Pak. J. Statist.
2024 Vol. 40(1), 105-122

A NOVEL OPTIMIZATION ALGORITHM FOR ESTIMATING THE PARAMETERS OF THE TRUNCATED DISTRIBUTION DEPENDING ON SURVIVAL FUNCTION

Noor Abdul Ameer Jabbar[1] and **Bayda Atiya Kalaf**[2]
Department of Mathematics
College of Education for Pure Sciences (Ibn Al-Haitham)
University of Baghdad, Iraq
Email: [1]naji198887@yahoo.com
[2]baydaa.a.k@ihcoedu.uobaghdad.edu.iq

ABSTRACT

This paper introduces Right Truncated Inverse Generalized Rayleigh distribution (RTIGRD) with two parameters λ and θ with some of its properties as; (Survival-Function, Probability Density Function, Hazard-Function, Cumulative Distribution Function, R-Th Moment, Mean, Variance, Median, Moment Generating Function, and Mode. In addition, we propose a new hybrid algorithm (Artificial Bee Colony Algorithm with Firefly Algorithm (ABC_FA)) to estimate Survival functions based on the parameters (λ, θ) of (RTIGRD). Simulation is utilized to compare the proposed algorithm with traditional methods (Maximum Likelihood Estimator and moment method) and standard algorithms (Artificial Bee Colony Algorithm and Firefly Algorithm). The results show proposed approach (ABC_FA) provides a 100% accurate estimate of the survival function for the cases selected in this research, as it has a less mean square error than other estimation methods.

KEY WORDS

Bee Colony Algorithm, Firefly Algorithm, Hybrid Algorithm, Right Truncated Inverse Generalized Rayleigh distribution, Maximum Likelihood Estimation Method, moment method.

1. INTRODUCTION

A Truncated Distribution is a conditional distribution on a specific range that restricted the full range. Also, it arise in large of fields, such as epidemiology, survival analysis, economics, and astronomy [1]. When a distribution is truncated, the domain of the resulting random variable is constrained based on the truncation points of interest, changing the distribution's shape in order to produce finer results. Consequently, many researchers are drawn to analyze such truncated data utilizing truncated versions of the standard statistical distributions in Survival Analysis. In recent years, statisticians have been interested in estimating survival functions [2-7] and Estimating survival by using Truncated Distribution [8]. They proposes the method of moments to compute the moment expression for two and three parameters, and truncated at (right, left and doubly) utilizing Weibull distributions. In [9] calculated an estimate for the value of the parameter in the truncated

Gamma probability distribution. Also, in [10] proposed skew-Cauchy and truncated skew-Cauchy at probability for modeling the exchange rate between the US Dollar from 1800 to 2003 and the UK Pound Sterling and results showed the truncated skew-Cauchy is a better probability as a function to model the dataset in skew-Cauchy. On the other hand, Rayleigh distribution is utilized as life time models and has applications including reliability theory, survival analysis, and especially in communication engineers [11], furthermore, the Right Truncated Inverse Generalized Rayleigh Distribution will be utilized to estimating its parameters by utilizing the Survival function. However, the problem of unknown parameters is estimated in the Truncated or statistical Distributions which are utilize to study certain phenomena in one of the important problems facing constantly those who are interested in applied statistics. The non-linearity of any model makes the estimation of the statistical analysis and the parameters are more difficult. In addition, classical methods fail to estimate the parameters of the distribution. The Meta heuristic and hybrid Meta heuristic algorithms are used to find solutions to these problems. For many reasons, not the least of which is that it is easy to put into practice while still being dependable, robust, and effective [12]. So, this study aimed to introduce a hybrid algorithms to estimate the parameters of Right Truncated Inverse Generalized Rayleigh Distribution based on Survival functions.

After this section, the paper is structured as follows: Section 2 describes the Right Truncated Inverse Generalized Rayleigh distribution. Section 3 presents some properties of Right Truncated Inverse Generalized Rayleigh distribution. Section 4 and section 5 introduce Maximum Likelihood Estimation and Moment Estimation Method, receptively. Section 6 describes the Hybrid Meta heuristic Algorithm, Simulation study, Numerical results and discussion are presented in Section 7 and Section 8. Section 9 conclusions and Section 10 Recommendation.

2. TRUNCATED INVERSE GENERALIZED RAYLEIGH DISTRIBUTION

The Rayleigh distribution is a good model for life-experimentation studies; it was derived from the Weibull distribution, which has only two parameters [13];

$$f(x) = \frac{2X}{\lambda}\ e^{-\frac{x^2}{\lambda}}. \tag{1}$$

$$F(x) = 1 - e^{-\frac{x^2}{\lambda}}. \tag{2}$$

Mudholkar and Srivastava [14] suggested a new method for generalization different distribution dependent on c.d.f, which we will be used to Generalized Rayleigh Distribution as follows:

$$\begin{aligned} G(x) &= [F(x)]^{\theta}. \\ &= \left(\left[1 - e^{-\frac{x^2}{\lambda}}\right]^{\theta}\right). \end{aligned} \tag{3}$$

$$g(x) = \left(\left[1 - e^{-\frac{x^2}{\lambda}}\right]^{\theta-1}\right) \frac{2x\,\theta}{\lambda}\ e^{-\frac{x^2}{\lambda}}. \tag{4}$$

The Generalized Raleigh Distribution illustrated by the random variable transformation. If T are the random variable (rv) and it has Generalized Raleigh

Distribution, then rv $X = \left(\frac{1}{T}\right)$ has an inverse Generalized Raleigh Distribution (IGRD). Suppose rv T is following inverse Generalized Rayleigh distribution at two parameters θ and λ. Then p.d.f, c.d.f function of Inverse Generalized Rayleigh Distribution are given for the equations (3) and (4), respectively by [15]

$$g(t) = \left[1 - e^{-\frac{1}{\lambda t^2}}\right]^{\theta-1} \frac{2\theta}{\lambda t^3}\, e^{-\frac{1}{\lambda t^2}}.$$

$$G(t) = 1 - \left[1 - e^{-\frac{1}{\lambda t^2}}\right]^{\theta}. \tag{5}$$

When $0 < t < \infty$ and $g(t) = 0$ o.w.

