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BAYESIAN PARAMETER ESTIMATION OF POWER FUNCTION DISTRIBUTION UNDER DIFFERENT LOSS FUNCTIONS

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1. ABSTRACT

In statistical inference and real world, the Bayesian analysis is a modern parameter estimation technique. We have used power function distribution (PFD) and informative prior (Gamma Distribution) to find the Bayes estimators of parameter ϕ under various loss functions: The square error loss function (SELF), the quadratic error loss function (QELF), the weighted square error loss function (WSELF), the precautionary error loss function (PELF), the K loss function (KLF), the entropy error loss function (EELF), the modified linear exponential error loss function (MLINEXELF) and the non-linear exponential error loss function (NLINEXLF). Furthermore, we compared the Bayes Estimators with the classical Maximum Likelihood Estimator (MLE) to evaluate their performance in terms of loss functions. Finally, the results have been shown graphically by using R software.

2. INTRODUCTION

The power function distribution has a wide range of applications across various fields, including statistics, reliability engineering, medical research, epidemiology, environmental science, geochemical data analysis, physical processes, wavelet analysis, and data transformation (Klein and Moeschberger, 2003) and (Rinne, 2008). It is used to model time to failure, survival times, reliability of components, spread of diseases, environmental variables, geochemical variables, physical processes, and wavelet coefficients (Johnson, Kotz and Balakrishnan, 1994). Additionally, it is employed as a data transformation technique to stabilize variance, make data more normal distribution-like, and improve the validity of measures of association (Box and Cox, 1964). Its flexibility and simplicity make it a valuable tool in many applications. Barray and Meniconi (Meniconi and Barry, 1996) computed the reliance of electrical components by contrasted the PFD with Weibull distribution, log normal distribution and Exponential distribution. The PFD is one of the most familiar distributions used to model real data sets in different areas containing but not restricted to life time, income, industry and environment (Alodat, Al-Rawwash and Al-Subh, 2018). Data analysis requires a reliable lifetime model which deal with real life problems in multidimensional areas. Univariate distribution has much importance to resolve these problems base on the reliability of the product. Among these univariate distribution PFD is a reasonably tractable model to evaluate the reliability of real-life data such as electrical components including semiconductor devices. The PFD is the special case of the Uniform and Beta distributions [Sultan, Childs and Balakrishnan (2000) and Hassan and Assar (2021)]. Statistical distributions have long been used to analyze the dependability of semiconductor devices and products. Most engineers employ the exponential distribution, which is usually chosen over mathematically more complicated distributions such as the Weibull and log normal, among others. As a result, it is suggested that the power function distribution be examined as a simpler option that, in some cases, may give a better match for failure data and more relevant information regarding failure (Kifayat, Aslam and Ali (2012). In the Bayesian approach, the loss function is used to evaluate the parameter estimation. The idea of the Loss function was first time introduced by Laplace and redefined by Abraham Wald in the middle of the 20^{th} century. The cost function or loss function is a function in decision theory and mathematical optimization that draw values or an event of one or more variables into a real number automatic representing some “cost” connected with the values or event and used for parameter estimation in Statistics (Wald, 1949). Han (2020) discussed the reliability and E-posterior risk of E-Bayesian estimations under various loss functions such as SELF, WSELF, PELF and K loss function with binomial distribution. Hasan and Baizid (2017) derived posterior distribution of Exponential distribution with informative Gamma prior. Further find Bayes estimators under SELF, QELF, MLINEXLF and NLINEXLF.

3. PRIOR AND POSTERIOR DENSITY FUNCTION OF PARAMETER ϕ

3.1 Power Function Distribution

The power function distribution was first introduced by Dallas, who developed a relationship between the Pareto and power function variables through an inverse transformation (Dallas, 1976). Let Y be a random variable having the following p.d.f. $f(y)$ for a power function distribution (PFD) with unknown parameter ϕ:

$$f(y) = \begin{cases} \phi y^{\phi-1} & ;\ 0 < y < 1,\ 0 < \phi < \infty, \\ 0 & ;\ otherwise. \end{cases}$$

3.2 Gamma Distribution

The Gamma distribution in Statistics and probability theory is a two-parameter family and widely used distribution. The Chi-square, Erlang and Exponential distributions are the applications of the Gamma distribution. The Gamma prior is also called conjugate prior (Raiffa and Schlaifer, 1961). Suppose gamma prior with parameter ϕ is

$$f(\phi) = \frac{b^a}{\Gamma(a)} (\phi)^{a-1} e^{-b\phi};\quad \phi > 0,\ a, b > 0.$$

By taking likelihood of Power Function Distribution

$$L(y, \phi) = \phi^n e^{(\phi-1)\sum_{i=1}^{n} \ln(y_i)}$$

$$f(\phi/y)=\frac{f(\phi).L(y,\phi)}{\int_{-\infty}^{\infty} f(\phi).L(y,\phi)d\phi}$$

$$\Rightarrow f(\phi/y)=\frac{(\phi)^{a+n-1}e^{-b\phi+\phi\sum_{i=1}^{n}\ln(y_i)}}{\int_{0}^{\infty}(\phi)^{a+n-1}e^{-b\phi+\phi\sum_{i=1}^{n}\ln(y_i)}d\phi}$$

$$\Rightarrow f(\phi/y)=\frac{\beta^{\alpha}}{\Gamma(\alpha)}\phi^{\alpha-1}e^{-\beta\phi}$$

$$f(\phi/y)=\text{Gamma}(\alpha,\beta)$$

$$\Rightarrow f(\phi/y)=\text{Gamma}\left(a+n,b-\sum_{i=1}^{n}\ln(y_i)\right).$$

4. DIFFERENT ESTIMATORS OF PARAMETER ϕ

Here, Bayes Estimators of ϕ for different loss functions along with maximum likelihood estimator has been determined.

