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NUMERICAL ESTIMATION FOR THE THREE-PARAMETER BURR-XII MODEL PARAMETERS BASED ON PICARD'S METHOD

M. Maswadah
Department of Mathematics, Faculty of Science
Aswan University, Aswan, Egypt
Email: maswadah@hotmail.com

ABSTRACT

In parameter estimation techniques, there are several methods for estimation in lifetime distributions and reliability theory. However, most of them are less efficient than the Bayesian method based on the informative prior. Thus, the main objective of this study is to present an optimal estimation technique using Picard's method for estimating the three-parameter Burr type-XII distribution parameters and compare them with the Bayesian estimates based on the informative gamma and kernel priors. A comparison between these estimators is provided using an extensive Monte Carlo simulation based on two criteria, namely, the average bias and the mean squared error. The simulation results indicated that Picard's method is highly favorable, which provides better estimates and outperforms the Bayesian estimates using different loss functions based on the generalized progressive hybrid censoring scheme. Finally, two real dataset analyses are presented to illustrate the efficiency of the proposed methods.

KEYWORDS

Bayes estimation; Informative prior; Kernel prior; LINEX loss function; Picard's method.

1. INTRODUCTION

In statistical inference, several methods have been used for estimating the distribution parameters. However, the most usable method in reliability analysis is the Bayes method, despite its subjectivity to prior information. Such subjectivity can mislead to subsequent inferences. Thus, the main objective of this study is to present an optimal estimation technique using Picard's method, for the first time in the literature, for estimating the distribution parameters. This method is more efficient compared to the Bayes method based on the informative and kernel priors using different loss functions. To illustrate that, we employed the proposed methods on one of the distributions employed in lifetime distributions and reliability theory, which is the three-parameter Burr type-XII distribution. Burr (1968) and Burr & Cislak (1968) showed that if one chooses the parameters appropriately, the Burr-XII distribution covers a large proportion of curve-shaped characteristics of types I, IV and VI in the Pearson family of distributions. This distribution is one of the distribution families of survival and lifetime data that have utility in survivorship applications.

Many authors have studied inferences on the Burr type-XII distribution. Soliman (2005) and Soliman et al. (2011) derived the maximum likelihood estimators (MLE) and Bayes estimators for some lifetime parameters, as well as the parameters of the Burr type-XII model based on progressive type-II censored samples. Wingo (1983) and Wingo (1993b) have described methods for fitting the Burr type-XII distribution on life test data based on type-II censoring using the MLE method. Wang, Keats & Zimmer (1996) and Wingo (1993a) derived the MLE based on censored and uncensored samples. Moore & Papadopoulos (2000) studied Bayesian inference based on various loss functions. Lee et al. (2009) has studied the Burr type-XII distribution as a failure model based on progressive type-II right-censored samples. Wu & Yu (2005) proposed pivotal quantities for testing the shape parameter and established the confidence interval for the shape parameter for the two-parameter Burr type-XII distribution under censored samples. Wu, Chen & Chang (2007) studied the statistical inference based on progressively censored samples with random removals from the Burr type XII distribution. Xiuchun et al. (2007) studied the empirical estimates of the reliability of the Burr type-XII distribution using the LINEX error loss function based on progressive type-II censored samples.

The Burr type-XII distribution is the one that has a great deal of application based on progressive order statistics because of its flexibility to describe life test data. Consider the three-parameter Burr-XII distribution with the (CDF) $F(x)$ and the (PDF) $f(x)$, which are defined respectively as:

$$F(x) = 1 - [1 + \beta x^{\alpha}]^{-\gamma}, \qquad x, \geq 0, \alpha, \beta, \gamma > 0 \tag{1}$$

$$f(x) = \alpha\beta\gamma x^{\alpha-1}[1 + \beta x^{\alpha}]^{-\gamma-1}, \qquad x \geq 0, \alpha, \beta, \gamma > 0 \tag{2}$$

where α and γ are form factors and beta is the scale factor.

In reliability analysis, the progressive Type-II censoring scheme is most applicable in life test experiments, it is useful for both industrial life test applications and clinical trials and allows removing some of the surviving experimental units at different stages before testing is terminated. Balakrishnan & Aggarwala (2000) and Balakrishnan & Cramer (2014) presented comprehensive studies on the topic of progressive censoring and its applications. However, the trial time can be quite long due to some highly reliable units. Thus, Kundu & Joarder (2006) recently proposed a censoring scheme called the Type-II progressive hybrid censoring scheme. However, the disadvantage of the progressive hybrid censoring scheme is that very few failures may occur before time point T. In order to provide a guarantee of the number of failures observed as well as the time to complete the test, Cho, Sun & Lee (2015a) and Cho, Sun & Lee (2015b) proposed the generalized progressive hybrid censoring scheme (GPHCS), which modifies the progressive hybrid censoring scheme. It allows the experiment to continue beyond time T to observe at least k failures if the number of failures is less than m. The GPHCS can be described as follows:

Consider that N identical items are placed on a test with considering $R_1, R_2, \ldots, R_m$ are the random removal units, which are fixed at the beginning of the experiment with $m < N$ such that $N = m + \sum_{i=1}^{m} R_i$. The terminated time T is also fixed beforehand with the integers k and m are pre-fixed such that $k < m$. In general, at the time of the ith failure, R_i units will be removed randomly from the remaining surviving units $S_i = n - i - \sum_{j=1}^{i-1} R_j$, where $i \in [1, m]$. Thus, we have three scenarios:

i. If the time of the m^{th} failure occurs before the time point T, then the experiment will stop at the time point $X_{m:m:N}$ and all the remaining surviving units $R_m = N - m - \sum_{j=1}^{m-1} R_j$ will be removed.
ii. If the m^{th} failure does not occur before the time point T and only k failures occur after the time point T, where $X_{m:m:N} > T$. Then at the time point $X_{k:m:N}$ the experiment terminates, and all the remaining surviving units $R_k = N - k - \sum_{j=1}^{k-1} R_j$ will be removed.
iii. If the m^{th} failure does not occur before the time point T and only l failures occur at the time point T, where $X_{k:m:N} < T < X_{m:m:N}$. Then at the time point $X_{l:m:N}$ the experiment terminates, and all the remaining surviving units $R_T^* = N - l - \sum_{j=1}^{l} R_j$ will be removed.

Thus, given a generalized progressive hybrid censored sample, the likelihood function for the three different cases can be written in a unified form as follows:

$$L(\bar{X};\theta) = C\prod_{i=1}^{N} f(x_i)\,[1 - F(x_i)]^{R_i}[1 - F(T)]^{R_T^*\delta}, \tag{3}$$

$$N = \begin{cases} m, & \delta = 0 \quad , \quad if \quad X_{k:m:N} \le \quad X_{m:m:N} < \quad T \\ k, & \delta = 0 \quad , \quad if \quad T < \quad X_{K:m:N} \le \quad X_{m:m:N}, \\ l, & \delta = 1 \quad , \quad if \quad X_{k:m:N} < \quad T < \quad X_{m:m:N} \end{cases}$$

where $\bar{X} = (X_1, X_2, \dots, X_N)$ and $R_T{}^*$ is the number of surviving units that are removed at the stopping time $T^* = max\{X_{k:m:N}, min\{X_{m:m:N}, T\}\}$.

The GPHCS has been applied to some distributions, such as the Weibull distribution, see Cho, Sun & Lee (2015b), the inverse Weibull distribution, see Maswadah (2021) and Mohie El-Din & Nagy (2017), the exponential distribution, see Cho, Sun & Lee (2015a), the Rayleigh distribution, see Cho, Sun & Lee (2014), and the shape-scale family, see Maswadah (2022).

2. ESTIMATION METHODS

2.1 Picard's Method

The MLE $\hat{\theta} = \hat{\theta}(\bar{x})$ of θ is the solution of the stationary equation, $\frac{\partial H(\underline{X};\theta)}{\partial\theta}|_{\theta=\hat{\theta}} = 0$, which is a function of $\underline{X}$ and $\hat{\theta}(\bar{x})$, where $H(\underline{X};\bar{\theta})$ is the log-likelihood function that depends on the unknown parameter $\hat{\theta} = (\alpha, \beta)$ and the data $\underline{X} = (x_1, x_2, \dots, x_N)$. Applying the implicit function theorem to the stationary equation, with considering all partial derivatives as well as the total derivatives are assumed to be evaluated at some known value of $\hat{\theta}(x) = \theta_0$, say. Taking the total derivative for the stationary equation with respect to $\in \underline{X}$, see Ramsay (2007), we obtain.

$$\frac{d}{dx}\left(\frac{\partial H(\underline{X};\theta)}{\partial\theta}\right)|_{\theta=\hat{\theta}} = \frac{\partial^2 H(\underline{X};\theta)}{\partial\theta\partial x}|_{\theta=\hat{\theta}} + \frac{\partial^2 H(\underline{X};\theta)}{\partial\theta^2}|_{\theta=\hat{\theta}}\frac{d\hat{\theta}}{dx} = 0. \tag{4}$$

Solving (4) we obtain the first derivative with respect to x for $\hat{\theta}$ at $\theta = \hat{\theta}$ as follows:

$$\frac{d\hat{\theta}}{dx} = f(x, \hat{\theta}), \tag{5}$$

where

$$f(x, \hat{\theta}) = -\left(\frac{\partial^2 H(\underline{X}; \theta)}{\partial \theta^2}\Big|_{\theta=\hat{\theta}}\right)^{-1} \frac{\partial^2 H(\underline{X}; \theta)}{\partial \theta \partial x}\Big|_{\theta=\hat{\theta}}.$$

Here $f(x, \hat{\theta})$ and $\frac{df(x,\hat{\theta})}{d\hat{\theta}}$ are defined and continuous functions at all points $(\underline{X}, \hat{\theta})$, which ensures the existence of a unique solution for (5). Using Picard's method, we can find an approximate solution given a trial set of parameter values and initial conditions. If the initial conditions are unavailable, they must be appended to the MLE $\hat{\theta}$ as quantities with respect to which the fit is optimized.

Thus, we can write (5) as a first-order ordinary differential equation in the maximum likelihood estimator $\hat{\alpha}(x)$ of α as

$$\frac{d\hat{\alpha}}{dx} = f(x, \hat{\alpha}, \hat{\beta}, \hat{\gamma}), \hat{\alpha}(x_0) = \alpha_0, \tag{6}$$

Similarly, the first order ordinary differential equation in the maximum likelihood estimator $\hat{\beta}(x)$ of β and $\hat{\gamma}(x)$ of γ as

$$\frac{d\hat{\beta}}{dx} = g(x, \hat{\alpha}, \hat{\beta}, \hat{\gamma}), \hat{\beta}(x_0) = \beta_0, \tag{7}$$

$$\frac{d\hat{\gamma}}{dx} = h(x, \hat{\alpha}, \hat{\beta}, \hat{\gamma}), \hat{\gamma}(x_0) = \gamma_0. \tag{8}$$

Using Picard's method, we can find the approximate solution for the parameters given a trial set of values for the first- order differential equations (6), (7), and (8). For the first-order differential equations (6), we can write the general iteration rule for Picard's method by integrating (6) with respect to x from x_0 to x as follows:

$$\hat{\alpha}_{n+1}(x) = \hat{\alpha}(x_0) + \int_{x_0}^{x} f(x, \hat{\alpha}_n, \hat{\beta}, \hat{\gamma}) dx = \alpha_0 + \int_{\hat{\alpha}(x_0)}^{\hat{\alpha}(x^*)} f(x, \hat{\alpha}_n, \hat{\beta}, \hat{\gamma}) \frac{dx}{d\hat{\beta}} d\hat{\beta}$$

From (7) we get $\frac{dx}{d\hat{\beta}} = 1/g(x, \hat{\alpha}, \hat{\beta}, \hat{\gamma})$, thus

$$\hat{\alpha}_{n+1}(x) = \alpha_0 + \int_{\alpha_0}^{\hat{\alpha}(x^*)} \left[\frac{f(x, \hat{\alpha}_n, \hat{\beta}, \hat{\gamma})}{g(x, \hat{\alpha}_n, \hat{\beta}, \hat{\gamma})}\right] d\hat{\beta}\,, \tag{9}$$

where α_0 is the initial point and $\hat{\alpha}(x^*)$ is the value for which the desired solution should be optimized. The iterative process based on (9) is continued until two consecutive numerical solutions are almost the same, that is, if $|\hat{\alpha}_{n+1} - \hat{\alpha}_n| < 1E-05$, for $n = 0,1,2,3,..$ Similarly, we can find the Picard estimates for the parameters β and γ.

For the Burr type-XII model, the likelihood function of the GPHCS (3) and its derivatives have been derived in Appendix A.