Hence for truncation for the Inverse Generalized Rayleigh Distribution, Right-Side Truncation for the Inverse Generalized Rayleigh Distribution to called Right Truncated Inverse Generalized Rayleigh Distribution (RTIGRD) on [0,1] by using $f_{RTIGRD}(t) = \frac{g(t)}{G(1)}$, When $t = 1$ in equation (5) $G(1) = \left(1 - \left[1 - e^{-\frac{1}{\lambda}}\right]^{\theta}\right)$

$$f_{RTIGRD}(t) = \frac{g(t)}{G(1)}$$

Then, The p.d.f of RTIGRD is

$$f_{RTIGRD}(t) = \frac{\frac{2\theta}{\lambda t^3}\left[1 - e^{-\frac{1}{\lambda t^2}}\right]^{\theta-1} e^{-\frac{1}{\lambda t^2}}}{1 - \left[1 - e^{-\frac{1}{\lambda}}\right]^{\theta}}, \quad 0 \le t \le 1.$$

The c.d.f of RTIGRD is

$$F_{RTIGRD}(t) = \int_0^t \frac{\left[1 - e^{-\frac{1}{\lambda t^2}}\right]^{\theta-1} \frac{2\theta}{\lambda t^3}\, e^{-\frac{1}{\lambda t^2}}}{1 - \left[1 - e^{-\frac{1}{\lambda}}\right]^{\theta}}\, dt$$

Therefore,

$$F_{RTIGRD}(t) = \frac{1 - \left[1 - e^{-\frac{1}{\lambda t^2}}\right]^{\theta}}{1 - \left[1 - e^{-\frac{1}{\lambda}}\right]^{\theta}}.$$

The Survival Function of RTIGRD is

$$S_{RTIGRD}(t) = 1 - F_{RTIGRD}(t)$$
$$= 1 - \frac{1 - \left[1 - e^{-\frac{1}{\lambda t^2}}\right]^{\theta}}{1 - \left[1 - e^{-\frac{1}{\lambda}}\right]^{\theta}} = \frac{1 - \left[1 - e^{-\frac{1}{\lambda}}\right]^{\theta} - 1 + \left[1 - e^{-\frac{1}{\lambda t^2}}\right]^{\theta}}{1 - \left[1 - e^{-\frac{1}{\lambda}}\right]^{\theta}}$$

$$S_{RTIGRD}(t) = \frac{\left[1 - e^{-\frac{1}{\lambda t^2}}\right]^{\theta} - \left[1 - e^{-\frac{1}{\lambda}}\right]^{\theta}}{1 - \left[1 - e^{-\frac{1}{\lambda}}\right]^{\theta}}. \qquad (6)$$

where, $0 < \mathrm{t} < 1;$ t: value of random variable, θ and λ: Shape parameter, Scale parameter $\theta, \lambda > 0$, respectively.

3. SOME PROPERTIES OF RIGHT TRUNCATED INVERSE GENERALIZED RAYLEIGH DISTRIBUTION

In this section, some properties gave for RTIGRD. However, some properties are complicated to be solved. Therefore, we made some simplification for the p.d.f. by using Binomial theorem and Tyler series

$$(a \mp x)^n = \sum_{j=0}^{n} \binom{n}{j} (\mp x)^j a^{n-j} \left[1 - e^{-\frac{1}{\lambda t^2}}\right]^{\theta-1} = \sum_{j=0}^{\theta-1} \binom{\theta-1}{j} (-e^{-\frac{1}{\lambda t^2}})^j$$

Thus,

$$f_{RTIGRD}(t) = \frac{\sum_{j=0}^{\theta-1} \binom{\theta-1}{j} (-1)^j \, e^{-\frac{j}{\lambda t^2}} \, \frac{2\theta}{\lambda t^3} \, e^{-\frac{1}{\lambda t^2}}}{1 - \left[1 - e^{-\frac{1}{\lambda}}\right]^{\theta}}$$

$$f_{RTIGRD}(t) = \frac{\sum_{j=0}^{\theta-1} \binom{\theta-1}{j} (-1)^j \, \frac{2\theta}{\lambda t^3} \, e^{-\frac{j+1}{\lambda t^2}}}{1 - \left[1 - e^{-\frac{1}{\lambda}}\right]^{\theta}}$$

Let

$$e^{-\frac{j+1}{\lambda t^2}} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \, \frac{(j+1)^k}{(\lambda t^2)^k}$$

$$f_{RTIGRD}(t) = \frac{\sum_{j=0}^{\theta-1} \binom{\theta-1}{j} (-1)^j \, \frac{2\theta}{\lambda t^3} \, \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \, \frac{(j+1)^k}{(\lambda t^2)^k}}{1 - \left[1 - e^{-\frac{1}{\lambda}}\right]^{\theta}}$$

$$= \frac{\sum_{k=0}^{\infty} \sum_{j=0}^{\theta-1} \frac{(-1)^{k+j}}{k!} \binom{\theta-1}{j} \frac{(j+1)^k}{(\lambda)^k \, t^{2k}} \, \frac{2\theta}{\lambda t^3}}{1 - \left[1 - e^{-\frac{1}{\lambda}}\right]^{\theta}}$$

$$= \frac{\sum_{k=0}^{\infty} \sum_{j=0}^{\theta-1} \frac{(-1)^{k+j}}{k!} \binom{\theta-1}{j} \frac{2\theta (j+1)^k}{(\lambda)^{k+1} \, t^{2k+3}}}{1 - \left[1 - e^{-\frac{1}{\lambda}}\right]^{\theta}}$$

$$f_{RTIGRD}(t) = \frac{\sum_{k=0}^{\infty}\sum_{j=0}^{\theta-1}\frac{(-1)^{k+j}}{k!}\binom{\theta-1}{j}\frac{2\theta(j+1)^k}{(\lambda)^{k+1}}\ t^{-2k-3}}{1-\left[1-e^{-\frac{1}{\lambda}}\right]^{\theta}}, 0 \leq t \leq 1$$