4.1 Maximum likelihood Estimator (MLE) of Parameter ϕ

The most essential analytical approach for determining the parameters of any probability distribution is maximum likelihood estimation. Suppose $Y_1,Y_2,Y_3,...,Y_n$ be random sample from power function distribution. Let $y_1,y_2,y_3,...,y_n$ is the observed value of $Y_1,Y_2,Y_3,...,Y_n$. Then the likelihood function of ϕ based on $y_1,y_2,y_3,...,y_n$ is given by (Mood, 1950)

$$L(y,\phi)=\phi^{n}e^{(\phi-1)\sum_{i=1}^{n}\ln(y_i)}$$

By taking *ln* on both sides

$$\Rightarrow \ln L(y,\phi)=n\ln\phi+\phi\sum_{i=1}^{n}\ln(y_i)-\sum_{i=1}^{n}\ln(y_i)$$

Differentiate it w.r.t ϕ

$$\frac{\partial}{\partial\phi}[\ln L(y,\phi)]=\frac{n}{\phi}+\sum_{i=1}^{n}\ln(y_i)$$

Put

$$\frac{\partial}{\partial\phi}[\ln L(y,\phi)]=0$$

$$\Rightarrow \phi = \frac{-n}{\sum_{i=1}^{n} \ln(y_i)}$$

Again, differentiating we have

$$\frac{\partial^2}{\partial \phi^2}[\ln L(y,\phi)] = \frac{-n}{\phi^2}$$

$$\frac{\partial^2}{\partial \phi^2}[\ln L(y,\phi)] = \frac{-\left(\sum_{i=1}^{n} \ln(y_i)\right)^2}{n} < 0$$

Hence

$$\hat{\phi} = \frac{-n}{\sum_{i=1}^{n} \ln(y_i)}$$

is the Maximum likelihood Estimator (MLE) of parameter ϕ.

4.2 Bayes Estimator of Parameter ϕ for Square Error Loss Function (SELF)

The Square Error Loss Function (SELF), Singh, Singh and Kumar (2011) is defined by

$$L(\phi,\hat{\phi}) = (\phi - \hat{\phi})^2$$

$$R(\phi,\hat{\phi}) = E_{\phi/y}\left[L(\phi,\hat{\phi})\right]$$

$$R(\phi,\hat{\phi}) = E_{\phi/y}\left(\phi^2 - 2\phi\hat{\phi} + \hat{\phi}^2\right)$$

Differentiate it w.r.t $\hat{\phi}$

$$\frac{\partial}{\partial \hat{\phi}}\left[R(\phi,\hat{\phi})\right] = 2\hat{\phi} - 2E_{\phi/y}(\phi)$$

Put

$$\frac{\partial}{\partial \hat{\phi}}\left[R(\phi,\hat{\phi})\right] = 0$$

$$\Rightarrow 2\hat{\phi} - 2E_{\phi/y}(\phi) = 0$$

$$\Rightarrow \hat{\phi} = E_{\phi/y}(\phi)$$

Now

$$E_{\phi/y}(\phi) = \int_{-\infty}^{\infty} \phi f(\phi/y)\, d\phi$$

$$E_{\phi/y}(\phi)=\int_0^\infty \phi.\frac{\beta^\alpha}{\Gamma(\alpha)}\phi^{\alpha-1}e^{-\beta\phi}d\phi$$

$$\Rightarrow E_{\phi/y}(\phi)=\frac{\alpha}{\beta}$$

$$\Rightarrow \hat{\phi}=\frac{a+n}{b-\sum_{i=1}^{n}\ln(y_i)}$$

i.e., Bayes estimator under SELF.

4.3 Bayes Estimator of Parameter ϕ for Quadratic Error Loss Function (QELF)

The Quadratic Error Loss Function, Bhuiyan, Roy and Imam (2007) is defined as

$$L(\phi,\hat{\phi})=\left[1-\frac{\hat{\phi}}{\phi}\right]^2$$

$$\Rightarrow R(\phi,\hat{\phi})=E_{\phi/y}\left[1-\frac{\hat{\phi}}{\phi}\right]^2$$

$$\Rightarrow \frac{\partial}{\partial\hat{\phi}}\left[R(\phi,\hat{\phi})\right]=-2E_{\phi/y}\left(\frac{1}{\phi}\right)+2\hat{\phi}E_{\phi/y}\left(\frac{1}{\phi^2}\right)$$

Put

$$\frac{\partial}{\partial\hat{\phi}}\left[R(\phi,\hat{\phi})\right]=0$$

$$\Rightarrow \hat{\phi}=\frac{E_{\phi/y}\left(\phi^{-1}\right)}{E_{\phi/y}\left(\phi^{-2}\right)}$$

