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ESTIMATING PARAMETERS IN ANALYSIS OF COVARIANCE MODEL WITH WEIBULL ERROR DISTRIBUTION

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ABSTRACT

This article aims to study the Analysis of Covariance (ANCOVA) techniques when the distribution of error terms are non-normal distributions. The Weibull Distribution is used for the error terms. A new method known as the modified maximum likelihood (MML) is used to derive the estimators of the model parameters. Simulations are performed to investigate the least squares (LS) method and the modified maximum likelihood method via two criteria, the bias, and the mean square error. The simulation results show that the proposed estimators are more efficient compared with the conventional least squares estimators.

KEYWORDS

Analysis of Covariance (ANCOVA); Weibull Distribution; Modified Maximum Likelihood (MML); Least Squares (LS).

1. INTRODUCTION

The main goal of this article is to introduce a new method to estimate the ANCOVA parameters when the distribution of error terms are non-normal distributions. This method is called modified maximum likelihood (MML). For the error terms, Weibull distribution is assumed. The analysis of covariance (ANCOVA) was originally developed by Fisher (1932) to reduce error variance in experimental studies. The value and use of ANCOVA have also received considerable attention in social science. ANCOVA is a linear model analysis of the design, which measures one or more concomitant continuous variables for each experimental unit along with the response variable. In order to reduce the residual variation. It is a general linear model that includes both the analysis of variance (categorical) predictors and regression (continuous) predictors. It combines one-way or two-way analysis of variance with a general linear regression model. See, [Birch and Myers (1982), Quinn and Keough (2002) and Shieh (2020)].

The usual form of the one-way classification ANCOVA model with a single covariate is:

$$Y_{ij} = \mu + \alpha_i + \beta\left(X_{ij} - \bar{X}_{ij}\right) + e_{ij} \tag{1}$$

where: Y_{ij} is the value of response variable, μ is the overall mean value of the response variable, α_i is the effect of the i^{th} level factor, β is the combined regression coefficient representing the pooling of the regression slopes of Y within each group; X_{ij} is the non-stochastic covariate value; $\bar{X}_{ij}$ is the general covariate mean, and e_{ij} is the independently and identically distributed (iid) random errors.

Alternatively, the model in (1) can also be written as

$$y_{ij} = \mu_i + \theta_i x_{ij} + e_{ij} \quad i = 1,2, \dots, C \; ; j = 1,2, \dots, n_i \tag{2}$$

via reparametrization of the model. μ_i is the effect associated with group i, and θ_i is the slope coefficient. For estimating the parameters in model (2), the error terms are traditionally assumed to be normally distributed with mean zero and variance σ^2.When the normality assumption is satisfied, least-squares estimators (LSE) are used. In practice, however, non-normal distributions are more prevalent. Much work has been interested to obtain efficient parametric estimators in ANCOVA under non-normal distributions. Senoglu (2007) considers the (ANCOVA) model with a single covariate when the distribution of error terms is short-tailed symmetric. See also [Senoglu and Tiku (2002), Senoglu and Avcioglu (2009) and Minsker and Wei (2017)].

The originality of this article is assuming Weibull distribution for the errors in the one-way ANCOVA model. Since the maximum likelihood method does not provide explicit estimators for the parameters in the model (2), the modified maximum likelihood (MML) methodology is obtained. Our aim is to obtain explicit estimators of the model parameters in the model (2) under the assumption of Weibull distribution of the error terms. The estimators obtained from the MML methodology are compared with the estimators of Least Squares (LS). Two criteria are used to obtain, the bias and relative efficiency of the mean square error (MSE). The results of the simulation show that the modified maximum likelihood (MML) estimators are more efficient than Least Squares (LS) estimators for the one-way ANCOVA model with Weibull distribution of the error terms.

The structure of this essay is as follows: The introduction is in Section 1, and the discussion of the Weibull Distribution is in Section 2. Modified Maximum Likelihood Estimation (MMLE) is described in Section 3. Section 4 introduces the Main Results. The Simulation Study, conclusion and future work are all found in sections five and six, respectively.

