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THE PARAMETER ESTIMATION AND RELIABILITY FUNCTION FOR THREE SYSTEMS STRESS – STRENGTH MODEL OF WEIBULL-EXPONENTIAL {RAYLEIGH} DISTRIBUTION

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ABSTRACT

This article investigates the parameters estimation and reliability systems of the stress-strength model, when x and y are independent and follow three parameters Weibull-exponential Rayleigh distribution (WED). Single, parallel, and series systems are derived for WED. The maximum likelihood estimator (MLE), Exact method of moments estimator (EMME), weighted least squares estimator (LSE), and shrinkage function (Shf) estimation methods are examined for its parameters. Different estimators are evaluated using three criteria: bias, mean squared error (MSE), and mean absolute percentage error (MAPE). The best four estimating strategies were compared with a simulation study.

KEYWORDS

Estimation method; MLE; MOM; Parallel reliability; Single reliability; Stress–Strength model; Series reliability; Weibull-exponential {Rayleigh}.

1. INTRODUCTION

Real-world data has been extensively modeled using conventional statistical distributions in numerous fields, including economics, business, industry, medicine, and agriculture. Furthermore, classical distributions in their many forms must be extended. New families of continuous distributions in recent years have been introduced in order to provide new models (Bhat and Ahmad, 2020 and Hamed, Famoye and Lee, 2018). These families have been created by modifying the baseline distribution with one or more shape parameters. One of these created families is Weibull-G, which was described by Bourguignon et al. (2014). Voda (1972) discussed the Raleigh distribution, while Gharraph (1993) provided closed-form formulas for several of its statistical properties. Early work on the exponential Rayleigh distribution was conducted by Rehman and Dar (2015). Because it only has two parameters, the exponential Raleigh distribution does not offer adequate versatility for assessing various forms of lifetime data. As a result, Hemeda and Hamoda (2018) introduced 2-parameters Weibull-exponential {Rayleigh} distribution (WED). This distribution is based on the exponential Raleigh distribution and the Weibull-generated family of distributions. Recently, Ibeh et al., (2021) suggested a new distribution to the family of generalized exponential distributions produced by the transformed-transformer approach called WED with three shape parameters, which are numerically estimated. Recently, Batah and Jalal (2022) constructed a new system of the version of X – Exponential distribution. Kalaf et al. (2024) advocates a new estimation technique related to Meta-heuristic (M-H) algorithms. Nomurous researchers compared some estimation methods of the parallel stress-strength model in many distribution [Raheem, et al. (2019), Hameed, et al., (2020), Jebur, et al, (2021), Kalaf, et al. (2023)]. Now, the probability density, cumulative distribution, reliability, and hazard functions of the WED are provided, respectively, below:

$$f(x,\theta,\mu,\rho) = \rho\mu\theta^{\mu}(2\rho x)^{\frac{\mu}{2}-1}e^{-\theta^{\mu}2\rho x^{\frac{\mu}{2}}}, \quad (1)$$

$$F(x,\theta,\mu,\rho) = 1 - e^{-\theta^{\mu}2\rho x^{\frac{\mu}{2}}}, \quad (2)$$

$$R\,(x,\theta,\mu,\rho) = e^{-\theta^{\mu}2\rho x^{\frac{\mu}{2}}}, \quad (3)$$

$$H(x,\theta,\mu,\rho) = \rho\mu\theta^{\mu}(2\rho x)^{\frac{\mu}{2}-1}. \quad (4)$$

where (θ,μ,ρ) are shape parameters**.** The organization of this article is given as follows. The single, parallel and series reliability systems in stress – strength (S-S) model are derived in section 2. The estimation of its parameters is investigated and presented in section 3. Section 4 demonstrates the flexibility and application of these systems through the usage of simulation study. Finally, some concluding remarks are outline in section 5.

2. THE PROPOSED RELIABILITY SYSTEMS IN THE STRESS-STRENGTH MODEL

In this section, we drive the single, parallel and series reliability systems in stress – strength (S-S) model based on the Weibull-exponential {Rayleigh} Distribution WED (θ,ρ,μ) as follows:

2.1 Single Reliability System

We obtain the single reliability system when x and y are independent and follow the Weibull-exponential Rayleigh distribution WED with three parameters as follows,

$$R_S = P(X > Y) = \int_0^{\infty}\int_0^{x} f(y)f(x)\;dy\,dx, \quad (5)$$

$$= \int_0^{\infty}\int_0^{\mathrm{x}} \rho\,\mu_2\,\theta^{\mu_2}\,(2\rho y)^{\frac{\mu_2}{2}-1}\;e^{-(\theta^{\mu_2}(2\rho y)^{\frac{\mu_2}{2}}}\,\rho\,\mu_1\,\theta^{\mu_1}$$

$$(2\rho x)^{\frac{\mu_1}{2}-1}e^{-(\theta^{\mu_1}(2\rho x)^{\frac{\mu_1}{2}}}\,\mathrm{dy\,dx},$$

$$= 1 - \int_0^{\infty} e^{-Z}\,e^{-C\,(z)^{\frac{\mu_2}{\mu_1}}}\,dz,$$

$$R_S = 1 - \sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\;c\,\Gamma\frac{\mu_2}{\mu_1}\;r+1 \text{ when } c = \theta^{\mu_2}\left(\frac{1}{\theta^{\mu_1}}\right)^{\frac{\mu_2}{\mu_1}} \quad (6)$$

