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STATISTICAL ANALYSIS OF INVERSE WEIBULL DISTRIBUTION BASED ON GENERALIZED PROGRESSIVE HYBRID TYPE-I CENSORING WITH COMPETING RISKS

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ABSTRACT

The competing risks model plays an important role in the statistical analysis of engineering, econometric, biological and other fields. A tested product may fail from different causes. Therefore, these failure causes as if competing with each other to bring about the failure of tested products (an experimental unit). Hence, in the statistical literature, this is known as the competing risk problem and it has been studied quite extensively by several researchers when the lifetime of the product is the latent failure time of the first failure cause among all the possible failure causes. In this paper, a competing risks model based on a generalized progressive hybrid type-I (GPH type-I) censoring is considered when the latent lifetime distributions of failure causes are inverse Weibull (IW) distributed and partially observed. We established various estimation methods including Maximum likelihood estimates (MLEs) and Bayes estimates (BEs). MLEs with the corresponding asymptotic confidence intervals are obtained. Bayes estimates of the parameters are obtained based on squared error loss (SEL) function under the assumption of independent gamma priors. Furthermore, we applied Markov Chain Monte Carlo (MCMC) techniques to compute BEs and to calculate the credible intervals. Finally, simulation studies and real data set are used for illustrative purpose.

KEYWORDS

Competing risks, GPH type-I censoring scheme, inverse Weibull, maximum likelihood estimation, Bayesian estimation, Markov Chain Monte Carlo approach.

1. INTRODUCTION

The most widely used censoring techniques in real-world test experiments are type-I and type-II censoring. The experimental time is fixed in type-I censoring scheme, but the number of observed failures is a random variable. On the other hand, in type-II censoring scheme, the experimental time is a random variable while the number of observed failures is fixed. Hybrid censoring is the combination of type-I and type-II censoring schemes. If the experiment ends at the time point $T_1^* = \min(x_{(r)}, T)$ it is referred to as hybrid type I proposed by Epstein (1954). The experiment is referred to as a hybrid type-II, suggested by Childs et al. (2003), if it ends at the time point $T_2^* = \max(x_{(r)}, T)$. These schemes have the disadvantage of not allowing the units to be removed from the experiment at any point other than the terminal point. To deal with this problem, a more general censoring scheme called progressive type-II censoring is used. Balakrishnan and Aggarwala (2000) and Wu (2002) proposed another type of censoring called progressive censoring allows removal of units from the test at times other than the final termination point.

Recently, progressive type-II censoring scheme has become very popular for analyzing life test data. Kundu and Joarder (2006) and Childs et al. (2008) introduced hybrid censoring schemes in the context of progressive censoring; they considered a type-I and type-II progressive hybrid censoring schemes, respectively, which are a combination of progressive type-II and hybrid censoring schemes. The drawback of the progressive hybrid Type-I censoring, similar to the conventional Type-I censoring, is that the number of failures is random and it can turn out to be a very small number (even equal to zero), and therefore the standard statistical inference procedures may not be applicable or they will have low efficiency. To overcome this drawback, Chandrasekar et al. (2004) proposed two versions of generalized hybrid censoring.

Cho et al. (2015a), introduced a new censoring scheme called GPH type-I censoring which can be explain as follows: suppose n identical units are placed on a life testing experiment with k and r pre-fixed with $1 \le k < r \le n$ and value of T is fixed as well. The censoring scheme $R_1, R_2, \ldots\ldots, R_r$ (where $R_i \ge 0, i = 1, \ldots., r$) are pre-fixed integers satisfies $\sum_{i=1}^{r} R_i + r = n$. Then, at the time of the first failure $x_{(1)}$, R_1 of the remaining units are randomly removed. Similarly at the time of the second failure $x_{(2)}$, R_2 of the remaining units are removed and so on. This process continues until, immediately following the terminated time $T_1^{\bullet} = \max(x_{(k)}, \min(x_{(r)}, T))$ all the remaining units are removed from the experiment as a schematic illustration in the Figure 1. Note that GPH type-I censoring provides the flexibility to continue the experiment beyond time point T if an appropriate number of failures are not observed. Under this scheme, the experimenter would ideally like to observe r failures, but is willing to accept a bare minimum of k failures. Let D denote the number of observed failures up to time T. In this scheme, the data is one of the following types

$$x_{(1)} < x_{(2)} < \ldots.. < x_{(k)} \quad \text{if} \quad T < x_{(k)} < x_{(r)} \quad \text{case1}$$

$$x_{(1)} < x_{(2)} < \ldots.. < x_{(D)} \quad \text{if} \quad x_{(k)} < T < x_{(r)} \quad \text{case2}$$

$$x_{(1)} < x_{(2)} < \ldots.. < x_{(k)} \ldots.. < x_{(r)} \quad \text{if} \quad x_{(k)} < x_{(r)} < T \quad \text{case3}$$

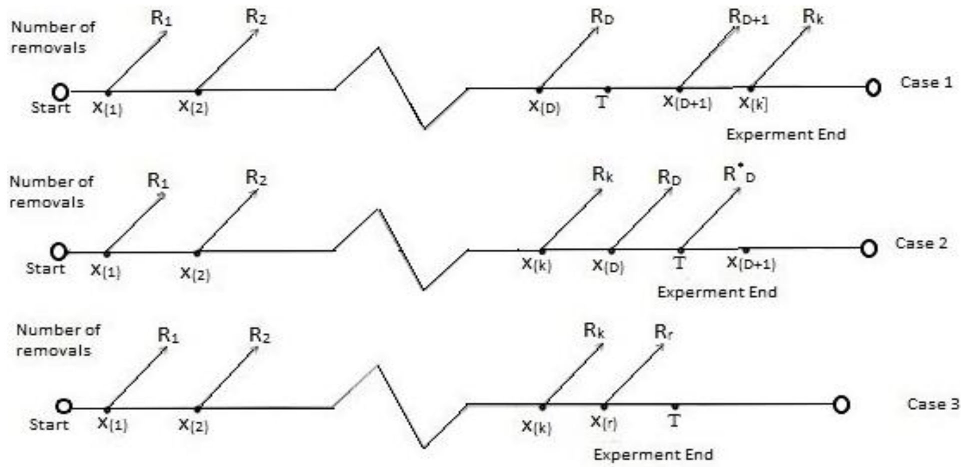

Figure 1: Schematic Illustration of GPH Type-I Censoring Scheme

Note that for case 1, $T < x_{(k)} < x_{(r)}$ and $x_{(k+1)} < ... < x_{(r)}$ are not observed. For case 2, $x_{(D)} < T < x_{(D+1)}$ and $x_{(D+1)} < ..... < x_{(r)}$ are not observed. The likelihood function of the GPH type-I censoring scheme can be found in Cho et al. (2015 a) as the following form

$$L = \begin{cases} \prod_{i=1}^{k} \sum_{V=i}^{r} (R_V + 1) \prod_{i=1}^{k} f(x_{(i)})(1 - F(x_{(i)}))^{R_i} & \text{case 1} \\ \prod_{i=1}^{D} \sum_{V=i}^{r} (R_V + 1) \prod_{i=1}^{D} f(x_{(i)})(1 - F(x_{(i)}))^{R_i} (1 - F(T))^{R_D^*} & \text{case 2} \\ \prod_{i=1}^{r} \sum_{V=i}^{r} (R_V + 1) \prod_{i=1}^{r} f(x_{(i)})(1 - F(x_{(i)}))^{R_i} & \text{case 3} \end{cases} \tag{1}$$

where $R_D^* = n - D - \sum_{i=1}^{D} R_i$.

Cho et al. (2015b), discussed the maximum likelihood and Bayes estimation of the Weibull distribution based on GPH type-I censoring. Otherwise, there are also some other similar generalized progressive hybrid censoring scheme proposed by Cho et al. (2016) and Kotb (2018). Lee et al. (2016), proposed the likelihood inference of the exponential parameter under GPH type-I censoring scheme.

Kundu and Koley (2017) analyzed the exponential GPH type-I censored data in the presence of competing risks. Wang (2018), introduced MLEs, approximate confidence intervals (ACIs), BEs and the highest posterior density (HPD) credible intervals are also constructed by using Metropolis-Hasting (MH) algorithm of the unknown parameters when the latent failure times follow the Weibull competing risks model based on GPH type-I censoring scheme. Moreover, various additional generalized progressive hybrid censoring schemes with similar properties have been presented in the literature, for further study see Wang and Li (2019), Wang et al. (2020), Wang et al. (2021) and Wang et al. (2022).

In this paper, we develop the statistical inference of the unknown parameters when the latent failure times follow the IW competing risks model based on GPH type-I censoring scheme, the MLEs and ACIs are introduced for the unknown parameters in the presence of competing risks when the cause of failure of each item is known. BE under SEL function and Bayes credible intervals (BCIs) are also generated by using the MH algorithm. A real data set has been provided for illustration.

The rest article will be organized as follows: in Section 2, we describe the model and discuss the basic notations used in this paper and the inference of MLEs for unknown parameters is covered. Gamma distribution is used as a prior distribution for the unknown parameters along with the MH algorithm in Section 3 to examine the Bayesian estimation approach. Theoretical findings are demonstrated in Section 4 with the use of a simulated research and a real data. Finally, Section 5 contains the conclusions.

2. MODEL DESCRIPTION AND LIKELIHOOD FUNCTION

Consider a lifetime experiment with n identical units where its lifetimes are described by identically independent and distributed (i.i.d) random variables $X_1, X_2, ...X_n$. Without loss of generality, we assume that there are only two causes of failure. Let X_{ji} denotes the lifetime of the i^{th} item under the j^{th} cause of failure for $i=1,2,...n,\ j=1,2$ and $X_i = min\left(X_{1i}, X_{2i}\right)$. The competing risks model assumes that the data consists of a failure time and an indicator denoting the cause of failure. We use the latent failure time modeling of Cox (1959) for analyzing competing risks data. In the latent failure time modeling, it is assumed that competing causes of failures are independent random variables.