Now

$$E_{\phi/y}(\phi)=\int_{-\infty}^{\infty}\phi f(\phi/y)\,d\phi$$

$$\Rightarrow E_{\phi/y}\left(\phi^{-1}\right)=\frac{\beta}{\alpha-1}$$

and similarly

$$E_{\phi/y}\left(\phi^{-2}\right)=\frac{\beta^2}{(\alpha-1)(\alpha-2)}$$

Hence Bayes estimator under QELF is

$$\hat{\phi}=\frac{a+n-2}{b-\sum_{i=1}^{n}\ln(y_i)}$$

4.4 Bayes Estimator of Parameter ϕ for Weighted Square Error Loss Function (WSELF)

The weighted square error loss function (WSELF), Berger (2013) is defined as

$$W(\phi,\hat{\phi})=\frac{\left(\phi-\hat{\phi}\right)^2}{\phi}$$

$$R(\phi,\hat{\phi})=E_{\phi/y}\left[\frac{\left(\phi-\hat{\phi}\right)^2}{\phi}\right]$$

$$\Rightarrow R(\phi,\hat{\phi})=E_{\phi/y}\left[\phi-2\hat{\phi}+\frac{\hat{\phi}^2}{\phi}\right]$$

$$\frac{\partial}{\partial\hat{\phi}}\left[R(\phi,\hat{\phi})\right]=-2+2\hat{\phi}E_{\phi/y}\left(\phi^{-1}\right)$$

Put

$$\frac{\partial}{\partial\hat{\phi}}\left[R(\phi,\hat{\phi})\right]=0$$

$$\Rightarrow \hat{\phi}=\frac{1}{E_{\phi/y}\left(\phi^{-1}\right)}$$

Now

$$\hat{\phi}=\frac{\alpha-1}{\beta}$$

Hence Bayes estimator under WSELF is

$$\hat{\phi}=\frac{a+n-1}{b-\sum_{i=1}^{n}\ln(y_i)}.$$

4.5 Bayes Estimator of Parameter ϕ for Precautionary Error Loss Function (PELF)

The Precautionary error loss function (PELF), Norstrom (1996) is given by

$$L(\phi,\hat{\phi})=\frac{\left(\phi-\hat{\phi}\right)^2}{\hat{\phi}}$$

$$R(\phi,\hat{\phi})=E_{\phi/y}\left[\frac{\phi^2+\hat{\phi}^2-2\phi\hat{\phi}}{\hat{\phi}}\right]$$

$$\Rightarrow \hat{\phi}=\sqrt{E_{\phi/y}\left(\phi^2\right)}$$

Now

$$\hat{\phi} = \sqrt{\frac{\alpha(\alpha+1)}{\beta^2}}$$

Hence Bayes estimator under PELF is

$$\hat{\phi} = \sqrt{\frac{(a+n)(a+n+1)}{(b-\sum_{i=1}^{n}\ln(y_i))^2}}\,.$$

4.6 Bayes Estimator of Parameter ϕ for Entropy Error Loss Function (EELF)

The entropy error loss function, James, Stein and Neyman (1961) is given by

$$L(\phi,\hat{\phi}) = \frac{\hat{\phi}}{\phi} - \ln\left(\frac{\hat{\phi}}{\phi}\right) - 1$$

$$R(\phi,\hat{\phi}) = E_{\phi/y}\left[\frac{\hat{\phi}}{\phi} - \ln\left(\frac{\hat{\phi}}{\phi}\right) - 1\right]$$

Differentiate it w.r.t $\hat{\phi}$

$$\frac{\partial}{\partial\hat{\phi}}\left[R(\phi,\hat{\phi})\right] = E_{\phi/y}\left[\frac{1}{\phi} - \frac{1}{\hat{\phi}}\right]$$

Put

$$\frac{\partial}{\partial\hat{\phi}}\left[R(\phi,\hat{\phi})\right] = 0$$

$$\Rightarrow \quad \hat{\phi} = \frac{1}{E_{\phi/y}\left(\phi^{-1}\right)}$$

Now

$$\hat{\phi} = \frac{\alpha-1}{\beta}$$

Hence Bayes estimator under EELF is

$$\hat{\phi} = \frac{a+n-1}{b-\sum_{i=1}^{n}\ln(y_i)}\,.$$

4.7 Bayes Estimator of Parameter ϕ for K Loss Function

The K loss function, Wasan (1970) is defined by

$$K(\phi,\hat{\phi})=\left[\sqrt{\frac{\hat{\phi}}{\phi}}-\sqrt{\frac{\phi}{\hat{\phi}}}\right]^2$$

Differentiate it w.r.t $\hat{\phi}$

$$\Rightarrow \frac{\partial}{\partial\hat{\phi}}\left[R(\phi,\hat{\phi})\right]=E_{\phi/y}\left(\phi^{-1}\right)-E_{\phi/y}\left(\phi\right).\frac{1}{\hat{\phi}^2}$$

$$\Rightarrow \hat{\phi}=\sqrt{\frac{E_{\phi/y}\left(\phi\right)}{E_{\phi/y}\left(\phi^{-1}\right)}}$$

Hence Bayes estimator under KLF is

$$\Rightarrow \hat{\phi}=\sqrt{\frac{(a+n)(a+n-1)}{\left(b-\sum_{i=1}^{n}\ln(y_i)\right)^2}}\,.$$

