0.02139{3}	0.02138{2}	0.02435{5}	0.23014{8}	0.02338{4}	0.02853{6}	0.02052{1}
		β	0.22936{7}	0.04217{4}	0.02679{2}	0.04674{5}	0.23540{8}	0.02505{1}	0.04879{6}	0.03603{3}
		λ	0.28214{7}	0.14991{6}	0.10172{1}	0.13359{4}	0.36576{8}	0.12192{3}	0.11808{2}	0.14582{5}
	MRE	θ	0.17048{6}	0.16021{5}	0.15213{2}	0.17991{8}	0.15356{3}	0.14449{1}	0.15442{4}	0.17211{7}
		α	0.14599{7}	0.05293{3}	0.05240{2}	0.05676{5}	0.16050{8}	0.05595{4}	0.06033{6}	0.04788{1}
		β	0.07472{8}	0.03746{4}	0.03171{2}	0.04091{5}	0.06906{7}	0.02967{1}	0.04210{6}	0.03485{3}
		λ	0.05232{7}	0.04299{6}	0.03481{1}	0.04044{4}	0.05990{8}	0.03779{2}	0.04008{3}	0.04290{5}
		$\sum R$	83{8}	55{4}	24{2}	66{6}	78{7}	21{1}	56{5}	49{3}
150	|BIAS|	θ	0.09275{5}	0.09518{6}	0.08816{3}	0.09769{8}	0.09162{4}	0.01090{1}	0.08720{2}	0.09702{7}
		α	0.18581{7}	0.07252{5}	0.06404{2}	0.07142{4}	0.20076{8}	0.00923{1}	0.07582{6}	0.06780{3}
		β	0.23700{8}	0.11764{4}	0.10035{2}	0.11956{5}	0.23299{7}	0.01530{1}	0.13077{6}	0.10421{3}
		λ	0.30458{7}	0.20912{3}	0.16787{2}	0.21195{4}	0.31955{8}	0.04439{1}	0.23241{6}	0.22390{5}
	MSE	θ	0.01427{6}	0.01406{5}	0.01258{3}	0.01540{7}	0.01289{4}	0.01090{1}	0.01208{2}	0.01576{8}
		α	0.08893{7}	0.00946{5}	0.00773{1}	0.00922{2}	0.09359{8}	0.00923{3}	0.01130{6}	0.00924{4}
		β	0.16408{8}	0.02177{4}	0.01589{2}	0.02333{5}	0.14027{7}	0.01530{1}	0.02692{6}	0.01861{3}
		λ	0.20071{8}	0.06506{3}	0.04725{2}	0.06696{4}	0.18654{7}	0.04439{1}	0.07414{6}	0.06941{5}
	MRE	θ	0.09275{5}	0.09518{6}	0.08816{3}	0.09769{8}	0.09162{4}	0.08373{1}	0.08720{2}	0.09702{7}
		α	0.09291{7}	0.03626{5}	0.03202{1}	0.03571{4}	0.10038{8}	0.03346{2}	0.03791{6}	0.03390{3}
		β	0.05925{8}	0.02941{4}	0.02509{2}	0.02989{5}	0.05825{7}	0.02335{1}	0.03269{6}	0.02605{3}
		λ	0.04351{7}	0.02987{3}	0.02398{2}	0.03028{4}	0.04565{8}	0.02385{1}	0.03320{6}	0.03199{5}
		$\sum R$	83{8}	53{3}	25{2}	60{5.5}	80{7}	15{1}	60{5.5}	56{4}