Consider a population, where every units failed due to one of the two known causes; 1 and 2. A unit is selected at random from the population. Let the variable Δ_i is the indicator denoting the cause of failure of the observation, i.e., failure due to cause 1 $(\Delta_i = 1)$, or failure due to cause 2 $(\Delta_i = 2)$, or failure due to one of the causes 1 or 2 but it is unknown $(\Delta_i = *)$. Here, $\Delta_i = j, j = 1,2$ means the unit i has failed due to cause j , while $\Delta_i = *$ means that the cause of failure of unit i is unknown. Under the GPH type-I censoring scheme in presence of competing risk data, we have one of the following types of observations

$$\left(x_{(1)}, \Delta_1, R_1\right), \left(x_{(2)}, \Delta_2, R_2\right), ..., \left(x_{(k)}, \Delta_k, R_k\right) \quad \text{if} \quad T < x_{(k)} < x_{(r)}, \quad \text{case}1$$

$$\left(x_{(1)}, \Delta_1, R_1\right), ..., \left(x_{(k)}, \Delta_k, R_k\right), ..., \left(x_{(D)}, \Delta_D, R_D\right) \quad \text{if} \quad x_{(k)} < T < x_{(r)}, \quad \text{case}2$$

$$\left(x_{(1)}, \Delta_1, R_1\right), ..., \left(x_{(k)}, \Delta_k, R_k\right), ..., \left(x\left(r\right), \Delta r, Rr\right) \quad \text{if} \quad x_{(k)} < x_{(r)} < T. \quad \text{case}3$$

where $I(\Delta_i = j) = \begin{cases} 1, & \text{if } j = 1,2 \\ 0, & \text{otherwise} \end{cases}$

and $I(\Delta_i = *) = \begin{cases} 1, & \text{if } \Delta_i = * \\ 0, & \text{otherwise}. \end{cases}$

The likelihood function of GPH type-I censored under competing risks when the cause of failure is known can be written as follows

$$L=\begin{cases}\gamma\prod_{i=1}^{k}h_{\Delta_i}\left(x_{(i)}\right)\prod_{j=1}^{2}\bar{F}(x_{(i)})(\bar{F}(x_{(i)}))^{R_i} & \text{case}1\\ \gamma\prod_{i=1}^{D}h_{\Delta_i}\left(x_{(i)}\right)\prod_{j=1}^{2}\bar{F}(x_{(i)})(\bar{F}(x_{(i)}))^{R_i}(\bar{F}(T))^{R_D^*} & \text{case}2\\ \gamma\prod_{i=1}^{r}h_{\Delta_i}\left(x_{(i)}\right)\prod_{j=1}^{2}\bar{F}(x_{(i)})(\bar{F}(x_{(i)}))^{R_i} & \text{case}3\end{cases} \tag{2}$$

where $\gamma=\sum_{V=i}^{j^*}(R_V+1)$, $j^*=k,\ j^*=D,\ j^*=r$ for case 1, 2 and 3 respectively $\bar{F}(T)=\bar{F}_1(T)\bar{F}_2(T)$, $h_{\Delta_i}\left(x_{(i)}\right)$ the hazard rate function under the cause of failure $\Delta_i=j$, Δ_i is the indicator for the cause of failure satisfying

$$\Delta_i=\begin{cases}1, & \text{units f ails due to cause 1}\\ 2, & \text{units f ails due to cause 2.}\end{cases}$$

Inverse Weibull distribution has been used to model degradation of mechanical components such as pistons, crankshafts of diesel engines, as well as breakdown of insulating fluid to mention just a few areas. The usefulness and applications of IW distribution in various areas including reliability, and branching processes can be seen in Keller et al. (1985). Keller and Kamath (1982) introduced IW distribution with two parameters β and λ. Many articles have considered IW distribution under different censoring schemes, see for example, Kundu and Howlader (2010), Sultan et al. (2014), Mohie El-Din and Nagy (2017) and Ateya (2020).

In this paper, we make inference under the assumption that the latent failure times follow two parameter IW distributions with different scale parameters $\lambda_1,\lambda_2>0$, $\lambda_1\neq\lambda_2$, the same shape parameter β and the cause of failure is known. The cumulative distribution function (cdf) and the probability density function (pdf) of the IW are, respectively, given by

$$F\left(x;\lambda,\beta\right)=e^{-\lambda x^{-\beta}} \tag{3}$$

$$f\left(x;\lambda,\beta\right)=\lambda\beta x^{-(\beta+1)}e^{-\lambda x^{-\beta}} \qquad x,\lambda,\beta>0$$

where β is the shape parameter and λ is the scale parameter.

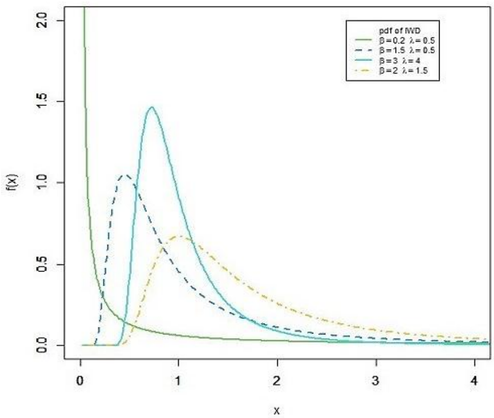

Figure 2: Density Function of IW Distribution for Some Value of β and λ

Figure 2 illustrate the behavior of IW distribution for some various values of β and λ. The pdf, cdf, survival function and hazard function of X_{ji} from j^{th} cause of failure are, respectively, given by

$$f_j\left(x;\lambda_j,\beta\right) = \lambda_j\beta x^{-(\beta+1)}e^{-\lambda_j x^{-\beta}} \quad ; \quad x,\lambda_j,\beta > 0 \tag{4}$$

$$F_j(x;\lambda_j,\beta) = e^{-\lambda_j x^{-\beta}} \tag{5}$$

$$\bar{F}_j(x;\lambda_j,\beta) = 1-e^{-\lambda_j x^{-\beta}} \tag{6}$$

and

$$h_j(x;\lambda_j,\beta) = \lambda_j \beta x^{-(\beta+1)}(e^{\lambda_j x^{-\beta}}-1)^{-1} \tag{7}$$

2.1 Maximum Likelihood Estimation

Based on equation (2), (6) and (7), the natural logarithm of the likelihood function of GPH type-I in presence of competing risks can be written as

$$\ln L = \ln\gamma + j_1\ln\lambda_1 + j_2\ln\lambda_2 + j^*\ln\beta - (\beta+1)\sum_{i=1}^{j^*}\ln(x_{(i)}) - \sum_{i=1}^{j^*}(\ln(A_{1i})+\ln(A_{2i})) + w \tag{8}$$

where

$$\gamma = \begin{cases} \prod_{i=1}^{k}\sum_{V=i}^{r}(R_V+1) & \text{case}\,1 \\ \prod_{i=1}^{D}\sum_{V=i}^{r}(R_V+1) & \text{case}\,2 \\ \prod_{i=1}^{r}\sum_{V=i}^{r}(R_V+1) & \text{case}\,3 \end{cases}$$

$$j^* = \begin{cases} k = \sum_{i=1}^{k} I(\Delta_i = j), \ \ j = 1,2 & \text{case} 1 \\ D = \sum_{i=1}^{D} I(\Delta_i = j), \ \ j = 1,2 & \text{case} 2 \\ r = \sum_{i=1}^{r} I(\Delta_i = j), \ \ j = 1,2 & \text{case} 3 \end{cases}$$

$$j_1 = \begin{cases} k_1 = \sum_{i=1}^{k} I(\Delta_i = 1), & \text{case} 1 \\ D_1 = \sum_{i=1}^{D} I(\Delta_i = 1), & \text{case} 2 \\ r_1 = \sum_{i=1}^{r} I(\Delta_i = 1), & \text{case} 3 \end{cases}$$

$$j_2 = \begin{cases} k_2 = \sum_{i=1}^{k} I(\Delta_i = 2), & \text{case} 1 \\ D_2 = \sum_{i=1}^{D} I(\Delta_i = 2), & \text{case} 2 \\ r_2 = \sum_{i=1}^{r} I(\Delta_i = 2), & \text{case} 3 \end{cases}$$

and

$$w = \begin{cases} \sum_{i=1}^{k} (R_i + 1)(\ln(U_{1i}) + \ln(U_{2i})), & \text{case} 1 \\ \sum_{i=1}^{D} (R_i + 1)(\ln(U_{1i}) + \ln(U_{2i})) + R_D^* (\ln(S_1) + \ln(S_2)), & \text{case} 2 \\ \sum_{i=1}^{r} (R_i + 1)(\ln(U_{1i}) + \ln(U_{2i})), & \text{case} 3 \end{cases}$$

where

$$R_D^* = n - D - \sum_{i=1}^{D} R_i \ , A_{1i} = \left(e^{\lambda_1 x_{(i)}^{-\beta}} - 1\right), \ A_{2i} = \left(e^{\lambda_2 x_{(i)}^{-\beta}} - 1\right), U_{1i} = \left(1 - e^{-\lambda_1 x_{(i)}^{-\beta}}\right),$$

$$U_{2i} = \left(1 - e^{-\lambda_2 x_{(i)}^{-\beta}}\right), \ S_1 = \left(1 - e^{-\lambda_1 T^{-\beta}}\right) \text{ and } S_2 = \left(1 - e^{-\lambda_2 T^{-\beta}}\right).$$

Differentiating equation (8) with respect to β, λ_1 and λ_2, respectively, we obtained

$$\frac{\partial \ ln \ L}{\partial \beta} = \frac{j^*}{\beta} - \sum_{i=1}^{j^*} \ln \ x_{(i)} + w'$$

$$\frac{\partial \ ln \ L}{\partial \lambda_1} = \frac{j_1}{\lambda_1} - w_1$$

$$\frac{\partial \ ln \ L}{\partial \lambda_2} = \frac{j_2}{\lambda_2} - w_2$$

where

$$w' = \begin{cases} \sum_{i=1}^{k} \ln x_{(i)}\, x_{(i)}^{-\beta} \ (\lambda_1 U_{1i}^{-1} + \lambda_2 U_{2i}^{-1}) - \sum_{i=1}^{k} (R_i+1) x_{(i)}^{-\beta} \ln x_{(i)} (\lambda_1 A_{1i}^{-1} + \lambda_2 A_{2i}^{-1}), & \text{case} 1 \\ \sum_{i=1}^{D} \ln x_{(i)}\, x_{(i)}^{-\beta} \ (\lambda_1 U_{1i}^{-1} + \lambda_2 U_{2i}^{-1}) - \sum_{i=1}^{D} (R_i+1) x_{(i)}^{-\beta} \ln x_{(i)} (\lambda_1 A_{1i}^{-1} + \lambda_2 A_{2i}^{-1}) & \\ - R_D^{*} T^{-\beta} \ \ln T (\lambda_1 Z_1^{-1} + \lambda_2 Z_2^{-1}), & \text{case} 2 \\ \sum_{i=1}^{r} \ln x_{(i)}\, x_{(i)}^{-\beta} \ (\lambda_1 U_{1i}^{-1} + \lambda_2 U_{2i}^{-1}) - \sum_{i=1}^{r} (R_i+1) x_{(i)}^{-\beta} \ln x_{(i)} (\lambda_1 A_{1i}^{-1} + \lambda_2 A_{2i}^{-1}), & \text{case} 3 \end{cases}$$

$$w_1 = \begin{cases} \sum_{i=1}^{k} x_{(i)}^{-\beta} \ U_{1i}^{-1} - \sum_{i=1}^{k} (R_i+1) x_{(i)}^{-\beta} \ A_{1i}^{-1} \ , & \text{case} 1 \\ \sum_{i=1}^{D} x_{(i)}^{-\beta} \ U_{1i}^{-1} - \sum_{i=1}^{D} (R_i+1) x_{(i)}^{-\beta} \ A_{1i}^{-1} \ - R_D^{*} T^{-\beta} \ Z_1^{-1} & \text{case} 2 \\ \sum_{i=1}^{r} x_{(i)}^{-\beta} \ U_{1i}^{-1} - \sum_{i=1}^{r} (R_i+1) x_{(i)}^{-\beta} \ A_{1i}^{-1} & \text{case} 3 \end{cases}$$

$$w_2 = \begin{cases} \sum_{i=1}^{k} x_{(i)}^{-\beta} \ U_{2i}^{-1} - \sum_{i=1}^{k} (R_i+1) x_{(i)}^{-\beta} \ A_{2i}^{-1} \ , & \text{case} 1 \\ \sum_{i=1}^{D} x_{(i)}^{-\beta} \ U_{2i}^{-1} - \sum_{i=1}^{D} (R_i+1) x_{(i)}^{-\beta} \ A_{2i}^{-1} \ - R_D^{*} T^{-\beta} \ Z_2^{-1} & \text{case} 2 \\ \sum_{i=1}^{r} x_{(i)}^{-\beta} \ U_{2i}^{-1} - \sum_{i=1}^{r} (R_i+1) x_{(i)}^{-\beta} \ A_{2i}^{-1} & \text{case} 3 \end{cases}$$

where $Z_1 = \left(e^{\lambda_1 T} - 1\right)$ and $Z_2 = \left(e^{\lambda_2 T} - 1\right)$. The MLEs of β, λ_j where $j = 1,2$ cannot be expressed in closed form. So we need to employ some required numerical approach for computing the MLEs of β, λ_j. As a result, it may be calculated for every given GPH type-I censored using numerical methods such the fixed point iterative approach to obtain the MLEs for β, λ_j where $j = 1,2$. We present the relative risk rates, RR_1 and RR_2 due to causes 1 and 2, respectively, in closed forms. The relative risk related to cause 1 is calculated as follows:

$$RR_1 = p\left(X_{1i} < X_{2i}\right) = \int_0^{\infty} f_1\left(x\right) \bar{F}_2\left(x\right) dx$$

$$= \int_0^{\infty} f_1\left(x\right) dx - \int_0^{\infty} f_1\left(x\right) e^{-\lambda_2 x_{(i)}^{-\beta}} dx$$

$$= 1 - \lambda_1 \beta \int_0^{\infty} x_{(i)}^{-(\beta+1)} \ e^{-(\lambda_1+\lambda_2) x_{(i)}^{-\beta}} dx$$