2. WEIBULL DISTRIBUTION

The Weibull probability distribution plays a very significant role in the statistical analysis and modeling of problems in the real life (Weibull (1951)). It is a very popular choice as a failure time distribution in life testing and reliability. Because of its flexibility, some modifications of the Weibull distribution have been made from several studies in order to best adjust the non-monotonic shapes. This distribution can be found with three parameters: scale, shape, and location. A random variable y would be expected to follow the Weibull distribution with the following density function:

$$f(y) = \frac{\beta}{\chi}\left(\frac{y-\gamma}{\beta}\right)^{\beta-1} \; e^{-\left(\frac{y-\gamma}{\chi}\right)^{\beta}} \qquad y > 0, \chi > 0, \beta > 0, \gamma > 0 \tag{3}$$

where: β is the shape parameter, also known as the Weibull slope.
χ is the scale parameter.
γ is the threshold parameter.

Frequently, the threshold parameter is not used, and the value for this parameter can be set to zero. When this is the case, the pdf equation reduces to that of the two-parameter Weibull distribution. It has the following cumulative distribution function (cdf):

$$F(y) = 1 - e^{-\left(\frac{y}{\chi}\right)^{\beta}} \tag{4}$$

The corresponding probability density function (pdf) is:

$$f(y) = \frac{\beta}{\chi}\left(\frac{y}{\beta}\right)^{\beta-1} e^{-\left(\frac{y}{\chi}\right)^{\beta}}, y > 0, \chi > 0, \beta > 0 \tag{5}$$

There are a number of methods for estimating the values of these parameters; some are graphical and others are analytical. Graphical methods include Weibull probability plotting and hazard plot. These methods are not very accurate, but they are relatively fast. The analytical methods include the maximum likelihood method, the least square method, and the method of moments. These methods are considered more accurate and reliable compared to the graphical method. See [Dolas (2014), Soumaya and Soufiane (2014), Ulrich et al. (2018), Ahmad et al. (2018) and Akram et al. (2022)].

3. MODIFIED MAXIMUM LIKELIHOOD ESTIMATION (MMLE)

Maximum Likelihood (ML) estimators of the parameters of the nonlinear function are difficult and lack explicit solutions. The modified maximum likelihood technique of estimating, according to Tiku (1967 and 1968), offers explicit solutions for estimators. Modified likelihood equations that are asymptotically equivalent to the likelihood equations are required. The MML estimates are asymptotically efficient, which is well known. a change to the maximum likelihood estimator that, under some circumstances, enables us to derive the estimators in closed-form formulations. The estimator has an asymptotic normal distribution and is consistent under moderate circumstances. It is also invariant under one-to-one transformations. Using the MML methodology, any non-normal distribution at the location scale can be examined.

The MML estimator is employed in a variety of contexts, including ranked set sampling (RSS), as described by Zheng and Al-Saleh (2002), and a modification of ranked set sampling which called moving extremes ranked set sampling (MERSS) Chen et al. (2021).

The modified maximum likelihood estimators are created in the following manner, step by step:

The likelihood equations must first be translated into ordered statistics as the initial step.

The second stage involves applying Taylor series expansions to linearize the difficult elements in the likelihood equations.

The second step's results equations must be solved in order to obtain the adjusted maximum likelihood estimators in the final step.

Modified Maximum Likelihood Estimation Characteristics

The MMLE estimates are obvious, but they are not direct functions of the sample observations. As a result, they can be computed faster than maximum likelihood estimates. Additionally, the MMLE is asymptotically identical to the maximum likelihood if the regularity conditions are satisfied. On the other hand, the MMLE estimates are fair and almost totally effective in terms of the minimal variance bounds (MVBs), even for small samples. The MMLE method is essentially self-censoring because it gives the extremes minimal weight. They are explicit and easier to calculate than maximum likelihood estimates. They are also fair, almost entirely effective in terms of the minimum variance bounds, and unbiased. See also [Bhattacharyya (1985), Tiku and Suresh (1992), Vaughan and Tiku (2000), Vaughan (2002), Yang and Lin (2007), Balci et al. (2013), Acitas et al. (2020) and Maswadah (2022)].