Table 7
Outcomes of Ranks for Different Estimation Methods

$(\theta,\alpha,\beta,\lambda)^T$	n	MLEs	LSEs	WLSEs	CVMEs	MPSEs	PCEs	ADEs	RADEs
$(2.5,4,2,3)^T$	20	8	1	2	3	7	5	4	6
	50	8	4	1.5	5	7	3	1.5	6
	150	7	5	1	4	8	3	2	6
$(5,2,1.5,1)^T$	20	8	2	3	1	7	5	4	6
	50	7	2	3	1	8	5	4	6
	150	7	2	3	1	8	4	5	6
$(4,0.5,3,2)^T$	20	8	2	4	1	7	6	5	3
	50	7	2	3	1	8	6	5	4
	150	7	2	3	1	8	5.5	4	5.5
$(1,2,4,7)^T$	20	8	4	2	6	7	1	3	5
	50	8	4	2	6	7	1	5	3
	150	8	3	2	5.5	7	1	5.5	4
$(0.75,2.75,6,3)^T$	20	8	3	2	6	7	1	4	5
	50	7	4.5	2	6	8	1	4.5	3
	150	7	3	2	4.5	8	1	6	4.5
$(1.2,5,5,2.5)^T$	20	8	5	1	6	7	3	2	4
	50	7	5	1	6	8	4	2.5	2.5
	150	7	3	1	4	8	2	6	5
$(2,3,1,0.5)^T$	20	8	1	3	2	7	5	4	6
	50	8	2.5	1	4	7	5	2.5	6
	150	7	3	2	4	8	6	1	5
$(1.1,0.9,0.5,0.3)^T$	20	8	2.5	1	4	7	6	2.5	5
	50	8	2.5	1	4	7	6	5	2.5
	150	8	2	1	4	7	6	3	5
$\sum R$		182	70	47.5	90	178	91.5	91	114
Overall rank		8	2	1	3	7	5	4	6

Table 8
The Outcomes of Goodness-of-Fit Measures for the First Data

Model	AIC	CAIC	BIC	HQIC	CVM	AD	$-\ell$	KS	p-value
BXMO-L	166.479	167.621	173.234	168.921	0.03121	0.19812	79.239	0.06884	0.99142
OEHL-L	167.597	168.740	174.352	170.039	0.03526	0.23203	79.798	0.07481	0.97859
E-L	166.684	167.351	171.751	168.516	0.03509	0.24693	80.342	0.08730	0.92069
MOBX-L	172.345	173.487	179.100	174.787	0.05791	0.40968	82.172	0.09048	0.89878
W-L	167.879	169.022	174.635	170.322	0.03163	0.23396	79.940	0.09063	0.89765
TW-L	169.877	171.642	178.321	172.930	0.03163	0.23346	79.938	0.09102	0.89480
OLL-L	173.085	174.228	179.840	175.527	0.07828	0.58156	82.542	0.10767	0.74267
KWL-L	173.249	174.392	180.005	175.692	0.08077	0.59810	82.625	0.10872	0.73175
BX-L	170.906	171.573	175.972	172.738	0.07971	0.59055	82.453	0.11109	0.70696
Mc-L	176.982	178.747	185.426	180.035	0.10777	0.77992	83.491	0.12563	0.55302
FTL-L	188.004	189.769	196.449	191.058	0.23552	1.53458	89.002	0.15219	0.31238
PO-L	185.202	185.869	190.269	187.034	0.25082	1.62178	89.601	0.15701	0.27757
L	230.641	230.965	234.018	231.862	0.20653	1.36895	113.32	0.36312	0.00005

Table 9
The Parameters Estimates (first line) and SEs (second line) for the First Data

BXMO-L$(\lambda, \beta, \alpha, \theta)$	129.693 1513.84	77.9627 830.063	108.563 505.012	0.41648 0.35671	–
OEHL-L(b, a, λ, α)	20116.1 525.858	12719.7 2764.16	0.01329 0.01647	0.79123 0.25202	–
E-L(β, α, λ)	6305.09 1838.70	4168.36 1294.41	0.00887 0.00491	–	–
MOBX-L(b, a, α, θ)	1024.10 1987.57	272.509 526.194	941.250 7704.47	0.02068 0.16588	–
W-L(β, α, a, b)	8680.07 2427.81	4531.04 4158.36	1.18585 0.90119	0.12643 0.01071	–
TW-L$(\beta, \alpha, a, b, \lambda)$	15396.2 1450.96	8187.68 10048.4	0.01243 0.01503	1.16769 1.26164	0.00100 0.51436
OLL-L(a, b, β, α)	93.4246 76.9751	3.88301 0.52176	106.648 298.273	0.00433 0.00127	–
KWL-L(β, α, a, b)	123813 253.314	1161.88 290.347	–	3.94793 0.36459	55728.8 26303.9
BX-L(λ, β, θ)	56448.9 472.220	6352.47 356.947	–	1.55061 0.31045	–
Mc-L$(\beta, \alpha, a, \eta, c)$	2629.35 3489.64	263.238 220.753	85.3710 65.6689	0.74746 0.20091	0.07847 0.09071
FTL-L$(b, a, \lambda, \beta, \alpha)$	12873.1 52.2671	11596.7 44.9849	1.11623 2.55741	10999.1 24386.95	0.26512 0.02996
PO-L(β, α, λ)	37.0154 19.0688	0.01380 0.00749	13.0214 3.69788	–	–
L(λ, β)	111284 581.645	17799.2 2797.88	–	–	–