3.1 R-Th Moment

The R-Th moment can be derived as follow:

$$E(t^r) = \int_0^1 t^r\ f_{RTIGRD}(t)\ dt$$

$$= \int_0^1 t^r \frac{\sum_{k=0}^{\infty}\sum_{j=0}^{\theta-1}\frac{(-1)^{k+j}}{k!}\binom{\theta-1}{j}\frac{2\theta(j+1)^k}{(\lambda)^{k+1}}\ t^{-2k-3}}{1-\left[1-e^{-\frac{1}{\lambda}}\right]^{\theta}}\ dt$$

$$= \frac{\sum_{k=0}^{\infty}\sum_{j=0}^{\theta-1}\frac{(-1)^{k+j}}{k!}\binom{\theta-1}{j}\frac{2\theta(j+1)^k}{(\lambda)^{k+1}}}{1-\left[1-e^{-\frac{1}{\lambda}}\right]^{\theta}}\int_0^1 t^{r-2k-3}dt$$

$$= \frac{\sum_{k=0}^{\infty}\sum_{j=0}^{\theta-1}\frac{(-1)^{k+j}}{k!}\binom{\theta-1}{j}\frac{2\theta(j+1)^k}{(\lambda)^{k+1}}}{1-\left[1-e^{-\frac{1}{\lambda}}\right]^{\theta}}\ \frac{t^{r-2k-2}}{r-2k-2}\Big|_0^1$$

$$E(t^r) = \frac{\sum_{k=0}^{\infty}\sum_{j=0}^{\theta-1}\frac{(-1)^{k+j}}{k!}\binom{\theta-1}{j}\frac{2\theta\ (j+1)^k}{(\lambda)^{k+1}\ (r-2k-2)}}{1-\left[1-e^{-\frac{1}{\lambda}}\right]^{\theta}}$$

When $r = 1$, the mean of RTIGRD equal to

$$\mu = E(t) = \frac{\sum_{k=0}^{\infty}\sum_{j=0}^{\theta-1}\frac{(-1)^{k+j}}{k!}\binom{\theta-1}{j}\frac{2\theta\ (j+1)^k}{(\lambda)^{k+1}\ (1-2k-2)}}{1-\left[1-e^{-\frac{1}{\lambda}}\right]^{\theta}}$$

$$\mu = E(t) = \frac{\sum_{k=0}^{\infty}\sum_{j=0}^{\theta-1}\frac{(-1)^{k+j}}{k!}\binom{\theta-1}{j}\frac{(j+1)^k\ 2\theta}{(\lambda)^{k+1}\ (-2k-1)}}{1-\left[1-e^{-\frac{1}{\lambda}}\right]^{\theta}}$$

When $r = 2$, we will get $E(t^2)$

$$E(t^2) = \frac{\sum_{k=0}^{\infty}\sum_{j=0}^{\theta-1}\frac{(-1)^{k+j}}{k!}\binom{\theta-1}{j}\frac{2\theta(j+1)^k}{(\lambda)^{k+1}\ (-2k)}}{1-\left[1-e^{-\frac{1}{\lambda}}\right]^{\theta}}$$

$$E(t^2) = \frac{\sum_{k=0}^{\infty}\sum_{j=0}^{\theta-1}\frac{(-1)^{k+j}}{k!}\binom{\theta-1}{j}\frac{\theta(j+1)^k}{(\lambda)^{k+1}\,(-k)}}{1-\left[1-e^{-\frac{1}{\lambda}}\right]^{\theta}}$$

When $r = 3$

$$E(t^3) = \frac{\sum_{k=0}^{\infty}\sum_{j=0}^{\theta-1}\frac{(-1)^{k+j}}{k!}\binom{\theta-1}{j}\frac{\theta(j+1)^k}{(\lambda)^{k+1}\,(1-2k)}}{1-\left[1-e^{-\frac{1}{\lambda}}\right]^{\theta}}$$

When $r = 4$

$$E(t^4) = \frac{\sum_{k=0}^{\infty}\sum_{j=0}^{\theta-1}\frac{(-1)^{k+j}}{k!}\binom{\theta-1}{j}\frac{\theta(j+1)^k}{(\lambda)^{k+1}\,(2-2k)}}{1-\left[1-e^{-\frac{1}{\lambda}}\right]^{\theta}}$$

3.2 Variance

The Variance (Var) of RTIGRD can be found as follows:

$$\sigma^2 = Var(t) = E(t^2) - [E(t)]^2$$

$$Var(t) = \frac{\sum_{k=0}^{\infty}\sum_{j=0}^{\theta-1}\frac{(-1)^{k+j}}{k!}\binom{\theta-1}{j}\frac{\theta(j+1)^k}{(\lambda)^{k+1}\,(-k)}}{1-\left[1-e^{-\frac{1}{\lambda}}\right]^{\theta}} - \left[\frac{\sum_{k=0}^{\infty}\sum_{j=0}^{\theta-1}\frac{(-1)^{k+j}}{k!}\binom{\theta-1}{j}\frac{2\theta(j+1)^k}{(\lambda)^{k+1}\,(-2k-1)}}{\left(1-\left[1-e^{-\frac{1}{\lambda}}\right]^{\theta}\right)}\right]^2$$

$$= \frac{\left[\left[1-\left[1-e^{-\frac{1}{\lambda}}\right]^{\theta}\right]\sum_{k=0}^{\infty}\sum_{j=0}^{\theta-1}\frac{(-1)^{k+j}}{k!}\binom{\theta-1}{j}\frac{\theta(j+1)^k}{(\lambda)^{k+1}\,(-k)} - \left[\sum_{k=0}^{\infty}\sum_{j=0}^{\theta-1}\frac{(-1)^{k+j}}{k!}\binom{\theta-1}{j}\frac{2\theta(j+1)^k}{(\lambda)^{k+1}\,(-2k-1)}\right]^2\right]}{\left[1-\left[1-e^{-\frac{1}{\lambda}}\right]^{\theta}\right]^2}$$