The relative risk related to cause 2 is calculated as follows:

$$RR_2 = p\left(X_{2i} < X_{1i}\right) = 1 - RR_1$$

$$RR_2 = \lambda_1 \beta \int_0^{\infty} x_{(i)}^{-(\beta+1)} \; e^{-(\lambda_1+\lambda_2)x_{(i)}^{-\beta}} dx$$

2.2 Confidence Interval

In this subsection, the asymptotic variance covariance matrix J_0^{-1} for the maximum likelihood estimators can be written as follows

$$J_0^{-1} \cong \begin{pmatrix} -\frac{\partial^2 \ln L}{\partial \lambda_1^2} & -\frac{\partial^2 \ln L}{\partial \lambda_1 \partial \lambda_2} & -\frac{\partial^2 \ln L}{\partial \lambda_1 \partial \beta} \\ -\frac{\partial^2 \ln L}{\partial \lambda_2 \partial \lambda_1} & -\frac{\partial^2 \ln L}{\partial \lambda_2^2} & -\frac{\partial^2 \ln L}{\partial \lambda_2 \partial \beta} \\ -\frac{\partial^2 \ln L}{\partial \beta \partial \lambda_1} & -\frac{\partial^2 \ln L}{\partial \beta \partial \lambda_2} & -\frac{\partial^2 \ln L}{\partial \beta^2} \end{pmatrix}^{-1}_{\beta=\hat{\beta},\ \lambda_1=\hat{\lambda}_1, \lambda_2=\hat{\lambda}_2}$$

$$\cong \begin{pmatrix} var(\lambda_1) & cov(\lambda_1,\lambda_2) & cov(\lambda_1,\beta) \\ cov(\lambda_2,\lambda_1) & var(\lambda_2) & cov(\lambda_2,\beta) \\ cov(\beta,\lambda_1) & cov(\beta,\lambda_2) & var(\beta) \end{pmatrix}_{\beta=\hat{\beta},\ \lambda_1=\hat{\lambda}_1, \lambda_2=\hat{\lambda}_2}$$

The second derivatives of $\ln L$ with respect to β, λ_1 and λ_2 are as follows

$$\frac{\partial^2 \ln L}{\partial \beta^2} = -\frac{j^*}{\beta^2} - w''$$

$$\frac{\partial^2 \ln L}{\partial \lambda_1^2} = -\frac{j_1}{\lambda_1^2} - w'_1$$

$$\frac{\partial^2 \ln L}{\partial \lambda_2^2} = -\frac{j_1}{\lambda_2^2} - w'_2$$

$$\frac{\partial^2 \ln L}{\partial \lambda_1 \partial \lambda_2} = \frac{\partial^2 \ln L}{\partial \lambda_2 \partial \lambda_1} = 0$$

$$\frac{\partial^2 \ln L}{\partial \lambda_1 \partial \beta} = \frac{\partial^2 \ln L}{\partial \beta \partial \lambda_1} = w_1''$$

$$\frac{\partial^2 \ln L}{\partial \lambda_2 \partial \beta} = \frac{\partial^2 \ln L}{\partial \beta \partial \lambda_2} = w_2''$$

where

$$w'' = \begin{cases} \sum_{i=1}^{k} (\ln x_{(i)})^2 \, x_{(i)}^{-\beta} \, (\lambda_1 U_{1i}^{-1} + \lambda_2 U_{2i}^{-1}) + \sum_{i=1}^{k} R_i \, x_{(i)}^{-2\beta} (\ln x_{(i)})^2 \, (\lambda_1^2 A_{1i}^{-1} U_{1i}^{-1} + \lambda_2^2 A_{2i}^{-1} U_{2i}^{-1}) \\ - \sum_{i=1}^{k} (R_i + 1) x_{(i)}^{-\beta} (\ln x_{(i)})^2 \, (\lambda_1 A_{1i}^{-1} + \lambda_2 A_{2i}^{-1}), & \text{case 1} \\ \sum_{i=1}^{D} (\ln x_{(i)})^2 \, x_{(i)}^{-\beta} \, (\lambda_1 U_{1i}^{-1} + \lambda_2 U_{2i}^{-1}) + \sum_{i=1}^{D} R_i \, x_{(i)}^{-2\beta} (\ln x_{(i)})^2 \, (\lambda_1^2 A_{1i}^{-1} U_{1i}^{-1} + \lambda_2^2 A_{2i}^{-1} U_{2i}^{-1}) \\ - \sum_{i=1}^{D} (R_i + 1) x_{(i)}^{-\beta} (\ln x_{(i)})^2 \, (\lambda_1 A_{1i}^{-1} + \lambda_2 A_{2i}^{-1}) - R_D^* T^{-\beta} (\ln T)^2 (\lambda_1 Z_1^{-1} + \lambda_2 Z_2^{-1}) \\ + R_D^* T^{-2\beta} (\ln T)^2 (\lambda_1 Z_1^{-1} S_1^{-1} + \lambda_2 Z_2^{-1} S_2^{-1}), & \text{case 2} \\ \sum_{i=1}^{r} (\ln x_{(i)})^2 \, x_{(i)}^{-\beta} \, (\lambda_1 U_{1i}^{-1} + \lambda_2 U_{2i}^{-1}) + \sum_{i=1}^{r} R_i \, x_{(i)}^{-2\beta} (\ln x_{(i)})^2 \, (\lambda_1^2 A_{1i}^{-1} U_{1i}^{-1} + \lambda_2^2 A_{2i}^{-1} U_{2i}^{-1}) \\ - \sum_{i=1}^{r} (R_i + 1) x_{(i)}^{-\beta} (\ln x_{(i)})^2 \, (\lambda_1 A_{1i}^{-1} + \lambda_2 A_{2i}^{-1}), & \text{case 3} \end{cases}$$

$$w'_1 = \begin{cases} \sum_{i=1}^{k} R_i \, x_{(i)}^{-2\beta} \, A_{1i}^{-1} U_{1i}^{-1}, & \text{case 1} \\ \sum_{i=1}^{D} R_i \, x_{(i)}^{-2\beta} \, A_{1i}^{-1} U_{1i}^{-1} + R_D^* T^{-\beta} \, Z_1^{-1} S_1^{-1} & \text{case 2} \\ \sum_{i=1}^{r} R_i \, x_{(i)}^{-2\beta} \, A_{1i}^{-1} U_{1i}^{-1} & \text{case 3} \end{cases}$$

$$w'_2 = \begin{cases} \sum_{i=1}^{k} R_i \, x_{(i)}^{-2\beta} \, A_{2i}^{-1} U_{2i}^{-1}, & \text{case 1} \\ \sum_{i=1}^{D} R_i \, x_{(i)}^{-2\beta} \, A_{2i}^{-1} U_{2i}^{-1} + R_D^* T^{-\beta} \, Z_2^{-1} S_2^{-1} & \text{case 2} \\ \sum_{i=1}^{r} R_i \, x_{(i)}^{-2\beta} \, A_{2i}^{-1} U_{2i}^{-1} & \text{case 3} \end{cases}$$

$$w_1'' = \begin{cases} \sum_{i=1}^{k} \ln x_{(i)} \, x_{(i)}^{-\beta} \, U_{1i}^{-1} (1 - \lambda_1 x_{(i)}^{-\beta} A_{1i}^{-1}) + \sum_{i=1}^{k} (R_i + 1) x_{(i)}^{-\beta} \ln x_{(i)} \, A_{1i}^{-1} (1 - \lambda_1 x_{(i)}^{-\beta} U_{1i}^{-1}) & \text{case 1} \\ \sum_{i=1}^{D} \ln x_{(i)} \, x_{(i)}^{-\beta} \, U_{1i}^{-1} (1 - \lambda_1 x_{(i)}^{-\beta} A_{1i}^{-1}) + \sum_{i=1}^{D} (R_i + 1) x_{(i)}^{-\beta} \ln x_{(i)} \, A_{1i}^{-1} (1 - \lambda_1 x_{(i)}^{-\beta} U_{1i}^{-1}) \\ - R_D^* T^{-\beta} \ln T \, Z_1^{-1} (1 - \lambda_1 T^{-\beta} S_1^{-1}) & \text{case 2} \\ \sum_{i=1}^{r} \ln x_{(i)} x_{(i)}^{-\beta} \, U_{1i}^{-1} (1 - \lambda_1 x_{(i)}^{-\beta} A_{1i}^{-1}) + \sum_{i=1}^{r} (R_i + 1) x_{(i)}^{-\beta} \ln x_{(i)} \, A_{1i}^{-1} (1 - \lambda_1 x_{(i)}^{-\beta} U_{1i}^{-1}) & \text{case 3} \end{cases}$$

and

$$w_2'' = \begin{cases} \sum_{i=1}^{k} \ln x_{(i)}\, x_{(i)}^{-\beta}\; U_{2i}^{-1}(1-\lambda_2\, x_{(i)}^{-\beta}\, A_{2i}^{-1}) + \sum_{i=1}^{k}(R_i+1)\, x_{(i)}^{-\beta} \ln x_{(i)}\, A_{2i}^{-1}(1-\lambda_2\, x_{(i)}^{-\beta} U_{2i}^{-1}) & \text{case}\,1 \\ \sum_{i=1}^{D} \ln x_{(i)}\, x_{(i)}^{-\beta}\; U_{2i}^{-1}(1-\lambda_2\, x_{(i)}^{-\beta}\, A_{2i}^{-1}) + \sum_{i=1}^{D}(R_i+1)\, x_{(i)}^{-\beta} \ln x_{(i)}\, A_{2i}^{-1}(1-\lambda_2\, x_{(i)}^{-\beta} U_{2i}^{-1}) & \\ - R_D^{*}\, T^{-\beta} \ln T\; Z_2^{-1}(1-\lambda_2\, T^{-\beta}\, S_2^{-1}) & \text{case}\,2 \\ \sum_{i=1}^{r} \ln x_{(i)} x_{(i)}^{-\beta}\; U_{2i}^{-1}(1-\lambda_2\, x_{(i)}^{-\beta}\, A_{2i}^{-1}) + \sum_{i=1}^{r}(R_i+1)\, x_{(i)}^{-\beta} \ln x_{(i)}\, A_{2i}^{-1}(1-\lambda_2\, x_{(i)}^{-\beta} U_{2i}^{-1}) & \text{case}\,3 \end{cases}$$

Also, we obtained the $100(1-\alpha)\%$ confidence intervals for the parameters β, λ_1 and λ_2 using the normal approximate of the MLEs and asymptotic variance covariance matrix, as

$$\hat{\beta} - Z_{\alpha/2}\sqrt{var(\hat{\beta})}\;,\; \hat{\beta} + Z_{\alpha/2}\sqrt{var(\hat{\beta})}$$

$$\hat{\lambda}_1 - Z_{\alpha/2}\sqrt{var(\hat{\lambda}_1)}\;,\; \hat{\lambda}_1 + Z_{\alpha/2}\sqrt{var(\hat{\lambda}_1)}$$

$$\hat{\lambda}_2 - Z_{\alpha/2}\sqrt{var(\hat{\lambda}_2)}\;,\; \hat{\lambda}_2 + Z_{\alpha/2}\sqrt{var(\hat{\lambda}_2)}$$

where $Z_{\alpha/2}$ is the percentile of the standard normal distribution with right-tail probability $\alpha/2$.

