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CONCOMITANTS FOR CASE-I OF GENERALIZED ORDER STATISTICS AND ITS DUAL FROM LAI AND XIE EXTENSIONS

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ABSTRACT

In this paper, we obtain the concomitants of the extensions of Morgenstern family (Lai and Xie extensions) for generalized order statistics and dual generalized order statistics. In addition, recurrence relation between moments for the required models is obtained. Furthermore, the joint distribution between two concomitants is derived for generalized order statistics and its dual. We finally present the Shannon entropy and Fisher information with exponential distribution as an example from Lai and Xie extensions.

KEYWORDS

Concomitants; Joint distribution; Shannon entropy; Fisher information.

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1. INTRODUCTION

The Farlie-Gumbel-Morgenstern family (FGM) is described by a parameter δ, and the marginal distribution functions $G_U(u)$ and $G_Z(z)$. By adding further parameters, Lai and Xie (2000) examined the bivariate Farlie-Gumbel-Morgenstern distribution as broader. They suggested cumulative distribution function cdf as

$$G(u,z) = G_U(u)G_Z(z) + \delta(1 - G_U(u))^{\alpha}(1 - G_Z(z))^{\alpha} G_U(u)^{\lambda} G_Z(z)^{\lambda}, \alpha, \lambda \geq 1 \tag{1.1}$$

for $0 \leq \delta \leq 1$. The related probability density function pdf is

$$g(u,z) = g_U(u)g_Z(z)(1 + \delta[\lambda - (\alpha+\lambda)G_U(u)][\lambda - (\alpha+\lambda)G_Z(z)] \\ [(1 - G_U(u))(1 - G_Z(z))]^{\alpha-1}(G_U(u)G_Z(z))^{\lambda-1}). \tag{1.2}$$

According to Bairamov et al. (2001), a bivariate copula for δ satisfying a wider range where

$$\min\left\{\frac{1}{[K^{+}(\alpha,\lambda)]^{2}},\frac{1}{[K^{-}(\alpha,\lambda)]^{2}}\right\}\leq\delta\leq\frac{1}{K^{+}(\alpha,\lambda)K^{-}(\alpha,\lambda)}.$$

where $K^{+}$and K^{-} are functions of α and λ.

Furthermore, the conditional cdf and pdf are:

$$G_{Z|U}(z|u)=G_Z(z)+\delta(1-G_U(u))^{\alpha}(1-G_Z(z))^{\alpha}G_U(u)^{\lambda-1}G_Z(z)^{\lambda},\alpha,\lambda\geq 1 \quad (1.3)$$

$$g_{Z|U}(z|u)=g_Z(z)(1+\delta[\lambda-(\alpha+\lambda)G_U(u)][\lambda-(\alpha+\lambda)G_Z(z)] \left[\left(1-G_U(u)\right)\left(1-G_Z(z)\right)\right]^{\alpha-1}\left(G_U(u)G_Z(z)\right)^{\lambda-1}). \quad (1.4)$$

Generalized order statistics (Gos), which we call case-I of Gos, is introduced by Kamps (1995). Accordingly, it has a variety of models that are ascendingly ordered random variables, including ordinary order statistics, record values, sequential order statistics, and Pfeifer's record model. On the other side, Pawlas and Szynal (2001) and later Burkschat et al. (2003) created the concept of lower Gos. They called it dual generalized order statistics (DGos). To enable a common method of descending ordered random variables like lower records models and reversed order statistics.

Based on $\widetilde{w}$, the following problems exist: Let $l\geq 1$, $i\in\mathbb{N}$, $w_1,\ldots,w_{i-1}\in\mathbb{R}$, $1\leq r\leq i-1$, $W_r=\sum_{j=r}^{i-1}w_j$, where parameters as $\gamma_r=l+i-r+W_r\geq 1$ for all r, and let $\widetilde{w}=(w_1,\ldots,w_{i-1})$.