2.2 Series Reliability System

Here, we will drive the series reliability systems as follows: Let $x_i\,(i = 1,2,\dots,k)$ are strength the WED with shape parameters (μ_i). Let (y) is the stress have the WED with shape parameter μ_s when the two parameters (ρ,θ) be known, we have $R_{SS} =$ p[x < z = min$(y_{1,}\, y_{2,}\, y_{3,} \dots y_r)$], then the series reliability system is

$$R_{SS} = p(z > x) = \int_0^\infty F_x(z)\, f(z)dz. \tag{7}$$

where $F_z(z) = p(Z < z) = 1 - p(Z > z) = 1 - p(\min(y_1, y_2, \dots y_r) > Z)$, then the series reliability system is given by

$$R_{SS} = 1 - \sum_{i=1}^{r} \int_0^\infty e^{(-t)} \sum_{r=0}^{\infty} \frac{-1^r\, C}{r!}\, (t)^{\frac{\mu_s}{\mu_i} r}\, dt$$

$$= 1 - \sum_{r=o}^{\infty} \sum_{i=1}^{r} \frac{-1^r\, C}{r!} \int_0^\infty e^{(-t)}\, (t)^{\frac{\mu_s}{\mu_i} r}\, dt$$

$$R_{SS} = 1 - \sum_{r=0}^{\infty} \sum_{i=1}^{r} \frac{-1^r\, C}{r!}\, \Gamma\left(\frac{\mu_s}{\mu_i}\, r + 1\right), \text{for } C = \theta^{\mu_s} \left(\frac{t}{\theta^{\mu_i}}\right)^{\frac{\mu_s}{\mu_i}} \tag{8}$$

2.3 Parallel Reliability System

In this subsection, we drive the parallel reliability stress strength systems as

$$R_{SP} = \text{p}[\text{x} < \text{z} = \max(y_{1,}\, y_{2,}\, y_{3,} \dots\dots . y_r) = \int_0^\infty \bar{F}\, z(x)\, f(x)dx \tag{9}$$

where $F_z(z) = p(Z < z) = p(y_1 < z)p(y_2 < z)p(y_3 < z)$, then the parallel reliability system is give as

$$R_{SP} = \int_0^\infty e^{-z} e^{-\theta^{\mu_2}\left(\frac{z}{\theta^{\mu_{k+1}}}\right)^{\frac{\mu_2}{\mu_{k+1}}}} dz + \int_0^\infty e^{-z}\, e^{-\theta^{\mu_1}\left(\frac{z}{\theta^{\mu_{k+1}}}\right)^{\frac{\mu_1}{\mu_{k+1}}}} dz$$

$$+ \int_0^\infty e^{-z}\, e^{-\theta^{\mu_3}\left(\frac{z}{\theta^{\mu_{k+1}}}\right)^{\frac{\mu_3}{\mu_{k+1}}}} dz$$

$$- \int_0^\infty e^{-z}\, e^{-\theta^{\mu_1}\left(\frac{z}{\theta^{\mu_{k+1}}}\right)^{\mu_1}} e^{-\theta^{\mu_2}\left(\frac{z}{\theta^{\mu_{k+1}}}\right)^{\frac{\mu_2}{\mu_{k+1}}}} dz$$

$$- \int_0^\infty e^{-z} e^{-\theta^{\mu_2}\left(\frac{z}{\theta^{\mu_{k+1}}}\right)^{\frac{\mu_2}{\mu_{k+1}}}} e^{-\theta^{\mu_3}\left(\frac{z}{\theta^{\mu_{k+1}}}\right)^{\frac{\mu_3}{\mu_{k+1}}}} dz$$

$$- \int_0^\infty e^{-z}\, e^{-\theta^{\mu_1}\left(\frac{z}{\theta^{\mu_{k+1}}}\right)^{\frac{\mu_1}{\mu_{k+1}}}} e^{-\theta^{\mu_3}\left(\frac{z}{\theta^{\mu_{k+1}}}\right)^{\frac{\mu_3}{\mu_{k+1}}}} dz$$

$$+ \int_0^\infty e^{-z}\, e^{-\theta^{\mu_1}\left(\frac{z}{\theta^{\mu_{k+1}}}\right)^{\frac{\mu_1}{\mu_{k+1}}}} e^{-\theta^{\mu_2}\left(\frac{z}{\theta^{\mu_{k+1}}}\right)^{\frac{\mu_2}{\mu_{k+1}}}} e^{-\theta^{\mu_3}\left(\frac{z}{\theta^{\mu_{k+1}}}\right)^{\frac{\mu_3}{\mu_{k+1}}}} dz$$