4.8 Bayes Estimator of Parameter ϕ for Modified Linear Exponential Loss Function

The modified linear exponential loss function, Debnath et al. (2021) is given by

$$L(\phi,\hat{\phi})=W\left[\left(\frac{\hat{\phi}}{\phi}\right)^c-c\ln\left(\frac{\hat{\phi}}{\phi}\right)-1\right],\qquad W>0,\ c\neq 0$$

where W and c are parameters.

$$R(\phi,\hat{\phi})=E_{\phi/y}\left[W\left\{\hat{\phi}^c.\phi^{-c}-c\ln\hat{\phi}+c\ln\phi-1\right\}\right]$$

$$\Rightarrow \frac{\partial}{\partial\hat{\phi}}\left[R(\phi,\hat{\phi})\right]=W\left[c\left(\hat{\phi}\right)^{c-1}.\,E_{\phi/y}\left(\phi^{-c}\right)-\frac{c}{\hat{\phi}}\right]$$

$$\Rightarrow \hat{\phi}=\left[E_{\phi/y}\left(\phi^{-c}\right)\right]^{\frac{-1}{c}}$$

Hence Bayes estimator under MLINEXLF is

$$\hat{\phi}=\left[\frac{\Gamma\left(a+n\right)}{\Gamma\left(a+n-c\right)}\right]^{\frac{1}{c}}.\frac{1}{b-\sum_{i=1}^{n}\ln(y_i)}$$

4.9 Bayes Estimator of Parameter ϕ for Non-Linear Exponential Loss Function

The non-linear exponential loss function, Islam, Roy and Ali (2004) is given by

$$L(\phi,\hat{\phi})=K\left[e^{c\left(\hat{\phi}-\phi\right)}+c\left(\hat{\phi}-\phi\right)^2-c\left(\hat{\phi}-\phi\right)-1\right],\quad K>0,\ c>0$$

where K and c are parameters.

$$R(\phi,\hat{\phi})=E_{\theta/x}\left[K\left\{e^{c\left(\hat{\phi}-\phi\right)}+c\left(\hat{\phi}-\phi\right)^2-c\left(\hat{\phi}-\phi\right)-1\right\}\right]$$

Differentiate it w.r.t ϕ

$$\frac{\partial}{\partial\hat{\phi}}\left[R(\phi,\hat{\phi})\right]=K\left[ce^{c\hat{\phi}}.\,E_{\phi/y}\left(e^{-c\phi}\right)+2c\left(\hat{\phi}\right)-2cE_{\phi/y}\left(\phi\right)-c\right]$$

Put

$$\frac{\partial}{\partial\hat{\phi}}\left[R(\phi,\hat{\phi})\right]=0$$

$$\Rightarrow\ K\left[ce^{c\hat{\phi}}.\,E_{\phi/y}\left(e^{-c\phi}\right)+2c\left(\hat{\phi}\right)-2cE_{\phi/y}\left(\phi\right)-c\right]=0$$

$$\Rightarrow\ e^{c\hat{\phi}}.\,E_{\phi/y}\left(e^{-c\phi}\right)=1-2E_{\phi/y}\left(\hat{\phi}-\phi\right)$$

By taking *ln* on both sides

$$\Rightarrow\ \ln\left[e^{c\hat{\phi}}.\,E_{\phi/y}\left(e^{-c\phi}\right)\right]=\ln\left[1-2E_{\phi/y}\left(\hat{\phi}-\phi\right)\right]$$

As

$$\ln\left(1-t\right)\approx -t$$

$$\Rightarrow\ t=2E_{\phi/y}\left(\hat{\phi}-\phi\right)$$

$$\Rightarrow\ \ln\left(E_{\phi/y}\left(e^{-c\phi}\right)\right)+\ln\left(e^{c\hat{\phi}}\right)=-2E_{\phi/y}\left(\hat{\phi}-\phi\right)$$

$$\Rightarrow\ \hat{\phi}=\frac{1}{c+2}\left[2E_{\phi/y}\left(\phi\right)-\ln\left(E_{\phi/y}\left(e^{-c\phi}\right)\right)\right]$$

Hence Bayes estimator under NLINEXLF is

$$E_{\phi/y}\left(e^{-c\phi}\right)=\left(1+\frac{c}{\beta}\right)^{-\alpha}$$

$$\Rightarrow\ \hat{\phi}=\frac{a+n}{c+2}\left[\frac{2}{b-\sum_{i=1}^{n}\ln(y_i)}+\ln\left(1+\frac{c}{b-\sum_{i=1}^{n}\ln(y_i)}\right)\right]$$

5. RESULTS AND DISCUSSION

In this research we have discussed the Bayes Estimators of different loss functions under BEL (EELF), BKL (KLF), BML (MLINEXLF), BNL (NLINEXLF), BPL (PELF), BQL (QELF), BSL (SELF) and BWL (WSELF). In order to compare Bayes estimators, we have considered mean square error (MSE) is defined as

$$MSE\left(\hat{\phi}\right) = E\left[\hat{\phi} - \phi\right]^2 = Var\left(\hat{\phi}\right) + \left[Bias\left(\hat{\phi}\right)\right]^2 .$$