Case-I of Gos:

Let $w_1=w_2=\cdots=w_{i-1}=w$, the pdf of $U_{(r,i,w,l)}$ is:

$$g_{(r,i,w,l)}(u) = \frac{m_{r-1}}{(r-1)!}(1-G_U(u))^{\gamma_r-1}g_U(u)h_w^{r-1}(G_U(u)), \quad (1.5)$$

where

$$m_{r-1}=\prod_{q=1}^{r}\gamma_q, k_w(y)=\begin{cases}\frac{-(1-y)^{w+1}}{w+1}, & w\neq -1,\\ -\ln(1-y), & w=-1.\end{cases}$$

$0<y<1$, $h_w(y)=k_w(y)-k_w(0)$.

We get the joint pdf of $U_{(r,i,w,l)}$ and $U_{(s,i,w,l)}$, $1\leq r<s\leq i$ as:

$$g_{(r,s,i,w,l)}(u_r,u_s)=\frac{m_{s-1}}{(r-1)!\,(s-r-1)!}(1-G_U(u_r))^{w}h_w^{r-1}(G_U(u_r))g_U(u_r) \times\left(k_w\left(G_U(u_s)\right)-k_w\left(G_U(u_r)\right)\right)^{s-r-1}(1-G_U(u_s))^{\gamma_s-1}g_U(u_s). \quad (1.6)$$

Case-I of DGos:

If $w_1=w_2=\cdots=w_{i-1}=w$, the pdf of $X_{d(r,i,w,l)}$ is defined by:

$$g_{d(r,i,w,l)}(u)=\frac{m_{r-1}}{(r-1)!}(G_U(u))^{\gamma_r-1}g_U(u)h_w^{r-1}(G_U(u)), \quad (1.7)$$

where

$$m_{r-1} = \prod_{q=1}^{r} \gamma_q, k_w(y) = \begin{cases} \frac{-1}{w+1} y^{w+1}, & w \neq -1, \\ -\ln y, & w = -1. \end{cases}$$

$0 \leq y < 1$, $h_w(y) = k_w(y) - k_w(1)$.

We get the joint pdf of $U_{d(r,i,w,l)}$ and $U_{d(s,i,w,l)}$, $1 \leq r < s \leq i$ as:

$$g_{d(r,s,i,w,l)}(u_r, u_s) = \frac{m_{s-1}}{(r-1)!\,(s-r-1)!} (G_U(u_r))^w h_w^{r-1}(G_U(u_r)) g_U(u_r) \times (k_w(G_U(u_s)) - k_w(G_U(u_r)))^{s-r-1} (G_U(u_s))^{\gamma_s - 1} g_U(u_s). \quad (1.8)$$

Beg and Ahsanullah (2008) applied the concept of concomitants in case-I of Gos. Let pdf and cdf of the concomitant of case-I of Gos $Z_{[r,i,w,l]}$, $1 \leq r \leq i$, are:

$$g_{[r,i,w,l]}(z) = \int_{-\infty}^{\infty} g_{Z|U}(z|u) g_{(r,i,w,l)}(u) du, \quad (1.9)$$

and

$$G_{[r,i,w,l]}(z) = \int_{-\infty}^{\infty} G_{Z|U}(z|u) g_{(r,i,w,l)}(u) du. \quad (1.10)$$

where $g_{(r,i,w,l)}(u)$ is the pdf of $U_{(r,i,w,l)}$ defined in (1.5).

Furthermore, Nayabuddin (2013) applied it also in case-I of $DGos$. Let pdf and cdf of the concomitant of case-I of $DGos$ $Z_{d[r,i,w,l]}$, $1 \leq r \leq i$, are:

$$g_{d[r,i,w,l]}(z) = \int_{-\infty}^{\infty} g_{Z|U}(z|u) g_{d(r,i,w,l)}(u) du, \quad (1.11)$$

and

$$G_{d[r,i,w,l]}(z) = \int_{-\infty}^{\infty} G_{Z|U}(z|u) g_{d(r,i,w,l)}(u) du. \quad (1.12)$$

where $g_{d(r,i,w,l)}(u)$ is the pdf of $U_{d(r,i,w,l)}$ defined in (1.7).