3.3 Moment Generating Function

$$M_t(ŧ) = E(e^{ŧt}) = \int_0^1 e^{ŧt}\, f_{RTIGRD}(t)\,dt$$

$$M_t(ŧ) = \int_0^1 e^{ŧt} \frac{\sum_{k=0}^{\infty}\sum_{j=0}^{\theta-1} \frac{(-1)^{k+j}}{k!} \binom{\theta-1}{j} \frac{2\theta(j+1)^k}{(\lambda)^{k+1}}\ t^{-2k-3}}{1-\left[1-e^{-\frac{1}{\lambda}}\right]^{\theta}}\ dt$$

$$= \int_0^1 e^{ŧt}\ t^{-2k-3} dt$$

Use,

$$e^{ŧt} = \sum_{n=0}^{\infty} \frac{(ŧt)^n}{n!}$$

$$= \frac{\sum_{k=0}^{\infty}\sum_{j=0}^{\theta-1} \frac{(-1)^{k+j}}{k!} \binom{\theta-1}{j} \frac{2\theta(j+1)^k}{(\lambda)^{k+1}}}{1-\left[1-e^{-\frac{1}{\lambda}}\right]^{\theta}} \int_0^1 \sum_{n=0}^{\infty} \frac{(ŧt)^n}{n!}\ t^{-2k-3}\ dt$$

$$= \frac{\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\theta-1} \frac{(-1)^{k+j}\ (ŧ)^n}{k!\ n!} \binom{\theta-1}{j} \frac{2\theta\ (j+1)^k}{(\lambda)^{k+1}}}{1-\left[1-e^{-\frac{1}{\lambda}}\right]^{\theta}} \int_0^1 t^{n-2k-3}\ dt$$

$$= \frac{\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\theta-1} \frac{(-1)^{k+j}\ (ŧ)^n}{k!\ n!} \binom{\theta-1}{j} \frac{2\theta(j+1)^k}{(\lambda)^{k+1}}}{1-\left[1-e^{-\frac{1}{\lambda}}\right]^{\theta}} \frac{t^{n-2k-2}}{n-2k-2}\Big|_0^1$$

$$M_t(ŧ) = \frac{\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\theta-1} \frac{(-1)^{k+j}\ (ŧ)^n}{k!\ n!} \binom{\theta-1}{j} \frac{2\theta(j+1)^k}{(\lambda)^{k+1}\ (n-2k-2)}}{1-\left[1-e^{-\frac{1}{\lambda}}\right]^{\theta}}$$

3.4 Median

$$F_{RTIGRD}(t) = \frac{1}{2} \Longrightarrow \frac{1-\left[1-e^{-\frac{1}{\lambda t^2}}\right]^{\theta}}{1-\left[1-e^{-\frac{1}{\lambda}}\right]^{\theta}} = \frac{1}{2},$$

$$t_{Median} = \sqrt{\frac{1}{\lambda\ ln\left[1-\left[\frac{1+\left[1-e^{-\frac{1}{\lambda}}\right]^{\theta}}{2}\right]^{\frac{1}{\theta}}\right]^{-1}}}.$$

3.5 Mode

$$f_{RTIGRD}(t)=\frac{\left[1-e^{-\frac{1}{\lambda t^2}}\right]^{\theta-1}\frac{2\theta}{\lambda t^3}\,e^{-\frac{1}{\lambda t^2}}}{1-\left[1-e^{-\frac{1}{\lambda}}\right]^{\theta}},0\le t\le 1,$$

$$t_{Mode}=\sqrt{2\sum_{j=0}^{\theta-1}\frac{j+1}{3\lambda}}.$$

4. MAXIMUM LIKELIHOOD ESTIMATION (MLE)

Let $t_1,t_2,\dots,t_n$ to be the random samples (rs) with size (n) from p.d.f $f(t;\lambda,\theta)$ with parameters (λ, θ), the likelihood function $L(\lambda,\theta)$ is the joint p.d.f of the (rs) is computed as:

$$L\,(t,t_2\,,\dots\dots,t_n,\lambda,\theta)=\prod_{i=1}^{n}f_{RTIGRD}(t_i)=\prod_{i=1}^{n}\frac{\left(1-e^{-\frac{1}{\lambda t_i^2}}\right)^{\theta-1}\frac{2\theta}{\lambda t_i^3}\,e^{-\frac{1}{\lambda t_i^2}}}{1-\left(1-e^{-\frac{1}{\lambda}}\right)^{\theta}}$$

$$=2^n\frac{\theta^n}{\lambda^n}\prod_{i=1}^{n}\left(\frac{\left(1-e^{-\frac{1}{\lambda t_i^2}}\right)^{\theta-1}\frac{1}{t_i^3}\,e^{-\frac{1}{\lambda t_i^2}}}{1-\left(1-e^{-\frac{1}{\lambda}}\right)^{\theta}}\right)$$

Taking the natural logarithm to the two sides to get:

$$LnL\;=Ln\,(2^n\frac{\theta^n}{\lambda^n})+\sum_{i=1}^{n}ln\left(\frac{\left(1-e^{-\frac{1}{\lambda t_i^2}}\right)^{\theta-1}\frac{1}{t_i^3}\,e^{-\frac{1}{\lambda t_i^2}}}{1-\left(1-e^{-\frac{1}{\lambda}}\right)^{\theta}}\right)$$

$$LnL=nLn(2)+nLn(\theta)-nLn(\lambda)-\sum_{i=1}^{n}Ln(t_i^{\,3})+\sum_{i=1}^{n}Ln\left(e^{-\frac{1}{\lambda t_i^2}}\right)$$
$$+\sum_{i=1}^{n}Ln\left(\left(1-e^{-\frac{1}{\lambda t_i^2}}\right)^{\theta-1}\right)-ln\left(1-\left(1-e^{-\frac{1}{\lambda}}\right)^{\theta}\right)^{n}$$

$$LnL = nLn(2) + nLn(\theta) - nLn(\lambda) - 3\sum_{i=1}^{n} Ln(t_i) + \sum_{i=1}^{n}\left(\frac{-1}{\lambda {t_i}^2}\right) + (\theta - 1)\sum_{i=1}^{n} Ln\left(1 - e^{-\frac{1}{\lambda {t_i}^2}}\right) - \left(1 - \left(1 - e^{-\frac{1}{\lambda}}\right)^{\theta}\right) \tag{7}$$

for equation (7) partial derivative with respect to the unknown parameters (λ, θ):

$$\frac{\partial\, Ln\, L}{\partial \lambda} = \frac{-n}{\lambda} + \sum_{i=1}^{n}\left(\frac{1}{\lambda^2\, {t_i}^2}\right) + (\theta - 1)\sum_{i=1}^{n}\left(\frac{\frac{-1}{\lambda^2\, {t_i}^2}\, e^{-\frac{1}{\lambda {t_i}^2}}}{\left(1 - e^{-\frac{1}{\lambda {t_i}^2}}\right)}\right) - \left(\frac{\theta\left(1 - e^{-\frac{1}{\lambda}}\right)^{\theta - 1}\frac{1}{\lambda^2}\, e^{-\frac{1}{\lambda}}}{1 - \left(1 - e^{-\frac{1}{\lambda}}\right)^{\theta}}\right)$$