$$R_{SP} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \left[\theta^{\mu_1} \left(\frac{1}{\theta^{\mu_{k+1}}}\right)^{\frac{\mu_1}{\mu_{k+1}}} \right] \Gamma\left(\frac{\mu_{1\,n}}{\mu_{k+1}} + 1\right)$$
$$+ \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \left[\theta^{\mu_2} \left(\frac{1}{\theta^{\mu_{k+1}}}\right)^{\frac{\mu_2}{\mu_{k+1}}} \right] \Gamma\left(\frac{\mu_{2\,m}}{\mu_{k+1}} + 1\right)$$
$$+ \sum_{w=1}^{\infty} \frac{(-1)^w}{w!} \left[\theta^{\mu_3} \left(\frac{1}{\theta^{\mu_{k+1}}}\right)^{\frac{\mu_3}{\mu_{k+1}}} \right] \Gamma\left(\frac{\mu_{3\,w}}{\mu_{k+1}} + 1\right)$$
$$- \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^n}{n!} \frac{(-1)^m}{m!} \left[\theta^{\mu_1} \left(\frac{1}{\theta^{\mu_{k+1}}}\right)^{\frac{\mu_1}{\mu_{k+1}}} \right] \left[\theta^{\mu_2} \left(\frac{1}{\theta^{\mu_{k+1}}}\right)^{\frac{\mu_2}{\mu_{k+1}}} \right] \Gamma\left(\frac{\mu_{1\,n} + \mu_2 m}{\beta_{k+1}} + 1\right)$$
$$- \sum_{n=0}^{\infty} \sum_{w=0}^{\infty} \frac{(-1)^n}{n!} \frac{(-1)^w}{w!} \left[\theta^{\mu_1} \left(\frac{1}{\theta^{\mu_{k+1}}}\right)^{\frac{\mu_1}{\mu_{k+1}}} \right] \left[\theta^{\mu_3} \left(\frac{1}{\theta^{\mu_{k+1}}}\right)^{\frac{\mu_3}{\mu_{k+1}}} \right] \Gamma\left(\frac{\mu_{1\,n} + \mu_3 w}{\beta_{k+1}} + 1\right)$$
$$- \sum_{m=0}^{\infty} \sum_{w=0}^{\infty} \frac{(-1)^m}{m!} \frac{(-1)^w}{w!} \left[\theta^{\mu_2} \left(\frac{1}{\theta^{\mu_{k+1}}}\right)^{\frac{\mu_2}{\mu_{k+1}}} \right] \left[\theta^{\mu_3} \left(\frac{1}{\theta^{\mu_{k+1}}}\right)^{\frac{\mu_3}{\mu_{k+1}}} \right] \Gamma\left(\frac{\mu_{2\,n} + \mu_3 w}{\beta_{k+1}} + 1\right)$$
$$+ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{w=0}^{\infty} \frac{(-1)^n}{n!} \frac{(-1)^m}{m!} \frac{(-1)^w}{w!} \left[\theta^{\mu_1} \left(\frac{1}{\theta^{\mu_{k+1}}}\right)^{\frac{\mu_1}{\mu_{k+1}}} \right] \left[\theta^{\mu_2} \left(\frac{1}{\theta^{\mu_{k+1}}}\right)^{\frac{\mu_2}{\mu_{k+1}}} \right] \left[\theta^{\mu_3} \left(\frac{1}{\theta^{\mu_{k+1}}}\right)^{\frac{\mu_3}{\mu_{k+1}}} \right] \Gamma\left(\frac{\mu_{1\,n} + \mu_2 m + \mu_3 w}{m_{k+1}} + 1\right) \quad (10)$$

3. ESTIMATION METHODS OF THREE SYSTEMS FOR WED(θ, μ, ρ). [Batah (2023), Batah and Abdulrazaq (2022) and Abdalrazaq and Batah (2022)]

This section will describe the estimation procedures for estimating the parameters of single, parallel and series reliability systems in stress – strength (S-S) model based on the Weibull-exponential {Rayleigh} Distribution. We will consider the maximum likelihood method, exact estimators of moment method, weighted least square estimators, and Shrinkage function (Shf).

3.1 Maximum Likelihood Estimator (MLE) [Abid (2014) and Ateeq, et al., (2019)].

To use the method of maximum likelihood for parameter estimation, let $x_{1_1}, x_{1_2}, \ldots, x_{1_{n1}}, x_{2_1}, x_{2_2}, x_{2_{n2}}, \ldots, x_{k_1}, x_{k_2}, \ldots, x_{k_{nk}}$ be a random strength sample of size

(n) from WED$(\theta,\mu,\rho\,)$, where is μ_i unknown parameter and θ,ρ, are known. Then the MLE function is given by:

$$L(x_i,\mu_i,\rho,\theta\,) = \prod_{j=1}^{ni}\left[\prod_{i=1}^{k} f(x_{i_j},\mu_i,\rho,\theta\,)\right]\prod_{h=1}^{m} f(y_h,\mu_s,\rho,\theta\,)$$

$$l = \prod_{i=1}^{ni}\left[\prod_{i=1}^{k}\rho\mu_i\,\theta^{\mu_i}(2\rho x_{i_j})^{\frac{\mu_i}{2}-1}\,e^{-\left(\theta^{\mu_i}(2\rho x_{i_j})^{\frac{\mu_i}{2}}\right)}\right]$$

$$\prod_{h=1\mu_i}^{m}\rho\mu_s\,\theta^{\mu_s}(2\rho y_h)^{\frac{\mu_s}{2}-1}\,e^{-\left(\theta^{\mu_s}(2\rho y_h)^{\frac{\mu_s}{2}}\right)}$$

$$= \prod_{i=1}^{k}\rho^{ni}\,\mu_i{}^{ni}\theta^{ni\mu_i}\prod_{i=1}^{k}\prod_{j=1}^{ni}(2\rho x_{i_j})^{\frac{\mu_i}{2}-1}\;e^{-\sum_{i=1}^{k}\sum_{j=1}^{ni}\theta^{\mu_i}(2\rho x)^{\frac{\mu_i}{2}}}$$

$$\mu_s{}^m\rho^m\theta^{m\mu_s}\prod_{h=1}^{m}(2\rho y_h)^{\frac{\mu_s}{2}-1}\,e^{-\sum_{h=1}^{m}(\theta^{\mu_s}(2\rho y_h)^{\frac{\mu_s}{2}}}$$