2.2 Bayes Estimations

In this section, the Bayes estimators for the parameters will be derived using gamma and kernel prior distributions based on three different loss functions:

Firstly, the squared error loss function (SLF), $L\big(g(\theta),\hat{g}(\theta)\big) = (g(\theta) - \hat{g}(\theta))^2$. For this loss function the Bayes estimator that minimizes the risk function is given by $\hat{g}(\theta) = E_\theta(g(\theta)|x)$.

Secondly, the compound LINEX loss function is defined as follows:

$$L(\Delta) = L_\delta(\Delta) + L_{-\delta}(\Delta) = e^{\delta\Delta} + e^{-\delta\Delta} - 2, \delta > 0.$$

It is named LINEX-based loss function, see Xiuchun (2007), where

$$L_\delta(\Delta) = exp[\,\delta\Delta] - \delta\Delta - 1, \Delta = \hat{g}(\theta) - g(\theta), \delta \neq 0,$$

is the LINEX loss function (LLF) that has been introduced in Wei, Wei &Su (2010) and Xiuchun et al. (2007). The Bayes estimator of the parameter θ that minimizes the risk function can be derived as follows:

$$\hat{g}_L(\theta) = \frac{1}{2\delta}\log\left(\frac{E\left(e^{\delta g(\theta)}\middle|X\right)}{E\left(e^{-\delta g(\theta)}\middle|X\right)}\right).$$

i. Informative Prior

We consider the unknown parameters α,β and γ have independent gamma prior distributions with the joint probability density function, which is given by:

$$h(\alpha,\beta,\gamma) \propto \alpha^{a-1}\beta^{c-1}\gamma^{e-1}e^{-b\alpha-d\beta-f\gamma}, \tag{10}$$

where the hyper-parameter a, b, c, d, e and f are assumed to be known and non-negative and chosen to reflect the prior belief about the unknown parameters.

ii. Kernel Prior

For deriving the kernel prior, we introduce the trivariate kernel density estimator for the unknown probability density function $g(\alpha,\beta,\gamma)$ with support on $(0,\infty)$, which is defined as

$$\hat{g}(\alpha,\beta,\gamma) = \frac{1}{Nh_1h_2h_3}\sum_{i=1}^{N} K\left(\frac{\alpha-\alpha_i}{h_1},\frac{\beta-\beta_i}{h_2},\frac{\gamma-\gamma_i}{h_3}\right), \tag{11}$$

$h_i, i = 1,2,3$ are called the bandwidths or smoothing parameters, which chosen such that $h_i \to 0$ and $Nh_i \to \infty$ as $N \to \infty$, where N is the sample size. The influence of the smoothing parameter h is critical because it determines the amount of smoothing. However, the optimal choice for h_i that minimizes mean squared errors is given by $h_i = 1.06\,S_iN^{-0.2}$, where S_i is the sample standard deviation. The optimal choice for the kernel function $K(.,.)$ can be used as the trivariate standard normal distribution for the parameters α, β and γ. Based on the properties of the MLE of the parameters, which converge in probability to the original parameters, the kernel prior estimate can be derived using the following algorithm:

a) Generate a random sample $\bar{X} = (X_1, X_2, X_3, \dots, X_N)$ from the parent distribution $f(x\,; \alpha\,, \beta, \gamma)$ with given specified values for the unknown parameters α, β and γ.
b) Bootstrapping with replacement N samples $x_1^*, x_2^*, x_3^*, \dots, x_N^*$, with size N each, where $x_i^* = (x_{i1}^*, x_{i2}^*, x_{i3}^*, \dots, x_{iN}^*)$ for $i = 1,2, \dots, N$ from the given random sample in step 1.
c) For each sample in step 2, calculate the MLEs for the parameters α, β and γ, thus we have the random variables $\underline{A} = (\hat{\alpha}_1, \hat{\alpha}_2, \dots, \hat{\alpha}_N)$, $\underline{B} = (\hat{\beta}_1, \hat{\beta}_2, \dots, \hat{\beta}_N)$ and $\underline{C} = (\hat{\gamma}_1, \hat{\gamma}_2, \dots, \hat{\gamma}_N)$ for the MLEs for α, β and γ respectively.
d) Finally, based on the random variables $\underline{A}$, $\underline{B}$ and $\underline{C}$, the kernel density estimator (11) can be used to derive the kernel prior density estimator $\hat{g}(\alpha, \beta, \gamma)$.

The kernel prior has been used in Ahsanullah, Maswadah & Seham (2013), Maswadah (2006), Maswadah (2007) and Maswadah (2010). Thus, using the joint priors of (10) and (11) with the likelihood function of the GPHCS (3) the posterior density for the parameters α, β and γ can be written in a unified form as follows:

$$f(\alpha, \beta, \gamma | \underline{x}) = Kq(\alpha, \beta, \gamma) L(\bar{X}\,; \alpha, \beta, \gamma),$$

where

$$\begin{aligned} q(\alpha, \beta, \gamma) &= h(\alpha, \beta, \gamma) \hat{g}(\alpha, \beta, \gamma) \\ &= \hat{g}_1^{p_1}(\alpha)\, \hat{g}_2^{p_2}(\beta)\, \hat{g}_3^{p_3}(\gamma)\, \alpha^{a-1} \beta^{c-1} \gamma^{e-1} e^{-b\alpha - d\beta - f\gamma}, \end{aligned}$$

is the general prior distribution function with $p_1 = p_2 = p_3 = 0$ for the informative prior (10), and $p_1 = p_2 = p_3 = 1$, $a = c = e = 1$, and $b = d = f = 0$ for the kernel prior (11). Thus, the log posterior density can be written as

$$\begin{aligned} H(\alpha, \beta, \gamma) = {} & p_1\, ln(\hat{g}_1(\alpha)) + p_2\, ln(\hat{g}_2(\beta)) + p_3\, ln(\hat{g}_3(\gamma)) \\ & + (N + a - 1)\, ln(\alpha) + (N + c - 1)\, ln(\beta) \\ & + (N + e - 1)\, ln(\gamma) - b\alpha \\ & - d\beta - f\gamma - \sum_{i=1}^{N} ln(1 + \beta x_i^{\alpha}) + (\alpha - 1) \sum_{i=1}^{N} ln(x_i) \\ & - \gamma \left[\sum_{i=1}^{N} (1 + R_i)\, ln(1 + \beta x_i^{\alpha}) + \delta R_T^* ln(1 + \beta T^{\alpha}) \right] \end{aligned} \tag{12}$$

Thus, based on (12), we can use the Tierney-Kadane approximation method to approximate all the Bayes estimators for the unknown parameters. Tierney & Kadane (1986) introduced an easily computable approximation for the posterior mean and variance of a non-negative parameter, or more generally, of a smooth function of the parameter that is non-zero on the interior of the parameter space. For detail, let $w(\alpha, \beta, \gamma)$ be a smooth, positive function on the parameter space. The posterior expectation of $w(\alpha, \beta, \gamma)$ can be obtained as follows:

$$\begin{aligned} w^* &= E(w(\alpha, \beta, \gamma) | \bar{X}) \\ &= \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} e^{NH^*(\alpha,\beta,\gamma)} d\alpha d\beta d\gamma \Big/ \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} e^{NH(\alpha,\beta,\gamma)} d\alpha d\beta d\gamma, \end{aligned} \tag{13}$$

where $= ln\, f\,(\alpha, \beta, \gamma | x)/N$, and $H^* = H + ln w(\alpha, \beta, \gamma)/N$.

For (α, β, γ) the Bayes estimator using Tierney-Kadane approximation for $w(\alpha, \beta, \gamma)$ can be obtained as

$$w^* = \sqrt{|\sum^{*}|/|\sum|}\, exp\left[N[H^*(\alpha, \beta, \gamma) - H(\alpha, \beta, \gamma)]\right]$$

where $(\hat{\alpha}, \hat{\beta}, \hat{\gamma})$ and $(\hat{\alpha}^*, \hat{\beta}^*, \hat{\gamma}^*)$ maximize the $H(\hat{\alpha}, \hat{\beta}, \hat{\gamma})$ and $H^*(\hat{\alpha}^*, \hat{\beta}^*, \hat{\gamma}^*)$, respectively. Let

$$|\sum| = \begin{vmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{vmatrix}^{-1} \text{ and } |\sum *| = \begin{vmatrix} H_{11}^* & H_{12}^* & H_{13}^* \\ H_{21}^* & H_{22}^* & H_{23}^* \\ H_{31}^* & H_{32}^* & H_{33}^* \end{vmatrix}^{-1}$$

denote the minus of inverse of Hessians of $H(\alpha, \beta, \gamma)$ and $H^*(\alpha, \beta, \gamma)$ at $(\hat{\alpha}, \hat{\beta}, \hat{\gamma})$ and $\left(\hat{\alpha}^*, \hat{\beta}^*, \hat{\gamma}^*\right)$ respectively. The derivatives of $H(\alpha, \beta, \gamma)$ and $H^*(\alpha, \beta, \gamma)$ have been derived in the Appendix B.

3. SIMULATION STUDY

The purpose of the simulation study is to compare the performance of the estimates using Picard and Bayes methods based on the informative gamma and the informative kernel priors for two different loss functions through two criteria: the average bias (AVB) and the mean squared error (MSE), as given by:

$$AVB = \frac{1}{L}\sum_{i=1}^{L} |\hat{\theta}_i - \theta|, \text{ and } MSE = \frac{1}{L}\sum_{i=1}^{L} \left(\hat{\theta}_i - \theta\right)^2$$

$\hat{\theta}$ is the estimate of θ and L is the number of replications.

In our simulation study we choose the hyperparameters of α, β and γ as follows: $a = c = e = 5, b = d = f = 3$ and the values for the parameters are $\alpha = (1,2)$, two values for the parameter $\beta = (0.75, 1.75)$ and two values for the parameter $\gamma = (1.5, 2.5)$ respectively. Using the above parameter values for generating different samples from the three-parameter Burr type-XII distribution with sizes $N = 20, 40$, and 60 to represent small, moderate and large sizes. To assess the performance of these estimates, the average bias (AVB) and the MSEs for each were calculated using 1000 replicates. An algorithm for generating the generalized progressive hybrid censoring scheme has been written, see Maswadah (2021) and Maswadah (2022).

From the simulation results in Tables 3, 4, 5, 6, 7, and 8, some of the points are quite clear based on these estimates, and the others have been summarized in the following main points:

i. It is clear that, generally, for both parameters β and γ the average bias values based on Picard's method outperform the corresponding values based on the Bayes method for different loss functions. However, for the parameter α both methods have average bias almost the same especially based on the LINEX loss function.
ii. In terms of MSE values, we can easily see that Picard's method has the smallest MSE values compared with its counterparts, which are based on the Bayes method.

iii. It is evident that the estimated AVB and MSE values decrease with increasing the hyperparameters, the termination time of the experiment T, and the sample sizes, as expected for all methods.
iv. For the parameter α, the estimated MSE values increase with increasing the value of α, while decreasing as the value of β and γ increase.
v. For the parameters β and γ the estimated MSE values decrease with increasing the values of β and γ, while decreasing as the value of α increases.
vi. In general, the estimated MSE values for the Bayes method based on the LINEX-based loss function are less than those based on the squared error function.

In conclusion, Picard's method competes and outperforms Bayes's method based on the informative and kernel priors.

4. REAL DATA ANALYSIS

In this section, we studied two real data sets to demonstrate the performance of the proposed methods on the Burr type XII model in practice and to illustrate that this distribution can be considered a good lifetime model for some new areas of applications, compared with many known distributions such as the Weibull distribution. We have fitted these data sets using some goodness of fit tests such as the Kolmogorov-Smirnov (K-S), Anderson-Darling (A-D) and Chi-Square (CH2) tests for significance level tests equal to 0.05.

4.1 The Reactor Pumps Data

In this section, a real data set for secondary nuclear pumps has been analyzed to illustrate the proposed methods. One of the most severe accidents in nuclear power generation is the loss of coolant, where the re-circulating coolant of the pressurized water reactor may flash into steam. Under such conditions, the reactor cooling pumps become unable to generate the same head as that of the single-phase flow case. Thus, the secondary reactor pump is a feedwater pump that takes from the desecrator storage tank feed water pressured up by the booster pump and pushes it into the steam generator through the high-pressure heater. Accordingly, the main feed pump must be a high- temperature and high-pressure pump since it requires a head larger than the pressure inside the steam generator. The secondary circulation pump differs slightly in design and has been developed specifically for cooling at higher temperatures. The following data set represents the times between the failures of the secondary reactor pumps. Varian (1975) and Wang, Keats & Zimmer (1996) have discussed the classical and Bayesian estimation methods under the Type-II censoring scheme of this data set. The times between failures of 23 secondary reactor pumps are as follows:

2.160, 0.746, 0.402, 0.954, 0.491, 6.560, 4.992, 0.347, 0.150, 0.358, 0.101, 1.359, 3.465, 1.060, 0.614, 1.921, 4.082, 0.199, 0.605, 0.273, 0.070, 0.062, 5.320.