$$\frac{-n}{\lambda} + \sum_{i=1}^{n}\left(\frac{1}{\lambda^2\, {t_i}^2}\right) + (\theta - 1)\sum_{i=1}^{n}\left(\frac{\frac{-1}{\lambda^2\, {t_i}^2}\, e^{-\frac{1}{\lambda {t_i}^2}}}{\left(1 - e^{-\frac{1}{\lambda {t_i}^2}}\right)}\right) - \left(\frac{\theta\left(1 - e^{-\frac{1}{\lambda}}\right)^{\theta - 1}\frac{1}{\lambda^2}\, e^{-\frac{1}{\lambda}}}{1 - \left(1 - e^{-\frac{1}{\lambda}}\right)^{\theta}}\right) = 0 \tag{8}$$

$$\frac{\partial\, Ln\, L}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^{n}\left(1 - e^{-\frac{1}{\lambda {t_i}^2}}\right)$$

$$\frac{n}{\theta} + \sum_{i=1}^{n}\left(1 - e^{-\frac{1}{\lambda {t_i}^2}}\right) = 0 \tag{9}$$

However, the two non-linear equations are complicated; the Newton-Raphson method was utilized to estimates λ and θ.

So, the estimated survival function $\hat{S}_{RTIGRD}(t)$ by MLE will be:

$$\hat{S}_{\substack{MLE\\ RTIGRD}}(t) = \frac{\left(1 - e^{-\frac{1}{\hat{\lambda}_{MLE}\, t^2}}\right)^{\hat{\theta}_{MLE}} - \left(1 - e^{-\frac{1}{\hat{\lambda}_{MLE}}}\right)^{\hat{\theta}_{MLE}}}{1 - \left(1 - e^{-\frac{1}{\hat{\lambda}_{MLE}}}\right)^{\hat{\theta}_{MLE}}} \tag{10}$$

Know, the objective function of FA and ABC algorithm depends on minimize the log likelihood function as objective function (fitness function).

$$F_{MLE} = nLn(2) + nLn(\theta) - nLn(\lambda) - 3\sum_{i=1}^{n} Ln(t_i) + \sum_{i=1}^{n}\left(\frac{-1}{\lambda {t_i}^2}\right) + (\theta - 1)\sum_{i=1}^{n} Ln\left(1 - e^{-\frac{1}{\lambda {t_i}^2}}\right) - nln\left(1 - \left(1 - e^{-\frac{1}{\lambda}}\right)^{\theta}\right) \quad (11)$$

5. MOMENTS ESTIMATION METHOD (MOM):

The MOM will be utilizes to estimating the parameters λ, and θ for Right Truncated Inverse Generalized Rayleigh Distribution on [0,1]. MOM is found by equating the sample moments to the corresponding population moment [16]

$$E(t^k) = \frac{1}{n}\sum_{i=1}^{n} t_i^k \text{, where } k = 1, 2, \dots$$

The first and second moment of population and sample for λ, and θ of RTIGRD is given depending on the general form of t^r moment respectively as follows:

$$E(t^r) = \frac{\sum_{k=0}^{\infty}\sum_{j=0}^{\theta-1} \frac{(-1)^{k+j}}{k!}\binom{\theta-1}{j}\frac{2\theta(j+1)^k}{(\lambda)^{k+1}(r-2k-2)}}{1-\left[1-e^{-\frac{1}{\lambda}}\right]^{\theta}}$$

When $r = 1$

$$E(t) = \frac{\sum_{k=0}^{\infty}\sum_{j=0}^{\theta-1} \frac{(-1)^{k+j}}{k!}\binom{\theta-1}{j}\frac{2\theta(j+1)^k}{(\lambda)^{k+1}(1-2k-2)}}{1-\left[1-e^{-\frac{1}{\lambda}}\right]^{\theta}}$$

Hence, when, $M_1 = E(t)$

$$\bar{t} = \frac{\sum_{k=0}^{\infty}\sum_{j=0}^{\theta-1} \frac{(-1)^{k+j}}{k!}\binom{\theta-1}{j}\frac{2\theta(j+1)^k}{(\lambda)^{k+1}(1-2k-2)}}{1-\left[1-e^{-\frac{1}{\lambda}}\right]^{\theta}}$$

$$\bar{t} - \frac{\sum_{k=0}^{\infty}\sum_{j=0}^{\theta-1}\frac{(-1)^{k+j}}{k!}\binom{\theta-1}{j}\frac{2\theta(j+1)^k}{(\lambda)^{k+1}(1-2k-2)}}{1-\left[1-e^{-\frac{1}{\lambda}}\right]^{\theta}} = 0 \qquad (12)$$

when $r = 2$ then,

$$E(t^2) = \frac{\sum_{k=0}^{\infty}\sum_{j=0}^{\theta-1}\frac{(-1)^{k+j}}{k!}\binom{\theta-1}{j}\frac{\theta(j+1)^k}{(\lambda)^{k+1}(-k)}}{1-\left[1-e^{-\frac{1}{\lambda}}\right]^{\theta}}$$

the second moment of samples and population are:

$$M_2 = \frac{1}{n}\sum_{i=1}^{n} t_i^2 \ , M_2 = E(t^2)$$

$$\frac{1}{n}\sum_{i=1}^{n} t_i^2 = \frac{\sum_{k=0}^{\infty}\sum_{j=0}^{\theta-1}\frac{(-1)^{k+j}}{k!}\binom{\theta-1}{j}\frac{\theta(j+1)^k}{(\lambda)^{k+1}(-k)}}{1-\left[1-e^{-\frac{1}{\lambda}}\right]^{\theta}}$$

$$\frac{1}{n}\sum_{i=1}^{n} t_i^2 - \frac{\sum_{k=0}^{\infty}\sum_{j=0}^{\theta-1}\frac{(-1)^{k+j}}{k!}\binom{\theta-1}{j}\frac{\theta(j+1)^k}{(\lambda)^{k+1}(-k)}}{1-\left[1-e^{-\frac{1}{\lambda}}\right]^{\theta}} = 0. \qquad (13)$$

The two non-linear equations are difficult to solve, the Newton-Raphson was utilized to estimate λ and θ.