4. MAIN RESULTS

Traditionally, the distribution of the error terms is assumed to be normal. However, in many applications, populations that are far from being normal are more prevalent. In this article, the Weibull distribution is assumed for the error in the one-way ANCOVA model.

Consider the error of the linear model in equation (2) as follows:

$$e_{ij} = y_{ij} - \mu_i - \theta_i x_{ij} \tag{6}$$

$$W(e,\gamma,\sigma) = \left(\frac{\gamma}{\sigma}\right)\left(\frac{e}{\gamma}\right)^{\gamma-1} exp\left\{-\left(\frac{e}{\sigma}\right)^{\gamma}\right\}, 0 < e < \infty \tag{7}$$

Assume that γ is known. Let the variate z_i be:

$$z_{i(j)} = \left(y_{i[j]} - \mu_i - \theta_i x_{i[j]}\right)/\sigma \tag{8}$$

The likelihood equations are given by:

$$\partial \mathrm{lnL}/\partial \mu_i = 0 \text{ , } \partial \mathrm{lnL}/\partial \theta_i = 0 \text{ and } \partial \mathrm{lnL}/\partial \sigma = 0$$

To derive modified likelihood equations that have explicit solutions, the first order is $w_i = y_i - \theta_i x_i$ (For a given θ_i) so that:

$$w_{i(1)} \le w_{i(2)} \le \cdots \le w_{i(n)} \text{ ; } w_{i(j)} = y_{i[j]} - \theta_i x_{i[j]} \text{ ; } z_{i(j)} = \frac{\{w_{i(j)} - \theta_0\}}{\sigma} \tag{9}$$

Are the ordered variates and $\left(y_{[i]}, x_{[i]}\right)$ is that pair of observations (y_i, x_i) which corresponds to $z_{i(j)}$; $\left(y_{i[j]}, x_{i[j]}\right)$ may be called the concomitant of $z_{i(j)}$.

To estimate μ_i, θ_i and σ the likelihood equations expressed in terms of $z_{i(j)}$ are:

$$\frac{\partial lnL}{\partial \mu_i} = -\frac{\gamma-1}{\sigma}\sum_{j=1}^{n_i} z_{i(j)}^{-1} + \frac{\gamma}{\sigma}\sum_{j=1}^{n_i} g\left(z_{i(j)}\right) = 0 \tag{10}$$

$$\frac{\partial lnL}{\partial \theta_i} = -\frac{\gamma-1}{\sigma}\sum_{j=1}^{n_i} z_{i(j)} z_{i(j)}^{-1} + \frac{\gamma}{\sigma}\sum_{j=1}^{n_i} z_{i(j)} g\left(z_{i(j)}\right) = 0 \tag{11}$$

$$\frac{\partial lnL}{\partial \sigma} = -\frac{n}{\sigma} - \frac{\gamma - 1}{\sigma}\sum_{j=1}^{n_i} z_{i(j)} z_{i(j)}^{-1} + \sum_{j=1}^{n_i} z_{i(j)} g\left(z_{i(j)}\right) = 0 \qquad (12)$$

where $g(z) = \frac{e^{-z}}{1-e^{-z}}$

To obtain the MMLEs, linearize the $g(z_{i(j)})$ function and consider

$$g\{z_{i(j)}\} \cong \alpha_{ij} - \beta_{ij} z_{i(j)} \qquad (13)$$

The coefficients α_{ij} and β_{ij} are obtained from the first two terms of a Taylor series expansion of $g\{z_{i(j)}\}$ around $t_{(j)}$. Approximate values of $t_{(j)}$ are used and obtained from the equations.