We found the Burr type-XII model to be a good fit for this dataset, as shown in Table 1 and Figure (1a). For studying the reliability of these reactor pumps based on this dataset, we found the estimates for the parameters that represent the shape of the failures between pumps using our model to determine the behavior of the failure pumps. We noticed that Picard and Bayes estimates for α lie in the interval [0.9, 1.1], for γ lies in the interval [1.0, 1.3], and for β lies in the interval [0.7, 0.9], which indicate that the above dataset is heavily right-skewed, and that means the failure rate decreases with increasing time, see Figure (1b), which indicates decreasing the reliability of safety mechanism with increasing time.

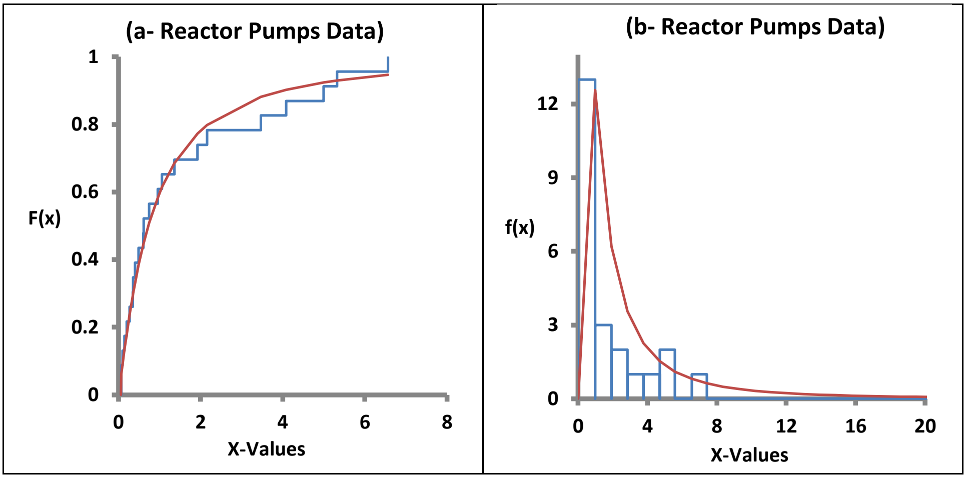

Figure 1: a) The Empirical CDF and the Fitted CDF for the Reactor Pumps Data
b) The Histogram and the Fitted PDF for the Reactor Pumps Data

4.2 The Vinyl Chloride Data

As vinyl chloride is a known human carcinogen, exposure to this compound should be avoided as far as practicable, and levels should be kept as low as technically feasible. It is known that a concentration of vinyl chloride in drinking water of 0.5 mg/liter was calculated as being associated with an excess risk of liver and brain tumors for exposure beginning in adulthood, and it would double the cancer risk for continuous exposure from birth. Therefore, we consider the dataset used by Bhaumik, Kapur & Gibbons (2009), which represents 34 data points in mg/L from the vinyl chloride obtained from clean upgrade monitoring wells, as follows:

5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8.0, 0.8, 0.4, 0.6, 0.9, 0.4, 2.0, 0.5, 5.3,
3.2, 2.7, 2.9, 2.5, 2.3, 1.0, 0.2, 0.1, 0.1, 1.8, 0.9, 2.0, 4.0, 6.8, 1.2, 0.4, 0.2.

We found the Burr type-XII model to be a very good fit for this dataset, as shown in Table 1 and Figure (2a). For studying the concentration of vinyl chloride in the water of these wells, based on this dataset, we find the parameter estimates that represent the shape of the concentration using our model to determine the average concentration in the water. We noticed that Picard and Bayesian estimates for α lie in the interval [0.5, 0.7] and for γ lie in the interval [1.9, 2.2], which indicate that the above dataset is moderately right skewed and that means the concentration decreases with increasing time, see Figures (2b). Also, Picard and Bayesian estimates for β i.e. in the interval [0.28, 0.4], which ensure the dataset is right-skewed and the vinyl chloride concentration will decrease with increasing time, therefore, monitoring these wells is very significant.

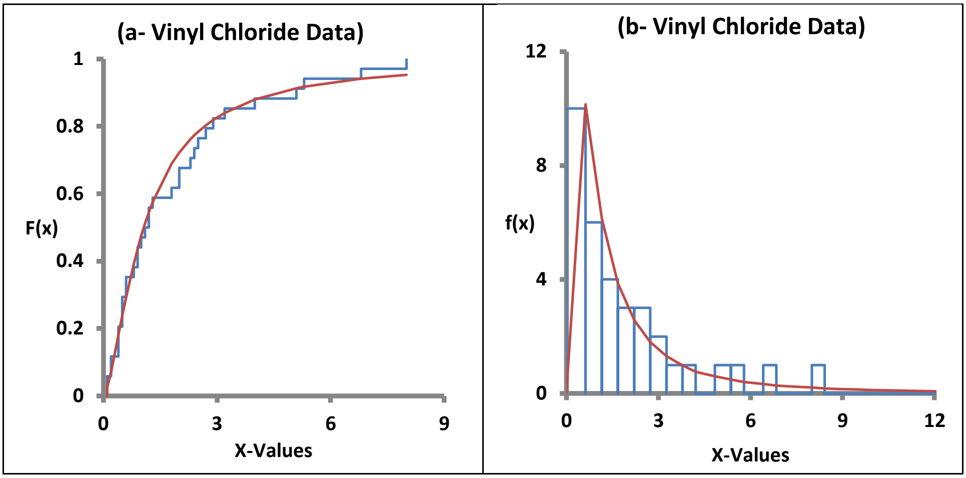

Figure 2: a) The Empirical CDF and the Fitted CDF for the Vinyl Chloride Data
b) The Histogram and the Fitted PDF for the Vinyl Chloride Data

Table 1
The Critical and Calculated Values for the K-S, A-D and CH2 Tests and their Powers (p-values) for the Burr Type-XII Model. The MLE for the Parameters for These Data Sets have been calculated.

Data	**The Tests**	**Calculated Value**	**Critical value**	**p-values**	**MLES**		
					α	**β**	**γ**
Reactor pumps Data N=23	K-S	0.4719	0.9001	0.8163	1.1072	0.9446	1.3754
	A-D	0.2466	0.7714	0.7563			
	CH2	9.5741	12.234	0.1355			
The vinyl Chloride data N=34	K-S	0.6064	0.9074	0.4872	0.6890	0.4159	2.2157
	A-D	0.3010	0.7711	0.6134			
	CH2	5.6472	15.086	0.3854			

Table 2
The Estimate and the Mean Squared Errors (MSEs) for the Parameter α, β and γ based on the Picard and Bayes Methods under Square Error Loss Function based on the GPHCS: for $m = n/2, k = m/2$.

Samples	T	Parameters	Picard Estamite		Gamma Prior		Kernel prior	
			Estimate	MSE	Estimate	MSE	Estimate	MSE
The reactor Pumps data n=23	0.25	α	1.0339	0.0054	0.9140	0.0373	0.9169	0.0362
		β	0.8825	0.00386	0.6951	0.0623	0.6918	0.0639
		γ	1.3249	0.00255	1.0345	0.1162	1.0648	0.0965
	4.0	α	1.0239	0.0069	0.9302	0.0313	0.9312	0.0309
		β	0.8877	0.0032	0.7479	0.0387	0.7486	0.0384
		γ	1.3229	0.0028	1.1063	0.0724	1.1175	0.0665
The vinyl Chloride data n=34	0.5	α	0.6532	0.0013	0.5625	0.0161	0.5621	0.0161
		β	0.2888	0.0162	0.3571	0.0346	0.3599	0.0314
		γ	2.1503	0.0043	1.8982	0.1008	1.9016	0.0987
	4.5	α	0.6553	0.0011	0.5679	0.0147	0.5681	0.0146
		B	0.4064	9.04E-4	0.3490	4.5E-03	0.3438	5.2E-03
		γ	2.0609	0.0239	1.8794	0.1130	1.8803	0.1124

From the results in Table 1, the three-parameter Burr type-XII model is a very good fit for these data sets since the calculated values for the goodness of fit tests are less than the critical values. Moreover, the power of the tests is greater than the significance level of 0.05. From the results in Table 2, the estimated values of MSEs based on Picard's method are smaller than those for the Bayes method for these data sets, considering the MLEs are the true values of the parameters. Thus, the results of these data sets ensure the simulation results.

CONCLUSIONS

This study presented Picard's method for the first time in the literature as an optimal estimation technique in the lifetime distributions and reliability theory, based on the generalized progressive hybrid censored data. This method is strongly unbiased and much more efficient than the Bayes method based on the informative and kernel priors for different loss functions. The Bayesian estimates based on the nonparametric kernel prior are more efficient than those based on the informative gamma prior and are close to those based on Picard estimates. Thus, the statistical significance of Picard's method is its efficiency compared to most estimation methods, and it is reliable and easy to apply, especially for social sciences and psychology researchers.

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DISCLOSURE STATEMENT

There is no potential conflict of interest.

REFERENCES

1. Ahsanullah, M., Maswadah, M. and Seham, A.M. (2013). Kernel Inference on the Generalized Gamma Distribution based on Generalized Order Statistics. *Journal of Statistical Theory and Applications*, 12(2), 152-172.
2. Balakrishnan, N. and Aggarwala, R. (2000). Progressive Censoring: Theory, Methods and Applications. Birkhãuser Publishers: Boston.
3. Balakrishnan, N. and Cramer, E. (2014). *The art of Progressive Censoring: Applications to Reliability and Quality, Statistics for Industry and Technology*, Springer, New York.
4. Bhaumik, D.K., Kapur, K. and Gibbons, R.D. (2009). Testing Parameters of a Gamma Distribution for Small Samples. *Technimetrics*, 51, 326-334.
5. Burr, I.W. (1968). On a general system of distributions III. The sample range. *Journal of the American Statistical Association*, 63, 636-643.
6. Burr, I.W. and Cislak, P.J. (1968). On a general system of distributions I. Its curve-shape characteristics II. The sample median. *Journal of the American Statistical Association*, 63(322), 627-635.
7. Cho, Y., Sun, H., and Lee, K. (2014). An estimation of the entropy for a Rayleigh distribution based on doubly generalized Type-II hybrid censored samples. *Entropy*, 16, 3655-3669.
8. Cho, Y., Sun, H., and Lee, K. (2015a). Exact likelihood inference for an exponential parameter under generalized progressive hybrid censoring scheme. *Stat. Method*, 23, 18- 34.
9. Cho, Y., Sun, H. and Lee, K. (2015b). Estimating the Entropy of a Weibull Distribution under Generalized Progressive Hybrid Censoring. *Entropy*, 17, 102-122. DOI: 10.3390/e17010102.
10. Kundu, D. and Joarder, A. (2006). Analysis of Type-II progressively hybrid censored data. *Comput. Stat Data Anal*., 50, 2509-2528.
11. Lee, W.C., Wu, J.W. and Hong, C.W. (2009). Assessing the Lifetime Performance Index of Products from Progressively Type II Right Censored Data Using Burr XII Model. *Mathematics and Computers in Simulation*, 79(7), 2167-2179.
12. Maswadah, M. (2006). Kernel inference on the inverse Weibull distribution. *The Korean Communications in Statistics*, 13(3), 503-512.
13. Maswadah, M. (2007). Kernel inference on the Weibull distribution. *Proc. of the Third National Statistical Conference*. Lahore, Pakistan, 14, 77-86.
14. Maswadah, M. (2010). Kernel inference on the type-II Extreme value distribution. *Proceedings of the Tenth Islamic Countries Conference on Statistical Sciences* (ICCS-X), Lahore, Pakistan, II: 870-880.
15. Maswadah, M. (2021). An optimal point estimation method for the inverse Weibull model parameters using the Runge-Kutta method. *Aligarh Journal of Statistics (AJS),* 41, 1-21.