So, the estimated survival function $\hat{S}_{RTIGRD}(t)$ by MOM will be:

$$\hat{S}_{\substack{MOM \\ RTIGRD}}(t) = \frac{\left(1-e^{-\frac{1}{\hat{\lambda}_{MOM}\, t^2}}\right)^{\hat{\theta}_{MOM}} - \left(1-e^{-\frac{1}{\hat{\lambda}_{MOM}}}\right)^{\hat{\theta}_{MOM}}}{1-\left(1-e^{-\frac{1}{\hat{\lambda}_{MOM}}}\right)^{\hat{\theta}_{MOM}}}. \qquad (14)$$

6. HYBRID ABC-FA ALGORITHM

Meta heuristics algorithms like the GA [17], PSO [18], SA [19] and ant colony algorithm [20] showed good performance for optimization problems of estimation parameters [21-24]. However, the hybrid meta heuristics results in robust solution methods [25] For an efficiency presented by (speed and quality) of the estimated algorithm, we must consider two major components in modern meta heuristics, namely diversification and intensification then balance between them.

We will combine two Meta-Heuristics Bee Colony algorithms according to recent assumptions and advanced modeling methods [26, 27] and firefly algorithm in [28, 29] to estimate the parameter, so the algorithm converges slowly, with solutions jumping around optimal ones. Diversification is powerful.

Artificial Bee Colony Algorithm with Firefly Algorithm (ABC_FA)

Steps of ABC_FA as follows:

Step 1: Generate values of parameters ABC and FA (α: randomization parameter, β_0: firefly attractiveness, $\mathbb{N}$: population size, γ: media absorption coefficient and maximum number of generation.

Step 2: Generate randomly solution set x_i.

Step 3: Evaluates the objective function (f) by minimize the log-likelihood of all solutions (X_i) in population.

$$f = nLn(2) + nLn(\theta) - nLn(\lambda) - 3\sum_{i=1}^{n} Ln(t_i) + \sum_{i=1}^{n}\left(\frac{-1}{\lambda {t_i}^2}\right) + (\theta - 1)\sum_{i=1}^{n} Ln\left(1 - e^{-\frac{1}{\lambda {t_i}^2}}\right) - nln\left(1 - \left(1 - e^{-\frac{1}{\lambda}}\right)^{\theta}\right).$$

Step 4: Save the best solution in the population.

Step 5: Generates new U_i, from oldest solutions, by using equation $U_{ij} = x_{ij} + \beta_0 e^{-\gamma\, r_{ij}^2}\, \emptyset_{ij}(x_{ij} - x_{kj}) + \alpha\,(r \text{ and } -\frac{1}{2})$, Where U is a new solution near Pi, $j \in \{1,2, \dots, D\}$, $k \in \{1,2,\dots, \mathbb{N}\}$, $\emptyset_{ij} \in [-1,1]$.

Step 6: Save the best solution between candidate and current solutions.

Step 7: Computes the probability (p_i) for X_i by utilizing the equation $p_i = \frac{fit_i}{\sum_{n=1}^{\mathbb{N}} fit_n}$ which equation computes fit_i; $fit_i = \begin{cases} \frac{1}{1+\sqrt{\sum_{n=1}^{\mathbb{N}} f_i}}, if\ f_i \geq 0 \\ 1 + \sqrt{\left|1 + \sqrt{\sum_{n=1}^{\mathbb{N}} f_i}\right|}, if\ f_i < 0 \end{cases}$

Step 8: Generates new solution $U_{i(new)}$ based on p_i. (Onlookers bees)

Where $U_{i(new)} = x_{ij} + \beta_0 e^{-\gamma\, r_{ij}^2}\, \emptyset_{ij}(x_{ij} - x_{kj}) + \alpha\,(r \text{ and } -\frac{1}{2})$

$$r_{ij} = \| x_{ij} - x_{kj} \| = \sqrt{\sum_{k=1}^{d} (x_{ij} - x_{kj})^2} \ , \gamma = 0.01 \ , \beta_0 = 1$$

Step 9: Evaluating the fitness function of all $U_{i(new)}$ in the population.

Step 10: If there is a solution that has been given up on, replace it with X_i. (Scout bee)

Step 11: Keep population's best solution.

7. SIMULATION STUDY

The Simulation study refers to the artificial method that may be utilized to solve the complicated problems for simulating purely in statistical problem [30].Replicate the simulation 1000 times. Various sample sizes are tested: 15, 30, 60, 90, 120, 160 and 200 for examining the effect of sample sizes.

The following steps of the simulation will be;

Step 1: Initialize all the parameters of FA, ABC, ABC_FA.

Step 2: Initialize random samples as $(u_1, u_2, \ldots, u_n)$, which are follows the uniform distribution (0, 1), than transforms it into random samples by applying the Right Truncated Inverse Generalized Rayleigh Distribution using $c.d.f.$ where shows as follows:

$$F_{RTIGRD}(t) = \frac{1 - \left[1 - e^{-\frac{1}{\lambda t^2}}\right]^{\theta}}{1 - \left[1 - e^{-\frac{1}{\lambda}}\right]^{\theta}}, u_i = \frac{1 - \left[1 - e^{-\frac{1}{\lambda t_i^2}}\right]^{\theta}}{1 - \left[1 - e^{-\frac{1}{\lambda}}\right]^{\theta}},$$

$$t_i = \frac{1}{\sqrt{\lambda \ ln \ ln \left(1 - \left(1 - u_i \left(1 - \left(1 - e^{-\frac{1}{\lambda}}\right)^{\theta}\right)\right)^{\frac{1}{\theta}}\right)^{-1}}}$$

Lets X_i is a vector utilizes for all parameters such as $X_i = [\lambda, \theta]$.

Step 3: Calculate the S from equations (6).

Step 4: Calculate $\hat{S}$ depending on MLE and MOM utilizing the equations (10), (14), respectively.

Step 5: Calculate the best solution of $(\hat{S})$ from FA, ABC, ABC_FA methods.