An index entropy measures dispersion, volatility risk and uncertainty. Shannon (1948) first suggested this idea in the information theory literature. A random variable average reduction in uncertainty of Z is measured by the Shannon entropy of Z, a mathematical measure of information. The following is the definition of entropy for a continuous random variable Z with pdf $g_Z(z)$ is:

$$H(z) = -E(\ln g_Z(z)) = -\int_{-\infty}^{\infty} g_Z(z) \ln g_Z(z) dz. \quad (1.13)$$

For a continuous random variable Z with pdf $g_Z(z)$, then the following definition Fisher information (FI) is:

$$I(u) = \int_{-\infty}^{\infty} \left[\frac{\partial \ln g_Z(z)}{\partial z}\right]^2 g_Z(z) dz \quad (1.14)$$

Additionally, it is employed to create a single idea of physical law known as the rule of "extreme physical information", see Frieden [(1988),(1998)].

In the literature, numerous authors have examined the concomitants of Gos, which is about information measures. Studying the FI for concomitants of Gos in the FGM family was done by Tahmasebi and Jafari (2013). In their study of the concomitants of Gos from the FGM family, Mohie El-Din et al. (2015) said that when $\gamma_i \neq \gamma_j, i \neq j, i, j = 1,2, n-1$. Moreover, Mohie El-Din et al. (2015) investigated the Shannon entropy and the FI for the concomitants of Gos from FGM family for some special known marginals. Recent research on the FI for concomitants of DGos in the HKFGM family was conducted by Abd Elgawad et al. (2020).

2. CONCOMITANTS OF Gos AND $DGos$

We obtain the pdf and cdf of concomitants for Lai and Xie extension in case-I of Gos and its DGos. The following theorems deal with this matter:

2.1 Case-I of Gos:

Theorem 2.1

For Lai and Xie extension, utilizing (1.3), (1.4) ,(1.5) in (1.9) and (1.10), we receive the pdf and cdf of the concomitant $Z_{[r,i,w,l]}$, of r-th case-I of Gos as:

$$\begin{aligned} g_{[r,i,w,l]}(z) &= g_Z(z)\ [1 + \delta R^*_{[r,i,w,l]}\big(\lambda - (\alpha+\lambda) G_Z(z)\big) \\ &\qquad \big(1 - G_Z(z)\big)^{\alpha-1} G_Z^{\lambda-1}(z)\ ], \end{aligned} \tag{2.1}$$

$$G_{[r,i,w,l]}(z) = G_Z(z)\big[1 + \delta \tau^*_{[r,i,w,l]}(1 - G_Z(z))^{\alpha} G_Z^{\lambda-1}(z)\big]. \tag{2.2}$$

where

$$R^*_{[r,i,w,l]} = m_{r-1}\left[\sum_{\epsilon=0}^{\lambda-1} \frac{\lambda\binom{\lambda-1}{\epsilon}(-1)^{\epsilon}}{\prod_{q=1}^{r}\left(\gamma_q + \epsilon + \alpha - 1\right)} - \sum_{j=0}^{\lambda} \frac{(\alpha+\lambda)\binom{\lambda-1}{\epsilon}(-1)^{j}}{\prod_{q=1}^{r}\left(\gamma_q + \epsilon + \alpha - 1\right)}\right] \tag{2.3}$$

and

$$\tau^*_{[r,i,w,l]} = m_{r-1} \sum_{\epsilon=0}^{\lambda-1} \frac{\binom{\lambda-1}{\epsilon}(-1)^{\epsilon}}{\prod_{q=1}^{r}\left(\gamma_q + \epsilon + \alpha\right)}. \tag{2.4}$$

Proof:

From (1.4) and $g_{(r,i,w,l)}(u)$ the pdf of case-I of $Gos\ U_{[r,i,w,l]}$ defined in (1.5), the pdf of the concomitant of r-th case-I of Gos, $Z_{[r,i,w,l]}$, is:

$$\begin{aligned} g_{[r,i,w,l]}(z) &= \int_{-\infty}^{\infty} g_{Z|U}(z|u)\, g_{(r,i,w,l)}(u)\,du \\ &= g_Z(z) + \delta\big(\lambda - (\alpha+\lambda)G_Z(z)\big)\big(1 - G_Z(z)\big)^{\alpha-1} g_Z(z) G_Z^{\lambda-1}(z) \\ &\qquad \int_{-\infty}^{\infty} [\lambda\Big(1 - \big(1 - G_U(u)\big)\Big)^{\lambda-1} \\ &\qquad -(\alpha+\lambda)(1 - (1 - G_U(u)))^{\lambda}]\big(1 - G_U(u)\big)^{\alpha-1} g_{(r,i,w,l)}(u)\,du \end{aligned}$$