$$\int_{-\infty}^{t_{(j)}} [(\gamma z^{\gamma-1}) exp(-z^{\gamma})] dz = \frac{j}{n_j + 1}, 1 \le j \le n_i \qquad (14)$$

where $\alpha_{ij} = (2-\gamma) t_{(j)}^{\gamma-1}$ and $\beta_{ij} = (\gamma - 1) t_{(j)}^{\gamma-1}$ $1 \le j \le n_i$

The following are the modified likelihood equations:

$$\frac{\partial lnL}{\partial \mu_i} \cong \frac{\partial lnL^*}{\partial \mu_i} = 0, \frac{\partial lnL}{\partial \theta_i} \cong \frac{\partial lnL^*}{\partial \theta_i} = 0, \frac{\partial lnL}{\partial \sigma} \cong \frac{\partial lnL^*}{\partial \sigma} = 0.$$

The modified likelihood equations are obtained by incorporating the linear approximation of $g\{z_{i(j)}\}$ from (13), in the likelihood equations given in (10), (11), and (12).

$$\frac{\partial \mathrm{lnL}}{\partial \mu_i} \cong \frac{\partial \mathrm{lnL}^*}{\partial \mu_i} = -\frac{\gamma - 1}{\sigma}\sum_{j=1}^{n_i} (\alpha_{i0} - \beta_{i0} z_{i(j)}) + \frac{\gamma}{\sigma}\sum_{j=1}^{n_i} (\alpha_{ij} - \beta_{ij} z_{i(j)}) = 0 \qquad (15)$$

$$\frac{\partial lnL}{\partial \theta_i} \cong \frac{\partial lnL^*}{\partial \theta_i} = -\frac{\gamma - 1}{\sigma}\sum_{j=1}^{n_i} z_{i(j)}\left(\alpha_{i0} - \beta_{i0} z_{i(j)}\right) + \frac{\gamma}{\sigma}\sum_{j=1}^{n_i} z_{i(j)} (\alpha_{ij} - \beta_{ij} z_{i(j)}) = 0$$

(16)

$$\frac{\partial lnL}{\partial \sigma} \cong \frac{\partial lnL^*}{\partial \sigma} = -\frac{n}{\sigma} - \frac{\gamma - 1}{\sigma}\sum_{j=1}^{n_i} z_{i(j)}\left(\alpha_{i0} - \beta_{i0} z_{i(j)}\right)$$

$$+ \sum_{j=1}^{n_i} z_{i(j)} (\alpha_{ij} - \beta_{ij} z_{i(j)}) = 0 \qquad (17)$$

The solution of these equations are the following $MMLE_s$ $(i = 1,2, \dots, c)$:

$$\hat{\mu}_i = \bar{y}_{i[.]} - \hat{\theta}_i \bar{x}_{i[.]} - \left(\frac{A}{m}\right) \hat{\sigma} \qquad (18)$$

$$\hat{\theta}_i = K - D\hat{\sigma} \qquad (19)$$

and

$$\hat{\sigma} = \frac{\left\{-B + \sqrt{B^2} + 4nC\right\}}{2\sqrt{n(n-2)}} \tag{20}$$

where

$$\delta_i = (\gamma - 1)\beta_{i0} + \gamma\beta_{ij}\,,\ A_i = (\gamma - 1)\alpha_{i0} - \gamma\alpha_{ij}\ ;$$

$$m = \sum_{i=1}^{n} \delta_i\,,\ A = \sum_{i=1}^{n} A_i\ ;$$

$$\bar{y}_{i[.]} = (1/m)\sum_{i=1}^{n} \delta_i y_{[i]}\,,\bar{x}_{i[.]} = (1/m)\sum_{i=1}^{n} \delta_i\, x_{[i]}\ ;$$

$$K = \sum_{i=1}^{n} \delta_i \left(x_{[i]} - \bar{x}_{i[.]}\right)\bar{y}_{i[.]} \Bigg/ \sum_{i=1}^{n} \delta_i \left(x_{[i]} - \bar{x}_{i[.]}\right)^2$$

