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TO APPROXIMATE DISTRIBUTIONS OF SOLUTIONS OF STOCHASTIC DIFFERENTIAL EQUATIONS

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ABSTRACT

Generalized Entropy Optimization Methods (GEOM's) are developed in over really investigations which allow to obtain in particularly distributions in the form Max-Max Ent, Min-Max Ent, Max-Minx Ent, and Min Minx Ent distributions so-called as Generalized Entropy Optimization distributions (GEOD's).

In the present study we have represented applications of mentioned methods in the theory of fundamental statistical distributions and the Stochastic Differential Equation Modelling.

Classical statistical methods can be also applied to obtain pdf and distributions of solutions of SDE at fixed time. However, there are random variables distributions of which cannot be expressed through classical theoretical distributions. Consequently, it is necessary to use GEOD's is more broadly and sufficiently.

Note that investigations to pdf of solutions of SDE are contained also by fact that mentioned pdf's can be considered as solutions of Kolmogorov Equation.

KEYWORDS

Generalized Entropy Optimization Methods, Stochastic Differential Equation, Max-Max Ent, Min-Max Ent, Max-Min Ent, Min-Minx Ent distributions.

1. INTRODUCTION

Generalized Entropy Optimization Methods (GEOM's) are represented and developed in [Shamilov (2006b), Shamilov (2006a), Shamilov (2007), Shamilov (2010) and Shamilov (2015)]. Their applications in several areas given in [Shamilov et al. (2008), Shamilov, Senturk and Yilmaz (2016) and Shamilov and Ince (2016)], GEOM's consist of following.

By starting from given fixed statistical distribution (discrete or continues) and entropy optimization measure L the special functional U on characterizing moment functions set K is defined. This functional allows to obtain distribution closest and distribution forest from the given fixed distribution. Mentioned distributions (GEOD's). For example, if $L = H$ is Shanon entropy measure then functional $U = H_{max}$. If $L = D$ is Kullback – Leibler entropy measure then mentioned functional $V = D_{min}$. GEOD's are Max Ent, Min Max Ent, Max Minx Ent, Min Minx Ent distributions.

Entropy optimization distributions MaxEnt distributions especially as Generalized Entropy Optimization distribution successfully describe distribution random variables (Kapur and Kesavan, 1992). Consequently, applications of GEOD's in modeling distributions of solutions of SDE also acquire significance.

For example, error distribution established as outcomes of apparatus measuring distance via radio waves cannot representable as classical theoretical distributions (Ventsel, 1969).

Modeling Physical systems by ordinary differential equations ignores stochastic effects (Shamilov, 2012).

During the past few decades, because of wide change applications of SDEs have become quite popular models in a variety of areas such as financial mathematics, actuarial sciences, physics, biology, geology, mechanics, astronomy and other fields of science and engineering [Allen (2007) and Öz (2013)]. In the literature, there are many interesting applications and models of stochastic differential equations in [Allen (1999), Allen (2010), Allen et al. (2007), Capasso and Bakstein (2005), Chernov et al. (2003), Evans (2015), Hayes and Allen (2005), Kloeden and Platen (1995), Korn and Korn (2001), Kunze (2012), Mikosch (1998) and Ross (1999)]. Methods the computational solution of SDEs are based on similar techniques for ordinary differential equations, but generalized to provide support for stochastic dynamics (Allen et al., 2007).

The present paper is organized as follows. In section 1 it is given basing of GEOM's to applications in approximately solution distributions of SDE. Section 2, rates to Generalities Entropy Optimization Methods for discreet and continues random variables. In section 3 it is given a brief explanation of SDE and Euler – Maruyama approximate solving SDE. In section 4, a goodness of fit criteria for SDE model is given. In section 5, distributions of approximate solution of SDE are represented. In section 6 GEOM to obtain approximate probability density function is given. In section 7 it is represented a method to obtain SDE model available to given statistical data. In section 8 are given stages to obtain approximate GEOM distribution for solution of SDE. In section 9 it is given Conclusion.

2. GENERALIZED ENTROPY OPTIMIZATION METHODS (GEOM's)

Entropy Optimization Problem (EOP) (Kapur and Kesavan, 1992) and Generalized Entropy optimization problem (GEOP) (Shamilov, 2006a), can be formulated in following form.