16. Maswadah, M. (2022). Improved maximum likelihood estimation of the shape-scale family based on the generalized progressive hybrid censoring scheme. *Journal of Applied Statistics*, 49(11), 4825-4849. DOI: 10.1080/02664763.2021.1924638.
17. Mohie El-Din, M.M. and Nagy, M. (2017). Estimation for Inverse Weibull distribution under Generalized Progressive Hybrid Censoring Scheme. *J. Stat. Appl. Pro. Lett.*, 4(3), 97-107.
18. Moore, D. and Papadopoulos, A.S. (2000). The Burr type XII distribution as a failure model under various loss functions. *Microelectronics Reliability*, 40(12), 2117-2122.
19. Soliman, A.A. (2005). Estimation of parameters of life from progressively censored data using Burr-XII model. *IEEE Transactions on Reliability*, 54(1), 34 -42.
20. Soliman, A.A, Abd Ellah, A.H., Abou-Elheggag, N.A. and Modhesh, A.A. (2011). Bayesian Inference and Prediction of Burr-XII Distribution for Progressive First Failure Censored Sampling. *Intelligent Information Management*, 3, 175-185.
21. Ramsay, J.O., Hooker, G., Campbell, D., Cao, J. (2007). Parameter estimation for differential equations: a generalized smoothing approach. *J. R. Statist. Soc*. B. 69(5), 741-796.
22. Tierney, L. and Kadane, J.B. (1986). Accurate Approximations for Posterior Moments and Marginal Densities. *Journal of the American Statistical Association*, 81(393), 82-86.
23. Varian, H.R. (1975). A Bayesian approach to real estate assessment. In SE Feinberge and A. Zellner (Eds.) *Studies in Bayesian Econometrics and Statistics in honor of LJ Savage*, North Holand, Amsterdam. 195-208.
24. Wang, F.K., Keats, J.B. and Zimmer, W.J. (1996). The maximum likelihood estimation of the Burr XII parameters with censored and uncensored data. *Microelectronics & Reliability*, 36, 395-362.
25. Wei, C.D., Wei, S. and Su, H. (2010). Bayes estimation and application of Poisson distribution parameter under compound LINEX symmetric loss [J], *Statistics and Decision*, 7, 156-157. (In Chinese).
26. Wingo, D.R. (1983). Maximum likelihood methods for fitting the Burr Type-XII distribution to life test data. *Biometrical J*., 25, 77-81.
27. Wingo, D.R. (1993a). Maximum likelihood estimation of the Burr type-XII distribution parameters under type-II censoring. *Microelectorn Reliab.*, 23, 1251-1257.
28. Wingo, D.R. (1993b). Maximum Likelihood Methods for Fitting the Burr Type-XII Distribution to Life Test Data. *Metrika*, 40(1), 203-210.
29. Wu, J.W. and Yu, H.Y. (2005). Statistical Inference about the Shape Parameter of the Burr Type XII Distribution under the Failure-Censored Sampling Plan. *Applied Mathematics and computation*, 163(1), 443-482.
30. Wu, S.J., Chen, Y.J. and Chang, C.T. (2007). Statistical inference based on progressively censored samples with random removals from the Burr type XII distribution. *Journal of Statistical Computation and Simulation*, 77(1), 19-27(9).
31. Xiuchun, L., Yimin, S., Jieqiong, W. and Jian, C. (2007). Empirical Bayes estimators of reliability performances using LINEX loss under progressively Type-II censored samples. *Mathematics and Computers in Simulation*, 73(5), 320-326.

Table 3
The Average Bias (AVB) and Mean Squared Errors (MSEs) in Parentheses for the Burr- XII Parameter α using Picard and Bayes methods with m = (n/2 and 3n/4) and k=(m/2, 3m/4) at T=0.75 and $\delta = 2$ for LINEX loss.

N	M	K	α	β	γ	Picard's Method	Gamma Prior		Kernel Prior	
							SQEL	LNXL	SQEL	LNXL
20	10	5	1	0.75	1.5	0.0876(0.0119)	0.1744(0.0314)	0.1179(0.0144)	0.1658(0.0288)	0.1205(0.0150)
				1.75	2.5	0.1481(0.0226)	0.3965(0.1600)	0.2722(0.0742)	0.3481(0.1395)	0.2779(0.0773)
			2	0.75	1.5	0.0639(0.0050)	0.1684(0.0302)	0.1104(0.0127)	0.1466(0.0224)	0.1126(0.0132)
				1.75	2.5	0.1193(0.0143)	0.4643(0.3120)	0.2706(0.0733)	0.3301(0.1440)	0.2746(0.0755)
		8	1	0.75	1.5	0.1242(0.1781)	0.1577(0.0274)	0.1183(0.0146)	0.1534(0.0241)	0.1211(0.0152)
				1.75	2.5	0.1562(0.0252)	0.3473(0.1214)	0.2734(0.0749)	0.3273(0.1078)	0.2793(0.0781)
			2	0.75	1.5	0.0679(0.0060)	0.1521(0.0244)	0.1154(0.0137)	0.1446(0.0220)	0.1176(0.0142)
				1.75	2.5	0.1222(0.0151)	0.3471(0.1291)	0.2718(0.0739)	0.3138(0.0990)	0.2760(0.0762)
	15	8	1	0.75	1.5	0.1009(0.0181)	0.1596(0.0263)	0.1208(0.0151)	0.1559(0.0248)	0.1234(0.0156)
				1.75	2.5	0.1552(0.0248)	0.3474(0.1219)	0.2738(0.0751)	0.3309(0.1179)	0.2797(0.0783)
			2	0.75	1.5	0.0911(0.3044)	0.1486(0.0233)	0.1130(0.0132)	0.1410(0.0204)	0.1152(0.0137)
				1.75	2.5	0.1219(0.0150)	0.3568(0.2960)	0.2721(0.0741)	0.3126(0.0979)	0.2762(0.0764)
		11	1	0.75	1.5	0.1064(0.0205)	0.1542(0.0262)	0.1239(0.0157)	0.1782(0.7657)	0.1263(0.0163)
				1.75	2.5	0.1620(0.0271)	0.3284(0.1088)	0.2746(0.0755)	0.3183(0.1019)	0.2805(0.0788)
			2	0.75	1.5	0.0751(0.0095)	0.1414(0.0210)	0.1157(0.0138)	0.1385(0.0196)	0.1178(0.0142)
				1.75	2.5	0.1250(0.0158)	0.3234(0.1082)	0.2720(0.0741)	0.3066(0.0941)	0.2765(0.0765)
40	20	10	1	0.75	1.5	0.1107(0.1014)	0.1454(0.0215)	0.1180(0.0143)	0.1459(0.0261)	0.1199(0.0147)
				1.75	2.5	0.1483(0.0223)	0.3298(0.1099)	0.2725(0.0743)	0.3146(0.1003)	0.2766(0.0765)
			2	0.75	1.5	0.0732(0.0265)	0.1390(0.0200)	0.1120(0.0128)	0.1322(0.0178)	0.1135(0.0131)
				1.75	2.5	0.1181(0.0140)	0.3454(0.1272)	0.2717(0.0738)	0.3045(0.0954)	0.2743(0.0753)
		15	1	0.75	1.5	0.1052(0.0408)	0.1413(0.0203)	0.1202(0.0147)	0.1411(0.0202)	0.1221(0.0151)
				1.75	2.5	0.1557(0.0247)	0.3137(0.0987)	0.2724(0.0743)	0.3047(0.0931)	0.2768(0.0767)
			2	0.75	1.5	0.0877(0.1146)	0.1320(0.0177)	0.1128(0.0130)	0.1295(0.0170)	0.1143(0.0133)
				1.75	2.5	0.1210(0.0147)	0.3126(0.0986)	0.2718(0.0739)	0.2978(0.0888)	0.2746(0.0755)
	30	15	1	0.75	1.5	0.1120(0.0426)	0.1403(0.0201)	0.1187(0.0144)	0.1392(0.0197)	0.1206(0.0148)
				1.75	2.5	0.1557(0.0247)	0.3137(0.0989)	0.2726(0.0744)	0.3054(0.0935)	0.2769(0.0767)
			2	0.75	1.5	0.0716(0.0109)	0.1336(0.0181)	0.1141(0.0132)	0.1306(0.0173)	0.1156(0.0135)
				1.75	2.5	0.1211(0.0147)	0.3117(0.0975)	0.2717(0.0739)	0.2979(0.0891)	0.2746(0.0754)
		23	1	0.75	1.5	0.1436(0.4027)	0.1372(0.0190)	0.1237(0.0155)	0.1379(0.0192)	0.1253(0.0159)
				1.75	2.5	0.1629(0.0270)	0.3001(0.0902)	0.2745(0.0754)	0.2988(0.0893)	0.2788(0.0778)
			2	0.75	1.5	0.0801(0.0172)	0.1299(0.0186)	0.1159(0.0136)	0.1282(0.0166)	0.1174(0.0139)
				1.75	2.5	0.1250(0.0157)	0.2961(0.0880)	0.2721(0.0741)	0.2924(0.0856)	0.2753(0.0758)
60	30	15	1	0.75	1.5	0.1123(0.0514)	0.1342(0.0182)	0.1193(0.0144)	0.1343(0.0182)	0.1208(0.0147)
				1.75	2.5	0.1514(0.0232)	0.3040(0.0925)	0.2728(0.0745)	0.2995(0.0900)	0.2762(0.0763)
			2	0.75	1.5	0.0680(0.0056)	0.1278(0.0165)	0.1133(0.0130)	0.1260(0.0160)	0.1144(0.0132)
				1.75	2.5	0.1198(0.0144)	0.3031(0.0921)	0.2716(0.0738)	0.2930(0.0862)	0.2739(0.0750)
		23	1	0.75	1.5	0.2071(0.0532)	0.1327(0.0178)	0.1188(0.0143)	0.1337(0.0183)	0.1203(0.0146)
				1.75	2.5	0.1550(0.0243)	0.2999(0.0900)	0.2729(0.0745)	0.2971(0.0884)	0.2764(0.0764)
			2	0.75	1.5	0.0712(0.0063)	0.1273(0.0169)	0.1139(0.0131)	0.1255(0.0159)	0.1151(0.0134)
				1.75	2.5	0.1211(0.0147)	0.2976(0.0887)	0.2717(0.0738)	0.2903(0.0844)	0.2740(0.0751)
	45	23	1	0.75	1.5	0.1503(0.4315)	0.1330(0.0179)	0.1187(0.0143)	0.1330(0.0179)	0.1202(0.0146)
				1.75	2.5	0.1559(0.0246)	0.2994(0.0897)	0.2727(0.0744)	0.2973(0.0893)	0.2762(0.0763)
			2	0.75	1.5	0.0750(0.0207)	0.1263(0.0163)	0.1133(0.0130)	0.1249(0.0158)	0.1145(0.0133)
				1.75	2.5	0.1207(0.0146)	0.2975(0.0888)	0.2721(0.0740)	0.2912(0.0849)	0.2744(0.0753)
		34	1	0.75	1.5	0.1868(0.0325)	0.1323(0.0177)	0.1229(0.0153)	0.1330(0.0178)	0.1243(0.0156)
				1.75	2.5	0.1636(0.0271)	0.2917(0.0851)	0.2741(0.0752)	0.2919(0.0852)	0.2777(0.0771)
			2	0.75	1.5	0.0882(0.0693)	0.1235(0.0154)	0.1148(0.0133)	0.1235(0.0154)	0.1159(0.0136)
				1.75	2.5	0.1249(0.0156)	0.2888(0.0840)	0.2720(0.0740)	0.2865(0.0821)	0.2745(0.0754)

Table 4
The Average Bias (AVB) and Mean Squared Errors (MSEs) in Parentheses for the Burr-XII Parameter α using Picard and Bayes methods with m = (n/2 and 3n/4) and k=(m/2 and 3m/4) at T=2 and $\delta = 2$ for LINEX loss.