$$D = \sum_{i=1}^{n} A_i \left(x_{[i]} - \bar{x}_{i[.]}\right) \Bigg/ \sum_{i=1}^{n} \delta_i \left(x_{[i]} - \bar{x}_{i[.]}\right)^2;$$

$$B = \sum_{i=1}^{n} A_i\left\{y_{[i]} - \bar{y}_{i[.]} - K\left(x_{[i]} - \bar{x}_{i[.]}\right)\right\},$$

$$C = \sum_{i=1}^{n} \delta_i\left\{y_{[i]} - \bar{y}_{i[.]} - K\left(x_{[i]} - \bar{x}_{i[.]}\right)\right\}^2$$

$$= \sum_{i=1}^{n} \delta_i \left(y_{[i]} - \bar{y}_{i[.]}\right)^2 - K\sum_{i=1}^{n} \delta_i \left(x_{[i]} - \bar{x}_{i[.]}\right)\bar{y}_{i[.]} \tag{21}$$

The Least Squares (LS) estimators for μ_i, θ_i, and σ are obtained from equations (21) by equating α_{ij} to zero and β_{ij} to one.

5. SIMULATION STUDY

All results were obtained by using (R) program version 3.2.2. The simulation study is provided to compare two methods (LS, MML) using different sample sizes. Three criteria are calculated, bias, mean square error (MSE), and relative efficiency to evaluate the estimator. Where x_{ij} and $y_{ij}(1 \leq \text{j} \leq n_i)$ were generated all the (10000) runs. This was done for each $i = 1,2,3$ the study design includes different sample sizes $(n_i's)$ leading to a find with the parameter values. Therefore, a comparison was made among these methods. The x_{ij} values were generated only once to be common to all the (10000) runs. This was done for every three groups. Different sample sizes crossed with six parameter values of Weibull error distribution

Generating Data:

Sample data are created using Monte Carlo simulations. The bias and mean square error of the μ_i and θ_i estimators are computed to study their effectiveness. To create sample data, we employed Monte Carlo simulations. As beginning values for the parameters in our experiment, we used the following settings:

$\beta = (1, 0.4277, 1.3792, 27.2913, 1.5974, 0.3240)$ (Charles (1994)).

The estimated values of the unknown parameters are obtained by fitting the data to the population model of the Weibull distribution. Without sacrificing generality, γ is set to equal 0.6, and σ is assumed to equal 1. Different instances are assumed depending on the sample size.

1. Small equal sample size when $n_1 = n_2 = n_3 = 8$
2. Moderate equal sample size when $n_1 = n_2 = n_3 = 15$
3. Large equal sample size when $n_1 = n_2 = n_3 = 20$
4. Small sample size when $n_1 = 2, n_2 = 8, n_3 = 12$
5. Moderate equal sample size when $n_1 = 7, n_2 = 10, n_3 = 14$
6. Large sample size when $n_1 = 10, n_2 = 15, n_3 = 20$

Table 1
The Bias and Relative Efficiency of the Methods (LS – MML) for Equal Sample Size

Sample Size		$\boldsymbol{\mu_1}$	$\boldsymbol{\mu_2}$	$\boldsymbol{\mu_3}$	$\boldsymbol{\theta_1}$	$\boldsymbol{\theta_2}$	$\boldsymbol{\theta_3}$
8	Bias LS	1.57184	0.61673	2.9179	3.57526	2.82809	1.67745
	Bias MML	1.9723	1.40056	1.41715	3.0864	1.4738	1.1762
	MSE(MML) / MSE(LS)	0.60009	0.75018	0.118949	0.747511	0.127906	0.23969
15	Bias LS	1.56913	0.610664	0.72305	3.41527	0.97776	1.66595
	Bias MML	1.68492	0.1208	0.2087	2.50667	0.4747	1.09531
	MSE(MML) / MSE(LS)	0.419448	0.756327	0.73418	0.99496	0.92007	0.10555
20	Bias LS	0.08564	0.03133	0.03793	1.00496	0.0521	0.01347
	Bias MML	0.07471	0.02179	0.03154	1.00557	0.02353	0.01367
	MSE(MML) / MSE(LS)	0.03536	0.05906	0.01816	0.01237	0.60009	0.75018