EOP: Let $f^{(0)}(x)$ be given probability density function of random Variable X, L be an entropy optimization measure and $g(x)$ be given moment vector function generating m moment constraints. It is required to obtain the distribution function $f(x)$ corresponded to $g(x)$ given extremum valve to L.

GEOP: Let $f^{(0)}(x)$ be an entropy optimization measure and K be a set of given moment vector functions. It is required to choice moment vector functions $g^{(1)}$, $g^{(2)}$ from K such that $g^{(1)}(x)$ generates distribution $f^{(2)}(x)$ closest to $f^{(0)}(x)$, $g^{(2)}$ generates distribution $f^{(2)}(x)$ farest from $f^{(0)}(x)$ which respect to entropy optimization measure L. If L is taken as Shanon entropy measure H, then $f^{(1)}(x)$ is called Min Max Ent distribution and $f^{(2)}(x)$

is called Max Ent distribution. If L is taken as Kullback – Leibler measure D, then $f^{(1)}(x)$ is called Min Max Ent distribution and $f^{(2)}(x)$ is called Max Min Ent distribution. The method of solving GEOP is called GEOM.

2.1 Definition of Max-Ent Functional

The problem of maximizing max Ent measure

$$H = -\sum_{i=1}^{n} p_i ln p_i, \tag{2.1}$$

subject to constraints

$$\sum_{i=1}^{n} p_i \;\; g_i(x_i) = \mu_j \; j = 0, 1, \ldots, m, g_i(x) \equiv 1. \tag{2.2}$$

where $\mu_0 = 1, g_0(x) \equiv 1,\;\; g_1(x), \ldots, g_m(x)$ linearly independent random variables $p_i > 0, i = 1, 2, \ldots, n, m + 1 < n,$ is entropy optimization problem.

Then

$$H_{max} = -\sum_{i=1}^{n} e^{-\sum_{j=0}^{m} \lambda_j (x_2)} \left[-\sum_{j=0}^{m} \lambda_j g_j(x_i) \right] = \sum_{j=0}^{m} \lambda_j \mu_j. \tag{2.4}$$

If the distribution $p^{(0)} = \left(p_1^{(0)}, \ldots, p_n^{(0)}\right)$ is given then it is possible to find $\mu = (1, \mu_1, \ldots, \mu_m)$ for the each moment vector function $g(x) = (1, g_1(x), \ldots, g_m(x))$. H_{max} can be considered as functional dependent on moment vector function $g(x)$. This functional is called Max Ent functional if (2.4) take into account in (2.2) then

$$\left.\begin{aligned} &\sum_{i=1}^{n} e^{-\lambda^0 - \sum_{j=1}^{m} \lambda_j g_j(x_i)} = 1 \\ &\sum_{i=1}^{n} g_j(x_i) e^{-\lambda_0 - \sum_{j=1}^{m} \lambda_j g_j(x_i)} = \mu_j \\ &\qquad j = 1, 2, \ldots, m \end{aligned}\right\} \tag{2.5}$$

where $\lambda_0, \lambda_1, \ldots, \lambda_m$ are Lagrange multiplievs.

From (2.5) follows that

$$\lambda^0 = ln \sum_{i=1}^{n} e^{-\sum_{j=1}^{m} \lambda_j g_j(x_i)}$$

$$f_j(\lambda_1, \ldots, \lambda_m) \equiv \sum_{i=1}^{n} g_j(x_i) \frac{e^{-\sum_{i=1}^{n} \lambda_j g_j(x_i)}}{\sum_{i=1}^{n} e^{-\sum_{j=1}^{m} \lambda_j g_j(x_i)}} = \mu_j \quad j = 1, 2, \ldots, m \tag{2.6}$$

In (2.2) $1, g_1(x), \dots, g_m(x)$ are linearly in depended characterizing moment functions. $\mu_0, \mu_1, \dots, \mu_m$ obtained when $p_1, \dots, p_n$ can be considered as frequencies of the given statistical data. H_{max} as functional is defined on the set K of moment functions. In applications K may be considered the set of finite number of elements.