$$= g_Z(z) + \delta\big(\lambda - (\alpha+\lambda)G_Z(z)\big)\big(1-G_Z(z)\big)^{\alpha-1} g_Z(z) G_Z^{\lambda-1}(z)\,\frac{m_{r-1}}{(r-1)!}$$
$$\times \int_{-\infty}^{\infty}\left\{\lambda\sum_{\epsilon=0}^{\lambda-1}\binom{\lambda-1}{\epsilon}(-1)^{\epsilon}\big(1-G_U(u)\big)^{\gamma_r+\epsilon+\alpha-2}\right.$$
$$\left.-(\alpha+\lambda)\sum_{j=0}^{\lambda}\binom{\lambda}{j}(-1)^{j}\big(1-G_U(u)\big)^{\gamma_r+j+\alpha-1}\right\}$$
$$\times\left[\frac{1}{w+1}\{1-(1-G_U(u))^{w+1}\}\right]^{r-1} g_U(u)du. \qquad (2.5)$$

From Beg and Ahsanullah (2008), we can write that

$$\int_{-\infty}^{\infty}(1-G_U(u))^{\gamma_r+j-1}\left[\frac{1}{w+1}\{1-(1-G_U(u))^{w+1}\}\right]^{r-1} g_U(u)du = \frac{(r-1)!}{\prod_{q=1}^{r}(\gamma_q+j)}. \qquad (2.6)$$

and the result follows. The cdf of case-I of Gos can be computed in the same way.

2.2 Case-I of $DGos$:

Theorem 2.2

For Lai and Xie extension, utilizing (1.3), (1.4) ,(1.7) in (1.11) and (1.12), we receive the pdf and cdf of the concomitant $Z_{[r,i,w,l]}$, of r-th case-I of $DGos$ as:

$$g_{d[r,i,w,l]}(z) = g_Z(z)\ \big[1+\delta R_d^*(r,i,w,l)\big(\lambda-(\alpha+\lambda)G_Z(z)\big) \big(1-G_Z(z)\big)^{\alpha-1}G_Z^{\lambda-1}(z)\big], \qquad (2.7)$$

$$G_{d[r,i,w,l]}(z) = G_Z(z)\big[1-\delta\tau_d^*(r,i,w,l)(1-G_Z(z))^{\alpha}G_Z^{\lambda-1}(z)\big]. \qquad (2.8)$$

where

$$R_d^*(r,i,w,l) = m_{r-1}\sum_{\epsilon=0}^{\alpha-1}\binom{\alpha-1}{\epsilon}(-1)^{\epsilon}\left[\frac{(\alpha+\lambda)}{\prod_{q=1}^{r}(\gamma_q+\epsilon+\lambda)}-\frac{\lambda}{\prod_{q=1}^{r}(\gamma_q+j+\lambda-1)}\right] \qquad (2.9)$$

and

$$\tau_d^*(r,i,w,l) = m_{r-1}\sum_{\epsilon=0}^{\alpha}\frac{\binom{\alpha}{\epsilon}(-1)^{\epsilon}}{\prod_{q=1}^{r}(\gamma_q+\epsilon+\lambda-1)}. \qquad (2.10)$$

3. MOMENT OF CONCOMITANTS FOR Gos AND $DGos$

We find the recurrence relations between moments of Case-I of Gos and its $DGos$

3.1 Case-I of Gos:

From the results of the preceding part, we may format the pdf of Case-I of Gos, $Z_{[r,i,w,l]}$ as follows