Then taking natural logarithm of both sides of the above equation, we get:

$$lnl = \sum_{i=1}^{k} ni\ln\rho + \sum_{i=1}^{k} ni\ln\mu_i + \sum_{i=1}^{k} ni\mu_i\ln\rho + \sum_{i=1}^{k}\frac{\mu_i}{2} - 1\sum_{j=1}^{ni} 2\rho x_{i_j}$$

$$-\sum_{i=1}^{k}\sum_{j=1}^{ni}\theta^{\mu_i}(2\rho x_{i_j})^{\frac{\mu_i}{2}}$$

$$.mln\mu_s + mln\rho + m_{\mu_s}ln\theta + \sum_{h=1}^{m}\frac{\mu_s}{2} - 1\,ln2\rho y_h - \sum_{h=1}^{m}(\theta^{\mu_s}(2\rho y_h)^{\frac{\mu_s}{2}} \qquad (11)$$

By differentiate equation (11) with respect μ_1 and equivalence of the results to zero, we obtained

$$\hat{\mu}_{i_{MLE}} = \frac{-\sum_{i=1}^{k} ni}{\left(\sum_{i=1}^{kl} niln(\theta) + \sum_{j=1}^{ni} 2\rho x_{i_j}\frac{1}{2}\right) - \sum_{i=1}^{k}\sum_{j=1}^{ni}\theta^{\mu_i}(2\rho x_{i_j})^{\frac{\mu_i}{2}}\ln\left(2\rho x_{i_j}\right)\frac{1}{2} + (2\rho x_{i_j})^{\frac{\mu_i}{2}}\,\theta^{\mu_i}\ln(\theta)} \qquad (12)$$

In the same way, let $y_1,\dots y_h$ be a random sample from the stress y which is distributed as WED$(\theta,\rho,\mu_s\,)$, then the likelihood function $f(y_h,\mu_s,\theta,\rho)$ in equation (1) is presented by

$$\hat{\mu}_{s_{M\,LE}} = \frac{-m}{mln\theta + \sum_{h=1}^{m}(2\rho y_h)\frac{1}{2} - \sum_{h=1}^{m}(\theta^{\mu_s}(2\rho y_h)^{\frac{\mu_s}{2}}ln(2\rho y_h)\frac{1}{2} + (2\rho y_h)^{\frac{\mu_s}{2}}\theta^{\mu_s}ln\theta} \qquad (13)$$

Now we substitute equation (12) and (13) in to equations (6, 8, 10), we get $(\hat{R}_{i_{M\,LE}},\hat{R}_{s_{M\,LE}},\hat{R}_{p_{M\,LE}}\,)$

3.2 The Exact Estimators of Moments Method (EMME) [Kao (1958)]

Given that the expected value $E(x)$, the variance $Var(x)$ and the coefficient of variation $CV(x)$ of the WED as follows:

$$E(x) = \frac{\Gamma\left(\frac{2}{\mu_i}+1\right)}{2\rho\theta^2}, Var(x) = \frac{\Gamma\left(\frac{4}{\mu_i}+1\right)}{4\rho^2\theta^4} - \left[\frac{\Gamma\left(\frac{2}{\mu_i}+1\right)}{2\rho\theta^2}\right]^2,$$

and

$$CV(x) = \frac{2\rho\theta^2\sqrt{\frac{\Gamma\left(\frac{4}{\mu_i}+1\right)}{4\rho^2\theta^4} - \left[\frac{\Gamma\left(\frac{2}{\mu_i}+1\right)}{2\rho\theta^2}\right]^2}}{\left(\frac{2}{\mu_i}-1\right)!\,\frac{s}{\bar{\bar{x}}}}$$

Then, the EMME is given below

$$\hat{\mu}_{i_{EMME}} = \frac{2\frac{s}{\bar{\bar{x}}}\left(\frac{2}{\mu_i}-1\right)!}{2\rho\theta^2\sqrt{\frac{\Gamma\left(\frac{4}{\mu_i}+1\right)}{4\rho^2\theta^4} - \left[\frac{\Gamma\left(\frac{2}{\mu_i}+1\right)}{2\rho\theta^2}\right]^2}} \tag{14}$$

$$\hat{\mu}_{s_{EMME}} = \frac{2\frac{s}{\bar{\bar{x}}}\left(\frac{2}{\mu_s}-1\right)!}{2\rho\theta^2\sqrt{\frac{\Gamma\left(\frac{4}{\mu_s}+1\right)}{4\rho^2\theta^4} - \left[\frac{\Gamma\left(\frac{2}{\mu_s}+1\right)}{2\rho\theta^2}\right]^2}} \tag{15}$$

If we substitute equations (14) and (15) into equations (6, 8, 10), we get $(\hat{R}_{i_{EMME}}, \hat{R}_{s_{EMMEE}}, \hat{R}_{PEMME})$.

3.3 Weighted Least Squares Estimators (LSE) [Hassan and Basheikh (2012) and Haddad and Batah (2021)].

Let $F(x_{i_j}, \rho, \theta, \mu_1)$ and $F(y_h, \rho, \theta, \mu_2)$ be two, C. D. F. for the random variables of strength and stress correspondingly

$$t_1 = \sum_{j=1}^{ni}\left(F\left(x_{i_j}\right) - q_j\right)^2, \text{ and } t_2 = \sum_{h=1}^{m}\left(F(y_h) - q_h\right)^2$$

where, $q_j = \frac{j}{ni+1}, j = 1,2,3 \dots n$, and $q_h = \frac{h}{m+1}$, $h = 1,2,3 \dots m$ denote the predictable values of $F(x_{i_j})$, and $F(y_h)$, respectively. From t_1 in equation (2) we get.