We have used R software to obtained MSE and Mean from Power function distribution. In our study 6000 samples have generated for every case. We have considered random sample sizes 5, 10, 15, 20, 25, 30, 35 in our study. The results and graphs are presented below. In Table 1 we perceived that the MSE of MLE and BWL gives high and low values at small sample size respectively but they are indistinguishable with others at large sample size (Figure 1). From Table 2 we noticed that the MSE of MLE is high at small sample size. Furthermore, BNL give least value for small sample size. For some cases the MSE of BEL, BPL and BWL give same values. Also at large sample size the Bayes estimators shows equally likely behaviors except classical estimator MLE (Figure 2). Table 3 indicates that BQL gives least value of MSE for small sample size. Also, MLE shows same behavior as above cases. Among at large sample size all the estimators are similar (Figure 3). Table 4 discuss the discrepancy of estimators for different sample sizes. The MSE of BPL is minimum for small sample but equally likely with others at large sample size. Also, MLE keeps same behaviors as previous cases (Figure 4). From Table 5 it is clear that BNL give smaller value of MSE for small sample size but it is identical with others for large sample size (Figure 5).

Table 1
Bayes Estimates at Different Loss Functions a = 0.5, b =1, ϕ = 1, c = 1

N	Criteria	BEL	BKL	MLE	BML	BNL	BPL	BQL	BSL	BWL
5	Mean	0.868	0.960	1.238	0.867	1.016	1.153	0.670	1.048	0.869
	MSE	0.140	0.146	0.551	0.139	0.151	0.233	0.178	0.170	0.135
10	Mean	0.938	0.982	1.117	0.943	1.023	1.090	0.836	1.038	0.943
	MSE	0.086	0.091	0.168	0.088	0.093	0.126	0.094	0.104	0.088
15	Mean	0.963	0.995	1.069	0.964	1.019	1.062	0.898	1.029	0.961
	MSE	0.059	0.064	0.094	0.062	0.066	0.079	0.063	0.069	0.061
20	Mean	0.969	0.999	1.053	0.968	1.020	1.046	0.918	1.025	0.971
	MSE	0.046	0.049	0.062	0.047	0.049	0.054	0.049	0.053	0.047
25	Mean	0.982	1.001	1.042	0.978	1.014	1.036	0.933	1.018	0.980
	MSE	0.038	0.039	0.050	0.038	0.040	0.042	0.039	0.040	0.038
30	Mean	0.981	0.999	1.034	0.983	1.007	1.034	0.948	1.016	0.980
	MSE	0.032	0.034	0.039	0.033	0.033	0.036	0.033	0.033	0.032
35	Mean	0.985	0.997	1.026	0.982	1.005	1.030	0.955	1.013	0.984
	MSE	0.028	0.027	0.032	0.028	0.027	0.030	0.027	0.029	0.027

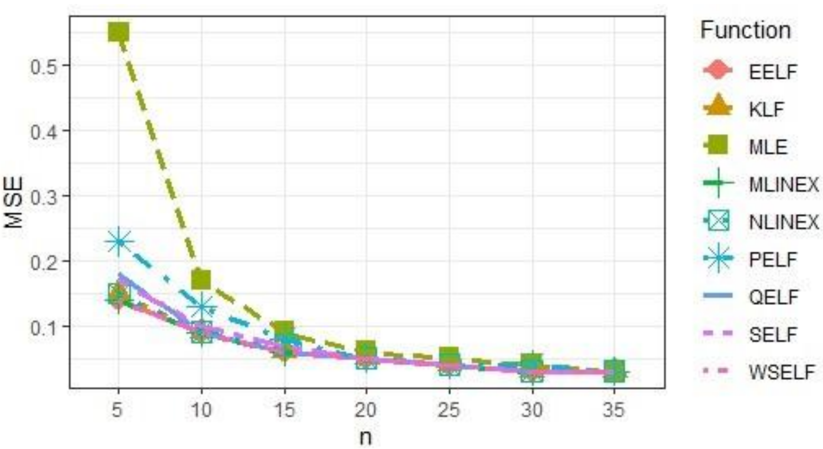

Figure 1: Mean Square Errors at Different Loss Functions at a = 0.5, b = 1, ϕ = 1, c = 1

Table 2
Bayes Estimates at Different Loss Functions at *a* = 1, *b* =2, ϕ = 1, *c* = 1

N	Criteria	BEL	BKL	MLE	BML	BNL	BPL	BQL	BSL	BWL
5	Mean	0.790	0.865	1.258	0.791	0.920	1.018	0.627	0.945	0.790
	MSE	0.107	0.093	0.592	0.106	0.089	0.107	0.178	0.092	0.107
10	Mean	0.889	0.940	1.108	0.898	0.969	1.020	0.801	0.982	0.894
	MSE	0.069	0.072	0.167	0.073	0.070	0.078	0.087	0.073	0.071
15	Mean	0.927	0.961	1.071	0.928	0.982	1.025	0.866	0.990	0.933
	MSE	0.053	0.054	0.092	0.052	0.052	0.060	0.061	0.055	0.050
20	Mean	0.950	0.971	1.050	0.944	0.985	1.019	0.897	0.990	0.949
	MSE	0.042	0.042	0.062	0.042	0.042	0.045	0.046	0.042	0.042
25	Mean	0.953	0.978	1.042	0.955	0.990	1.020	0.920	0.999	0.959
	MSE	0.036	0.035	0.050	0.034	0.035	0.037	0.036	0.036	0.035
30	Mean	0.963	0.983	1.036	0.967	0.992	1.013	0.935	1.000	0.961
	MSE	0.030	0.029	0.040	0.030	0.030	0.031	0.031	0.031	0.030
35	Mean	0.972	0.985	1.028	0.975	0.993	1.013	0.945	0.999	0.971
	MSE	0.026	0.026	0.031	0.025	0.026	0.027	0.026	0.026	0.025