N	M	K	α	β	γ	Picard's Method	Gamma Prior		Kernel Prior	
							SQEL	LNXL	SQEL	LNXL
20	10	5	1	0.75	1.5	0.1074(0.0528)	0.1561(0.0250)	0.1211(0.0152)	0.1547(0.0246)	0.1238(0.0158)
				1.75	2.5	0.1605(0.0267)	0.3373(0.1142)	0.2724(0.0744)	0.3238(0.1068)	0.2789(0.0779)
			2	0.75	1.5	0.0704(0.0066)	0.1458(0.0220)	0.1126(0.0131)	0.1411(0.0207)	0.1151(0.0136)
				1.75	2.5	0.1230(0.0152)	0.3346(0.1130)	0.2713(0.0737)	0.3109(0.0969)	0.2759(0.0762)
		8	1	0.75	1.5	0.1058(0.0279)	0.1548(0.0245)	0.1194(0.0148)	0.1532(0.0239)	0.1222(0.0154)
				1.75	2.5	0.1575(0.0256)	0.3463(0.1203)	0.2725(0.0744)	0.3263(0.1083)	0.2788(0.0778)
			2	0.75	1.5	0.0718(0.0083)	0.1481(0.0226)	0.1110(0.0128)	0.1410(0.0204)	0.1137(0.0134)
				1.75	2.5	0.1219(0.0150)	0.3460(0.1219)	0.2713(0.0737)	0.3133(0.0986)	0.2758(0.0761)
	15	8	1	0.75	1.5	0.1034(0.0222)	0.1555(0.0246)	0.1208(0.0151)	0.1537(0.0241)	0.1236(0.0157)
				1.75	2.5	0.1589(0.0260)	0.3380(0.1145)	0.2730(0.0747)	0.3222(0.1040)	0.2794(0.0781)
			2	0.75	1.5	0.0759(0.0270)	0.1461(0.0222)	0.1132(0.0133)	0.1406(0.0202)	0.1156(0.0138)
				1.75	2.5	0.1229(0.0152)	0.3351(0.1133)	0.2714(0.0737)	0.3126(0.1004)	0.2760(0.0762)
		11	1	0.75	1.5	0.1099(0.0421)	0.1527(0.0241)	0.1237(0.0157)	0.1514(0.0232)	0.1262(0.0162)
				1.75	2.5	0.1635(0.0279)	0.3269(0.1077)	0.2737(0.0750)	0.3165(0.1003)	0.2800(0.0785)
			2	0.75	1.5	0.0822(0.0424)	0.1411(0.0208)	0.1138(0.0134)	0.1381(0.0195)	0.1163(0.0139)
				1.75	2.5	0.1254(0.0158)	0.3211(0.1046)	0.2713(0.0737)	0.3062(0.0938)	0.2761(0.0763)
40	20	10	1	0.75	1.5	0.1895(0.0436)	0.1370(0.0191)	0.1202(0.0148)	0.1377(0.0192)	0.1222(0.0152)
				1.75	2.5	0.1618(0.0266)	0.3133(0.1823)	0.2729(0.0745)	0.3012(0.0912)	0.2775(0.0771)
			2	0.75	1.5	0.0956(0.2090)	0.1291(0.0170)	0.1132(0.0131)	0.1282(0.0167)	0.1149(0.0134)
				1.75	2.5	0.1238(0.0154)	0.3015(0.0949)	0.2713(0.0736)	0.2937(0.0863)	0.2746(0.0754)
		15	1	0.75	1.5	0.1151(0.0229)	0.1381(0.0195)	0.1209(0.0148)	0.1381(0.0192)	0.1228(0.0153)
				1.75	2.5	0.1610(0.0263)	0.3045(0.0934)	0.2731(0.0747)	0.3006(0.0904)	0.2777(0.0772)
			2	0.75	1.5	0.0762(0.0087)	0.1283(0.0168)	0.1129(0.0130)	0.1279(0.0166)	0.1146(0.0134)
				1.75	2.5	0.1240(0.0155)	0.2984(0.0894)	0.2711(0.0735)	0.2932(0.0861)	0.2744(0.0753)
	30	15	1	0.75	1.5	0.1110(0.0170)	0.1377(0.0194)	0.1217(0.0150)	0.1373(0.0190)	0.1236(0.0155)
				1.75	2.5	0.1628(0.0271)	0.3020(0.0913)	0.2731(0.0746)	0.2997(0.0899)	0.2777(0.0772)
			2	0.75	1.5	0.0847(0.0283)	0.1308(0.0220)	0.1135(0.0131)	0.1279(0.0166)	0.1152(0.0135)
				1.75	2.5	0.1243(0.0155)	0.2989(0.0899)	0.2713(0.0736)	0.2932(0.0860)	0.2746(0.0755)
		23	1	0.75	1.5	0.7102(0.1532)	0.1370(0.0190)	0.1227(0.0153)	0.1374(0.0191)	0.1244(0.0157)
				1.75	2.5	0.1635(0.0273)	0.3008(0.0907)	0.2739(0.0751)	0.2987(0.0892)	0.2784(0.0776)
			2	0.75	1.5	0.0766(0.0117)	0.1281(0.0167)	0.1147(0.0134)	0.1282(0.0166)	0.1164(0.0137)
				1.75	2.5	0.1252(0.0157)	0.2955(0.0875)	0.2714(0.0737)	0.2919(0.0852)	0.2747(0.0755)
60	30	15	1	0.75	1.5	0.1227(0.0280)	0.1334(0.0194)	0.1206(0.0147)	0.1328(0.0178)	0.1222(0.0151)
				1.75	2.5	0.1611(0.0262)	0.2938(0.0864)	0.2730(0.0746)	0.2932(0.0860)	0.2768(0.0767)
			2	0.75	1.5	0.0758(0.0078)	0.1248(0.0158)	0.1137(0.0131)	0.1245(0.0156)	0.1151(0.0134)
				1.75	2.5	0.1235(0.0153)	0.2904(0.0844)	0.2714(0.0737)	0.2878(0.0828)	0.2740(0.0751)
		23	1	0.75	1.5	0.1348(0.1186)	0.1323(0.0177)	0.1207(0.0147)	0.1328(0.0178)	0.1223(0.0151)
				1.75	2.5	0.1616(0.0265)	0.2948(0.0889)	0.2729(0.0745)	0.2930(0.0859)	0.2767(0.0766)
			2	0.75	1.5	0.0763(0.0078)	0.1236(0.0154)	0.1128(0.0129)	0.1236(0.0154)	0.1142(0.0132)
				1.75	2.5	0.1235(0.0153)	0.2908(0.0847)	0.2714(0.0737)	0.2877(0.0828)	0.2740(0.0751)
	45	23	1	0.75	1.5	0.1378(0.1768)	0.1320(0.0176)	0.1218(0.0150)	0.1325(0.0177)	0.1233(0.0153)
				1.75	2.5	0.1628(0.0268)	0.2928(0.0860)	0.2735(0.0749)	0.2922(0.0854)	0.2773(0.0769)
			2	0.75	1.5	0.0786(0.0090)	0.1240(0.0156)	0.1139(0.0131)	0.1239(0.0155)	0.1152(0.0134)
				1.75	2.5	0.1249(0.0157)	0.2880(0.0831)	0.2711(0.0735)	0.2864(0.0821)	0.2738(0.0750)
		34	1	0.75	1.5	0.1300(0.0952)	0.1335(0.0183)	0.1228(0.0152)	0.1334(0.0179)	0.1242(0.0156)
				1.75	2.5	0.1642(0.0273)	0.2918(0.0853)	0.2735(0.0748)	0.2917(0.0851)	0.2772(0.0769)
			2	0.75	1.5	0.0793(0.0088)	0.1238(0.0156)	0.1139(0.0131)	0.1237(0.0154)	0.1152(0.0134)
				1.75	2.5	0.1248(0.0156)	0.2881(0.0831)	0.2714(0.0737)	0.2863(0.0820)	0.2742(0.0752)

Table 5
The Average Bias (AVB) and Mean Squared Errors (RMSEs) in Parentheses for the Burr-XII Parameter β using Picard and Bayes Methods with m = (n/2 and 3n/4) and k=(m/2 and 3m/4) at T=0.75 and δ = 2 for LINEX loss.

N	M	K	α	β	γ	Picard's Method	Gamma Prior		Kernel Prior	
							SQEL	LNXL	SQEL	LNXL
20	10	5	1	0.75	1.5	03254(0.0342)	0.293(0.2442)	0.1307(0.0171)	0.4716(0.2428)	0.1301(0.0169)
				1.75	2.5	0.0447(0.0020)	0.741(0.8654)	0.1224(0.0150)	0.4838(0.3392)	0.1221(0.0149)
			2	0.75	1.5	0.1119(0.0131)	0.269(0.1475)	0.3083(0.0951)	0.4065(0.6730)	0.2813(0.0791)
				1.75	2.5	0.0931(0.0088)	0.743(0.6574)	0.2562(0.0657)	0.3462(0.3606)	0.2461(0.0606)
		8	1	0.75	1.5	0.2341(0.0342)	0.246(0.1242)	0.1242(0.0154)	0.4418(0.2026)	0.1242(0.0154)
				1.75	2.5	0.0436(0.0019)	0.643(0.8997)	0.1200(0.0144)	0.4390(0.1965)	0.1199(0.0144)
			2	0.75	1.5	0.1029(0.0109)	0.262(0.2381)	0.2766(0.0765)	0.3050(0*****)	0.2624(0.0688)
				1.75	2.5	0.0901(0.0082)	0.612(0.9324)	0.2487(0.0618)	0.2367(0.4349)	0.2423(0.0587)
	15	8	1	0.75	1.5	0.1979(0.0324)	0.249(0.1277)	0.1242(0.0154)	0.4362(0.1952)	0.1242(0.0154)
				1.75	2.5	0.0437(0.0019)	0.681(0.8943)	0.1200(0.0144)	0.4339(0.1915)	0.1199(0.0144)
			2	0.75	1.5	0.1020(0.0109)	0.248(0.2368)	0.2767(0.0766)	0.2560(0.1388)	0.2626(0.0690)
				1.75	2.5	0.0901(0.0082)	0.3462(0.7658)	0.2486(0.0618)	0.2226(0.1315)	0.2423(0.0587)
		11	1	0.75	1.5	0.2429(0.0325)	0.226(0.0917)	0.1213(0.0147)	0.4232(0.1860)	0.1214(0.0147)
				1.75	2.5	0.0430(0.0019)	0.596(0.6758)	0.1187(0.0141)	0.4143(0.1732)	0.1187(0.0141)
			2	0.75	1.5	0.0979(0.0100)	0.223(0.1241)	0.2643(0.0699)	0.1978(0.1381)	0.2545(0.0648)
				1.75	2.5	0.0880(0.0078)	0.596(0.7684)	0.2446(0.0598)	0.1523(0.1584)	0.2400(0.0576)
40	20	10	1	0.75	1.5	0.0518(0.0106)	0.236(0.1342)	0.1287(0.0166)	0.4344(0.1976)	0.1291(0.0167)
				1.75	2.5	0.0438(0.0019)	0.612(0.6175)	0.1217(0.0148)	0.4448(0.2617)	0.1219(0.0149)
			2	0.75	1.5	0.1066(0.0164)	0.254(0.3152)	0.2990(0.0894)	0.3007(0.2617)	0.2839(0.0806)
				1.75	2.5	0.0917(0.0084)	0.585(0.9723)	0.2545(0.0648)	0.2981(0.3850)	0.2485(0.0618)
		15	1	0.75	1.5	0.0461(0.0021)	0.205(0.1003)	0.1247(0.0156)	0.4193(0.1815)	0.1251(0.0157)
				1.75	2.5	0.0430(0.0019)	0.554(0.5182)	0.1202(0.0144)	0.4198(0.1951)	0.1204(0.0145)
			2	0.75	1.5	0.0981(0.0101)	0.209(0.1035)	0.2793(0.0780)	0.2129(0.1955)	0.2700(0.0729)
				1.75	2.5	0.0893(0.0080)	0.558(0.6498)	0.2492(0.0621)	0.1768(0.0948)	0.2451(0.0601)
	30	15	1	0.75	1.5	0.1234(0.0153)	0.228(0.4255)	0.1248(0.0156)	0.4200(0.1818)	0.1252(0.0157)
				1.75	2.5	0.0430(0.0018)	0.556(0.6224)	0.1201(0.0144)	0.4138(0.1738)	0.1204(0.0145)
			2	0.75	1.5	0.0977(0.0097)	0.245(0.6574)	0.2791(0.0779)	0.2139(0.1030)	0.2696(0.0727)
				1.75	2.5	0.0893(0.0080)	0.607(0.8794)	0.2492(0.0621)	0.1928(0.3422)	0.2451(0.0601)
		23	1	0.75	1.5	0.0659(0.4653)	0.192(0.1094)	0.1209(0.0146)	0.3999(0.1620)	0.1213(0.0147)
				1.75	2.5	0.0423(0.0018)	0.464(0.2937)	0.1185(0.0140)	0.3908(0.1535)	0.1187(0.0141)
			2	0.75	1.5	0.0922(0.0086)	0.173(0.0470)	0.2620(0.0687)	0.1172(0.0538)	0.2568(0.0659)
				1.75	2.5	0.0871(0.0076)	0.478(0.5447)	0.2440(0.0595)	0.0951(0.0988)	0.2414(0.0583)
60	30	15	1	0.75	1.5	0.0459(0.0022)	0.257(0.5643)	0.1253(0.0157)	0.4068(0.1740)	0.1260(0.0159)
				1.75	2.5	0.0432(0.0019)	0.565(0.6064)	0.1207(0.0146)	0.4194(0.2225)	0.1211(0.0147)
			2	0.75	1.5	0.0980(0.0097)	0.232(0.2433)	0.2857(0.0816)	0.2092(0.1088)	0.2775(0.0770)
				1.75	2.5	0.0898(0.0081)	0.538(0.9512)	0.2509(0.0630)	0.1950(0.1803)	0.2476(0.0613)
		23	1	0.75	1.5	0.2134(0.0325)	0.191(0.1506)	0.1245(0.0155)	0.4064(0.1759)	0.1251(0.0156)
				1.75	2.5	0.0428(0.0018)	0.513(0.4854)	0.1200(0.0144)	0.4030(0.1646)	0.1204(0.0145)
			2	0.75	1.5	0.0953(0.0092)	0.183(0.0515)	0.2779(0.0772)	0.1906(0.2415)	0.2714(0.0736)
				1.75	2.5	0.0889(0.0079)	0.461(0.3567)	0.2488(0.0619)	0.1703(0.4085)	0.2460(0.0605)
	45	23	1	0.75	1.5	0.0454(0.0023)	0.182(0.0991)	0.1245(0.0155)	0.4048(0.1680)	0.1251(0.0157)
				1.75	2.5	0.0428(0.0018)	0.499(0.4124)	0.1200(0.0144)	0.3994(0.1612)	0.1204(0.0145)
			2	0.75	1.5	0.0955(0.0093)	0.191(0.0992)	0.2779(0.0772)	0.1812(0.1035)	0.2714(0.0737)
				1.75	2.5	0.0891(0.0079)	0.538(0.8954)	0.2489(0.0619)	0.1570(0.1450)	0.2460(0.0605)
		34	1	0.75	1.5	0.2134(0.0352)	0.157(0.0279)	0.1210(0.0146)	0.3913(0.1565)	0.1215(0.0148)
				1.75	2.5	0.0421(0.0018)	0.454(0.6220)	0.1185(0.0140)	0.3808(0.1453)	0.1188(0.0141)
			2	0.75	1.5	0.0929(0.0115)	0.154(0.0259)	0.2625(0.0689)	0.1094(0.2642)	0.2589(0.0670)
				1.75	2.5	0.0866(0.0075)	0.456(0.5056)	0.2440(0.0595)	0.0495(0.0105)	0.2422(0.0587)

Table 6
The Average Bias (AVB) and Mean Squared Errors (RMSEs) in Parentheses for the Burr-XII Parameter β using Picard and Bayes Methods with m = (n/2 and 3n/4) and k=(m/2 and 3m/4)at T=2 and δ = 2 for LINEX loss.