$g_1(x), \dots, g_m(x)$ characterizing moment functions assumed as random variables being linearly independent. According to definition of linearly independency the inequality

$$E\{|a, g_1 + \cdots + a_m g_m|^2\} > 0 \tag{2.7}$$

for each $a = [a_1, \dots, a_m] \neq 0$ is satisfied.

If equality

$$a_1 g_1(x) + \cdots + a_m g_m(x) = 0$$

for at least one $a = [a_1, \dots, a_m] \neq 0$ is satisfied then $g_1(x), \dots, g_m(x)$ are said linearly depended.

From (2.7) follows that

$$E\{|a, g_1 + \cdots + a_m g_m|^2\} = a^T R a,$$
$$R = \{R_{ij}\}, R_{ij} = E\{g_i g_j\}$$
$$a^T R a = 0$$

only when $a = 0$.

Consequently if $g_1(x), \dots, g_m(x)$ are linearly independent then

$$a_1 g_1(x) + \cdots + a_m g_m(x) = 0$$

because R is positive defined matrix.

It is proved that system (2.6) has unique solution with respect to $\lambda_1, \lambda_2, \dots, \lambda_m$.

For (2.6) it is proved that

$$\frac{D(f_1, \dots, f_m)}{D(\lambda_1, \dots, \lambda_m)} \neq 0$$

since this is determinant of variance – covariance matrix of linearly independent random variables $g_1(x), \dots, g_m(x)$. Consequently $\lambda_i, \dots, \lambda_m$ can be obtained from (2.6) according to existence of implicit function theorem. Mentioned solution can be find for example Newton method (Shamilov, 2006b).

2.2 Definition of MinxEnt Functional

The problem of minimizing MinxEnt measure

$$\mathrm{D}(p; q) = \sum_{i=0}^{n} p_i \ln \frac{p_i}{q_i} \tag{2.2.1}$$

subject to constraints (2.2), gives MinxEnt distribution.

This problem has solution

$$p_i = q_i e^{-\sum_{j=0}^{m} \lambda_j g_j(x_i)} \; i = 1, 2, \dots, n \tag{2.2.2}$$

where $\lambda_j, j = 1, 2, \dots, m$ are Lagrange multipliers

$$D_{min} = \sum_{i=0}^{n} q_i \, e^{-\sum_{j=0}^{m} \lambda_j g_j(x_i)} \left[- \sum_{j=0}^{m} \lambda_j g_i(x_i) \right] = - \sum_{j=0}^{m} \lambda_j \mu_j. \tag{2.2.3}$$

According to constraints (2.2)

$$\left. \begin{aligned} \sum_{i=0}^{n} q_i \, e^{-\lambda_0 - \sum_{j=1}^{m} \lambda_j g_j(x_i)} &= 1 \\ \sum_{i=0}^{n} g_j(x_i) q_i \, e^{-\lambda_0 - \sum_{j=1}^{m} \lambda_j g_j(x_i)} &= \mu_j \end{aligned} \right\} \tag{2.2.4}$$

From this equations

$$\lambda_0 = \ln q_i \, e^{-\sum_{j=1}^{m} \lambda_j g_j(x_i)}$$

$$\sum_{i=1}^{n} g_j(x_i) q_i \frac{e^{-\sum_{j=1}^{m} \lambda_j g_j(x_i)}}{\ln q_i e^{-\sum_{j=1}^{m} \lambda_j g_j(x_i)}} = \mu_j; \, j = 1, \dots, m \tag{2.2.5}$$

From (2.2.5) it is possible to find $\lambda_1, \dots, \lambda_m$ from (2.2.5). Mentioned solution can be obtained by Newton Method as (2.6) in other words from (2.2.5) be expressed by $\mu_1, \dots, \mu_m$. Mentioned staitments realisied analogues statements using on definition of H_{max} functional (2.4). It is possible use (2.2.5) for D_{min} (2.2.3).

2.3 Geod's with Finite Number of Moment Functions

Let $1, g_1(x), \dots, g_r(x)$ are linearly independent moment functions and $1, \mu_1, \dots, \mu_m$ are moment valves established by statistical data.

Let us $K = \{g_1(x), \dots, g_r(x)\}$, $K^{(r,m)}$ is set off all m combinations of elements of K. Eech combination taken from $K^{(r,m)}$ and the given statistical data according to (2.2) represents moment constraints for functionals H_{max}, D_{min}. Consequenty mentioned values allow to define MinMaxEnt, MaxMaxEnt for H_{max} and MinMinEnt, MinMinxEnt for functional D_{min}.