Table 2
The Bias and Relative Efficiency of the Methods (LS – MML) for Unequal Sample Size

Sample Size		μ_1	μ_2	μ_3	θ_1	θ_2	θ_3
2,8,12	Bias LS	2.66761	2.35915	2.57359	1.61408	2.27116	1.46044
	Bias MML	1.455	1.5633	2.0723	1.1208	1.7913	0.9026
	$\frac{MSE(MML)}{MSE(LS)}$	0.00643	0.00675	0.00468	0.00514	0.0047	0.005486
7,10,14	Bias LS	1.90076	2.0745	2.07925	1.23285	2.18991	1.40327
	Bias MML	1.185	1.081	1.2147	1.0816	1.3756	0.0749
	$\frac{MSE(MML)}{MSE(LS)}$	0.00389	0.003696	0.00381	0.004082	0.003619	0.004855
10,15,20	Bias LS	0.01475	0.01302	0.03792	0.04188	0.13073	0.07973
	Bias MML	0.00957	0.00939	0.0093	0.01119	0.05076	0.03646
	$\frac{MSE(MML)}{MSE(LS)}$	0.001626	0.00161	0.0014	0.00153	0.0074	0.001632

The bias and relative efficiency of LS and MML methods, to the ANCOVA coefficients for unequal slopes with equal variance, are shown in Table (1) using an equal sample size. For unequal sample size the results are shown in Table (2). The bias of LS and MML estimators will be compared. Besides, the relative efficiency (R.E) using the mean square error of the LS and the MML estimators will be discussed.

According to the simulation study, many results can be concluded.

First, regardless of whether the sample size is equal or unequal, the bias using the MML approach is lower than the bias using the LS method for all model parameters. Also, it can be deduced that the MML estimator's bias is not high. On the other hand, when sample size grows, the bias of the MML technique lessens.

Second, the relative efficiency using the mean square error of the MML method compared to the LS method is less than one for different cases. It means that the MML estimators are more efficient (less variance) than the LS estimators.

6. CONCLUSION AND FUTURE WORK

ANOVA and multiple linear regression are two extensively used processes, and ANCOVA offers a good method for combining their merits. The normal distribution is typically believed to be the error term. However, populations are not typical in many applications. When the error term is distributed according to the Weibull distribution, the goal of this article is to estimate the ANCOVA model parameters. Modified maximum likelihood (MML) and least squares (LS) are two techniques that are contrasted. Our findings indicate that, for Weibull error distributions, MML estimators are more effective than traditional LS estimators.

Future Work

It is advised to compare various scenarios of slopes with variances under a non-normal distribution in future study and to employ an analysis of covariance with pre-treatment in measuring randomized trials. The parameters of the Weibull distribution can be estimated using a variety of techniques. Comparing estimate techniques, like Bayesian estimation, with other techniques might produce useful results.