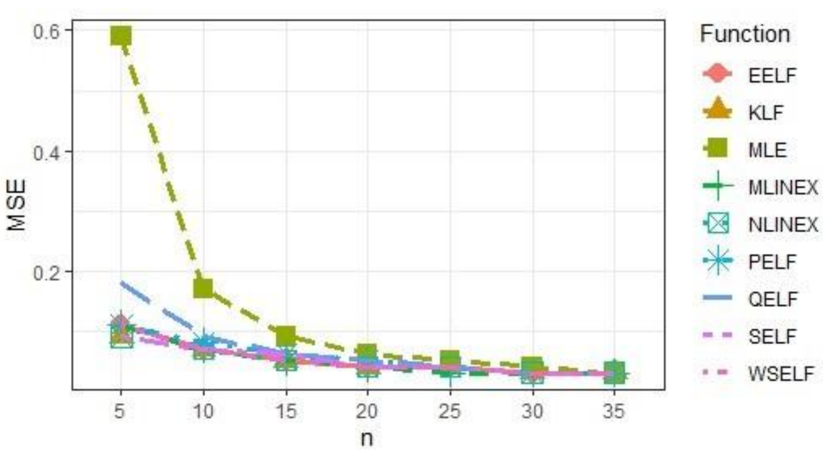

Figure 2: Mean Square Errors at Different Loss Functions at *a* = 1, *b* =2, ϕ = 1, *c* = 1

Table 3
Bayes Estimates at Different Loss Functions at *a* = 0.5, *b* =1, ϕ = 0.5, *c* = 1

N	Criteria	BEL	BKL	MLE	BML	BNL	BPL	BQL	BSL	BWL
5	Mean	0.483	0.536	0.622	0.483	0.580	0.641	0.377	0.586	0.482
	MSE	0.050	0.062	0.146	0.048	0.072	0.104	0.044	0.078	0.048
10	Mean	0.497	0.522	0.550	0.495	0.542	0.575	0.443	0.548	0.497
	MSE	0.026	0.030	0.040	0.026	0.033	0.041	0.025	0.034	0.027
15	Mean	0.500	0.517	0.536	0.497	0.529	0.547	0.466	0.532	0.497
	MSE	0.018	0.019	0.023	0.017	0.020	0.021	0.016	0.021	0.016
20	Mean	0.499	0.511	0.526	0.498	0.525	0.538	0.472	0.524	0.500
	MSE	0.012	0.013	0.016	0.012	0.015	0.016	0.012	0.015	0.013
25	Mean	0.498	0.509	0.518	0.501	0.519	0.527	0.478	0.520	0.500
	MSE	0.010	0.010	0.012	0.010	0.011	0.012	0.010	0.011	0.010
30	Mean	0.501	0.507	0.516	0.501	0.515	0.524	0.483	0.516	0.499
	MSE	0.008	0.008	0.009	0.008	0.009	0.010	0.008	0.009	0.008
35	Mean	0.500	0.508	0.515	0.498	0.512	0.522	0.485	0.512	0.499
	MSE	0.007	0.007	0.008	0.007	0.008	0.008	0.007	0.007	0.007

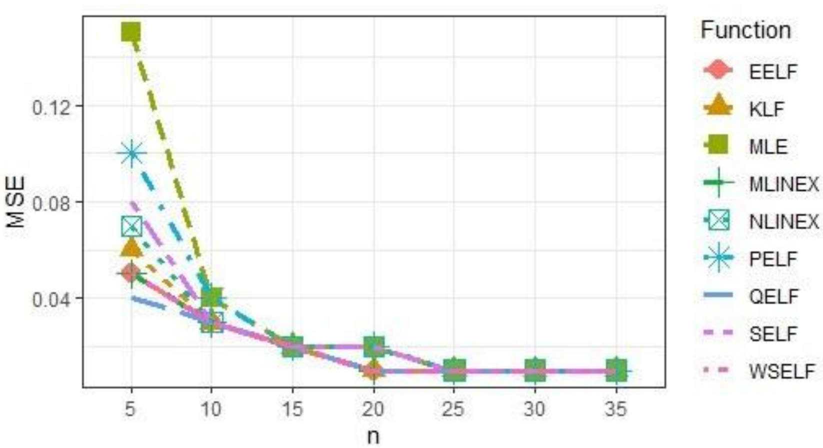

Figure 3: Mean Square Errors at Different Loss Functions
at *a* = 0.5, *b* = 1, ϕ = 0.5, *c* = 1

Table 4
Bayes Estimates at Different Loss Functions at *a* = 1.5, *b* =2, ϕ =1.5, *c* = 2