The number of elements of set $K^{(r,m)}$is equel to $\binom{r}{m} = l$. For this risen minimum and maximum values of functionals H_{max}, D_{min} can be established, so if $L = H$, $U = H_{max}$.

$$\min_{1 \le j \le l} U\left(g^{(j)}\right) = U(\tilde{g}) \; \max_{1 \le j \le l} U\left(g^{(j)}\right) = U\left(\tilde{\tilde{g}}\right)$$

Moment vector function $\tilde{g}$ defines MinMaxEnt distribution and $\tilde{\tilde{g}}$ defines MaxmaxEnt distributions

If $L = D(p, q)$, then $V = D_{min}$

$$\min_{1 \le j \le l} V\left(g^{(j)}\right) = V(\tilde{g}), \max_{1 \le j \le l} V\left(g^{(j)}\right) = V\left(\tilde{\tilde{g}}\right)$$

then vector function $\tilde{g}$ defines MinMinxEnt distribution and $\tilde{\tilde{g}}$ defins MaxMinxEnt distribution.

Remark:

GEOD'S with finite number y moment constraints for discrete random variables can be obtained for continuous variables random variables/ If Set K consist of $r = r = 6$ number characterizing moment functions if is necessary to consider all cases $m = 1, \dots, 6$. Because goodness - fit dependents not only on number of characterizing functions dependents also on class of each function.

It is known that fundamental classical distributions can be established as Entropy optimization distributions by virtue of corresponding characterizing moment functions. Consequently, in order to make set K participated in formulation of GEOM it is necessary to take into account mentioned characterizing moment functions equality with other know characterizing moment functions (Kapur and Kesavan, 1992).

2.4 Generalized Entropy Optimization Distributions for Continuous Variate Random Variables

Let $f^{(0)}(x)$ be given probability density function the problem of maximizing continuous variate MaxEnt measure

$$H = -\int_a^b f(x) \ln g(x)\, dx \tag{2.4.1}$$

subject to constraints

$$\int_a^b f(x) g_j(x) dx = \mu_j \;;\; j = 0, \dots, m \tag{2.4.2}$$

$$g_0(x) = 1, \mu_0 = 1$$

has solution

$$f(x) = e^{-\sum_{j=0}^{m} \lambda_j g_j(x)} \; j = 0,1, \dots, m \tag{2.4.3}$$

where $\lambda_j, j = 0,1, \dots, m$ are Lagrange multipliers consequently

$$H_{max} = -\int_a^b e^{-\sum_{j=0}^{m} \lambda_j g_j(x)} \left(-\sum_{j=0}^{m} \lambda_j g_j(x) \right) dx = \sum_{j=0}^{m} \lambda_j \mu_j. \tag{2.4.4}$$

If the pdf $f^0(x)$ is given, then one can obtain moment value $(1, \mu_1, \dots, \mu_m)$ for each moment vector function $g(x) = \big(1, g_1(x), \dots, g_m(x)\big)$ and H_{max} can be considered as a functional dependent on moment functions $g_1(x), \dots, g_m(x)$.

Let us $K = \{g_1(x), \dots, g_r(x)\}$, $K^{(r,m)}$ is set of all m combinations of elements of K. Each combination taken from $K^{(r,m)}$ and the given statistical data according to (2.4.2) represent moment constraints for functional (2.4.1) L (H or D). The number of corresponding values L are finite. Consequently, mentioned values allow to define MinMaxEnt, MaxMaxEnt distributions.

So if $L = H, U = H_{max}$

$$\min_{1 \le j \le l} U\left(g^{(j)}\right) = U(\tilde{g}) \ \max_{1 \le j \le l} U\left(g^{(j)}\right) = U\left(\tilde{\tilde{g}}\right)$$

moment vector function $\tilde{g}$ defines MinMaxEnt, and $\tilde{\tilde{g}}$ defines MaxMaxEnt distributions if

$$L = D(p;q) = \int_a^b P(x) \ln \frac{P(x)}{q(x)} dx \quad (2.4.6)$$

and constraints are (2.4.2) if $V = D_{min}$ then

$$\min_{1 \le j \le l} V\left(g^{(j)}\right) = V\left(\tilde{g}^{(1)}\right), \max_{1 \le j \le l} V\left(g^{(j)}\right) = V\left(\tilde{\tilde{g}}^{(2)}\right)$$

$\tilde{g}^{(1)}$ defines MinMinxEnt, $\tilde{\tilde{g}}^{(2)}$ defines MaxMinxEnt distributions.