Step 6: MSE is calculated as follows for $L = 1000$ iterations:

$$MSE = \frac{1}{L} \sum_{i=1}^{L} \left((\hat{S}_i - S)^2 \right).$$

8. NUMERICAL RESULTS

To determine the best method for the proposed hybrid algorithm (ABC-FA) with standard algorithms (ABC, FA) and classical methods (MLE, MOM) by estimating Survival function based on the scale and shape parameters (λ, θ) of RTIGRD, Seven sample sets (15, 30, 60, 90, 120, 160, 200) were used. Then, the simulation results for all methods (ABC, MLE, FA, MOM, ABC_FA) as shown in Tables 1 to 3 depend on Survival analyses and MSE. In Table 1, the two parameters for this model $(\lambda_1, \theta_1) = (2,1)$, the parameters $(\lambda_2, \theta_2) = (3,2)$ in Table 2, and the two parameters $(\lambda_3, \theta_3) = (1,1)$ in Table 3. Tables 1-3 showed that the Hybrid (ABC_FA) algorithms provided less Mean Square Error than standard algorithms and classical methods for the Survival function. This implies that the (ABC_FA) method was the best of the other estimators. Then come the standard algorithms and then the traditional methods. Additionally, we notice in Table 1 when the two parameters are different i.e. the (shape parameter) is less than the (Scale parameter) (first case) the ABC algorithm showed that it has less MSE when $n = 30$, 90, 120, 160, 200, but in the case of $n = 15$, 60, it was shown that FA possesses Less MSE, so it is better than ABC, then MLE, then MOM in this paper. While in Table 2, we notice that the best algorithm comes after the ABC_FA algorithm. The ABC algorithm showed that it has less MSE then FA, then MLE and then MOM, in all the different samples that were taken in this paper, While in Table 3, when the parameters of shape and Scale are equal and their value is equal to 1, we note that the standard algorithms (ABC, FA) provided reasonably suitable solutions, as they have a MSE is less than classical methods (MLE, MOM).

Table 1: MSE Values of $\hat{s}$ when $\lambda_1 = 2$ and $\theta_1 = 1$

n	S_{RTIGRD}	MSE (MLE)	MSE (MOM)	MSE (FA)	MSE (ABC)	MSE (ABC-FA)	Best
15	0.69536823	0.000279428	0.0003502758	0.0000033965	0.0000484441	0.0000016402	**ABC-FA**
30	0.20364897	0.000032544	0.0000389562	0.0000242187	0.0000167972	0.00001116591	**ABC-FA**
60	0.71328565	0.000010345	0.0003598742	0.000004126	0.0000078178	0.00000000897	**ABC-FA**
90	0.97661764	0.000014824	0.000140916	0.000209093	0.000004434	0.00000054673	**ABC-FA**
120	0.93607551	0.000004147	0.000271967	0.000031009	0.000008273	0.0000040863	**ABC-FA**
160	0.84751029	0.000027631	0.000378042	0.0000182925	0.0000103616	0.00000002866	**ABC-FA**
200	0.52514897	0.000019746	0.000231699	0.000006818	0.0000029052	0.0000014012	**ABC-FA**

Table 2: MSE Values of $\hat{s}$ when $\lambda_2 = 3$ and $\theta_2 = 2$

n	S_{RTIGRD}	MSE (MLE)	MSE (MOM)	MSE (FA)	MSE (ABC)	MSE (ABC-FA)	Best
15	0.53549829	0.000022734	0.000250536	0.0000656504	0.0000150243	0.00000340529	**ABC-FA**
30	0.55905762	0.000236549	0.000268477	0.0000900801	0.0000042203	0.00000001841	**ABC-FA**
60	0.56906055	0.000025471	0.0002455957	0.000037199	0.0000042911	0.00000259583	**ABC-FA**
90	0.43882727	0.000022511	0.0001774321	0.0000814238	0.0000014214	0.00000064302	**ABC-FA**
120	0.62688707	0.000010996	0.0003176245	0.000065546	0.0000059413	0.00000400296	**ABC-FA**
160	0.87010244	0.000025286	0.0003401512	0.000016753	0.000009915	0.0000026157	**ABC-FA**
200	0.90374701	0.000005384	0.000226576	0.000025907	0.0000089267	0.00000351602	**ABC-FA**

Table 3: MSE Values of $\hat{s}$ when $\lambda_3 = 1$ and $\theta_3 = 1$

n	S_{RTIGRD}	MSE (MLE)	MSE (MOM)	MSE (FA)	MSE (ABC)	MSE (ABC-FA)	Best
15	0.03760081	0.000001406	0.0000014111	0.0000498446	0.0000013982	0.00000045506	**ABC-FA**
30	0.09199902	0.000008404	0.000008445	0.0000250173	0.0000084159	0.00000837323	**ABC-FA**
60	0.85374337	0.000709516	0.0007145423	0.0001264311	0.0003359395	0.00000197689	**ABC-FA**
90	0.26951253	0.000072468	0.0000725243	0.0000039169	0.0000477501	0.00000133984	**ABC-FA**
120	0.06754231	0.000003255	0.0000045577	0.0000263238	0.0000039054	0.00000254381	**ABC-FA**
160	0.16555507	0.000023419	0.000027376	0.000066912	0.0000184527	0.00001582867	**ABC-FA**
200	0.07357868	0.000037617	0.0000540878	0.0000231682	0.0000371159	0.00000474864	**ABC-FA**

9. CONCLUSIONS

In this paper proposes three methods (ABC), (FA) and (ABC_FA) to estimate Survival functions based on the parameters (λ, θ) of Right Truncated Inverse Generalized Rayleigh distribution. Simulation is utilized to compared the suggested method with classical method which are includes (MLE and MOM). The results shows that (In the first, second, third and fourth cases, it was found that (ABC_FA) algorithm is better than the other algorithms in all the different sample sizes that were randomly selected in this paper, where the proposed algorithms were strengthened by taking the strengths of each proposed algorithm in this paper and merging them with Some in order to give better and stronger results, as the results proved that the new algorithms are the best because they have less MSE.

10. RECOMMENDATION

In this section, some recommendations for further studies were presented as follows:

1. Different optimization algorithms have different strengths and weaknesses. Thus, our recommendation is to propose other hybrid algorithms to estimate the Survival functions based on the parameters of any distribution.
2. Using our proposed methods to estimate the parameters of any distribution containing more parameters to generalize the algorithm.

REFERENCES

[1] Khalifa, E.H., Ramadan, D.A. and El-Desouky, B. (2021). Statistical inference of truncated weibull-rayleigh distribution: its properties and applications. *Open Journal of Modelling and Simulation*, 9(3), 281-298.