N	M	K	α	β	γ	Picard's Method	Gamma Prior		Kernel Prior	
							SQEL	LNXL	SQEL	LNXL
20	10	5	1	0.75	1.5	0.0624(0.1107)	0.277(0.8922)	0.1224(0.0150)	0.4325(0.1914)	0.1225(0.0150)
				1.75	2.5	0.0431(0.0019)	0.680(0.8139)	0.1192(0.0142)	0.4254(0.1826)	0.1193(0.0142)
			2	0.75	1.5	0.0990(0.0101)	0.301(0.6570)	0.2679(0.0718)	0.2330(0.2030)	0.2575(0.0663)
				1.75	2.5	0.0892(0.0080)	0.644(0.9926)	0.2461(0.0606)	0.2317(0.2134)	0.2411(0.0581)
		8	1	0.75	1.5	0.0848(0.0352)	0.246(0.1193)	0.1229(0.0151)	0.4343(0.1996)	0.1230(0.0151)
				1.75	2.5	0.0433(0.0019)	0.637(0.8604)	0.1197(0.0143)	0.4412(0.2448)	0.1197(0.0143)
			2	0.75	1.5	0.1018(0.0128)	0.244(0.1515)	0.2725(0.0742)	0.2372(0.0836)	0.2606(0.0679)
				1.75	2.5	0.0901(0.0082)	0.637(0.6843)	0.2477(0.0614)	0.2172(0.0924)	0.2420(0.0586)
	15	8	1	0.75	1.5	0.0586(0.0834)	0.238(0.1090)	0.1225(0.0150)	0.4319(0.1940)	0.1226(0.0150)
				1.75	2.5	0.0532(0.1054)	0.620(0.8569)	0.1193(0.0142)	0.4276(0.1876)	0.1193(0.0142)
			2	0.75	1.5	0.0997(0.0105)	0.245(0.1165)	0.2677(0.0717)	0.2179(0.1359)	0.2575(0.0663)
				1.75	2.5	0.0893(0.0080)	0.704(0.8943)	0.2462(0.0606)	0.1930(0.1304)	0.2411(0.0581)
		11	1	0.75	1.5	0.1352(0.0432)	0.222(0.0796)	0.1210(0.0146)	0.4215(0.1842)	0.1211(0.0147)
				1.75	2.5	0.0429(0.0019)	0.597(0.7394)	0.1185(0.0141)	0.4179(0.2067)	0.1186(0.0141)
			2	0.75	1.5	0.1226(0.6731)	0.228(0.1290)	0.2616(0.0684)	0.1945(0.1155)	0.2531(0.0641)
				1.75	2.5	0.0876(0.0077)	0.574(0.8885)	0.2436(0.0594)	0.1259(0.0315)	0.2395(0.0574)
40	20	10	1	0.75	1.5	0.0504(0.0214)	0.207(0.0720)	0.1216(0.0148)	0.4043(0.1720)	0.1220(0.0149)
				1.75	2.5	0.0424(0.0018)	0.568(0.5306)	0.1188(0.0141)	0.3972(0.1594)	0.1191(0.0142)
			2	0.75	1.5	0.0935(0.0088)	0.225(0.1065)	0.2639(0.0696)	0.1448(0.0485)	0.2585(0.0668)
				1.75	2.5	0.0876(0.0077)	0.555(0.7568)	0.2447(0.0599)	0.1006(0.0748)	0.2420(0.0586)
		15	1	0.75	1.5	0.1300(0.0325)	0.206(0.0916)	0.1216(0.0148)	0.4030(0.1647)	0.1220(0.0149)
				1.75	2.5	0.0425(0.0018)	0.556(0.5927)	0.1188(0.0141)	0.3977(0.1591)	0.1191(0.0142)
			2	0.75	1.5	0.0955(0.0121)	0.207(0.0858)	0.2639(0.0696)	0.1427(0.0623)	0.2585(0.0668)
				1.75	2.5	0.0875(0.0077)	0.639(0.8954)	0.2447(0.0599)	0.1095(0.0893)	0.2420(0.0586)
	30	15	1	0.75	1.5	0.0482(0.0166)	0.225(0.2787)	0.1209(0.0146)	0.4002(0.1627)	0.1213(0.0147)
				1.75	2.5	0.0423(0.0018)	0.566(0.9381)	0.1186(0.0141)	0.4017(0.2105)	0.1189(0.0141)
			2	0.75	1.5	0.0929(0.0087)	0.197(0.0692)	0.2624(0.0688)	0.1612(0.3132)	0.2573(0.0662)
				1.75	2.5	0.0874(0.0077)	0.561(0.8331)	0.2442(0.0597)	0.0972(0.0457)	0.2417(0.0584)
		23	1	0.75	1.5	0.0839(0.0675)	0.170(0.0355)	0.1206(0.0146)	0.3968(0.1600)	0.1210(0.0147)
				1.75	2.5	0.0423(0.0018)	0.530(0.8932)	0.1184(0.0140)	0.3900(0.1527)	0.1186(0.0141)
			2	0.75	1.5	0.0931(0.0096)	0.185(0.1333)	0.2599(0.0676)	0.1227(0.0409)	0.2554(0.0652)
				1.75	2.5	0.0868(0.0076)	0.487(0.8768)	0.2432(0.0592)	0.0708(0.0140)	0.2409(0.0580)
60	30	15	1	0.75	1.5	0.3488(0.0342)	0.174(0.0452)	0.1216(0.0148)	0.3970(0.1628)	0.1221(0.0149)
				1.75	2.5	0.0423(0.0018)	0.484(0.4113)	0.1188(0.0141)	0.3871(0.1507)	0.1192(0.0142)
			2	0.75	1.5	0.0919(0.0085)	0.179(0.0874)	0.2637(0.0696)	0.1325(0.1056)	0.2602(0.0677)
				1.75	2.5	0.0874(0.0076)	0.529(0.7843)	0.2447(0.0599)	0.0784(0.0250)	0.2429(0.0590)
		23	1	0.75	1.5	0.0484(0.0188)	0.174(0.0452)	0.1216(0.0148)	0.3944(0.1607)	0.1221(0.0149)
				1.75	2.5	0.0423(0.0018)	0.481(0.3270)	0.1188(0.0141)	0.3860(0.1501)	0.1192(0.0142)
			2	0.75	1.5	0.0916(0.0088)	0.180(0.1042)	0.2637(0.0696)	0.1258(0.1885)	0.2602(0.0677)
				1.75	2.5	0.0874(0.0076)	0.507(0.9330)	0.2447(0.0599)	0.0811(0.0329)	0.2429(0.0590)
	45	23	1	0.75	1.5	0.0487(0.0136)	0.180(0.0590)	0.1206(0.0146)	0.3873(0.1513)	0.1212(0.0147)
				1.75	2.5	0.0422(0.0018)	0.456(0.2632)	0.1185(0.0140)	0.3848(0.1490)	0.1188(0.0141)
			2	0.75	1.5	0.0911(0.0084)	0.182(0.0827)	0.2609(0.0681)	0.0981(0.0372)	0.2578(0.0665)
				1.75	2.5	0.0867(0.0075)	0.517(0.8943)	0.2436(0.0593)	0.0566(0.0213)	0.2420(0.0586)
		34	1	0.75	1.5	0.2113(0.0432)	0.151(0.0251)	0.1206(0.0145)	0.3872(0.1507)	0.1211(0.0147)
				1.75	2.5	0.0421(0.0018)	0.482(0.9996)	0.1184(0.0140)	0.3813(0.1458)	0.1187(0.0141)
			2	0.75	1.5	0.0915(0.0102)	0.153(0.0264)	0.2601(0.0676)	0.0916(0.0445)	0.2571(0.0661)
				1.75	2.5	0.0867(0.0075)	0.445(0.4512)	0.2434(0.0592)	0.0606(0.0279)	0.2419(0.0585)

Table 7
The Average Bias (AVB) and Mean Squared Errors (RMSEs) in Parentheses for the Burr-XII Parameter γ using Picard and Bayes Methods with m = (n/2 and 3n/4) and k=(m/2 and 3m/4) at T=0.75 and δ = 2 for LINEX loss.