3. BAYESIAN ESTIMATION

In this section, we consider Bayesian inference of the unknown parameters when the latent failure times follow the IW competing risks model using GPH type-I censoring scheme with competing risks. The BEs and HPD credible intervals were obtained for the unknown parameters, HPD is one of the methods for defining a credible interval in Bayesian statistics.

3.1 Prior Distributions

In this subsection, we assumed that the priors of β, λ_1 and λ_2 are independent and follow Gamma (a_1,b_1), Gamma (a_2,b_2) and Gamma (a_3,b_3), respectively. Therefore, the priors for β, λ_1 and λ_2 are of the forms

$$\pi(\beta) = \frac{b_1^{a_1}}{\Gamma a_1}\beta^{a_1-1} e^{-b_1\beta} \qquad \beta > 0 \text{ and } a_1,\ b_1 > 0,$$

$$\pi(\lambda_1) = \frac{b_2^{a_2}}{\Gamma a_2}\lambda_1^{a_2-1} e^{-b_2\lambda_1} \qquad \lambda_1 > 0 \text{ and } a_2,\ b_2 > 0,$$

and

$$\pi(\lambda_2) = \frac{b_3^{a_3}}{\Gamma a_3}\lambda_2^{a_3-1} e^{-b_3\lambda_2} \qquad \lambda_2 > 0 \text{ and } a_3,\ b_3 > 0,$$

then, the joint prior is given by

$$\pi(\beta,\lambda_1,\lambda_2) = \pi(\beta)\pi(\lambda_1)\pi(\lambda_2) \ . \tag{9}$$

The elicitation of the hyper-parameters will depend on the prior knowledge of the unknown parameters. These informative priors will be obtained from the MLEs for $(\beta,\lambda_1,\lambda_2)$ by equating the mean and variance of $\left(\hat{\beta}^t,\hat{\lambda}_1^t,\hat{\lambda}_2^t\right)$ with the mean and variance of the priors under consideration (Gamma priors), where $t=1,2...,M$ and M is the number of samples available from the IW distribution. Thus, on equating mean and variance of $\left(\hat{\beta}^t,\hat{\lambda}_1^t,\hat{\lambda}_2^t\right)$ with the mean and variance of Gamma priors, one can obtain (Dey et al. (2016))

$$\frac{a_1}{b_1}=\frac{1}{M}\sum_{t=1}^{M}\hat{\beta}^t \quad , \quad \frac{a_1}{b_1^2}=\frac{1}{M-1}\sum_{t=1}^{M}\left(\hat{\beta}^t-\frac{1}{M}\sum_{t=1}^{M}\hat{\beta}^t\right)^2$$

$$\frac{a_2}{b_2}=\frac{1}{M}\sum_{t=1}^{M}\hat{\lambda}_1^t \quad , \quad \frac{a_2}{b_2^2}=\frac{1}{M-1}\sum_{t=1}^{M}\left(\hat{\lambda}_1^t-\frac{1}{M}\sum_{t=1}^{M}\hat{\lambda}_1^t\right)^2$$

$$\frac{a_3}{b_3}=\frac{1}{M}\sum_{t=1}^{M}\hat{\lambda}_2^t \quad \text{and} \quad \frac{a_3}{b_3^2}=\frac{1}{M-1}\sum_{t=1}^{M}\left(\hat{\lambda}_2^t-\frac{1}{M}\sum_{t=1}^{M}\hat{\lambda}_2^t\right)^2$$

By solving the above equations, the estimated hyper-parameters can be written as

$$a_1=\frac{\left(\frac{1}{M}\sum_{t=1}^{M}\hat{\beta}^t\right)^2}{\frac{1}{M-1}\sum_{t=1}^{M}\left(\hat{\beta}^t-\frac{1}{M}\sum_{t=1}^{M}\hat{\beta}^t\right)^2}$$

$$b_1=\frac{\frac{1}{M}\sum_{t=1}^{M}\hat{\beta}^t}{\frac{1}{M-1}\sum_{t=1}^{M}\left(\hat{\beta}^t-\frac{1}{M}\sum_{t=1}^{M}\hat{\beta}^t\right)^2}$$

$$a_2=\frac{\left(\frac{1}{M}\sum_{t=1}^{M}\hat{\lambda}_1^t\right)^2}{\frac{1}{M-1}\sum_{t=1}^{M}\left(\hat{\lambda}_1^t-\frac{1}{M}\sum_{t=1}^{M}\hat{\lambda}_1^t\right)^2}$$

$$b_2=\frac{\frac{1}{M}\sum_{t=1}^{M}\hat{\lambda}_1^t}{\frac{1}{M-1}\sum_{t=1}^{M}\left(\hat{\lambda}_1^t-\frac{1}{M}\sum_{t=1}^{M}\hat{\lambda}_1^t\right)^2}$$

$$a_3 = \frac{\left(\frac{1}{M}\sum_{t=1}^{M}\hat{\lambda}_2^t\right)^2}{\frac{1}{M-1}\sum_{t=1}^{M}\left(\hat{\lambda}_2^t - \frac{1}{M}\sum_{t=1}^{M}\hat{\lambda}_2^t\right)^2}$$

and

$$b_3 = \frac{\frac{1}{M}\sum_{t=1}^{M}\hat{\lambda}_2^t}{\frac{1}{M-1}\sum_{t=1}^{M}\left(\hat{\lambda}_2^t - \frac{1}{M}\sum_{t=1}^{M}\hat{\lambda}_2^t\right)^2} \tag{10}$$

Based on equation (9), the posterior density function of the parameters β, λ_1 and λ_2 can be written as follows:

$$\pi\left(\beta,\lambda_1,\lambda_2 \,|\, \underline{x}\right) = \frac{L(x_1, x_2, \ldots.., x_n \,|\, \underline{\lambda})\ \ \pi\left(\beta,\lambda_1,\lambda_2\right)}{\int_0^\infty\int_0^\infty\int_0^\infty L(x_1, x_2, \ldots.., x_n \,|\, \underline{\lambda})\ \ \pi\left(\beta,\lambda_1,\lambda_2\right)\ \partial\beta\,\partial\lambda_1\,\partial\lambda_2}$$

$$\propto \beta^{\,a_1 + j^* - 1}\,\lambda_1^{\,a_2 + j_1 - 1}\ \ \lambda_2^{\,a_3 + j_2 - 1}\ \ e^{-b_1\beta - b_2\lambda_1 - b_3\lambda_2}\ \ \psi \tag{11}$$

where

$$\psi = \begin{cases} \prod_{i=1}^{k} x_{(i)}^{-(\beta+1)} A_{1i}^{-1}\, A_{2i}^{-1}\ \ (U_{1i}^{-1}\, U_{2i}^{-1})^{R_i+1} & \text{case}\,1 \\ \prod_{i=1}^{D} x_{(i)}^{-(\beta+1)} A_{1i}^{-1}\, A_{2i}^{-1}\ \ (U_{1i}^{-1}\, U_{2i}^{-1})^{R_i+1}\ (S_1^{-1}S_2^{-1})^{R_D^*} & \text{case}\,2 \\ \prod_{i=1}^{r} x_{(i)}^{-(\beta+1)} A_{1i}^{-1}\, A_{2i}^{-1}\ \ (U_{1i}^{-1}\, U_{2i}^{-1})^{R_i+1} & \text{case}\,3 \end{cases}$$

Bayes estimator of any function of β, λ_1 and λ_2, say $\zeta\left(\beta,\lambda_1,\lambda_2\right)$ under SEL function is the posterior mean, denoted by $\tilde{\zeta}\left(\beta,\lambda_1,\lambda_2\right)$ and can be obtained as follows:

$$\tilde{\zeta}\left(\beta,\lambda_1,\lambda_2\right) = \frac{\int_0^\infty\int_0^\infty\int_0^\infty \zeta\left(\beta,\lambda_1,\lambda_2\right) L(x_1, x_2, \ldots.., x_n \,|\, \underline{\lambda})\ \ \pi\left(\beta,\lambda_1,\lambda_2\right)\ \partial\beta\,\partial\lambda_1\,\partial\lambda_2}{\int_0^\infty\int_0^\infty\int_0^\infty L(x_1, x_2, \ldots.., x_n \,|\, \underline{\lambda})\ \ \pi\left(\beta,\lambda_1,\lambda_2\right)\ \partial\beta\,\partial\lambda_1\,\partial\lambda_2} \tag{12}$$

Bayes estimator of $\zeta\left(\beta,\lambda_1,\lambda_2\right)$ cannot be expressed in closed form, so we need to employ some approximation method to compute the estimate given in (12). We propose to use MCMC method to obtain the BEs and HPD credible intervals of the unknown parameters. This method is particularly useful in Bayesian inference as a result of focusing on subsequent distributions that are often difficult to work with through mathematical analysis. The MH algorithm starts with simulating a candidate sample θ^* from the proposal distribution $q(.)$. Note that samples from the proposal distribution are not accepted automatically as posterior samples. These candidate samples are accepted probabilistically based on the acceptance probability. To draw samples from a distribution using MCMC:

1. Start with an initial guess: a single value that could be derived from the distribution.
2. Generating a series of new samples from this initial guess. Two steps are generated as a result of each new sample:

Proposal: the most recent sample is disturbed with a small random perturbation to provide a proposal for the new sample.

Acceptance: A new proposal will be approved as a new sample or rejected (in this case the old sample is recorded).

The best known of these are the Gibbs sampling algorithm, MH presented by Metropolis et al. (1953) and Hastings (1970).

The MH algorithm sampling can be described as follows:

Step 1: Start with any initial guess $\theta^{(0)} = \theta$ which $\theta = (\beta, \lambda_1, \lambda_2)$.

Step 2: Set $t = 1$.

Step 3: Generate θ^* with normal proposal distribution $q(\theta) = N(\hat{\theta}, var(\hat{\theta}))$.

Step 4: Given the candidate point θ^*, calculate the acceptance probability

$$A\left(\theta^{(t-1)}, \theta^*\right) = \min\left[1, \frac{\pi\left(\theta^* | \underline{x}\right) q(\theta^{(t-1)} | \theta^*)}{\pi(\theta^{(t-1)} | \underline{x}) q\left(\theta^* | \theta^{(t-1)}\right)}\right]$$

Step 5: Generate a sample from uniform distribution, i.e., $u \sim u(0,1)$.

$$if \begin{cases} u \le A\left(\theta^{(t-1)}, \theta^*\right) & Accept \quad \theta^* = \theta^{(t)} \\ u \ge A\left(\theta^{(t-1)}, \theta^*\right) & Accept \quad \theta^{(t-1)} = \theta^{(t)} \end{cases}$$

Step 6: Set $t = t + 1$, and repeat steps 2-5 M times until get M samples and obtain $\left(\beta^t, \lambda_1^t, \lambda_2^t\right)$, $t = 1, 2, \ldots M$

From the random samples of size M drawn from the posterior density, some of the initial samples can be discarded (burn-in), and remaining samples can be carried out to calculate Bayes estimates. Then, BEs of $\beta, \lambda_1, \lambda_2$, with respect to SEL function is

$$\tilde{\zeta}(\beta, \lambda_1, \lambda_2) = \sum_{t=\Upsilon+1}^{M} \zeta\left(\beta^t, \lambda_1^t, \lambda_2^t\right) \Big/ M - \Upsilon$$

where Υ represent the number of burn-in samples and $t = \Upsilon + 1, \ldots\ldots M$.