$$t_1 = \sum_{j=1}^{ni} \left(\left(\theta^{\mu_i} (2\rho x)^{\frac{\mu_i}{2}} \right) + ln(1 - q_j) \right)^2 .$$

Now, we consider partial derivative of equation with respect to μ_i and the correspondence of the results to zero are:

$$\theta^{\mu_i} = - \frac{\sum_{j=1}^{ni} \left(\left(\theta^{\mu_i} \left(2\rho x_{i_j} \right)^{\frac{\mu_i}{2}} \right) + ln(1 - q_j) \right) \left(\theta^{\mu_i} \left(2\rho x_{i_j} \right)^{\frac{\mu_i}{2}} ln \left(2\rho x_{i_j} \right) \frac{1}{2} \right)}{\sum_{j=1}^{ni} \left(\left(\theta^{\mu_i} \left(2\rho x_{i_j} \right)^{\frac{\mu_i}{2}} \right) + ln(1 - q_j) \right) \left(\left(2\rho x_{i_j} \right)^{\frac{\mu_i}{2}} ln\theta \right)}$$

Let $w_i = \frac{(n+1)^2 \ (n+2)}{(i(n-i+1))}$ and $w_j = \frac{(m+1)^2 \ (m+2)}{(j(m-j+1))}$, we get

$$\hat{\mu}_{i_{WLs}} = \frac{\frac{(n+1)^2 \ (n+2)}{\left(i(n-i+1)\right)} \left[- \frac{\sum_{j=1}^{ni} \left(\left(\theta^{\mu_{i_0}} \left(2\rho x_{i_j} \right)^{\frac{\mu_{i_0}}{2}} \right) + ln(1 - q_j) \right) \left(\theta^{\mu_{i_0}} \left(2\rho x_{i_j} \right)^{\frac{\mu_{i_0}}{2}} ln \left(2\rho x_{i_j} \right) \frac{1}{2} \right)}{\sum_{j=1}^{ni} \left(\left(\theta^{\mu_{i_0}} \left(2\rho x_{i_j} \right)^{\frac{\mu_{i_0}}{2}} \right) + ln(1 - q_j) \right) \left(\left(2\rho x_{i_j} \right)^{\frac{\mu_{i_0}}{2}} ln\theta \right)} \right]}{ln\theta} \tag{16}$$

Let $y_1, \dots, y_h$ be a random sample of size h from the stress (y) which is WED(θ, ρ, μ_s) with shape parameter μ_s we can get the following.

$$\hat{\mu}_{s_{WLS}} = \frac{\frac{(m+1)^2 \ (m+2)}{(j(m-j+1))} ln \left[- \frac{\sum_{j=1}^{ni} \left(\left(\theta^{\mu_{s_0}} (2\rho y_h)^{\frac{\mu_{s_0}}{2}} \right) + ln(1 - q_j) \right) \left(\theta^{\mu_{s_0}} (2\rho y_h)^{\frac{\mu_{s_0}}{2}} ln(2\rho y_h) \frac{1}{2} \right)}{\sum_{j=1}^{ni} \left(\left(\theta^{\mu_{s_0}} (2\rho y_h)^{\frac{\mu_{s_0}}{2}} \right) + ln(1 - q_j) \right) \left((2\rho y_h)^{\frac{\mu_{s_0}}{2}} ln\theta \right)} \right]}{ln\theta} \tag{17}$$

Now we substitute equations (16) and (17) in to equations (6, 8, 10) we get $(\hat{R}_{i_{WLE}}, \hat{R}_{s_{WLE}}, \hat{R}_{p_{WLE}})$.

3.4 Shrinkage function (Shf) [Ibeh et al. (2012) and Tahir et al. (2016)]

The shrinkage estimator using shrinkage function of $\hat{\mu}_i$ and $\hat{\mu}_s$ which is defined as follows,

$$\hat{\mu}_{sh} = \Omega(\hat{\mu})\mu_{MLE} + \left(1 - \Omega(\hat{\mu})\right)\mu_0 \tag{18}$$

The shrinkage function of the sizes x and y in this situations such that, $\Omega(\hat{\mu}_i) = e^{-n}$ and $\Omega(\hat{\mu}_s) = e^{-m}$, where $\Omega(\hat{\mu}) = 0 \leq \Omega(\hat{\mu}) \leq 1$. $\hat{\mu}_i$ and $\hat{\mu}_s$ which is defined in equation (18) as.

$$\hat{\mu}_{i_{shf}} = (e^{-n})\hat{\mu}_{i_{MLE}} + (1 - (e^{-n}))\mu_{i_0} \tag{19}$$

Let $y_1, y_2 \ldots\ldots y_h$ be a random sample of size of size h from the stress (y) which is PRD($\mathfrak{z}$, μ_s) with shape parameter μ_s, we can get the following shf

$$\hat{\mu}_{s_{shf}} = (e^{-m})\hat{\mu}_{s_{MLE}} + \left(1 - (e^{-m})\right)\mu_{s_0} \tag{20}$$

Now we substitute equation (19) and (20) in to equations (6, 8, 10) we get $(\hat{R}_{i_{shf}}, \hat{R}_{s_{shf}}, \hat{R}_{p_{shf}})$.

4. SIMULATION STUDY

Because a theoretical comparison of the estimators is challenging, the performance of the estimators in this section was compared via a simulation study.

4.1 Simulation Design

Step 1: Create a random sample that corresponds to the continuous uniform distribution specified on the interval (0,1) as $u_{i_1}, u_{i_2}, \ldots, u_{i_{ni}}$.