REFERENCES

1. Acitas, S., Arslan, T. and Senoglu, B. (2020). Slash Maxwell Distribution: Definition, Modified Maximum Likelihood Estimation and Applications. *Journal of Science*, 33(1), 249-263.
2. Ahmad, Z., Elgarhy, M. and Hamedani, G. (2018). A new Weibull-X family of distributions: properties, characterizations and applications. *Journal of Statistical Distributions and Applications*, 5(1), 1-18.
3. Akram, K., Shirin, S. and Saralees, N. (2022). Multi-component stress-strength model for Weibull distribution in progressively censored samples. *Statistics & Risk Modeling*, 39(1/2), 1-21.
4. Balci, S., Akkaya, A.D. and Ulgen, B.E. (2013). Modified maximum likelihood estimators using ranked set sampling. *Journal of Computational and Applied Mathematics*, 238, 171-179.
5. Bhattacharyya, G. (1985). The asymptotics of maximum likelihood and related estimators based on type II censored data. *Journal of the American Statistical Association*, 80, 398-404.
6. Birch, J. and Myers, R. (1982). Robust analysis of covariance. *Biometrics*, 38, 699-713.
7. Charles, J., Emet, D. and Stephen, M. (1994). ANCOVA for nonparallel slopes: The Johnson-Neyman technique. *International Journal of Bio-Medical Computing*, 37, 273-286.
8. Chen, W., Long, C., Yang, R. and Yao, D. (2021). Maximum Likelihood Estimator of the Location Parameter under Moving Extremes Ranked Set Sampling Design. *Acta Mathematicae Applicatae Sinica, English Series*, 37, 101-108.
9. Dolas, D., Jaybhaye, M. and Deshmukh, S. (2014). Estimation the System Reliability using Weibull Distribution. *Proc. of Economics Development and Research*, 75-79
10. Fisher, R. (1932). *Statistical methods for research workers* (4th ed.). Edinburgh, Oliver and Boyd.
11. Maswadah, M. (2022). Improved maximum likelihood estimation of the shape-scale family based on the generalized progressive hybrid censoring scheme. *Journal of Applied Statistics*, 49(11), 2825-2844.
12. Minsker, S. and Wei, X. (2017). Estimation of the covariance structure of heavy-tailed distributions. In *Proceedings of the 31st Conference on Neural Information Processing (NIPS)*. arXiv:1708.00502v3
13. Quinn, G. and Keough, M. (2002). *Experimental design and data analysis for biologists*. Cambridge University.

14. Şenoğlu, B. (2007). Estimating parameters in one-way analysis of covariance model with short-tailed symmetric error distributions. *Journal of Computational and Applied Mathematics,* 201, 275-283.
15. Senoğlu, B. and Tiku, M.L. (2002). Linear contrasts in experimental design with non-identical error distribution. *Biometrical Journal*, 44, 359-374.
16. Senoglu, B. and Avcioglu, M. (2009). Analysis of Covariance with Non-normal Errors. *International Statistical Review*, 77, 366-377.
17. Shieh, G. (2020). Power Analysis and Sample Size Planning in ANCOVA Designs. *Psychometrika,* 85(1), 101-120.
18. Soumaya, G. and Soufiane, G. (2014). Parameters Estimations for Some Modification of the Weibull Distribution. *Open Journal of Statistics*, 4, 597-610.
19. Tiku, M. (1967). Estimating the mean and standard deviation from a censored normal sample, *Biometrika,* 54, 155-165.
20. Tiku, M. (1968). Estimating the parameters of Normal and Logistic distributions from censored samples. *Australian and New Zealand Journal of Statistics*, 10, 64-74.
21. Tiku, M. and Suresh, R. (1992). A new method of estimation for location and scale parameters. *Journal of Statistical Planning and Inference*, 30, 281-92.
22. Ulrich, W., Nakadai, R., Matthews, T. and Kubota, Y. (2018). The two-parameter Weibull distribution as a universal tool to model the variation in species relative abundances. *Ecological Complexity*, 36, 110-116.
23. Vaughan, D. and Tiku, M. (2000). Estimation and hypothesis testing for a non-normal bivariate distribution and applications. *Mathematical and Computer Modelling*, 32(1-2), 53-67.
24. Vaughan, D. (2002). The generalized secant hyperbolic distribution and properties. *Communication in Statistics-Theory and Methods*, 31(2), 219-238.
25. Weibull, W. (1951). A Statistical Distribution Function of Wide Applicability. *Journal of Applied Mechanics*, 18, 293-297.
26. Yang, Z. and Lin, D. (2007). Improved maximum-likelihood estimation for the common shape parameter of several Weibull populations. *Applied Stochastic Models in Business and Industry*, 23, 373-383.
27. Zheng, G. and Al-Saleh, M. (2002). Modified Maximum Likelihood Estimators Based on Ranked Set Samples. *Annals of the Institute of Statistical Mathematics*, 54, 641-658.