N	M	K	α	β	γ	Picard's Method	Gamma Prior		Kernel Prior	
							SQEL	LNXL	SQEL	LNXL
20	10	5	1	0.75	1.5	0.0650(0.0043)	0.530(0.9134)	0.3348(0.1121)	0.4683(0.3860)	0.3179(0.1011)
				1.75	2.5	0.0391(0.0016)	0.573(0.6784)	0.5069(0.2570)	0.5779(0.3644)	0.4679(0.2189)
			2	0.75	1.5	0.0819(0.0067)	0.491(0.5028)	0.3787(0.1435)	0.4758(0.2502)	0.3402(0.1157)
				1.75	2.5	0.0945(0.0089)	0.564(0.7843)	0.5378(0.2894)	0.5885(0.3555)	0.4777(0.2282)
		8	1	0.75	1.5	0.0691(0.0048)	0.437(0.5634)	0.3206(0.1028)	0.4248(0.2144)	0.3119(0.0973)
				1.75	2.5	0.0495(0.0025)	0.735(0.9694)	0.4896(0.2398)	0.5578(0.3215)	0.4649(0.2162)
			2	0.75	1.5	0.0842(0.0071)	0.423(0.2895)	0.3438(0.1182)	0.4414(0.3054)	0.3255(0.1060)
				1.75	2.5	0.0999(0.0100)	0.837(0.9843)	0.5111(0.2613)	0.5618(0.3241)	0.4731(0.2238)
	15	8	1	0.75	1.5	0.0692(0.0048)	0.446(0.5385)	0.3205(0.1027)	0.4272(0.2406)	0.3118(0.0972)
				1.75	2.5	0.0496(0.0025)	0.761(0.8943)	0.4897(0.2398)	0.5609(0.3316)	0.4650(0.2162)
			2	0.75	1.5	0.0840(0.0071)	0.438(0.6345)	0.3444(0.1187)	0.4188(0.1848)	0.3257(0.1061)
				1.75	2.5	0.0997(0.0100)	0.801(0.7864)	0.5110(0.2612)	0.5538(0.3100)	0.4730(0.2238)
		11	1	0.75	1.5	0.0728(0.0053)	0.386(0.2833)	0.3139(0.0985)	0.4305(0.8622)	0.3087(0.0953)
				1.75	2.5	0.0598(0.0036)	0.665(0.6776)	0.4802(0.2306)	0.5387(0.3027)	0.4627(0.2141)
			2	0.75	1.5	0.0862(0.0074)	0.427(0.6758)	0.3299(0.1088)	0.3921(0.1597)	0.3186(0.1015)
				1.75	2.5	0.1051(0.0111)	0.715(0.8943)	0.4963(0.2463)	0.5485(0.4373)	0.4696(0.2205)
40	20	10	1	0.75	1.5	0.0658(0.0043)	0.396(0.2733)	0.3309(0.1095)	0.4075(0.1787)	0.3194(0.1020)
				1.75	2.5	0.0391(0.0015)	0.818(0.9673)	0.5032(0.2532)	0.5583(0.3353)	0.4725(0.2233)
			2	0.75	1.5	0.0823(0.0068)	0.434(0.6182)	0.3694(0.1365)	0.4346(0.2035)	0.3434(0.1180)
				1.75	2.5	0.0950(0.0090)	0.768(0.8546)	0.5332(0.2844)	0.5688(0.3474)	0.4853(0.2355)
		15	1	0.75	1.5	0.0683(0.0047)	0.374(0.4089)	0.3219(0.1036)	0.4002(0.2239)	0.3146(0.0990)
				1.75	2.5	0.0467(0.0022)	0.774(0.8943)	0.4919(0.2420)	0.5469(0.4601)	0.4696(0.2206)
			2	0.75	1.5	0.0838(0.0070)	0.361(0.2206)	0.3476(0.1208)	0.3990(0.1642)	0.3317(0.1101)
				1.75	2.5	0.0983(0.0097)	0.673(0.8214)	0.5147(0.2650)	0.5444(0.2986)	0.4804(0.2308)
	30	15	1	0.75	1.5	0.0684(0.0047)	0.380(0.3225)	0.3221(0.1037)	0.3982(0.2628)	0.3147(0.0990)
				1.75	2.5	0.0463(0.0022)	0.689(0.8275)	0.4918(0.2419)	0.5342(0.2883)	0.4696(0.2205)
			2	0.75	1.5	0.0837(0.0070)	0.370(0.2492)	0.3475(0.1208)	0.4003(0.1641)	0.3317(0.1100)
				1.75	2.5	0.0983(0.0097)	0.693(0.7684)	0.5147(0.2649)	0.5444(0.3000)	0.4804(0.2308)
		23	1	0.75	1.5	0.0732(0.0054)	0.301(0.1011)	0.3132(0.0981)	0.3541(0.1263)	0.3095(0.0958)
				1.75	2.5	0.0607(0.0037)	0.578(0.4972)	0.4791(0.2295)	0.5131(0.2642)	0.4655(0.2167)
			2	0.75	1.5	0.0864(0.0075)	0.327(0.2945)	0.3283(0.1078)	0.3644(0.1340)	0.3204(0.1027)
				1.75	2.5	0.1056(0.0112)	0.585(0.5602)	0.4947(0.2448)	0.5260(0.2886)	0.4738(0.2245)
60	30	15	1	0.75	1.5	0.0680(0.0046)	0.367(0.1931)	0.3233(0.1046)	0.3805(0.1750)	0.3166(0.1002)
				1.75	2.5	0.0433(0.0019)	0.727(0.8832)	0.4963(0.2463)	0.5418(0.3817)	0.4737(0.2244)
			2	0.75	1.5	0.0831(0.0069)	0.416(0.4854)	0.3548(0.1259)	0.4012(0.1833)	0.3386(0.1147)
				1.75	2.5	0.0967(0.0094)	0.730(0.7563)	0.5214(0.2719)	0.5486(0.3060)	0.4867(0.2368)
		23	1	0.75	1.5	0.0689(0.0048)	0.321(0.1539)	0.3214(0.1033)	0.3761(0.1624)	0.3154(0.0995)
				1.75	2.5	0.0468(0.0022)	0.616(0.6121)	0.4913(0.2414)	0.5448(0.6305)	0.4720(0.2228)
			2	0.75	1.5	0.0839(0.0070)	0.323(0.1361)	0.3462(0.1198)	0.3890(0.1717)	0.3333(0.1111)
				1.75	2.5	0.0985(0.0097)	0.825(0.6573)	0.5138(0.2640)	0.5367(0.2892)	0.4840(0.2343)
	45	23	1	0.75	1.5	0.0686(0.0047)	0.313(0.1133)	0.3215(0.1034)	0.3706(0.1440)	0.3155(0.0995)
				1.75	2.5	0.0467(0.0022)	0.632(0.7159)	0.4912(0.2413)	0.5252(0.2827)	0.4719(0.2227)
			2	0.75	1.5	0.0839(0.0070)	0.352(0.2695)	0.3462(0.1199)	0.3844(0.1505)	0.3333(0.1111)
				1.75	2.5	0.0985(0.0097)	0.613(0.6896)	0.5138(0.2640)	0.5384(0.2930)	0.4840(0.2343)
		34	1	0.75	1.5	0.0734(0.0054)	0.282(0.0870)	0.3134(0.0982)	0.3463(0.1221)	0.3102(0.0963)
				1.75	2.5	0.0594(0.0036)	0.563(0.8452)	0.4795(0.2299)	0.5037(0.2540)	0.4674(0.2185)
			2	0.75	1.5	0.0865(0.0075)	0.285(0.0930)	0.3287(0.1081)	0.3562(0.1275)	0.3220(0.1037)
				1.75	2.5	0.1049(0.0110)	0.540(0.4336)	0.4953(0.2453)	0.5139(0.2642)	0.4768(0.2273)

Table 8
The Average Bias (AVB) and Mean Squared Errors (MSEs) in Parentheses for the Burr-XII Parameter γ using Picard and Bayes Methods with m = (n/2 and 3n/4) and k=(m/2 and 3m/4) at T=2 and δ = 2 for LINEX loss.

N	M	K	α	β	γ	Picard's Method	Gamma Prior		Kernel Prior	
							SQEL	LNXL	SQEL	LNXL
20	10	5	1	0.75	1.5	0.0703(0.0050)	0.452(0.4017)	0.3171(0.1005)	0.4179(0.2873)	0.3102(0.0963)
				1.75	2.5	0.0527(0.0029)	0.896(0.7685)	0.4851(0.2354)	0.5536(0.3294)	0.4640(0.2153)
			2	0.75	1.5	0.0848(0.0072)	0.483(0.8004)	0.3365(0.1132)	0.4120(0.1783)	0.3221(0.1038)
				1.75	2.5	0.1015(0.0103)	0.995(0.8796)	0.5037(0.2537)	0.5507(0.3058)	0.4716(0.2224)
		8	1	0.75	1.5	0.0700(0.0049)	0.601(0.8976)	0.3181(0.1012)	0.4167(0.2013)	0.3107(0.0966)
				1.75	2.5	0.0489(0.0024)	0.765(0.6785)	0.4886(0.2387)	0.5731(0.7594)	0.4647(0.2160)
			2	0.75	1.5	0.0841(0.0071)	0.418(0.4050)	0.3421(0.1170)	0.4214(0.1833)	0.3248(0.1055)
				1.75	2.5	0.0999(0.0100)	0.738(0.7854)	0.5092(0.2593)	0.5589(0.3164)	0.4728(0.2236)
	15	8	1	0.75	1.5	0.0701(0.0049)	0.401(0.2422)	0.3173(0.1007)	0.4091(0.2052)	0.3104(0.0963)
				1.75	2.5	0.0527(0.0028)	0.775(0.8954)	0.4852(0.2354)	0.5596(0.4604)	0.4640(0.2153)
			2	0.75	1.5	0.0846(0.0072)	0.460(0.6486)	0.3367(0.1134)	0.4081(0.1717)	0.3222(0.1038)
				1.75	2.5	0.1016(0.0103)	0.745(0.9087)	0.5038(0.2539)	0.5626(0.3906)	0.4716(0.2224)
		11	1	0.75	1.5	0.0727(0.0053)	0.377(0.3869)	0.3136(0.0984)	0.3971(0.2664)	0.3085(0.0952)
				1.75	2.5	0.0598(0.0036)	0.708(0.8954)	0.4796(0.2300)	0.5350(0.2979)	0.4626(0.2140)
			2	0.75	1.5	0.0863(0.0074)	0.403(0.3543)	0.3287(0.1081)	0.3903(0.1544)	0.3181(0.1012)
				1.75	2.5	0.1046(0.0110)	0.709(0.7854)	0.4949(0.2449)	0.5389(0.3022)	0.4693(0.2202)
40	20	10	1	0.75	1.5	0.2103(0.0354)	0.356(0.1780)	0.3153(0.0994)	0.3711(0.1464)	0.3108(0.0966)
				1.75	2.5	0.0554(0.0031)	0.758(0.8943)	0.4824(0.2327)	0.5247(0.2840)	0.4667(0.2178)
			2	0.75	1.5	0.0855(0.0073)	0.404(0.6054)	0.3321(0.1103)	0.3763(0.1442)	0.3229(0.1042)
				1.75	2.5	0.1027(0.0106)	0.697(0.9381)	0.4993(0.2493)	0.5326(0.3067)	0.4756(0.2262)
		15	1	0.75	1.5	0.0716(0.0051)	0.364(0.4275)	0.3154(0.0995)	0.3672(0.1410)	0.3108(0.0966)
				1.75	2.5	0.0553(0.0031)	0.672(0.6537)	0.4824(0.2327)	0.5172(0.2682)	0.4667(0.2178)
			2	0.75	1.5	0.0855(0.0073)	0.371(0.3149)	0.3321(0.1103)	0.3741(0.1417)	0.3229(0.1042)
				1.75	2.5	0.1027(0.0106)	0.679(0.8643)	0.4992(0.2492)	0.5306(0.2936)	0.4756(0.2262)
	30	15	1	0.75	1.5	0.0729(0.0053)	0.358(0.4785)	0.3137(0.0984)	0.3612(0.1462)	0.3098(0.0960)
				1.75	2.5	0.0571(0.0033)	0.703(0.8435)	0.4810(0.2314)	0.5173(0.2685)	0.4662(0.2174)
			2	0.75	1.5	0.0859(0.0074)	0.358(0.1977)	0.3302(0.1091)	0.3732(0.1426)	0.3217(0.1035)
				1.75	2.5	0.1037(0.0108)	0.627(0.5230)	0.4973(0.2473)	0.5257(0.2772)	0.4749(0.2256)
		23	1	0.75	1.5	0.0737(0.0055)	0.321(0.3593)	0.3129(0.0979)	0.3556(0.1291)	0.3093(0.0957)
				1.75	2.5	0.0605(0.0037)	0.565(0.4689)	0.4786(0.2291)	0.5123(0.2632)	0.4654(0.2166)
			2	0.75	1.5	0.0865(0.0075)	0.303(0.1024)	0.3272(0.1071)	0.3644(0.1335)	0.3198(0.1023)
				1.75	2.5	0.1054(0.0111)	0.573(0.5245)	0.4935(0.2436)	0.5197(0.2704)	0.4735(0.2242)
60	30	15	1	0.75	1.5	0.0718(0.0052)	0.305(0.1060)	0.3153(0.0994)	0.3512(0.1242)	0.3116(0.0971)
				1.75	2.5	0.0551(0.0031)	0.594(0.5217)	0.4825(0.2328)	0.5110(0.2620)	0.4687(0.2197)
			2	0.75	1.5	0.0855(0.0073)	0.325(0.2149)	0.3320(0.1103)	0.3645(0.1335)	0.3243(0.1052)
				1.75	2.5	0.1026(0.0105)	0.598(0.6201)	0.4994(0.2494)	0.5215(0.2729)	0.4786(0.2291)
		23	1	0.75	1.5	0.0716(0.0051)	0.332(0.3457)	0.3154(0.0995)	0.3536(0.1288)	0.3116(0.0971)
				1.75	2.5	0.0551(0.0031)	0.599(0.6540)	0.4824(0.2327)	0.5113(0.2628)	0.4687(0.2197)
			2	0.75	1.5	0.0855(0.0073)	0.327(0.3408)	0.3321(0.1103)	0.3674(0.1529)	0.3244(0.1052)
				1.75	2.5	0.1027(0.0105)	0.641(0.7648)	0.4995(0.2495)	0.5212(0.2721)	0.4787(0.2291)
	45	23	1	0.75	1.5	0.0737(0.0054)	0.317(0.1225)	0.3130(0.0980)	0.3436(0.1184)	0.3100(0.0961)
				1.75	2.5	0.0585(0.0034)	0.612(0.7994)	0.4799(0.2303)	0.5059(0.2562)	0.4676(0.2187)
			2	0.75	1.5	0.0862(0.0074)	0.358(0.6537)	0.3287(0.1080)	0.3583(0.1292)	0.3220(0.1037)
				1.75	2.5	0.1043(0.0109)	0.632(0.6573)	0.4953(0.2454)	0.5157(0.2662)	0.4769(0.2274)
		34	1	0.75	1.5	0.0735(0.0054)	0.273(0.0768)	0.3129(0.0979)	0.3438(0.1186)	0.3099(0.0961)
				1.75	2.5	0.0596(0.0036)	0.534(0.3298)	0.4791(0.2295)	0.5028(0.2534)	0.4673(0.2183)
			2	0.75	1.5	0.0864(0.0075)	0.323(0.4563)	0.3277(0.1074)	0.3551(0.1263)	0.3214(0.1033)
				1.75	2.5	0.1050(0.0110)	0.618(0.7853)	0.4943(0.2443)	0.5167(0.2689)	0.4764(0.2270)