Step 7: To obtain the HPD credible intervals of $\zeta\left(\beta^{t},\lambda_1^{t},\lambda_2^{t}\right)$ first arrange all the estimates $\zeta_t=\zeta\left(\beta^{t},\lambda_1^{t},\lambda_2^{t}\right)$, for $t=1,2,...M$ in an ascending order, as $\zeta^{(1)},\zeta^{(2)},...\zeta^{(M)}$, after burn in as $\zeta^{(\Upsilon+1)},\zeta^{(\Upsilon+2)},...\zeta^{(M)}$, then for arbitrary $0<\alpha<1$, the $100(1-\alpha)\%$ two sided HPD credible interval of $\zeta\left(\beta^{t},\lambda_1^{t},\lambda_2^{t}\right)$ can be obtained as

$$\left(\zeta^{[t^*]},\zeta^{[t^*+(1-(\alpha/2))(M-\Upsilon)]}\right),\quad t^*=\Upsilon+1,\Upsilon+2.....\left[(\alpha/2)M\right]$$

where t^* is chosen such that

$$\left(\zeta^{[t^*]},\zeta^{[t^*+(1-(\alpha/2))(M-\Upsilon)]}\right)=\min_{1\le t\le(\alpha/2)(M-\Upsilon)}\left(\zeta^{[t]},\zeta^{[t+(1-(\alpha/2))(M-\Upsilon)]}\right)$$

where $[x]$ denotes the greatest integer less than or equal to x. Then the HPD credible interval of ζ_t is that interval which has the smallest width.

4. SIMULATION STUDY AND DATA ANALYSIS

The aim of this section is to compare the performance of the different methods of estimation discussed in the previous sections. Monte Carlo study is employed to check the behavior of the proposed methods as well as to evaluate the statistical performances of the estimators under GPH type-I censoring scheme in the presence of competing risks model. Also, a real data set is analyzed for illustrative purpose. R-statistical programming language will be used for calculation. Further, one can utilize bbmle and HD Interval packages to compute MLEs and HPD interval in R-language.

4.1 Simulation Study

In this subsection, we present some simulation results mainly to see how the different methods proposed in this paper behave in practice. Monte Carlo process is performed using two methods of estimation; namely maximum likelihood estimation and Bayesian estimation. To generate generalized progressive hybrid censored competing risks data from IW distribution, do the following algorithm:

Step 1: Set the parameter values of β,λ_1 and λ_2

Step 2: Generate an ordinary progressive Type-II censored sample $x_{(i)}^{R}, i=1,2,....r$ using the algorithm outlined by Balakrishnan and Sandhu (1995), as follows:

a) Generate w independent observations of size r as $w_1,w_2,....w_r$.

b) For a given values of n,r,T and R_i, $i=1,2,...,r$ put $V_i=w^{(i+\sum_{j=r-i+1}^{r}R_j)^{-1}}$.

c) Set $U_i=1-V_rV_{r-1}...V_{r-i+1}$ for $i=1,2,...,r$. Then U_i is a progressive type-II censored sample of size r from $U(0,1)$ distribution.

d) Generate progressive type-II censored competing risks sample from IW (β,λ_j) by $x_{(i)}=F(x;\beta,\lambda_j)^{-1}$, $j=1,2$, $i=1,2,....r$.

Step 3: If $T < x_{(k)} < x_{(r)}$ the experiment stops at k^{th} failure, we get the generalized progressively hybrid censored sample of case 1 which is $x_{(1)}, x_{(2)}, ...x_{(k)}$.

Step 4: If $x_{(k)} < T < x_{(r)}$ the experiment stops at pre-fixed time T with D^{th} the number of observed failures up to time T. In this case, we get the generalized progressive hybrid censored sample of case 2 which is $x_{(1)}, x_{(2)}, ...x_{(D)}$.

Step 5: If $x_{(k)} < x_{(r)} < T$ the experiment stops at r^{th} failure, we get the generalized progressively hybrid censored sample of case 3 which is $x_{(1)}, x_{(2)}, ...x_{(r)}$.

Using each simulated data, the MLEs and associated ACIs of β, λ_1 and λ_2 are computed. Here, When the actual values of the parameters $(\beta, \lambda_1, \lambda_2)$ are taken as (1.5,.5,.75) and (2,1.2,1.5), a large number of 1000 GPH type-I censored samples are generated from IW distribution using various mixtures of sample sizes n (for each cause of failure,) r (effective sample size), k (minimum effective number of failures) and T (threshold time point). Also, three different progressive censoring schemes are considered as

Scheme I: $R_1 = n - r, R_2 = ... = R_r = 0$,

Scheme II: $R_{r/2} = n - r, R_1 = ... = R_{r/2-1} = R_{r/2+1} = ... = R_r = 0$,

Scheme III: $R_1 = ... = R_{r-1} = 0, R_r = n - r$.

Consider the following different cases in Table 1.

Table 1
Different Cases Considered for Simulation when $T = (0.6, 0.8)$

Scheme I	Scheme II	Scheme III
$n = 20, r = 18, k = 10$	$n = 20, r = 18, k = 10$	$n = 20, r = 18, k = 10$
$n = 20, r = 18, k = 15$	$n = 20, r = 18, k = 15$	$n = 20, r = 18, k = 15$
$n = 40, r = 25, k = 10$	$n = 40, r = 25, k = 10$	$n = 40, r = 25, k = 10$
$n = 40, r = 25, k = 15$	$n = 40, r = 25, k = 15$	$n = 40, r = 25, k = 15$

Based on the generated data, MLEs and associated 95% ACIs / HPD are computed. Note that the initial guess values are considered to be the same as the true parameter values while obtaining MLEs and subsequently get the hyper parameter values from equation (10). These values, hyper-parameters, are then plugged-in to calculate the desired estimates. We have obtained various estimates based on 1000 replications. At the end, using MH algorithm to calculate Bayesian estimators, 2000 burn-in samples are discarded among the total 10000 MCMC samples generated from the posterior density. The performance of 95% ACIs / HPD are compared using their average interval lengths (AILs) and related coverage probabilities (CPs) (in brackets), respectively. The average estimates (AEs) and related Mean squared errors (MSEs) (in brackets) for both methods are reported in Table 2-5.

Specifically, the average estimates (AEs) of β, λ_1 and λ_2 (say θ) is given by

$$\bar{\tilde{\theta}} = \frac{1}{N} \sum_{\tau=1}^{N} \tilde{\theta}^{\tau} \text{ for } i = 1,2,3$$

where N denotes the number of replications in the Monte Carlo simulation and $\theta_1 = \beta, \theta_2 = \lambda_1, \theta_3 = \lambda_2$.

A comparison between point estimates of θ was made based on their MSE as

$$\text{MSE}(\tilde{\theta}) = \frac{1}{N} \sum_{\tau=1}^{N} (\tilde{\theta}^{\tau} - \theta)^2 \text{ for } i = 1,2,3$$

The smaller value of MSE represents an estimator with better accuracy.

On the other hand, the comparison between the interval estimates of θ was made based on their AILs and CP, respectively, as

$$\text{AIL}(\theta_i) = \frac{1}{N} \sum_{\tau=1}^{N} (U_{\tilde{\theta}^{\tau}} - L_{\tilde{\theta}^{\tau}}) \text{ and } \text{CP}(\theta_i) = \frac{1}{N} \sum_{\tau=1}^{N} I_{(U_{\tilde{\theta}^{\tau}};\, L_{\tilde{\theta}^{\tau}})}(\theta_i)$$

where $I(.)$ is the indicator function, $L(.)$ and $U(.)$ denote the lower and upper bounds, respectively, of $100(1-\alpha)\%$ asymptotic (or credible) interval.

From Tables 2-5, it is observed that:

- All of the average estimates and related MSEs for both methods are showed.
- As the effective increases (i.e., *n* or *r* or *k*, or their combinations), the AEs and MSEs of both MLEs and BEs for parameters β, λ_1 and λ_2 decreases in most cases.
- Under fixed censoring schemes, the AEs decreases and MSEs increases of MLEs, when *T* increases.
- Further, the corresponding AILs and CPs (in brackets) for all the proposed confidence intervals, namely; ACIs and HPD interval are presented when $T = 0.6$ and $T = 0.8$, respectively.
- For all cases of confidence intervals, HPD interval of the Bayes SE has the smallest length among the approximate confidence interval of the MLEs.
- In most cases, as the value of β, λ_1 and λ_2 increases, the associated MSEs of the MLEs of β, λ_1 and λ_2 increase (in the case of ACIs) and decrease (in the case of credible intervals).

To sum up, it is clear from the simulation results that the performance of both Bayes point and credible interval estimates behave superior than the traditional estimates obtained under the maximum likelihood approach in terms of minimum MSEs for point estimates and in terms of lowest AILs and largest CPs for interval estimates. Finally, the Bayes MCMC method using MH algorithm to estimate the unknown parameters of the IW distribution under GPH type-I censored data with competing risks is recommended.

Moreover, as further illustration, the trace and density plots for all parameters in an MCMC trace with their histograms for each parameter and the convergence of MCMC estimation for β, λ_1 and λ_2 of GPH type-I using MH algorithm are showed in Figure 3.

Table 2
AEs, MSEs (in bracket), AILs and CPs (in brackets) of the MLEs and BEs at $(\beta,\lambda_1,\lambda_2)=(1.5,.5,.75)$ and $T=$ 0.6