Table 10
The Outcomes of Goodness-of-Fit Measures for the Second Data

Model	AIC	CAIC	BIC	HQIC	CVM	AD	$-\ell$	KS	p-value
BXMO-L	607.048	608.190	613.803	609.490	0.01902	0.15523	299.524	0.06434	0.99642
OEHL-L	607.574	608.717	614.329	610.016	0.02209	0.17513	299.787	0.06721	0.99361
E-L	609.487	610.153	614.553	611.319	0.03948	0.30542	301.743	0.08483	0.93572
MOBX-L	611.218	612.364	617.973	613.660	0.04988	0.36519	301.609	0.07682	0.97229
W-L	611.275	612.418	618.031	613.718	0.06143	0.43048	301.638	0.08547	0.93200
TW-L	620.257	622.022	628.702	623.310	0.15670	0.99971	305.129	0.13164	0.49216
OLL-L	616.932	618.075	623.688	619.375	0.13798	0.89056	304.466	0.12420	0.56795
KWL-L	617.577	618.720	624.332	620.019	0.14745	0.94606	304.789	0.12895	0.51912
BX-L	610.401	611.068	615.467	612.233	0.08434	0.56892	302.200	0.11983	0.61398
Mc-L	626.671	628.435	635.115	629.724	0.25462	1.56185	308.335	0.18859	0.11623
FTL-L	626.956	628.720	635.400	630.009	0.24947	1.52135	308.478	0.14995	0.32952
PO-L	651.260	651.927	656.327	653.092	0.55094	3.12565	322.630	0.31064	0.00089
L	646.899	647.224	650.277	648.121	0.26609	1.61853	321.450	0.30298	0.00129

The values in Tables 8 and 10 indicate that the BXMO-L distribution offers a superior fit compared to several competing Lomax distributions, including MOBX-L, OEHL-L, W-L, OLL-L, E-L, TW-L, KW-L, BX-L, Mc-L, FTL-L, PO-L, and L distributions. The BXMO-L distribution consistently achieves the lowest values for all goodness-of-fit criteria and exhibits the highest p-value among all the fitted models.

Table 11
The parameters estimates (first line) and SEs (second line) for the second data.

BXMO-L($\lambda,\beta,\alpha,\theta$)	2054.70 925.294	8.5754 4.3831	150.6153 376.6894	0.32698 0.16479	–
OEHL-L (b,a,λ,α)	1870788 1101.624	4654.31 1691.13	0.03952 0.06638	0.71376 0.29501	–
E-L(β,α,λ)	1785.53 1945.89	7.54739 5.28960	0.01489 0.01197	–	–
MOBX-L(b,a,α,θ)	675228.8 1251.342	776.770 72.9307	13.2423 15.8381	0.43372 0.39179	–
W-L(β,α,a,b)	1054.400 920.9241	4.39255 2.89681	1.17614 0.53818	0.01599 0.01716	–
TW-L(β,α,a,b,λ)	0.01326 0.00497	0.05434 0.00173	11.0726 17.5375	21.28142 3.956403	0.99999 0.29337
OLL-L(a,b,β,α)	31609.2 940.712	2.575759 0.338573	160.712 368.082	0.04087 0.09635	–
KWL-L(β,α,a,b)	27972.5 1427.21	1.16837 1.39750	–	2.64223 0.38055	2669.867 7412.170
BX-L(λ,β,θ)	7783362 904.568	4100.26 271.146	–	0.90758 0.16884	–
Mc-L(β,α,a,η,c)	56.6371 128.482	0.61612 0.23867	12.8430 15.5885	163.0923 139.7054	42.6220 44.8365
FTL-L(b,a,λ,β,α)	13000.6 44.3140	62.77444 25.35453	3.35908 7.00559	0.09306 1854.55	0.21690 0.09321
PO-L(β,α,λ)	0.00781 0.00366	13.1539 6.06651	1.30187 0.17735	–	–
L(λ,β)	5210537 1152.345	4583.71 724.717	–	–	–

Figures 6 and 7 display some estimated functions of the BXMO-L distribution for both data sets, respectively. The plots support the values in Tables 8 and 10, showing the superior fit of the proposed BXMO-L distributions.