3. STOCHASTIC DIFFERENTIAL EQUATION (SDE) MODELS IN STATISTICS

Many Stochastic Differential Equation Models can be developed by using procedures analogies procedures using to develop ordinary differential equations models. Moreover, there are many stochastic differential equations models developed in corresponded scientific areas. Mentioned stochastic differential equation models can be used in applications.

A typical one-dimensional stochastic differential equation has the form

$$X(t,\omega) = X(0,\omega) + \int_0^t f(s, X(s,\omega)\, ds + \int_0^t g(s, X(s,\omega)\, dW(s) \quad (3.1)$$

and differential form

$$dX(t) = f(t,X)dt + g(t,X)dW(t) \quad (3.2)$$

for $0 \le t \le T$ where $X(0,\omega)\epsilon H_{RV}$, $X(t,\omega)$ is a stochastic process not a deterministic function.

$W(t,0) = W(t)$ is a Winer process or Brownian motion and since it is no differentiable $W(t)$, $t \ge 0$ is a continuous stochastic process with stationary independent increments such that $W(0) = 0, \int_c^d dW(s) = W(d) - W(s) \sim N(0, d-s)$ for $0 \le c \le d$.

The function f is often called the drift coefficient, of the stochastic differential equation while g is referred to as the diffusion coefficient. It is assumed that the functions f and g are non-anticipating and satisfy the following conditions (c_1) and (c_2) for some constant $k \ge 0$ of existenc and uniqueness theorem of solution of SDE (Allen, 2007).

Condition (c_1): $|f(t,x) - f(s,y)|^2 \le k(t-s) + (x-y)^2$, $s \ge 0$, $T \ge t$, $x, y\epsilon R$
Condition (c_2): $|f(t,x)|^2 \le k(1+|x|^2)$, $0 \le t \le T$, $x\epsilon R$.

There are many approximate methods to solving SDE. One of mentioned methods is Euler – Maruyama (EM) method. This method is represented in following form.

3.1 Euler – Maruyama Method

Suppose that stochastic process is observed at times $t_0, t_1, \dots, t_{N-1}$, where $t_i = i\Delta t$ for a constant $\Delta t > 0$.

Let $x_0, x_1, \dots, x_{N-1}$ denote N observations $x(t_t) = x_i, i = 0, \dots, N-1$ of the stochastic process and the SDE (3.1) for the process is given.

In the future investigations, it is required to approximate solve SDE (3.1) using Euler – Maruyama (EM) method.

3.2 Forward Euler – Maruyama Method

This method allows to obtain approximate trajectory according to observation x_i on interval $[t_i, T]$ by formula

$$X_i = X(t_i, \omega) = X(t_{i-1}, \omega) + f\big(t_{i-1} + i\Delta t, X(t_{i-1}, \omega)\big)\Delta t \\ + + g\big(t_{i-1} + i\Delta t, X(t_{i-1}, \omega)\big)\sqrt{\Delta t}\eta^{(0,1)}, \quad (3.2.1)$$

where $t_0 = 0,\ t\epsilon[0, T],\ t_i = i\Delta t,\ i = 1, \dots, K-1,\ \Delta t = \frac{T}{K},\ K$ is number of steps using Euler-Maruyama method.

3.3 Backward Euler – Maruyama Method

As showed above trajectory according to x_i is not obtained on interval $[0, t_i]$. This range can be removed by backward Euler – Maruyama method in the following form

$$X(t_{i-1}) = X(t_i) - f\big(t_i; X(t_i - i\Delta t)\big)\Delta t \\ - - g\big(t_i; X(t_i - i\Delta t)\big)\sqrt{\Delta t}\eta(t_i - i\Delta t), \quad (3.2.2)$$

where $t_0 = 0, t\epsilon[0, T], t_i = i\Delta t, i = l, l-1, \dots \Delta t = \frac{T}{K}$.