n	r	k	Sc	par	MLE AEs(MSE)	MLE ACI (CP)	Bayes AEs(MSE)	Bayes HPD (CP)
20	18	10	I	β	1.4314(0.4784)	1.3539(87.0)	1.6603(0.0427)	0.4653(98.8)
				λ_1	0.9954(0.4286)	1.0549(90.1)	0.6041(0.0174)	0.2509(98.1)
				λ_2	1.2863(0.4389)	1.4598(96.8)	0.8276(0.0107)	0.2432(96.3)
			II	β	1.3996(0.4089)	1.3208(88.1)	1.6610(0.0400)	0.4399(97.9)
				λ_1	1.0159(0.4448)	1.0615(92.0)	0.6142(0.0195)	0.2539(98.9)
				λ_2	1.2893(0.4206)	1.4597(97.4)	0.8159(0.0079)	0.2329(97.5)
			III	β	1.4151(0.4796)	1.3299(87.7)	1.6554(0.0422)	0.4709(97.2)
				λ_1	1.0116(0.4459)	1.0492(90.8)	0.6121(0.0196)	0.2488(98.8)
				λ_2	1.2769(0.4176)	1.4443(97.4)	0.8105(0.0076)	0.2162(97.7)
		15	I	β	1.6016(0.3352)	1.2433(87.9)	1.6252(0.0282)	0.4249(97.2)
				λ_1	0.7019(0.1235)	0.7607(89.2)	0.5419(0.0049)	0.1989(98.0)
				λ_2	1.1721(0.3167)	1.1338(93.2)	0.8537(0.0158)	0.2708(96.8)
			II	β	1.5912(0.3324)	1.2420(87.4)	1.6372(0.0336)	0.4529(98.1)
				λ_1	0.7096(0.1218)	0.7679(90.2)	0.5512(0.0059)	0.2096(97.2)
				λ_2	1.1777(0.3130)	1.1391(93.6)	0.8395(0.0121)	0.2386(95.8)
			III	β	1.5825(0.3319)	1.2429(87.1)	1.6256(0.0296)	0.4296(97.5)
				λ_1	0.7205(0.1356)	0.7722(89.0)	0.5520(0.0066)	0.2059(97.3)
				λ_2	1.1675(0.3096)	1.1338(92.9)	0.8572(0.0168)	0.2720(96.3)
40	25	10	I	β	1.1647(0.2862)	0.8536(85.9)	1.6475(0.0411)	0.5407(97.7)
				λ_1	1.2627(0.8597)	0.9508(83.3)	0.6847(0.0517)	0.4119(98.2)
				λ_2	1.5262(0.9063)	1.3214(92.3)	0.8523(0.0138)	0.2194(98.0)
			II	β	1.1582(0.2830)	0.8217(84.8)	1.6232(0.0283)	0.4473(96.0)
				λ_1	1.2777(0.9063)	0.9125(80.4)	0.7014(0.0595)	0.4089(96.5)
				λ_2	1.4723(0.7541)	1.2076(91.2)	0.8491(0.0123)	0.1956(96.6)
			III	β	1.1505(0.2777)	0.8118(84.4)	1.6260(0.0297)	0.4486(96.9)
				λ_1	1.2953(0.9224)	0.8935(84.1)	0.7151(0.0664)	0.4052(97.9)
				λ_2	1.4728(0.7181)	0.9201(85.2)	1.6359(0.0316)	0.4524(98.7)
		15	I	β	1.0372(0.5249)	0.8381(71.6)	0.6284(0.0309)	0.3586(96.2)
				λ_1	1.3684(0.5472)	1.1742(93.2)	0.8638(0.0174)	0.2553(96.0)
				λ_2	1.2136(0.2772)	0.8591(85.9)	1.6377(0.0364)	0.4939(97.3)
			II	β	1.1721(0.7129)	0.8595(78.0)	0.6714(0.0489)	0.4020(97.5)
				λ_1	1.4219(0.6279)	1.1685(93.4)	0.8517(0.0135)	0.2058(97.5)
				λ_2	1.2213(0.2843)	0.8556(84.1)	1.6399(0.0377)	0.5264(96.6)
			III	β	1.1961(0.7679)	0.8457(76.2)	0.6927(0.0602)	0.4241(96.7)
				λ_1	1.4103(0.6176)	1.1274(94.5)	0.8562(0.0139)	0.2082(96.0)
				λ_2	1.0372(0.5249)	0.9201(85.2)	1.6359(0.0316)	0.4524(98.7)

Table 3
AEs, MSEs(in bracket), AILs and CPs(in brackets) of the MLEs and Bayes Estimates at $(\beta,\lambda_1,\lambda_2)=(1.5,.5,.75)$ and $T=0.8$

n	r	k	Sc	par	MLE AEs(MSE)	MLE ACI (CP)	Bayes AEs(MSE)	Bayes HPD (CP)
20	18	10	I	β	1.3147(0.4020)	1.0216(85.5)	1.6190(0.0310)	0.5023(97.8)
				λ_1	1.1659(0.8141)	0.9181(65.8)	0.6584(0.0463)	0.4028(97.1)
				λ_2	1.3727(0.6082)	1.2464(91.4)	0.8441(0.0123)	0.2332(96.9)
			II	β	1.3371(0.3238)	1.0276(85.2)	1.6197(0.0312)	0.4969(98.9)
				λ_1	1.1272(0.7629)	0.8940(63.7)	0.6555(0.0465)	0.4056(97.8)
				λ_2	1.3247(0.5333)	1.1967(91.4)	0.8398(0.0115)	0.2277(96.6)
			III	β	1.2861(0.3475)	0.9966(84.0)	1.6058(0.0303)	0.5355(97.6)
				λ_1	1.2059(0.8757)	0.9214(69.0)	0.6970(0.0658)	0.4386(97.1)
				λ_2	1.3646(0.5753)	1.2104(93.9)	0.8477(0.0126)	0.2154(96.5)
		15	I	β	1.4769(0.3014)	1.0874(86.3)	1.6193(0.0316)	0.4487(98.0)
				λ_1	0.8579(0.3120)	0.7830(77.4)	0.5855(0.0182)	0.3376(96.8)
				λ_2	1.2492(0.3975)	1.1143(93.4)	0.8484(0.0139)	0.2354(97.5)
			II	β	1.4671(0.3423)	1.0890(86.5)	1.6068(0.0264)	0.4788(98.6)
				λ_1	0.9069(0.3903)	0.7974(72.1)	0.6073(0.0249)	0.3503(98.6)
				λ_2	1.2524(0.4173)	1.1142(92.4)	0.8556(0.0153)	0.2566(96.5)
			III	β	1.4597(0.2808)	1.0838(85.3)	1.6122(0.0258)	0.4514(97.1)
				λ_1	0.9052(0.3914)	0.7927(72.1)	0.6079(0.0269)	0.3657(99.1)
				λ_2	1.2440(0.4038)	1.1064(93.9)	0.8422(0.0122)	0.2300(96.8)
40	25	10	I	β	1.4566(0.1302)	0.8625(88.6)	1.6087(0.0247)	0.4281(96.7)
				λ_1	0.7330(0.1772)	0.6042(91.1)	0.5619(0.0145)	0.2981(95.2)
				λ_2	1.2127(0.3278)	0.8993(92.1)	0.8914(0.0256)	0.2841(96.9)
			II	β	1.4923(0.1441)	0.9058(88.8)	1.6230(0.0272)	0.4249(96.6)
				λ_1	0.7097(0.1651)	0.5948(91.4)	0.5568(0.0152)	0.2931(95.3)
				λ_2	1.1873(0.3049)	0.9029(91.5)	0.9029(0.0294)	0.2949(96.6)
			III	β	1.5053(0.1424)	0.9336(88.5)	1.6204(0.0264)	0.4128(97.8)
				λ_1	0.6881(0.1546)	0.5908(92.7)	0.5533(0.0153)	0.2061(95.2)
				λ_2	1.1558(0.2735)	0.9018(92.1)	0.8832(0.0224)	0.2587(97.8)
		15	I	β	1.5025(0.1712)	0.9344(86.7)	1.6231(0.0296)	0.4765(98.7)
				λ_1	0.7081(0.1913)	0.5948(92.2)	0.5567(0.0186)	0.5690(95.7)
				λ_2	1.1651(0.2962)	0.9034(90.5)	0.8923(0.0254)	0.2797(97.6)
			II	β	1.4681(0.1437)	0.8913(88.6)	1.6141(0.0275)	0.4467(96.7)
				λ_1	0.7204(0.1685)	0.6005(91.1)	0.5577(0.0149)	0.2611(95.2)
				λ_2	1.1864(0.2958)	0.8971(91.3)	0.8949(0.0265)	0.2764(96.7)
			III	β	1.5157(0.1542)	0.9423(88.4)	1.6161(0.0268)	0.4477(98.2)
				λ_1	0.6819(0.1610)	0.5875(91.4)	0.5501(0.0155)	0.2152(95.5)
				λ_2	1.1448(0.2772)	0.8969(91.5)	0.8894(0.0249)	0.2887(97.3)

Table 4
AEs, MSEs(in bracket), AILs and CPs(in brackets) of the MLEs and BEs at $(\beta,\lambda_1,\lambda_2)=(2,1.2,1.5)$ and $T=0.6$

n	r	k	Sc	par	MLE		Bayes	
					AEs(MSE)	ACI (CP)	AEs(MSE)	HPD (CP)
20	18	10	I	β	2.9485(4.5877)	2.7659(85.7)	2.1389(1.4941)	0.4619(99.4)
				λ_1	1.3476(0.3466)	1.2592(88.0)	1.2761(0.2028)	0.3955(97.8)
				λ_2	2.0789(1.0667)	1.8241(89.5)	0.0599(0.0107)	0.5047(96.9)
			II	β	2.9571(3.9806)	2.8011(85.3)	2.1168(0.1179)	0.4622(98.6)
				λ_1	1.3609(0.3551)	1.2603(86.5)	1.2674(0.0328)	0.4201(97.5)
				λ_2	2.0619(1.0585)	1.7943(89.2)	1.6534(0.0568)	0.4922(97.4)
			III	β	2.9875(5.6183)	2.8567(86.4)	2.1567(2.3894)	0.4797(98.3)
				λ_1	1.3181(0.3196)	1.2382(88.4)	1.2804(0.5277)	0.4111(97.7)
				λ_2	2.0142(1.1305)	1.7831(91.0)	1.6453(0.0493)	0.4949(98.7)
		15	I	β	2.3092(0.9241)	1.7330(87.6)	2.0327(0.0192)	0.4956(99.2)
				λ_1	0.8579(0.3120)	0.7830(77.4)	0.5855(0.0182)	0.3376(96.8)
				λ_2	2.0142(1.1305)	1.7831(91.0)	1.6453(0.0493)	0.4949(98.7)
			II	β	2.3092(0.9241)	1.7330(87.6)	2.0327(0.0192)	0.4956(99.2)
				λ_1	1.6197(0.4157)	1.2227(91.1)	1.3306(0.0299)	0.4089(96.9)
				λ_2	2.3191(1.2365)	1.7105(89.9)	1.7404(1.6837)	0.5051(98.6)
			III	β	2.2841(0.9668)	1.7338(86.8)	2.0275(0.0203)	0.5261(98.5)
				λ_1	1.5944(0.3926)	1.1971(91.2)	1.3275(0.0276)	0.4073(98.1)
				λ_2	2.2491(1.1021)	1.6447(92.0)	1.6925(0.0532)	0.4965(96.8)
40	25	10	I	β	3.0759(3.3156)	2.9106(86.2)	2.2380(2.6287)	0.46567(97.7)
				λ_1	1.0505(0.3061)	1.1307(83.7)	1.2399(0.0236)	0.4684(98.0)
				λ_2	1.6835(0.4404)	1.6543(93.0)	1.6593(0.0552)	0.5338(97.1)
			II	β	3.2049(4.6596)	3.1257(86.0)	2.2685(4.0862)	0.5048(97.5)
				λ_1	1.0379(0.3064)	1.1391(84.5)	1.2263(0.0327)	0.4560(98.4)
				λ_2	1.6207(0.4209)	1.6248(91.4)	1.6467(0.0627)	0.5408(97.7)
			III	β	3.2178(4.3261)	3.2225(84.7)	2.2431(4.4589)	0.4949(97.4)
				λ_1	1.0049(0.3045)	1.1407(85.0)	1.2163(0.0508)	0.4847(98.6)
				λ_2	1.5587(0.4484)	1.6114(83.2)	1.6446(0.0669)	0.5631(98.7)
		15	I	β	2.4483(1.1216)	1.8440(86.7)	2.1288(0.0375)	0.5523(97.9)
				λ_1	1.3574(0.2375)	1.01756(85.5)	1.2910(0.0233)	0.4681(96.5)
				λ_2	2.0143(0.5546)	1.3901(90.2)	1.7402(0.0820)	0.5985(97.1)
			II	β	2.5570(1.4522)	1.9789(86.0)	2.1268(0.0388)	0.5196(98.2)
				λ_1	1.3045(0.1976)	1.0051(88.3)	1.2932(0.0214)	0.4337(97.2)
				λ_2	1.9419(0.4689)	1.3629(92.9)	1.7286(0.0712)	0.5135(96.5)
			III	β	2.5224(1.3869)	1.9925(85.5)	2.1187(0.0383)	0.5199(98.2)
				λ_1	1.2988(0.1819)	1.0057(88.7)	1.2921(0.0195)	0.3998(95.3)
				λ_2	1.9271(0.4551)	1.3656(92.0)	1.7291(0.0809)	0.5035(96.1)