Step 2: Create a random sample from the continuous uniform distribution over (0, 1) as follows: $t_1, t_2, \ldots, t_m$.

Step 3: Transform the uniform random samples to follow the WED and then applying the inverse cumulative distribution function as bellow:

$$F\left(x_{i_j}\right) = 1 - e^{-\left(\theta^{\mu_i}\left(2\rho x_{i_j}\right)^{\frac{\mu_i}{2}}\right)}$$

$$u = 1 - e^{-\left(\theta^{\mu_i}\left(2\rho x_{i_j}\right)^{\frac{\mu_i}{2}}\right)}, \qquad x_{i_j} = e^{\frac{ln\left[\frac{-ln\left(1-u_{i_j}\right)}{\theta^{\mu_i}}\right]}{\frac{\mu_i}{2}} - \ln 2\rho}$$

And by the same way, calculate t_h from Step 2 to obtain the random variable y_h:

$$y_h = e^{\frac{ln\left[\frac{-ln(1-u_h)}{\theta^{\mu_{k+1}}}\right]}{\frac{\mu_{k+1}}{2}} - \ln 2\rho}$$

Step 4: Recall the R_S, R_{SS}, R_{SP} in equations (6, 8, 10).

Step 5: Calculate the MLE for R_S, R_{SS}, R_{SP}

Step 6: Calculate the WLS for R_S, R_{SS}, R_{SP}.

Step 7: Evaluate (Shf) for R_S, R_{SS}, R_{SP}

Step 8: Compute the EMME for R_S, R_{SS}, R_{SP}

Step 9: Founded on (L=1000) replication, the Bias, MSE and MAPE for all proposed estimation methods of R_S, R_{SS}, R_{SP} are calculated as follows.

$$MSE = \frac{1}{L}\sum_{i=1}^{L}(\hat{R}_{3_p} - R_{3_p})^2\,, \text{MAPE} = \frac{1}{l}\sum_{i=1}^{l}\frac{\left|\hat{R}_{i_p - R_{3_p}}\right|}{\left|R_{3_p}\right|}.$$

We consider a random sample for x_{i_j} and y_h of size (n, m) equal from (25, 25), to (100, 100). The simulated reliabilities and their bias, MSE and MAPE are documented in Tables 1 & 2 for single system, in Tables 3 & 4 for series system and in Tables 5 & 6 for parallel system. This simulation was done using MATLAB software 2013.

4.4 Simulation Results Discussion

If we review Table 2, we can see that most of the time EMME performed better than the MLE, WLS, Shf in the sense of smaller absolute bias, smaller MSE and smaller MAPE for the single system. Table 4 indicates that most of the time EMME performed better than the MLE, WLS, Shf in the sense of smaller absolute bias, smaller MSE and smaller MAPE for the series system. We also observed from Table 6 is that most of the time EMME performed better than the MLE, WLS, Shf in the sense of smaller absolute bias, smaller MSE and smaller MAPE for the parallel system.

Table 1

Single System of WED, R=(0.4969648301) θ=0.9 μ_1=5, μ_2=2

n, m	$\hat{R}_{Mle}$	$\hat{R}_{wLse}$	$\hat{R}_{Shf}$	$\hat{R}_{Emm}$
25,25	0.496881810787670	0.502139632342966	0.503626746483009	0.507542809998262
25,50	0.481133293388918	0.484735472186824	0.477507628393329	0.496373601764112
25,75	0.486825387496598	0.507319361090431	0.506202846510810	0.496282309764201
25,100	0.493040531228113	0.489903953556536	0.484642104631163	0.501686911400476
50,25	0.496632796424815	0.489708755131653	0.484457633991397	0.521140053012613
50,50	0.507698124600420	0.511402798865845	0.503078461904262	0.497073243421505
50,75	0.507786708524176	0.476249456218032	0.501666961273627	0.491929774551222
50,100	0.501777953378363	0.502787853755074	0.513783462127467	0.510143436125335
75,25	0.504096684121769	0.503077266049149	0.494064061439343	0.500954834148570
75,50	0.512115039980116	0.507848445756123	0.503559035645450	0.507385588819983
75,75	0.509812646593619	0.513962724850232	0.468797835817926	0.482912711232943
75,100	0.508021085576570	0.506719071171193	0.502721634492269	0.499148061021255
100,25	0.49498680905804	0.49444924528375	0.495821522525393	0.494852811568519
100,50	0.48859255076294	0.50860308419086	0.509264455559704	0.492404005423283
100,75	0.49587288231098	0.48842766903469	0.497385923571654	0.492340464245674
100,100	0.48884931662002	0.48895683797265	0.496812507802477	0.507056371439961

Table 2
Bias, Mse, and Mape of Single System of WED, R=(0.4969648301) θ=0.9 μ_1=5, μ_2=2