By taking into account above expressed, we have stated that forward and backward EM methods allow to obtain at least N trajectories according to N observations $x_0, x_1, \dots, x_{N-1}$ of stochastic process represented by SDE (3.2).

It should be noted that each of mentioned trajectories derives N number approximate values of random variables $X(t)$ wich is solution of SDE (3.2) at time t_i

$$\hat{X}(t_i) = \big(x_1^{(i)}, x_2^{(i)}, \dots, x_k^{(i)}\big), i = 0, 1, 2, \dots, k; k = N, 2N, \dots, t_i = i\varDelta t, \varDelta t = T/K.$$

Note 3.1.1.

By using forward and back ward EM methods, it is possible to obtain N trajectories according to N observation $x_0, x_1, \dots, x_{N-1}$ of stochastic process appropriated to SDE model (3.2).

In order to obtain pdf of random variable $X(t_i)$ the number of valves can be increased by simulations in order words by using vales $K = 2N, 3N, \dots$

Many other problems are considered in [Evans (2015), Hayes and Allen (2005), Kloeden and Platen (1995), Korn and Korn (2001), Kunze (2012), Mikosch (1998), Ross

(1999), Vajargah and Asghari (2014), Kempthorne et al. (2013), Higham (2001) and Bak, Nielsen and Madsen (1999)].

4. A COODNES – OF – FIT TEST FOR AN SDE MODEL

A simple goodness – of-fit test in this section is described. To test if there is a lack – of – fit between the stochastic differential equation model and the data (Allen, 2007).

Assume that $\{X(t), t > 0\}$ the process is observed at time points $t_i = 1,2,\dots,N$. Let $x_1, x_2, \dots, x_N$ denote the observations. For each $t_i, i = 1,2,\dots,N$ simulations are performed to obtain M trajectories from time $t-1$ untiel t starting at x_{t-1}. It is known that Euler – Maruyama (EM) method with R steps to approcsimaity solving SDE (3.2) can be represented by formula

$$X_{j+1,1}^{(m)} = X_{j,i}^{(m)} + f\left(t_{i-1} + j\frac{\Delta t}{K}, X_{j,i}^{(m)}\right)\frac{\Delta t}{K} + g\left(t_{i-1} + j\frac{\Delta t}{K}, X_{j,i}^{(m)}\right)\sqrt{\frac{\Delta t}{K}}\eta_{j,i}^{(m)}, \tag{3.4.1}$$

where $j = 0,1,\dots,K-1$, $m = 1,2,\dots,M$; $t_i = i\Delta t$, $i = 0,1,\dots,K-1$
$X_{0,i}^{(m)} = X_{i-1}$, $\eta_{j,i}^{(m)} \sim N(0,1)$ for each i, j and m.

Moreover $X_{j,i}^{(m)}$ and $\eta_{j,i}^{(m)}$ are calculated at point $\left(t_{i-1} + j\frac{\Delta t}{K}, \omega\right)$ where ω is an elementary outcome $\omega \epsilon \Omega$, Ω is the set of elementary outcomes.

Notices that in formula (3.4.1) for Winer process $W(t_i)$, it used the following relations

$$W(t_{i+1}) - W(t_i) \sim N(0, t_{i+1} - t_i) = N\left(0, \frac{\Delta t}{K}\right) \text{ and } N\left(0, \frac{\Delta t}{K}\right) \equiv \sqrt{\frac{\Delta t}{K}}\eta(0,1)$$

where η is random variable with normal distribution having "0" mean and "j" variance. Indeed, let $\eta(0,1) \sim X$, $N\left(0, \frac{\Delta t}{K}\right) \sim Y$, $\sqrt{\frac{\Delta t}{K}} = \alpha$ and $Y = \alpha x$. Then distribution function $F(y)$ of random variable Y can be defined in the following form

$$F(y) = P(Y < y) = P\left(x < \frac{y}{\alpha}\right) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} \bar{e}^{\frac{x^2}{2}} dx.$$

Substituting $\frac{y}{\alpha} = x, , d\alpha = \frac{dy}{\alpha}$

$$F(y) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2\alpha^2}} \frac{dy}{\alpha} = \int_{-\infty}^{x} \frac{1}{\alpha\sqrt{2\pi}} e^{-\frac{y^2}{2\alpha^2}} dy = N(0, \alpha^2),$$

consequently, $N(0, \alpha^2) = \alpha N(0,1)$.