Table 5
AEs, MSEs(in bracket), AILs and CPs(in brackets) of the MLEs and BE $(\beta,\lambda_1,\lambda_2)=(2,1.2,1.5)$ and $T=$ 0.8

n	r	k	Sc	par	MLE AEs(MSE)	MLE ACI (CP)	Bayes AEs(MSE)	Bayes HPD (CP)
20	18	10	I	β	2.9039(4.4882)	2.7212(86.1)	2.1405(1.4944)	0.4575(99.4)
				λ_1	1.3803(0.3397)	1.2738(88.6)	1.2838(0.2031)	0.3753(97.3)
				λ_2	2.0948(1.0549)	1.8291(89.9)	1.6644(0.0600)	0.5172(98.2)
			II	β	2.8885(3.7866)	2.7329(85.9)	2.1188(0.1181)	2.3064(98.6)
				λ_1	1.4061(0.3461)	1.2812(87.9)	1.2803(0.0312)	0.3864(97.9)
				λ_2	2.0859(1.0405)	1.8058(89.6)	1.6544(0.0553)	0.4950(97.1)
			III	β	2.8812(5.2868)	2.7419(86.8)	2.1582(2.3866)	0.4783(97.6)
				λ_1	1.3843(0.3019)	1.2685(91.1)	1.2992(0.5255)	0.3575(97.5)
				λ_2	2.0485(1.1012)	1.7946(91.6)	1.6458(0.0469)	0.4982(98.8)
		15	I	β	2.3092(0.9241)	1.7330(87.6)	2.0327(0.0192)	0.4956(99.2)
				λ_1	1.6197(0.4157)	1.2227(91.1)	1.3306(0.0299)	0.4089(96.8)
				λ_2	2.3195(1.2365)	1.7105(89.9)	1.7404(1.6837)	0.5051(98.6)
			II	β	2.2164(0.7702)	1.6695(86.9)	2.0187(0.0193)	0.5214(98.9)
				λ_1	1.6549(0.4399)	1.2345(91.6)	1.3391(0.0309)	0.4178(96.1)
				λ_2	2.3243(1.2280)	1.6997(90.6)	1.7028(0.0578)	0.5089(97.1)
			III	β	2.2817(0.9639)	1.7320(86.9)	2.0275(0.0203)	0.5261(98.5)
				λ_1	1.5962(0.3917)	1.1979(91.2)	1.3280(0.0276)	0.4035(98.1)
				λ_2	2.2503(1.1010)	1.6451(92.0)	1.6925(0.0532)	0.4974(96.9)
40	25	10	I	β	2.6563(2.8187)	2.4786(84.6)	2.1956(0.0619)	0.5679(97.6)
				λ_1	1.4570(0.4384)	1.2458(82.7)	1.3537(0.2446)	0.4793(97.1)
				λ_2	1.9053(0.4361)	1.6954(95.2)	0.5225(0.0138)	0.5328(96.2)
			II	β	2.5202(2.3547)	2.3791(85.4)	2.1814(0.0565)	0.4991(98.7)
				λ_1	1.5715(0.5067)	1.2758(85.1)	1.4019(0.1119)	0.4859(98.4)
				λ_2	1.9139(0.4142)	1.6600(94.5)	1.6677(0.0498)	0.5153(97.5)
			III	β	2.4293(2.5072)	2.3258(84.8)	2.2106(0.1833)	0.5509(98.7)
				λ_1	1.6836(0.6179)	1.2960(87.8)	1.4495(0.8427)	0.4478(98.6)
				λ_2	1.9357(0.4068)	1.6540(96.3)	1.6499(0.0557)	0.4676(96.5)
		15	I	β	2.4471(1.0683)	1.8384(86.0)	2.1276(0.0783)	0.5097(98.7)
				λ_1	1.3624(0.2145)	1.0185(88.1)	1.3089(0.0259)	0.4412(96.5)
				λ_2	2.0124(0.5363)	1.3952(90.5)	1.7145(0.0668)	0.5619(97.2)
			II	β	2.4369(1.2939)	1.8823(86.3)	2.1269(0.0743)	0.5077(97.6)
				λ_1	1.3881(0.2246)	1.0262(88.3)	1.3219(0.0280)	0.4154(97.1)
				λ_2	1.9819(0.4589)	1.3781(92.7)	1.7112(0.0633)	0.5427(96.8)
			III	β	2.4322(1.2849)	1.9219(85.7)	2.1545(0.5373)	0.4799(99.1)
				λ_1	1.4192(0.2221)	1.0382(89.7)	1.3412(0.0388)	0.3959(97.6)
				λ_2	1.9705(0.4383)	1.3664(93.4)	1.7089(0.0623)	0.5292(96.5)

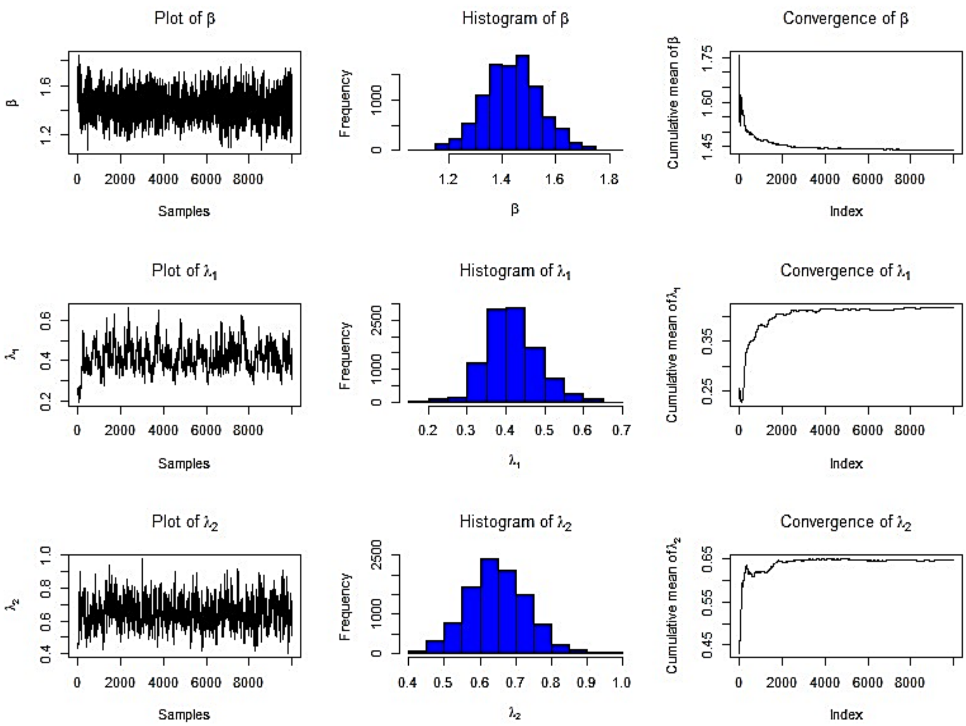

Figure 3: Convergence of MCMC Estimates for β, λ_1 and λ_2 using MH Algorithm

4.2 Real Data

A real data set is analyzed in this section to illustrate the proposed competing risk model. The studied data was first analyzed by Meintanis (2007) using Marshall-Olkin distributions. The data employed here corresponds to the soccer (football) data where at least one goal was scored directly by a kick by any team and at least one goal was scored by the home team. In this data analysis, random variable X_{1i} denotes the time in hours (divided by 60) of the first kick goal scored by any team and X_{2i} represents the time in hours (divided by 60) of the first goal of any type scored by the home team, and the detail data is provided in Table 4, the random variables, X_{1i} and X_{2i} are treated as two competing risks as cause 1 and cause 2, respectively, and the associated complete competing risks data are also reported. Here we are analysing the time, say $X_i = min\left(X_{1i}, X_{2i}\right)$, taken to score the first goal of any type by a team in the UEFA Champion League (football game) as competing risks data. To see how the causes are competing to each other, there are three possibilities (i) $X_{1i} < X_{2i}$, (ii) $X_{1i} > X_{2i}$ and (iii) $X_{1i} = X_{2i}$. So, we have $X_i = X_{1i}$ if $X_{1i} \leqslant X_{2i}$ and $X_i = X_{2i}$ if $X_{1i} \geqslant X_{2i}$. Thus causes are competing among each other. From Table 6, three groups of generalized progressive hybrid type-I censored competing risks are generated which are listed in Table 7 using the same initial sample size setting $n = 37$ and items R_i removed at the time censoring T_i where $i = 1,2,....r$. It is seen that the finally three groups of failure samples correspond to types of observations as case 1, case 2 and case 3, respectively, and these different schemes can be described as follow:

Case 1 ($R_1 = R_2 = ... = R_{29} = 0, R_{30} = 7$, this is can be written as: $R = (0^{*29}, 7), T = .5$ and $K = 25$), Case 2 ($R_1 = 7, R_2 = R_3 = ... = R_{30} = 0$, this is can be written as: $R = (7, 0^{*29}), T = 1$ and $D = 27$) and Case 3 ($R_1 = 4, R_2 = R_3 = ... = R_{29} = 0, R_{30} = 3$, this is can be written as: $R = (4, 0^{*28}, 3), T = 1.3$ and $r = 30$).

Since there is no information about the unknown parameters, the non-informative priors (NIPs) with $a_1 = b_1 = a_2 = b_2 = a_3 = b_3 = 0$ are adopted in this illustrative example.