APPENDIX A

For the Burr type-XII model the likelihood function of the GPHCS (3) and its derivatives can be derived as

$$H(\alpha,\beta,\gamma) = N\,ln(\alpha) + N\,ln(\beta) + N\,ln(\gamma) - \sum_{i=1}^{N} ln(1+\beta x_i^{\alpha}) + (\alpha-1)\sum_{i=1}^{N} ln(x_i) - \gamma\left[\sum_{i=1}^{N}(1+R_i)\,ln(1+\beta x_i^{\alpha}) + \delta R_T^{*}\,ln(1+\beta T^{\alpha})\right].$$

$$\frac{\partial H}{\partial\alpha} = N/\alpha + \sum_{i=1}^{N} lnx_i - \sum_{i=1}^{N}\frac{\beta {x_i}^{\alpha}\,ln\,x_i}{1+\beta {x_i}^{\alpha}} - \gamma\left[\sum_{i=1}^{N}(1+R_i)\frac{\beta {x_i}^{\alpha}\,ln\,x_i}{1+\beta {x_i}^{\alpha}} + \frac{\beta T^{\alpha}\,ln\,T}{1+\beta T^{\alpha}}\right]$$

$$\frac{\partial^2 H}{\partial\alpha\partial x} = \sum_{i=1}^{N} 1/x_i - \sum_{i=1}^{N}\frac{(1+\beta {x_i}^{\alpha})[\alpha\beta {x_i}^{\alpha-1}\,ln\,x_i + \beta x_i^{\alpha-1}] - \alpha\beta^2 x_i^{\alpha-1} lnx_i}{(1+\beta {x_i}^{\alpha})^2} - \gamma\left[\sum_{i=1}^{N}(1+R_i)\frac{(1+\beta {x_i}^{\alpha})[\alpha\beta {x_i}^{\alpha-1}\,ln\,x_i + \beta x_i^{\alpha-1}] - \alpha\beta^2 x_i^{\alpha-1} lnx_i}{(1+\beta {x_i}^{\alpha})^2} + \delta R_T^{*}\frac{(1+\beta T^{\alpha})[\alpha\beta T^{\alpha-1}\,ln\,T + \beta T^{\alpha-1}] - \alpha\beta^2 T^{\alpha} lnT}{(1+\beta T^{\alpha})^2}\right]$$

$$\frac{\partial^2 H}{\partial\alpha^2} = \left[-N/\alpha^2 - \sum_{i=1}^{N}\frac{(1+\beta {x_i}^{\alpha})\beta {x_i}^{\alpha}(ln\,x_i)^2 - (\beta {x_i}^{\alpha}\,ln\,x_i)^2}{(1+\beta {x_i}^{\alpha})^2} - \gamma\left[\sum_{i=1}^{N}(1+R_i)\frac{(1+\beta {x_i}^{\alpha})\beta {x_i}^{\alpha}(ln\,x_i)^2 - (\beta {x_i}^{\alpha}\,ln\,x_i)^2}{(1+\beta {x_i}^{\alpha})^2} + \delta R_T^{*}\frac{(1+\beta T^{\alpha})\beta T^{\alpha}(ln\,T)^2 - (\beta T^{\alpha}\,ln\,T)^2}{(1+\beta T^{\alpha})^2}\right]\right]$$

$$\frac{\partial H}{\partial\beta} = N/\beta - \sum_{i-1}^{N}\frac{x_i^{\alpha}}{1+\beta x_i^{\alpha}} - \gamma\left[\sum_{i-1}^{N}(1+R_i)\frac{x_i^{\alpha}}{1+\beta x_i^{\alpha}} + \delta R_T^{*}\frac{T^{\alpha}}{1+\beta T^{\alpha}}\right]$$

$$\frac{\partial^2 H}{\partial\beta\partial x} = -\sum_{i-1}^{N} \frac{(1+\beta x_i{}^{\alpha})\alpha x_i^{\alpha-1} - \alpha\beta x_i^{2\alpha-1}}{(1+\beta x_i{}^{\alpha})^2} - \gamma\left[\sum_{i-1}^{N}(1+R_i)\frac{(1+\beta x_i{}^{\alpha})\alpha x_i^{\alpha-1} - \alpha\beta x_i^{2\alpha-1}}{(1+\beta x_i{}^{\alpha})^2} + \delta R_T^* \frac{(1+\beta T^{\alpha})\alpha T^{\alpha-1} - \alpha\beta T^{2\alpha-1}}{(1+\beta x_i{}^{\alpha})^2}\right]$$

$$\frac{\partial^2 H}{\partial\beta^2} = -N/\beta^2 + \sum_{i-1}^{N}\frac{x_i^{2\alpha}}{(1+\beta x_i^{\alpha})^2} + \gamma\left[\sum_{i-1}^{N}(1+R_i)\frac{x_i^{2\alpha}}{(1+\beta x_i^{\alpha})^2} + \delta R_T^* \frac{T^{2\alpha}}{(1+\beta T^{\alpha})^2}\right]$$

$$\frac{\partial H}{\partial\gamma} = \frac{N}{\gamma} - \left[\sum_{i=1}^{N}(1+R_i) ln\,(1+\beta x_i^{\alpha}) + \delta R_T^* ln(1+\beta T^{\alpha})\right]$$

$$\frac{\partial^2 H}{\partial\gamma\partial x} = \sum_{i=1}^{N}(1+R_i)\frac{\alpha\beta x_i^{\alpha-1}}{1+\beta x_i^{\alpha}}, \frac{\partial^2 H}{\partial\gamma^2} = -\frac{N}{\gamma^2}\ .$$

APPENDIX B

$$H(\alpha,\beta,\gamma)=\Bigg[p_1\,ln\big(\hat{g}_1(\alpha)\big)+p_2\,ln\big(\hat{g}_2(\beta)\big)+p_3\,ln\big(\hat{g}_3(\gamma)\big)+(N+a-1)\,ln(\alpha)+(N+c-1)\,ln(\beta)+(N+e-1)\,ln(\gamma)-b\alpha-d\beta-f\gamma-\sum_{i=1}^{N}ln(1+\beta x_i^{\alpha})+(\alpha-1)\sum_{i=1}^{N}ln(x_i)-\frac{\gamma\left[\sum_{i=1}^{N}(1+R_i)\,ln(1+\beta x_i^{\alpha})+\delta R_T^*\,ln(1+\beta T^{\alpha})\right]}{N}\Bigg].$$

$$\frac{\partial H}{\partial\alpha}=\left[p_1\frac{\hat{g}_1'(\alpha)}{\hat{g}_1(\alpha)}+(N+a-1)/\alpha-b+\sum_{i=1}^{N}lnx_i-\sum_{i=1}^{N}\frac{\beta x_i{}^{\alpha}\,ln\,x_i}{1+\beta x_i{}^{\alpha}}-\gamma\left(\sum_{i=1}^{N}(1+R_i)\frac{\beta x_i{}^{\alpha}\,ln\,x_i}{1+\beta x_i{}^{\alpha}}+\frac{\beta T^{\alpha}\,ln\,T}{1+\beta T^{\alpha}}\right)\right]/N$$

$$\frac{\partial^2 H}{\partial\alpha^2}=\Bigg[p_1\frac{\hat{g}_1(\alpha)\hat{g}_1''(\alpha)-\hat{g}'^{2}_{1}(\alpha)}{\hat{g}_1^2(\alpha)}-(N+a-1)/\alpha^2-\sum_{i=1}^{N}\frac{(1+\beta x_i{}^{\alpha})\beta x_i{}^{\alpha}(ln\,x_i)^2-(\beta x_i{}^{\alpha}\,ln\,x_i)^2}{(1+\beta x_i{}^{\alpha})^2}-\gamma\left[\sum_{i=1}^{N}(1+R_i)\frac{(1+\beta x_i{}^{\alpha})\beta x_i{}^{\alpha}(ln\,x_i)^2-(\beta x_i{}^{\alpha}\,ln\,x_i)^2}{(1+\beta x_i{}^{\alpha})^2}+\delta R_T^*\frac{(1+\beta T^{\alpha})\beta T^{\alpha}(ln\,T)^2-(\beta T^{\alpha}\,ln\,T)^2}{(1+\beta T^{\alpha})^2}\right]\Bigg]/N$$

$$\frac{\partial^2 H}{\partial\alpha\partial\beta}=\Bigg[-\sum_{i=1}^{N}\frac{(1+\beta x_i{}^{\alpha})x_i{}^{\alpha}\,ln\,x_i-\beta x_i{}^{2\alpha}\,ln\,x_i}{(1+\beta x_i{}^{\alpha})^2}-\gamma\left[\sum_{i=1}^{N}(1+R_i)\frac{(1+\beta x_i{}^{\alpha})x_i{}^{\alpha}\,ln\,x_i-\beta x_i{}^{2\alpha}\,ln\,x_i}{(1+\beta x_i{}^{\alpha})^2}+\delta R_T^*\frac{(1+\beta T^{\alpha})T^{\alpha}\,ln\,T-\beta T^{2\alpha}\,ln\,T}{(1+\beta T^{\alpha})^2}\right]\Bigg]/N$$

$$\frac{\partial H}{\partial \beta}=\left[p_2\frac{\hat{g}_2'(\beta)}{\hat{g}_2(\beta)}+(N+c-1)/\beta-d-\sum_{i-1}^{N}\frac{x_i^{\alpha}}{1+\beta x_i^{\alpha}}\right.$$
$$\left.-\gamma\left[\sum_{i-1}^{N}(1+R_i)\frac{x_i^{\alpha}}{1+\beta x_i^{\alpha}}+\delta R_T^*\frac{T^{\alpha}}{1+\beta T^{\alpha}}\right]\right]/N$$

$$\frac{\partial^2 H}{\partial \beta^2}=\left[p_2\frac{\hat{g}_2(\beta)g_2''(\beta)-\hat{g}_2'^{\,2}(\beta)}{\hat{g}_2^{\,2}(\beta)}-(N+c-1)/\beta^2+\sum_{i-1}^{N}\frac{x_i^{2\alpha}}{(1+\beta x_i^{\alpha})^2}\right.$$
$$\left.+\gamma\left[\sum_{i-1}^{N}(1+R_i)\frac{x_i^{2\alpha}}{(1+\beta x_i^{\alpha})^2}+\delta R_T^*\frac{T^{2\alpha}}{(1+\beta T^{\alpha})^2}\right]\right]/N.$$

$$\frac{\partial H}{\partial \gamma}=\left[p_3\frac{\hat{g}_3'(\gamma)}{\hat{g}_3(\gamma)}+\frac{D+e-1}{\gamma}-f-\left[\sum_{i=1}^{N}(1+R_i)ln\,(1+\beta x_i^{\alpha})+\delta R_T^* ln(1+\beta T^{\alpha})\right]\right]/N$$

$$\frac{\partial^2 H}{\partial \gamma^2}=\left[p_3\frac{\hat{g}_3(\gamma)g_3''(\gamma)-\hat{g}_3'^{\,2}(\gamma)}{\hat{g}_3^{\,2}(\gamma)}-(N+e-1)/\gamma^2\right]/N$$

$$\frac{\partial H}{\partial \gamma\partial \alpha}=-\left[\sum_{i=1}^{N}(1+R_i^*)\frac{\beta x_i{}^{\alpha}\,ln\,x_i}{1+\beta x_i{}^{\alpha}}+\delta R_T^*\frac{\beta T^{\alpha}\,ln\,T}{1+\beta T^{\alpha}}\right]/N$$

$$\frac{\partial H}{\partial \gamma\partial \beta}=-\left[\sum_{i=1}^{N}(1+R_i^*)\frac{x_i{}^{\alpha}}{1+\beta x_i{}^{\alpha}}+\delta R_T^*\frac{T^{\alpha}}{1+\beta T^{\alpha}}\right]/N$$

where the rth derivative of the kernel density estimation can be defined as

$$\frac{d^r\hat{g}_1(\alpha)}{d\alpha^r}=\hat{g}_1{}^r(\alpha)=\frac{1}{Nh_1^{r+1}}\sum_{i=1}^{N}K^r\left(\frac{\alpha-\hat{\alpha}_i}{h_1}\right),\text{where } r=0,1,2,3,---. \quad (11)$$

Using the Gaussian kernel and (11), we have

$$\hat{g}_1(\alpha)=\frac{1}{Nh_1\sqrt{2\pi}}\sum_{i=1}^{N}e^{-0.5\left(\frac{\alpha-\hat{\alpha}_i}{h_1}\right)^2},$$

$$\hat{g}_1'(\alpha)=-\frac{1}{Nh_1^2\sqrt{2\pi}}\sum_{i=1}^{N}\left(\frac{\alpha-\hat{\alpha}_i}{h_1}\right)e^{-0.5\left(\frac{\alpha-\hat{\alpha}_i}{h_1}\right)^2}$$

$$\hat{g}_1''(\alpha)=\frac{1}{Nh_1^3\sqrt{2\pi}}\sum_{i=1}^{N}\left[\left(\frac{\alpha-\alpha_i}{h_1}\right)^2-1\right]e^{-0.5\left(\frac{\alpha-\alpha_i}{h_1}\right)^2}.$$

Similarly for the kernel priors $\hat{g}_2(\beta)$ and $\hat{g}_3(\gamma)$.