Here after the rank V_t of valve x_t as compared to the endpoints M simulated trajectories is calculated. With R_t being the stochastic vaviable covvesponding to the observation V_t it hold under H_0, that

$$P(R_t = q) = P_{tq} = \frac{1}{M+1}; q = 1, 2, \dots, M+1; t = 2,3, \dots, N$$

In general, the dependence of P_{tq} on t and q is of interest. However, for every $t = 2,3, \dots, N$ only one observation of R_t is available and therefore we most assume that the probability is independent of time; i.e.: $P_{tq} = P_q$. Under this assumption

$$P_q = \frac{\Omega_{N-1}(a)}{N-1};\ q = 1, 2, \dots, M+1$$

with $\Omega_{N-1}(a) = \sum_{t=2}^{N} I(R_t = q);\ q = 1, 2, \dots, M+1$

where $I(R_t = q) = 1$ if $R_t = q$ and "0" otherwise, under H_0, it is clear that

$$E(P_q) = E\left[\frac{\Omega_{N-1}(a)}{N-1}\right] = \frac{1}{M+1}$$

The Pearson test statistics for the hypotheses that $P_q = \frac{1}{M+1}$; $q = 1,2, \dots, M+1$

$$\mathcal{X}^2 = \sum_{q=1}^{M+1} \frac{\left(\Omega_{N-1}(q) \frac{N-1}{M+1}\right)^2}{\frac{N-1}{M+1}},$$

witch under H_0, as asymptotically is distributed as $\mathcal{X}^2(M)$. The approximation fails when the frequencies expected under H_0 are small in (Allen, 2007). Many researchers have used the role that no expected frequencies should be less than 5. Therefore, in this case $\frac{N-1}{M+1} \geq 5$, yielding an upper bound of $\frac{N-C}{5}$ on M. In practices the number of simulated trajectories will be well below that bound if for instance $N = 500$ the rule requires that no more than 98 inter observation trajectories are simulated for each $t = 2, \dots, N$ (Bak, Nielsen and Madsen, 1999).

5. DISTRIBUTIONS OF APPROXIMATE SOLUTION OF STOCHASTIC DIFFERENTIAL EQUATION

Let

$$\hat{X}(t_i) = \left(x_1^{(i)}, x_2^{(i)}, \dots, x_k^{(i)}\right), i = 0, 1, 2, \dots, K; K = N, 2N, \dots,$$

$$t_i = i\Delta t, \Delta t = \frac{T}{K}, \tag{5.1}$$

where K is used number of steps approximating Euler – Maruyama method, N is number of values of given statistical data. Values $x_1^{(i)}, x_2^{(i)}, \dots, x_k^{(i)}$ $i = 0, 1, 2, \dots, K$; $K = N, 2N, \dots$ of random variable $\hat{X}(t_i)$ are evaluated by starting given statistical data and Euler – Maruyama approximation method. Values of random variables $\hat{X}(t_i)$ are obtained as EM approximations according to located at straight line $t = t_j$, $j = 0,1, \dots, K$; $K = N, 2N, \dots$ successive modal points of sample path. Consequently points $\left(t_i, x_j^{(i)}\right), j = 0,1, \dots, k$; $K = N, 2N, \dots$ Can be considered as nodal points of some approximate trajectories of SDE (3.2).

It is known that the mean square error in EM method satisfies inequality (Allen, 2007):

$$E\left|X(t_i)-\hat{X}(t_i)\right|^2 < \hat{c}\Delta t, \tag{5.2}$$

where $X(t_i)$ is random variable as solution of SDE (3.2),

$$\hat{c}=\frac{1}{2}(1+c)e^{(1+4k)T}, c=2k(T+1)(1+M),$$

$$M=3(E|X(0)|^2+KT^2+KT)e^{3K(T+T^2)}$$

K is represented number in (c_1), $F(c_2)$ conditions. According to Lyapunov inequality from (5.2) follows

$$E\left|X(t_i)-\hat{X}(t_i)\right| \le E(\left|X(t_i)-\hat{X}(t_i)\right|^2)^{\frac{1}{2}} < \hat{c}\Delta t \tag{5.3}$$

From (5.3), it is seen that $\{\hat{X}(t_i)\}$ strongly converges to $X(t_i)$, when $\Delta t \to 0$.