N	Criteria	BEL	BKL	MLE	BML	BNL	BPL	BQL	BSL	BWL
5	Mean	1.108	1.208	1.852	1.005	1.208	1.412	0.909	1.314	1.111
	MSE	0.242	0.193	1.168	0.317	0.176	0.152	0.406	0.162	0.241
10	Mean	1.294	1.344	1.678	1.220	1.325	1.472	1.159	1.408	1.289
	MSE	0.144	0.136	0.397	0.172	0.128	0.136	0.199	0.132	0.149
15	Mean	1.354	1.394	1.604	1.306	1.384	1.482	1.268	1.439	1.357
	MSE	0.110	0.104	0.211	0.118	0.100	0.107	0.131	0.108	0.109
20	Mean	1.390	1.414	1.582	1.350	1.407	1.489	1.319	1.461	1.391
	MSE	0.087	0.086	0.151	0.093	0.084	0.090	0.100	0.084	0.091
25	Mean	1.403	1.439	1.560	1.378	1.428	1.493	1.352	1.459	1.413
	MSE	0.075	0.073	0.105	0.079	0.070	0.074	0.084	0.073	0.073
30	Mean	1.426	1.450	1.552	1.399	1.434	1.499	1.377	1.476	1.422
	MSE	0.062	0.066	0.086	0.066	0.060	0.062	0.070	0.062	0.062
35	Mean	1.438	1.456	1.540	1.415	1.446	1.498	1.394	1.477	1.435
	MSE	0.055	0.054	0.075	0.056	0.052	0.057	0.060	0.056	0.056

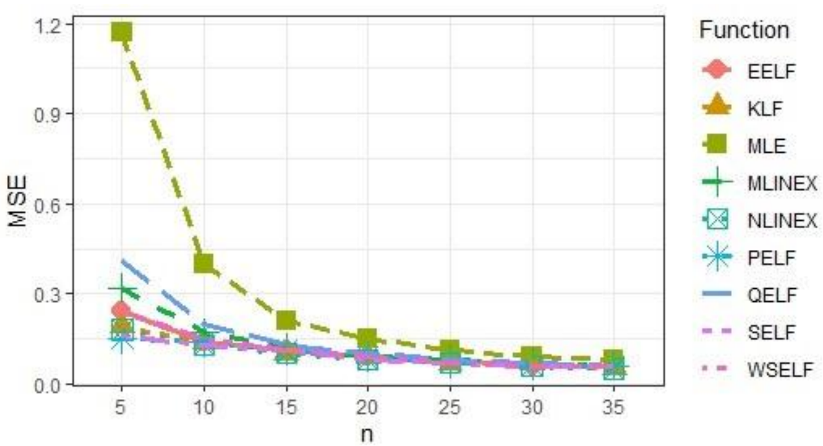

Figure 4: Mean Square Errors at Different Loss Functions at *a* = 1.5, *b* = 2, ϕ = 1.5, *c* = 2

Table 5
Bayes Estimates at Different Loss Functions at $a = 1$, $b = 1.5$, $\phi = 1$, $c = 2$

N	Criteria	BEL	BKL	MLE	BML	BNL	BPL	BQL	BSL	BWL
5	Mean	0.865	0.947	1.247	0.771	0.954	1.112	0.696	1.035	0.862
	MSE	0.112	0.113	0.607	0.125	0.097	0.160	0.153	0.136	0.144
10	Mean	0.940	0.984	1.109	0.889	0.986	1.086	0.845	1.031	0.937
	MSE	0.073	0.080	0.182	0.078	0.073	0.110	0.087	0.083	0.077
15	Mean	0.966	0.991	1.068	0.921	0.996	1.057	0.900	1.025	0.959
	MSE	0.059	0.057	0.093	0.057	0.056	0.069	0.057	0.064	0.058
20	Mean	0.971	0.990	1.055	0.943	0.993	1.045	0.924	1.020	0.978
	MSE	0.045	0.044	0.063	0.044	0.043	0.053	0.045	0.049	0.045
25	Mean	0.974	0.994	1.038	0.954	0.999	1.035	0.937	1.015	0.976
	MSE	0.037	0.037	0.048	0.035	0.037	0.041	0.037	0.038	0.036
30	Mean	0.982	0.997	1.035	0.965	0.996	1.030	0.950	1.015	0.981
	MSE	0.031	0.032	0.038	0.030	0.030	0.034	0.030	0.031	0.029
35	Mean	0.988	0.995	1.028	0.970	0.999	1.026	0.957	1.014	0.982
	MSE	0.028	0.028	0.032	0.026	0.026	0.029	0.027	0.029	0.028

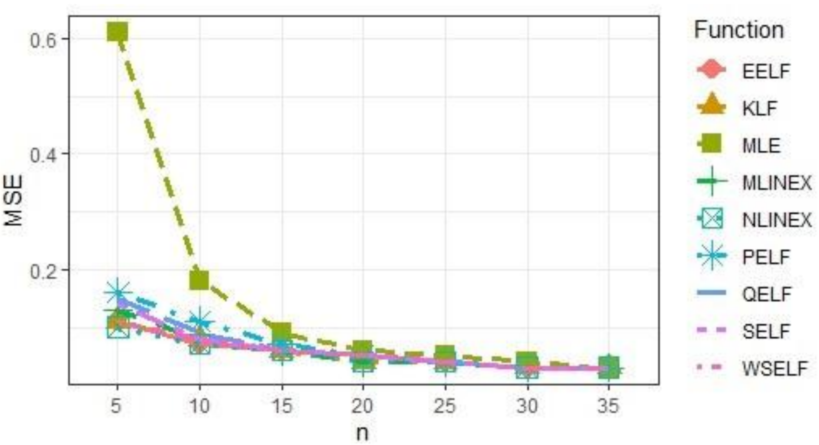

Figure 5: Mean Square Errors at Different Loss Functions at $a = 1$, $b = 1.5$, $\phi = 1$, $c = 2$