Table 6
Complete Competing Risks Failure Data (X_{1i}, X_{2i})
for the Proportion Time of Football Game

Origin Football Game Proportion Time			
(0.0333, 0.0333)	(0.6833, 0.0500)	(0.7000, 0.0500)	(0.9000, 0.1167)
(0.1333, 0.1333)	(0.4167, 0.1500)	(0.9166, 0.1833)	(0.7333, 0.2167)
(0.3667, 0.2333)	(0.4167, 0.2333)	(1.0667, 0.2500)	(0.2667, 0.2667)
(0.2667, 1.2500)	(1.0500, 0.3000)	(0.3000, 0.3000)	(0.3167, 0.3167)
(0.4333, 0.3333)	(0.4000, 0.4000)	(0.4333, 0.8000)	(0.4500, 0.7833)
(0.4667, 0.4667)	(0.8500, 0.4667)	(0.7333, 0.5000)	(0.5667, 0.5667)
(0.6000, 0.8667)	(0.6500, 0.6500)	(0.8833, 0.6500)	(0.6667, 0.6667)
(0.7000, 0.7000)	(1.3667, 0.8000)	(0.8167, 0.8167)	(0.8167, 0.8167)
(1.1000, 1.0333)	(1.2667, 1.0667)	(1.1000, 1.4167)	(1.1500, 1.1833)
(1.2000, 1.2000)			

Complete Competing Risks Failure Data						
(0.0333, 1)	(0.0500, 2)	(0.0500, 2)	(0.1167, 2)	(0.1333, 1)	(0.1500, 2)	(0.1833, 2)
(0.2167, 2)	(0.2333, 2)	(0.2333, 2)	(0.2500, 2)	(0.2667, 2)	(0.2667, 1)	(0.3000, 2)
(0.3000, 1)	(0.3167, 1)	(0.3333, 2)	(0.4000, 1)	(0.4333, 1)	(0.4500, 1)	(0.4667, 1)
(0.4667, 2)	(0.5000, 2)	(0.5667, 1)	(0.6000, 1)	(0.6500, 1)	(0.6500, 2)	(0.6667, 1)
(0.7000, 1)	(0.8000, 2)	(0.8167, 1)	(0.8167, 1)	(1.0333, 2)	(1.0667, 2)	(1.1000, 1)
(1.1500, 1)	(1.2000, 1)					

Table 7
GPH Type-I Censored Samples from Football Game Data

Case 1: $n = 37,\ r = 30,\ R = (0^{*29}, 7),\ T = .5,\ k = 25$						
(.0333, 1)	(.0500, 2)	(.1167, 2)	(.0333, 1)	(.1333, 1)	(.1500, 2)	(.1833, 2)
(.2167, 2)	(.2333, 2)	(.2333, 2)	(.2500, 2)	(.2667, 1)	(.2666, 1)	(.3000, 2)
(.3000, 1)	(.3167, 1)	(.3333, 2)	(.4000, 1)	(.4667, 2)	(.4333,1)	(.4500, 1)
(.4667, 1)	(.5000, 2)	(.5667, 1)	(.6000, 1)			
Case 2: $n = 37,\ r = 30,\ R = (7, 0^{*29}), T = 1,\ D = 27$						
(.0333, 1)	(.0500, 2)	(.1167, 2)	(.1333, 1)	(.1500, 2)	(.1833, 2)	(.2167, 2)
(.2333, 2)	(.2500, 2)	(.2667, 1)	(.2667, 1)	(.3000, 1)	(.3000, 1)	(.3167, 1)
(.3333, 2)	(.4000, 1)	(.4333, 1)	(.4500, 1)	(.4667, 1)	(.5000, 2)	(.5667, 1)
(.6000, 1)	(0.6500, 2)	(.6667, 1)	(.7000, 1)	(.8000, 2)	(.8167, 1)	
Case 3: $n = 37,\ r = 30,\ R = (4, 0^{*28}, 3), T = 1.3,\ r = 30$						
(0.0333, 1)	(0.0500, 2)	(.1167, 2)	(.1333, 1)	(.1500, 2)	(.1833, 2)	(.2167, 2)
(0.2333, 2)	(0.2500, 2)	(.2667, 2)	(.2667, 1)	(.3000, 1)	(.3000, 1)	(.3167, 1)
(0.3333, 2)	(0.4000, 1)	(.4333, 1)	(.4500, 1)	(.4667, 1)	(.5000, 2)	(.5667, 1)
(0.6000, 1)	(0.6500, 2)	(.6667, 1)	(.7000, 1)	(.8000, 2)	(.8167, 1)	(.8167, 1)
(1.0333, 2)	(1.0667, 2)					

Before analyze the data, we investigate whether the IW distribution can be employed or not to analyze these data. Kolmogorov-Smirnov test statistic values (K-S) and the corresponding p-values are provided, saying that the data fit the IW distribution with the parameters given in Table 8. Moreover, as further illustration, the empirical cumulative distributions plot and the fitted densities plot with a histogram of probability are graphed in Figure 4, which also imply that the IW distribution provides a reasonable fit for these data. Moreover, the curves of profile log-likelihood function are plotted in Figure 5, which shows that the MLE of β is unique and exist for all cases. The MLEs and BEs (with their standard errors) based on both case 1, 2 and 3 are calculated and reported in Table 9. It is observed from this table that the point estimates obtained by maximum likelihood and Bayesian methods of the unknown parameters β, λ_1 and λ_2 are quite close to each other and the relative risk at each censoring scheme are computed and reported in Table 9. The results of Table 10 indicate that the HPDs are slightly shorter than the other confidence intervals in respect of their interval lengths.

Table 8
MLEs, Kolmogorov-Smirnov Test and p-value Results for Data

Data	β	λ_1	λ_2	**K-S**	**p-value**
Cause 1 for failure	0.9098716	0.3239284	-	0.2399817	0.2238614
Cause 2 for failure	1.1279799	-	0.1416295	0.1738660	0.6479818

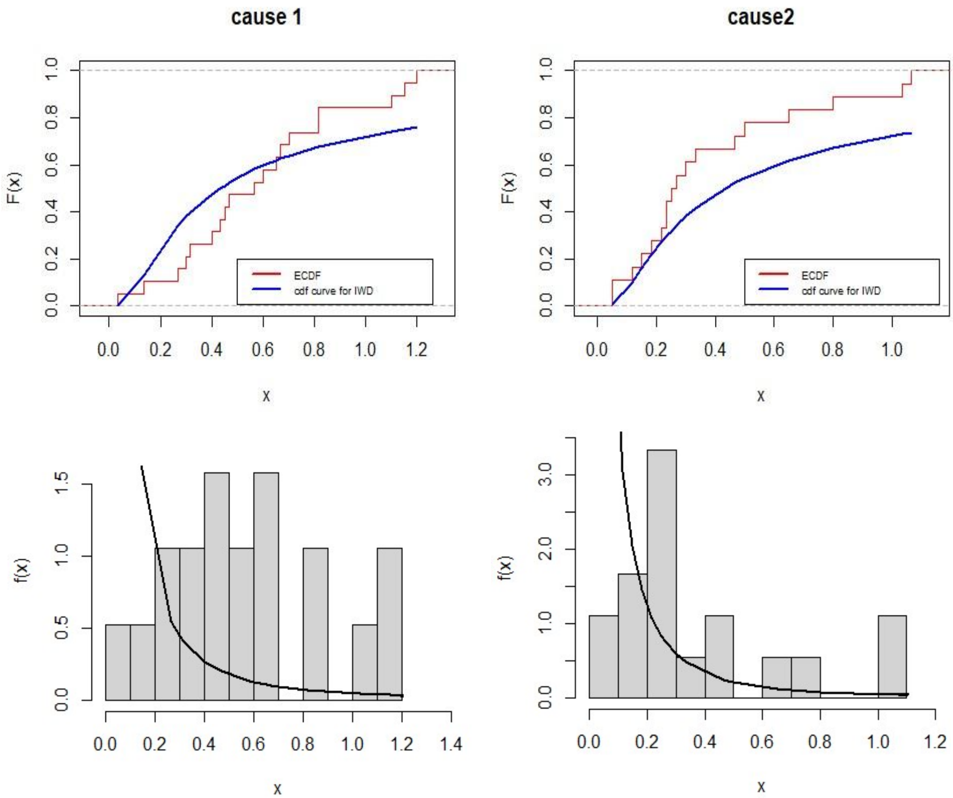

Figure 4: Fitted the cdf and pdf of IW for Data

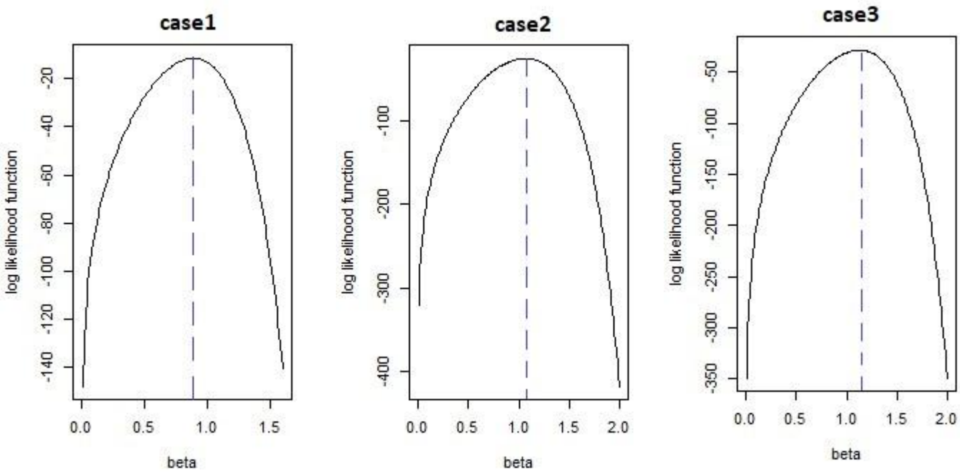

Figure 5: Plots of the Profile Log-Likelihood of β from Real Data

Table 9
Point Estimates of β, λ_1 and λ_2 for Real Data and Relative Risk

Case	Parameter	MLE		Bayes		RR_1	RR_2
		Estimate	sd.error	Estimate	sd.error		
1	β	0.885468	0.117363	0.908368	0.118657	0.4275	0.5725
	λ_1	0.367085	0.112515	0.350686	0.113485		
	λ_2	0.274121	0.089564	0.255262	0.083335		
2	β	1.068541	0.097822	0.745556	0.091513	0.2467	0.7533
	λ_1	0.485360	0.128200	0.306153	0.084629		
	λ_2	0.349940	0.102160	0.279963	0.076142		
3	β	1.127046	0.095993	1.128659	0.093544	0.1872	0.8128
	λ_1	0.262253	0.072281	0.260797	0.073632		
	λ_2	0.259646	0.068907	0.262037	0.068134		

Table 10
Interval Estimates for MLEs and HPD Credible Interval for Real Data based on GPH Type-I Censored

Case	par	ACI			HPD		
		lower	upper	AIL	lower	upper	AIL
1	β	0.655441	1.115495	0.460054	0.6831695	1.1379157	0.4547463
	λ_1	0.146559	0.587609	0.441051	0.6831695	1.1379157	0.4547463
	λ_2	0.098578	0.449664	0.351086	0.1128373	0.4234993	0.3106620
2	β	0.555328	0.918450	0.363122	0.5678666	0.9211730	0.3533065
	λ_1	0.434291	1.038141	0.603849	0.4455977	1.0149608	0.5693631
	λ_2	0.229722	0.717220	0.487498	0.2411288	0.7265564	0.4854276
3	β	0.938903	1.315189	0.376286	0.9615474	1.3445845	0.3830371
	λ_1	0.120585	0.403921	0.283335	0.1240977	0.4005367	0.2764395
	λ_2	0.124591	0.394701	0.270109	0.1493062	0.3488508	0.1995447

ACI: approximate confidence interval, par: parameter

5. CONCLUSIONS

In this paper, we considered making statistical inference for GPH type-I in presence of competing risks. We obtained both point and interval estimates of the parameters using MLE and Bayesian approaches when latent failure times follow IW distribution with the same shape and different scale parameters. We propose to apply MCMC technique to carry out a Bayesian estimation procedure. The performance of the proposed methods was also studied and we noticed that the Bayesian method provides better estimation results compared to the MLE method. A real data is also discussed in support of the proposed competing risks model. In most instances, the MLEs and the Bayes estimates are almost similar and the HPDs have shorter interval lengths than those of associated ACIs in most cases. The Bayes estimates are obtained under non-informative prior because we do not have any prior information about the unknown parameters. In the literature, a limitation of the progressive hybrid control system is that it cannot be applied when few failures occur before time T. So, the GPH type-I censoring scheme proposed that allows us to observe a pre-specified number of failures. More work is needed in this way. As an extension of the current work is the inference of unknown parameters based on data from GPH type-I in the presence of competing risks when latent failure times follow IW distributions with different scale and shape parameters or a different shape and common scale parameters. Also, assumption is made in this paper that the competing risks are statistically independent. The case, however, where the competing risks are dependent, is very common in practice and related statistical inference with dependent competing risks model is possible future work. We hope that the results and methodologies proposed here will be beneficial to reliability practitioners and extended to other censoring plans.

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