n,m	Criteria	$\hat{R}_{Mle}$	$\hat{R}_{wLse}$	$\hat{R}_{Shf}$	$\hat{R}_{Emm}$	Best
25,25	Bias	-0.1529505446	-0.1576927230	-0.1562056089	-0.1622895454	MLE
	Mse	0.10521153851	0.10804617382	0.10357644771	0.11017478001	Shf
	Mape	0.40755268011	0.41704876841	0.40818413252	0.42093270894	MLE
25,50	Bias	-0.1786990620	-0.1750968832	-0.1623247264	-0.1834587536	Shf
	Mse	0.11272724041	0.11048987582	0.11752428802	0.10865597951	EMME
	Mape	0.42835783891	0.42147762061	0.43725628852	0.41487500435	EMME
25,75	Bias	-0.1630069679	-0.1525129943	-0.1536295088	-0.1735500456	Shf
	Mse	0.11461964691	0.10676963584	0.10769082821	0.10358342121	EMME
	Mape	0.42929780023	0.41597953762	0.41620737592	0.4061641364	EMME
25,100	Bias	-0.1667918242	-0.1699284018	-0.1551902507	-0.1781454440	Shf
	Mse	0.11050491281	0.11534500842	0.11501768691	0.10561943261	EMME
	Mape	0.42216518195	0.43612522481	0.42684386861	0.41157045071	EMME
50,25	Bias	-0.1631995584	-0.1701236002	-0.1653747214	-0.1786923023	MLE
	Mse	0.11039674033	0.11042889311	0.11391840102	0.09835132572	EMME
	Mape	0.41739025562	0.41885981591	0.42747874572	0.39222891513	EMME
50,50	Bias	-0.1521342303	-0.1484295561	-0.1567538935	-0.1627591119	MLE
	Mse	0.10325376493	0.10475593571	0.10385051123	0.10846900112	EMME
	Mape	0.40399648562	0.40662221412	0.40315776232	0.00570747501	EMME
50,75	Bias	-0.1520456466	-0.1835828992	-0.1581653941	-0.1679025808	WLS
	Mse	0.10692668915	0.11528751962	0.10998700811	0.11736742815	Shf
	Mape	0.41270485564	0.43404371892	0.42060461583	0.03445185852	EMME
50,100	Bias	-0.1580544023	-0.1570445013	-0.1460488932	-0.1606889191	Shf
	Mse	0.10669432382	0.10919971733	0.10697202302	0.10188864645	EMME
	Mape	0.41299202564	0.41822886883	0.41437843562	0.40325923331	EMME
75,25	Bias	-0.1557356712	-0.1567550893	-0.1576829396	-0.1688775212	MLE
	Mse	0.11095587983	0.10774795762	0.10717628262	0.10617864155	EMME
	Mape	0.42269949884	0.41124236292	0.41794728951	0.40114858083	EMME
75,50	Bias	-0.1477173154	-0.1519839092	-0.1562733197	0.1524467665	MLE
	Mse	0.10495014555	0.10310809164	0.10402378053	0.10612196565	EMME
	Mape	0.40792738715	0.4090533268	0.40963265753	0.41151479645	EMME
75,75	Bias	-0.1500197081	-0.1458696304	-0.1910345195	-0.1969196441	WLS
	Mse	0.10689478804	0.10133791042	0.12009701945	0.11466119907	WLS
	Mape	0.41000617873	0.39804453243	0.44488816024	0.00639184832	EMME
75,100	Bias	-0.1518112692	-0.1531132842	-0.1571107209	-0.1106842943	EMME
	Mse	0.10943735902	0.10755657484	0.10729363233	0.00871376784	EMME
	Mape	0.42197080121	0.41565245283	0.41520308981	0.01817113813	EMME
100,25	Bias	-0.1648455461	-0.1653831102	-0.1640108328	- 0.154979543	EMME
	Mse	0.11099647012	0.10894426323	0.10932301513	0.11179030872	WLS
	Mape	0.41962190506	0.41915671104	0.41637797513	0.12412526413	EMME
100,50	Bias	- 0.171239804	-0.151122927	0.11505678998	- 0.167428349	Shf
	Mse	0.11701418045	0.10757031172	0.10386754754	0.01111431189	EMME
	Mape	0.43551686494	0.41659746603	0.40056008873	0.11894391955	EMME
100,75	Bias	0.1639594730	- 0.171404686	- 0.162446431	- 0.167491891	Shf
	Mse	0.11415458803	0.11478911024	0.11040738003	0.00826042548	EMME
	Mape	0.42972923722	0.43025098144	0.42366888754	0.01434965481	EMME
100,100	Bias	- 0.170983038	- 0.170875517	- 0.163019847	0.152775983	EMME
	Mse	0.10833089031	0.11074437933	0.10386711044	0.01292428743	EMME
	Mape	0.41799505801	0.42137214763	0.40456312105	0.02848338211	EMME

Table 3:
Series System of WED R=(0.5064832202), θ=0.5 μ_1=2, μ_2=2.5, μ_3=3, m_{k+1}=4

n1,n2,n3,n4	$\hat{R}_{Mle}$	$\hat{R}_{Ls}$	$\hat{R}_{Shf}$	$\hat{R}_{Emm}$
25, 25,25, 25	0.4942293207	0.5062535341	0.5066056735	0.6309688982
25, 50,25, 75	0.4679502653	0.4749738278	0.4736578668	0.6336044887
25,75,100,50	0.4661658450	0.4733292034	0.4736578668	0.6336408402
25,75,100,75	0.4668042287	0.4741235513	0.4736578668	0.6336240375
50,50,50, 50	0.4675270223	0.4732308910	0.4736578668	0.6335458754
75,100,25,75	0.4667196403	0.4745634416	0.4736578668	0.6336051300
75,75,75, 75	0.4674406675	0.4734993256	0.4736578668	0.6336110243
100,100,100,100	0.4690556203	0.4736082933	0.4736578668	0.6335163028