6. GEOM TO OBTAIN APPROCSIMATE PDF FOR SOLUTION OF SDE

Classical method of statistics can be applied to establish pdf of solution $\hat{X}(t_i)$ of SDE. Howerer there are random variables pdf of which cannot be expressed through classical statistical distributions (Ventsel, 1969). Consequentli it is necessary to use GEOM'S to obtain pdf of $X(t_i)$. Since set of GEOD'S is more broadly and sufficiently. It should be noted that if $X(t,\omega)$ is solution of SDE (3.2) then pdf of $X(t,\omega)$ is solution of Kolmogorov – Fokker- Plank equation (Allen, 2007). Consequently, by starting statistical data approximate Euler – Maruyama method with GEOM of obtaining pdf allows to obtain approximate solution $\varphi(t,x)$ of Kolmogorov equation

Let us consider

$$\hat{X}(t_i)=\left(x_1^{(i)},x_2^{(i)},\dots,x_k^{(i)}\right), i=0,1,2,\dots,K; K=N,2N,\dots,t_i=i\Delta t\,,\Delta t=T/K, \tag{5.1}$$

where $\hat{X}(t_i)$ is established by Euler – Maruyama method (3.1) by starting from statistical data $x_0,x_1,\dots,x_{N-1}$ at times $t_0,t_1,\dots,t_{N-1}$.

$$x_1^{(0)}=x_0, x_2^{(0)}=x_1, \dots, x_N^{(0)}=x_{N-1} \text{ , or } x_l^{(0)}=x_{l-1}, l=1,2,\dots,N.$$

As it is state in section 5 points $\left(t_i,x_j^{(i)}\right)$, $j=0,1,2,\dots,K; K=N,2N,\dots$, can be considered as nodal points of approximate trajectories of SDE (3.2). Consequently points $\left(t_i,x_j^{(i)}\right)$, $j=0,1,2,\dots,K$ are values of random variable $X(t_i)$ of approximate solution of SDE (3.2).

7. STOCHASTIC DIFFERENTIAL EQUATION MODEL AVAILABLE TO STATISTICAL DATA

Many stochastic Differential Equations models can be developed by using procedures analogical procedures using to develop ordinary differential models (Allen, 2007). Moreover, there are many stochastic differential equations developed in corresponded statistical areas. Mentioned stochastic differential models can be used in order to obtain SDE model available to statistical data. These problems can be solved by using goodness-of-fit test described in section 4.

If there is goodness-of-fit between statistical data and a stochastic differential equation model, then stochastic differential equation model describes the statistical data.

8. APPLICATION STAGES

1. By using goodness-of-fit test establish stochastic differential equation model available to given statistical data.
2. By using Euler – Maruyama method or other method obtain random variable $\hat{X}(t_i)$ (5.1).
3. By using GEOM'S obtain pdf or distribution at fixed time of approximate solution of SDE (3.1), (3.2).

9. CONCLUSION

In the present study many necessary aspects of GEOM'S and SDE are represented. It is showed that GEOM'S are important models to obtain distributions of approximate solution of SDE. Moreover, application stages of approximate solving SDE and obtaining distributions of solution of SDE are given.

In order to consider applications of GEOM'S in statistical modeling we have formulated and investigated several aspects illustrated in literature.

Approximate solutions of SDE established by using Euler-Maruyama method or other methods are represented by GEOD's.

There are considered MaxEnt, MinxEnt functionals defined on the set of characterizing moment functions. Mentioned functionals allow to define GEOM's: MaxMaxEnt, MinMaxEnt, MaxMinxEnt and MinMinxEnt distributions.

The use GEOD'S in this problem fakes into account that these distributions are broadly and sufficiently with respect to other distributions.

Moreover, EOD's especially as GEOD'S successfully describe distributions of random variables (Shamilov et al., 2008). Consequently, applications of GEOD'S in modeling distributions of approximate solution of SDE acquire significance.

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