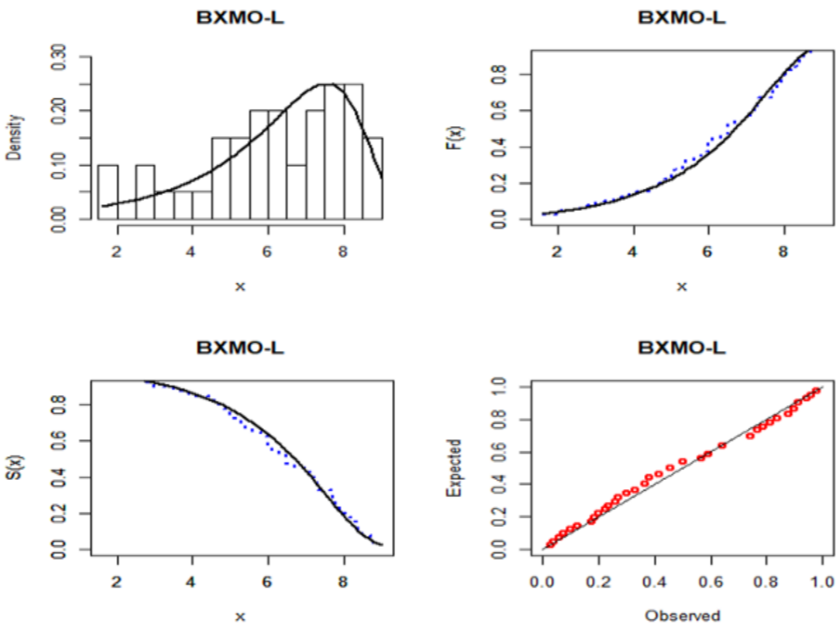

Figure 6: The Fitted Density, cdf, sf and PP Plots of the BXMO-L Distribution for First Data

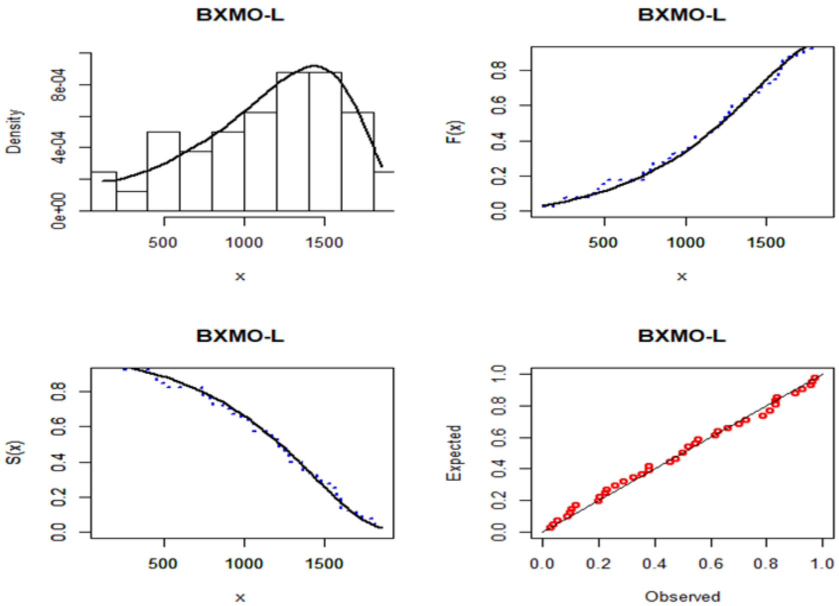

Figure 7: The Fitted Density, cdf, sf and PP Plots of the BXMO-L Distribution for Second Data