6. CONCLUSION

In this work, we derived a posterior distribution of the Power function distribution with conjugate prior under different loss functions. We have evaluated Bayes estimates and their posterior risks using the square error loss function (SELF), the quadratic error loss function (QELF), the weighted square error loss function (WSELF), the precautionary error loss function (PELF), the K loss function (KLF) the entropy error loss function (EELF), the modified linear exponential error loss function (MLINEXELF) and the non-linear exponential error loss function (NLINEXLF). In the context of data analysis, we investigate the efficacy of the proposed Maximum Likelihood Estimation (MLE) and Bayesian estimators across various loss functions. Our findings suggest that larger Bayesian samples offer more accurate insights into the study's true nature compared to smaller classical samples. Furthermore, the simulation outcomes demonstrate the superior accuracy of Bayesian estimates compared to MLE in terms of MSE. From above results and discussion, we have presumed that the Bayes estimator under WSELF is best from others estimators. The Bayes estimator can be employed to forecast insurance loss payments, enabling a more precise calculation of potential claims and informing data-driven decisions in risk assessment and policy pricing.

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R-Coding for EELF

```
rm(list=ls())
library("ggplot2")
samp<-seq(5,35,by=5)
nalpha<-seq(0.5,0.5,by=0)
ntheta<-seq(1,1,by=0)
nbeta<-seq(1,1,by=0)
c<-1
s<-6000
Mean<-NULL
MSE<-NULL
mydata<-data.frame()
tot<-length(samp)
bse<-matrix(rep(0,1*s),nrow=s)
for(tlf in 1:length(ntheta))
{
```

```
theta<-ntheta[tlf]
for(blf in 1:length(nbeta))
{
beta<-nbeta[blf]
for(alf in 1:length(nalpha))
{
alpha<-nalpha[alf]
for(kn in 1:tot)
{
 n<-samp[kn]
for(i in 1:s)
 {
 u<-runif(n,min=0,max=1)
 x<-u^(1/theta)
 BEL<-(alpha+n-1)/(beta-sum (log(x)))
 bse[i,]<-BEL
 }
bMean<-mean(bse[,1])
bbias<-(bMean-theta)
bMSE<-mean((bse[,1]-theta)^2)
ndata<-data.frame(n,alpha,beta,theta,bMean,bMSE)
 mydata<-rbind(mydata,ndata)
}
}
}
}
# Mean
tdata<-mydata[,c(1,2,3,4,5)]
tdata$func<-"BEL"
tdata$BR<-"Bayes Estimators"
head(tdata)
names(tdata)[5]<-"Mean"
data3<-tdata
# MSE
bmse<-mydata[,c(1,2,3,4,7)]
bmse$func<-"BEL"
bmse$BR<-"Bayes Estimators"
head(bmse)
names(bmse)[5]<-"MSE"
data<-bmse
data3$MSE<-data$MSE
data3$Function<-"EELF"
write.csv(data3,"EELF.csv")
p <- ggplot(data3, aes(n, Mean))
p<-p + geom_point(aes(colour=func))
p1<-p + geom_point(aes(n,mean(MSE)))
p
```

R-Coding for MLE

```
rm(list=ls())
library("ggplot2")
samp<-seq(5,35,by=5)
nalpha<-seq(0.5,0.5,by=0)
ntheta<-seq(1,1,by=0)
nbeta<-seq(1,1,by=0)
s<-6000
Mean<-NULL
MSE<-NULL
mydata<-data.frame()
tot<-length(samp)
bse<-matrix(rep(0,1*s),nrow=s)
for(tlf in 1:length(ntheta))
{
theta<-ntheta[tlf]
for(blf in 1:length(nbeta))
{
beta<-nbeta[blf]
for(alf in 1:length(nalpha))
{
alpha<-nalpha[alf]
for(kn in 1:tot)
{
 n<-samp[kn]
for(i in 1:s)
 {
 u<-runif(n,min=0,max=1)
 x<-u^(1/theta)
 MLE<-(-n)/sum (log(x))
 bse[i,]<-MLE
 }
bMean<-mean(bse[,1])
bbias<-(bMean-theta)
bMSE<-mean((bse[,1]-theta)^2)
ndata<-data.frame(n,alpha,beta,theta,bMean,bMSE)
 mydata<-rbind(mydata,ndata)
}
}
}
}
# Mean
tdata<-mydata[,c(1,2,3,4,5)]
 tdata$BR<-"Bayes Estimators"
tdata$func<-"MLe"
head(tdata)
names(tdata)[5]<-"Mean"
```

```
data5<-tdata
# MSE
bmse<-mydata[,c(1,2,3,4,7)]
bmse$BR<-"Bayes Estimators"
bmse$func<-"MLe"
head(bmse)
names(bmse)[5]<-"MSE"
data<-bmse
data5$MSE<-data$MSE
data5$Function<-"MLE"
write.csv(data5,"MLE.csv")
p <- ggplot(data5, aes(n, Mean))
p<-p + geom_point(aes(colour=func))
p1<-p + geom_point(aes(n,mean(MSE)))
p1
```