[2] Zain, S.A. and Hussein, I.H. (2021). Estimate for Survival and Related Functions of Weighted Rayleigh Distribution. *Ibn AL-Haitham Journal for Pure and Applied Sciences*, 34(1), Doi: 10.30526/34.1.2558

[3] Ibrahim, A. and Kalaf, B.A. (2022). Estimation of the survival function based on the log-logistic distribution. *International Journal of Nonlinear Analysis Applications*, 13(1), 127-141.

[4] Bayda, A.K. (2021). A new algorithm to estimate the parameters of log-logistic distribution based on the survival functions. *Journal of Physics: Conference Series*, 1879(3), 032037: IOP Publishing.

[5] Hussein, I.H. and Hamas, H.A. (2019). Fuzzy Survival and Hazard functions estimation for Rayleigh distribution. *Iraqi Journal of Science,* 624-632.

[6] Hameed, B.A., Salman, A.N. and Kalaf, B.A. (2020). On Estimation of in Cased Inverse Kumaraswamy Distribution. *Iraqi Journal of Science*, 61(4), 845-853.

[7] Kalaf, B.A., Raheem, S.H. and Salman, A.N. (2021). Estimation of the reliability system in model of stress-strength according to distribution of inverse Rayleigh. *Periodicals of Engineering Natural Sciences,* 9(2), 524-533.

[8] McEwen, R.P. and Parresol, B.R. (1991). Moment expressions and summary statistics for the complete and truncated Weibull distribution. *Communications in Statistics-Theory and Methods*, 20(4), 1361-1372.

[9] Chapman, D.G. (1956). Estimating the parameters of a truncated gamma distribution. *The Annals of Mathematical Statistics*, 27(2), 498-506.

[10] Jamalizadeh, A., Pourmousa, R. and Balakrishnan, N. (2009). Truncated and limited skew-normal and skew-t distributions: properties and an illustration. *Communications in Statistics-Theory and Methods*, 38(16-17), 2653-2668.

[11] Ardianti, F. (2018). Estimating parameter of Rayleigh distribution by using maximum likelihood method and Bayes method. *IOP Conference Series: Materials Science and Engineering*, 300(1), 012036: IOP Publishing.

[12] AdilabdAlkareem, Z. and Kalaf, B.A. (2022). Comparison Different Estimation Methods for the Parameters of Non-Linear Regression. *Iraqi Journal of Science*, 1662-1680.

[13] Raheem, S.H., Mansor, H.K., Kalaf, B.A. and Salman, A.N. (2019). A Comparison for Some of the estimation methods of the Parallel Stress-Strength model in the case of Inverse Rayleigh Distribution. In *2019 First International Conference of Computer and Applied Sciences* (CAS) (pp. 22-27). IEEE.

[14] Mudholkar, G.S. and Srivastava, D.K. (1993). Exponentiated Weibull family for analyzing bathtub failure-rate data. *IEEE Transactions on Reliability*, 42(2), 299-302.

[15] Jebur, I.G. Kalaf, B.A. and Salman, A.N. (2021). An efficient shrinkage estimators for generalized inverse Rayleigh distribution based on bounded and series stress-strength models. *Journal of Physics: Conference Series*, 1897(1), 012054: IOP Publishing.

[16] Bowman, K.O. and Shenton, L. (2004). Estimation: Method of moments. *Encyclopedia of Statistical Sciences*, https://doi.org/10.1002/0471667196.ess1618

[17] Holland, J.H. (1992). *Adaptation in natural and artificial systems: an introductory analysis with applications to biology, control, and artificial intelligence*. MIT Press.

[18] Eberhart, R. and Kennedy, J. (1995). A new optimizer using particle swarm theory. *MHS'95. Proceedings of the Sixth International Symposium on Micro Machine and Human Science*, 39-43. IEEE.

[19] Kirkpatrick, S. Gelatt Jr, C.D. and Vecchi, M.P.J.s. (1983). Optimization by simulated annealing. *Science*, 220(4598), 671-680.

[20] Colorni, A., Dorigo, M. and Maniezzo, V. (1991). Distributed optimization by ant colonies. *Proceedings of the first European Conference on Artificial Life*, 142, 134-142: Paris, France.

[21] Özsoy, V.S. Ünsal, M.G. and Örkcü, H.H. (2020). Use of the heuristic optimization in the parameter estimation of generalized gamma distribution: comparison of GA, DE, PSO and SA methods. *Computational Statistics*, 35(4), 1895-1925.

[22] Louzada, F., Ramos, P.L. and Perdoná, G.S. (2016). Different estimation procedures for the parameters of the extended exponential geometric distribution for medical data. *Computational Mathematical Methods in Medicine*, Article ID 8727951, https://doi.org/10.1155/2016/8727951

[23] Li, Z., Cui, J., Li, W. and Cui, Y. (2018). Three parameter Weibull distribution estimation based on particle swarm optimization. *13th IEEE Conference on Industrial Electronics and Applications (ICIEA)*, pp. 1892-1899: IEEE.

[24] Chi, K., Kang, J., Wu, K. and Wang, X. (2016). Bayesian parameter estimation of weibull mixtures using cuckoo search. *International Conference on Intelligent Networking and Collaborative Systems (INCoS)*, 411-414: IEEE.

[25] Kalaf, B.A., Mohammed, G.J. and Salman, M.D. (2021). A New Hybrid Meta-Heuristics Algorithms to Solve APP Problems. *Journal of Physics: Conference Series*, 1897(1), 012011: IOP Publishing.

[26] Karaboga, D. and Basturk, B. (2007). Artificial bee colony (ABC) optimization algorithm for solving constrained optimization problems. *International Fuzzy Systems Association World Congress*, 789-798: Springer.

[27] Karaboga, D. (2005). *An idea based on honey bee swarm for numerical optimization*. Technical Report-TR06, Erciyes University, Engineering Faculty, Computer Engineering Department.2005, vol. 200.

[28] Yang, X.-S. (2010). *Nature-inspired metaheuristic algorithms*. Luniver Press.

[29] Kumar, D. Gandhi, B. and Bhattacharjya, R.K. (2020). Firefly algorithm and its applications in engineering optimization. *Nature-Inspired Methods for Metaheuristics Optimization*: Springer, 93-103.

[30] Patowary, A., Sriwastav, G. and Hazarika, J. (2016). Inference of $R = P(X < Y < Z)$ for n-Standby System: A Monte-Carlo Simulation Approach. *J. Math.*, 12, 18-22.