9. CONCLUSIONS

In this paper, we introduce the Burr-X Marshall–Olkin-F (BXMO-F) family, a flexible two-parameter class of continuous distributions. This family encompasses various special models with densities that can exhibit symmetrical and asymmetrical shapes, and accommodate a range of hazard rates, including bathtub, upside-down bathtub, J-shape, reversed J-shape, unimodal, decreasing, and increasing. We focus on a specific sub-model within this family, the BXMO–Lomax (BXMO-L) distribution, and derive several key risk measures for it. We evaluate the performance of eight frequentist estimation methods for estimating the parameters of the BXMO-L distribution. Detailed simulations reveal that all estimators perform well, but the weighted least-squares approach stands out with the highest overall performance score of 47.5, demonstrating superior accuracy. The flexibility of the BXMO-L distribution is further illustrated through its application to two real-life data sets. Our analysis shows that the BXMO-L distribution provides a better fit compared to several established Lomax extensions, including the exponential Lomax, Weibull–Lomax, Kumaraswamy–Lomax, McDonald–Lomax, Fréchet Topp–Leone Lomax, odd log-logistic Lomax, Burr-X Lomax, and Poisson–Lomax.

Data Availability:

This work is mainly a methodological development and has been applied on secondary data, but, if required, data will be provided.

Conflicts of Interest:

All authors declare no conflicts of interest.

REFERENCES

1. Marshall, A.W. and Olkin, I. (1997). A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families. *Biometrika*, 84(3), 641-652.
2. Eugene, N., Lee, C. and Famoye, F. (2002). Beta-normal distribution and its applications. *Communications in Statistics-Theory and Methods*, 31(4), 497-512.
3. Zografos, K., and Balakrishnan, N. (2009). On families of beta-and generalized gamma-generated distributions and associated inference. *Statistical Methodology*, 6(4), 344-362.
4. Cordeiro, G.M. and de Castro, M. (2011). A new family of generalized distributions. *Journal of Statistical Computation and Simulation*, 81(7), 883-898.
5. Alexander, C., Cordeiro, G.M., Ortega, E.M. and Sarabia, J.M. (2012). Generalized beta-generated distributions. *Computational Statistics and Data Analysis*, 56(6), 1880-1897.
6. Alzaatreh, A., Lee, C. and Famoye, F. (2013). A new method for generating families of continuous distributions. *Metron*, 71(1), 63-79.
7. Cordeiro, G.M., Ortega, E.M. and da Cunha, D.C. (2013). The exponentiated generalized class of distributions. *Journal of Data Science*, 11(1), 1-27.
8. Cordeiro, G.M., Alizadeh, M. and Ortega, E.M. (2014). The exponentiated half-logistic family of distributions: properties and applications. *Journal of Probability and Statistics*, 2014(1), 1-21.