Table 4
Bias, Mse, and Mape of Series System of WED R=(0.5064832202),
θ=0.5, $\mu_1 = 2$, $\mu_2 = 2.5$, $\mu_3 = 3$, m_{k+1}=4

n,m	Criteria	$\hat{R}_{Mle}$	$\hat{R}_{wLs}$	$\hat{R}_{Shf}$	$\hat{R}_{Emm}$	Best
25,25, 25,25	**Bias**	-0.01225389	-0.00022968600	0.00122453299	0.0000448567	**EMME**
	Mse	0.00224916	0.00003711178	0.00000014994	0.0550154197	**EMME**
	Mape	0.06662207	0.00931576813	0.002417716810	0.0007843985	**EMME**
25,50, 25,75	**Bias**	-0.00555310	0.001470461491	0.00154500521	0.00010112244	**EMME**
	Mse	0.00140178	0.00003104179	0.00000023870	-0.06371369804	**EMME**
	Mape	0.05896163	0.00936984053	0.00326292339	0.00012034682	**EMME**
25,75, 100,75	**Bias**	-0.00669375	0.00062018503	0.00154500521	0.00012067128	**EMME**
	Mse	0.00102723	0.00002379375	0.00000023870	0.0564167955	**Shf**
	Mape	0.05119807	0.00806199549	0.00326292339	0.33816163237	**Shf**
75,100, 25,75	**Bias**	-0.00837259	0.00106007530	0.00154500519	0.16010176371	**Mle**
	Mse	0.00119577	0.00002643188	0.00000023870	0.02563636221	**Sh3**
	Mape	0.05515270	0.00864121725	0.00326292335	0.00012170114	**EMME**
75,75, 75,75	**Bias**	-0.00606269	-0.0000040406	0.00154500519	0.00010765808	**EMME**
	Mse	0.00107014	0.00002455427	0.00000023870	0.02563761710	**Shf**
	Mape	0.05190733	0.00808042788	0.00326292336	0.00003414957	**EMME**

Table 5
Parallel System of WED R=(0.4177237353),
θ =0.2, $\mu_1 = 0.5$, $\mu_2 = 2$,μ_3 =1, m_{k+1}=3

n1,n2,n3,n4	$\hat{R}_{Mle}$	$\hat{R}_{wLs}$	$\hat{R}_{Shf}$	$\hat{R}_{Emm}$
25,25,25, 25	0.3887180387	0.40619435508	0.4178663352	0.6837593399
25,50,25, 75	0.4015858491	0.4164441020	0.4178663352	0.7415737420
25,75,100,50	0.4077553118	0.4112169251	0.4178663352	0.7439737987
25,75,100,75	0.4091800472	0.4149210490	0.4178663352	0.7451729641
50,50,50, 50	0.4026787250	0.4108642679	0.4178663352	0.7415859331
75,100,25, 75	0.4074052677	0.4169956042	0.4178663352	0.7436130739
75,75,75, 75	0.4048310801	0.4155629399	0.4178663352	0.7442241477
100,100,100,100	0.4093262937	0.4148659529	0.4178663352	0.7470745915

Table 6
Bias, Mse, and Mape of Parallel System of WED R=(0.4177237353)
θ =0.2, $\mu_1 = 0.5$, $\mu_2 = 2$, $\mu_3 = 1$, m_{k+1}=3

n,m	Criteria	$\hat{R}_{Mle}$	$\hat{R}_{wLs}$	$\hat{R}_{Shf}$	$\hat{R}_{Emm}$	Best
25,25, 25,25	**Bias**	-0.02900569	-0.01152938	0.00142599912	0.26603560455	**Mle**
	Mse	0.00000000	0.008762584	0.00000020334	1.34982420848	**Mle**
	Mape	0.00186024	0.135528199	0.00341373737	0.0008777368	**EMME**
25,50, 25,75	**Bias**	-0.01613788	-0.001279638	0.00142599912	0.00000000661	**EMME**
	Mse	0.00000000	0.002038790	0.00000020334	0.10686402709	**Mle**
	Mape	0.000 69175	0.084548032	0.00341373735	0.00007317503	**EMME**
25,75, 100,75	**Bias**	-0.00854368	-0.002802686	0.00142599912	0.32744922876	**LS**
	Mse	0.00546883	0.001709268	0.00000020334	0.10887306389	**Shf**
	Mape	0.13621512	0.077776601	0.00341373737	0.00009812687	**EMME**
75,100, 25,75	**Bias**	-0.01031846	-0.000728131	0.00142599911	0.32588933858	**Mle**
	Mse	0.00531634	0.001767524	0.00000020334	0.10766140059	**Sh3**
	Mape	0.00056431	0.078930929	0.00341373734	0.00005518626	**EMME**
75,75, 75,75	**Bias**	-0.0128926	-0.002160787	0.00142599911	0.32650041232	**Mle**
	Mse	0.00495224	0.001366483	0.00000020334	0.10767131235	**Sh3**
	Mape	0.00046805	0.069981000	0.00341373733	0.00001805200	**EMME**

5. SOME CONCLUDING REMARKS

In this article, we developed the stress-strength model for the Weibull-exponential Rayleigh distribution for single, series, and parallel systems. The parameters of the systems are estimated using maximum likelihood method, exact technique of moment, weighted least squares, and shrinkage approach. To compare the performance of the estimators, a simulation study has been conducted. When we look at Table 2, and Table 4, we can see that EMME outperformed the MLE, WLS, and Shf in terms of absolute bias, MSE, and MAPE for the single and series systems the common of the time. Table 6 further shows that EMME outperformed the MLE, WLS, and Shf in terms of absolute bias, MSE, and MAPE for the parallel system most of the time. The exact estimators of moments (EMME) is appeared to be the best methods among four and can be recommended for the practitioners. The statistical inference about the parameters of the WED would be interesting, which is an open problem for the researchers.

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