9. Yousof, H.M., Afify, A.Z., Hamedani, G.G. and Aryal, G.R. (2017). The Burr X generator of distributions for lifetime data. *J. Stat. Theory Appl.*, 16(3), 288-305.
10. Mahdavi, A. and Kundu, D. (2017). A New Method for Generating Distributions with an Application to Exponential Distribution. *Communications in Statistics-Theory and Methods*, 46(13), 6543-6557.
11. Mead, M.E., Afify, A. and Butt, N.S. (2020). The Modified Kumaraswamy Weibull Distribution: Properties and Applications in Reliability and Engineering Sciences. *Pakistan Journal of Statistics and Operation Research*, 16(3), 433-446.
12. Afify, A.Z., Cordeiro, G.M., Ibrahim, N.A., Jamal, F., Elgarhy, M. and Nasir, M.A. (2021). The Marshall-Olkin Odd Burr III-G Family: Theory, Estimation and Engineering Applications. *IEEE Access*, 2021(9), 4376-4387.
13. Afify, A.Z., Al-Mofleh, H., Aljohani, H.M. and Cordeiro, G.M. (2022). The Marshall–Olkin–Weibull-H Family: Estimation, Simulations and Applications to COVID-19 Data. *Journal of King Saud University-Science*, 34(5), 102-115.
14. Alqawba, M., Altayab, Y., Zaidi, S.M. and Afify, A.Z. (2023). The Extended Kumaraswamy Generated Family: Properties, Inference and Applications in Applied Fields. *Electronic Journal of Applied Statistical Analysis*, 16(3), 740-763.
15. Mahran, H., Mansour, M.M., Abd Elrazik, E.M. and Afify, A.Z. (2024). A New One-Parameter Flexible Family with Variable Failure Rate Shapes: Properties, Inference and Real-Life Applications. *AIMS Mathematics*, 9(5), 11910-11940.
16. Al Abbasi, J.N., Afify, A.Z., Alnssyan, B. and Shama, M.S. (2024). The Lambert-G Family: Properties, Inference and Applications. *CMES-Computer Modeling in Engineering & Sciences*, 140(1), 514-536.
17. Jamal, F., Alqawba, M., Altayab, Y., Iqbal, T. and Afify, A.Z. (2024). A unified exponential-H family for modeling real-life data: Properties and inference. *Heliyon*, 10(6), 1-21.
18. Bowley, A.L. (1901). The Statistics of Wages in the United Kingdom during the Last Hundred Years. (Part VIII.) Wages in the Building Trades--Concluded. London. *Journal of the Royal Statistical Society*, 64(1), 102-112.
19. Moors, J.J.A. (1988). A quantile alternative for kurtosis. *Journal of the Royal Statistical Society: Series D (The Statistician)*, 37(1), 25-32.
20. Artzner, P., Delbaen, F., Eber, J.M. and Heath, D. (1999). Coherent measures of risk. *Mathematical Finance*, 9(3), 203-228.
21. Landsman, Z. (2010). On the tail mean–variance optimal portfolio selection. *Insurance: Mathematics and Economics*, 46(3), 547-553.
22. Bakar, S.A., Hamzah, N.A., Maghsoudi, M. and Nadarajah, S. (2015). Modeling loss data using composite models. *Insurance: Mathematics and Economics*, 61, 146-154.
23. Gupta, R.D. and Kundu, D. (2001). Generalized exponential distribution: different method of estimations. *Journal of Statistical Computation and Simulation*, 69(4), 315-337.
24. Xu, K., Xie, M., Tang, L.C. and Ho, S.L. (2003). Application of neural networks in forecasting engine systems reliability. *Applied Soft Computing*, 2(4), 255-268.
25. Abouammoh, A.M., Abdulghani, S.A. and Qamber, I.S. (1994). On partial orderings and testing of new better than renewal used classes. *Reliability Engineering and System Safety*, 43(1), 37-41.

26. Afify, A.Z., Altun, E., Alizadeh, M., Ozel, G. and Hamedani, G.G. (2017). The odd exponentiated half-logistic-G family: properties, characterizations and applications. *Chilean Journal of Statistics*, 8(2), 65-91.
27. El-Bassiouny, A.H., Abdo, N.F. and Shahen, H.S. (2015). Exponential Lomax distribution. *International Journal of Computer Applications*, 121(13), 24-29.
28. Jamal, F., Tahir, M.H., Alizadeh, M. and Nasir, M.A. (2017). On Marshall-Olkin Burr X family of distribution. *Tbilisi Mathematical Journal*, 10(4), 175-199.
29. Tahir, M.H., Cordeiro, G.M., Mansoor, M. and Zubair, M. (2015). The Weibull-Lomax distribution: properties and applications. *Hacettepe Journal of Mathematics and Statistics*, 44(2), 455-474.
30. Lemonte, A.J. and Cordeiro, G.M. (2013). An extended Lomax distribution. *Statistics*, 47(4), 800-816.
31. Afify, A.Z., Nofal, Z.M., Yousof, H.M., El Gebaly, Y.M. and Butt, N.S. (2015). The transmuted Weibull Lomax distribution: properties and application. *Pakistan Journal of Statistics and Operation Research*, 11(1), 135-152.
32. Gleaton, J.U. and Lynch, J.D. (2006). Properties of generalized log-logistic families of lifetime distributions. *Journal of Probability and Statistical Science*, 4(1), 51-64.
33. Yousof, H.M., Afify, A.Z., Hamedani, G.G. and Aryal, G.R. (2017). The Burr X generator of distributions for lifetime data. *J. Stat. Theory Appl.*, 16(3), 288-305.
34. Reyad, H., Korkmaz, M.Ç., Afify, A.Z., Hamedani, G.G. and Othman, S. (2021). The Fréchet Topp Leone-G family of distributions: properties, characterizations and applications. *Annals of Data Science*, 8(2), 345-366.
35. Al-Zahrani, B. and Sagor, H. (2014). The Poisson-Lomax distribution. *Revista Colombiana de Estadística*, 37(1), 225-245.