$$g_{[r,i,w,l]}(z) = g_Z(z)\left[1+\delta R^*(r,i,w,l)\big(\lambda-(\alpha+\lambda)G_Z(z)\big)\big(1-G_Z(z)\big)^{\alpha-1}G_Z^{\lambda-1}(z)\right]$$
$$= g_Z(z)\,[1+\delta R^*(r;i,w,l)\big(\lambda-(\alpha+\lambda)G_Z(z)\big)$$
$$G_Z^{\lambda-1}(z)\sum_{i_1=0}^{\alpha-1}\binom{\alpha-1}{i_1}(-1)^{i_1}G_Z^{i_1}(z)]$$
$$= g_Z(z)+\delta\lambda R^*(r;i,w,l)\sum_{i_1=0}^{\alpha-1}I_1(i_1+\lambda)g_Z(z)G_Z^{i_1+\lambda-1}(z)$$
$$-(\alpha+\lambda)\delta R^*(r;i,w,l)\sum_{i_1=0}^{\alpha-1}I_2(i_1+\lambda+1)g_Z(z)G_Z^{i_1+\lambda}(z)$$
$$= g_Z(z)+\delta R^*(r;i,w,l)\left[\lambda\sum_{i_1=0}^{\alpha-1}I_1 g_{V_1}(z)-(\alpha+\lambda)\sum_{i_1=0}^{\alpha-1}I_2 g_{V_2}(z)\right]. \quad (3.1)$$

where

$$I_1=\frac{\binom{\alpha-1}{i_1}(-1)^{i_1}}{i_1+\lambda}, I_2=\frac{\binom{\alpha-1}{i_1}(-1)^{i_1}}{i_1+\lambda+1}, \quad (3.2)$$

$$g_{V_1}(z)=(i_1+\lambda)G_Z^{i_1+\lambda-1}(z)g_Z(z), g_{V_2}(z)=(i_1+\lambda+1)G_Z^{i_1+\lambda}(z)g_Z(z), \quad (3.3)$$

$$V_1\sim G_Z^{i_1+\lambda-1}(z), V_2\sim G_Z^{i_1+\lambda}(z), \quad (3.4)$$

Hence, the moment generating function of $Z_{[r;i,w,l]}$ is

$$M_{[r,i,w,l]}(t) \quad = M_Z(t)+\delta R^*(r,i,w,l)\left[\lambda\sum_{i_1=0}^{\alpha-1}I_1M_{V_1}(t)-(\alpha+\lambda)\sum_{i_1=0}^{\alpha-1}I_2M_{V_2}(t)\right]. \quad (3.5)$$

Theorem 3.1

Let $2\le r\le i$, we get recurrence relation between moments of concomitants of case-I of Gos from Lai and Xie extension as

$$M_{[r,i,w,l]}(t)-M_{[r-1,i,w,l]}(t)=\delta m_{r-2}\left[\lambda\sum_{i_1=0}^{\alpha-1}I_1M_{V_1}(t)-(\alpha+\lambda)\sum_{i_1=0}^{\alpha-1}I_2M_{V_2}(t)\right]$$
$$\times\left[\sum_{j=0}^{\lambda}\frac{(\alpha+\lambda\binom{\lambda}{j}(-1)^j(j+\alpha-1)}{\prod_{q=1}^{r}(\gamma_q+j+\alpha-1)}-\sum_{\epsilon=0}^{\lambda-1}\frac{\lambda\binom{\lambda-1}{\epsilon}(-1)^\epsilon(\epsilon+\alpha-1)}{\prod_{q=1}^{r}(\gamma_q+\epsilon+\alpha-1)}\right]. \quad (3.6)$$

Proof:

We using the following relation $m_{r-2}=\frac{m_{r-1}}{\gamma_r}$, $a_\epsilon(r)=\frac{1}{\prod_{q=1}^{r}(\gamma_q+\epsilon+\alpha-1)}$,

$(\gamma_r+\epsilon+\alpha-1)\,a_\epsilon(r)=a_\epsilon(r-1)$, $a_j(r)=\frac{1}{\prod_{q=1}^{r}\left(\gamma_q+j+\alpha-1\right)}$,

$(\gamma_r+j+\alpha-1)a_j(r)=a_j(r-1)$

$$
\begin{aligned}
&R^*(r,i,w,l)-R^*(r-1,i,w,l)\\
&\quad= m_{r-1}\left[\sum_{\epsilon=0}^{\lambda-1}\frac{\lambda\binom{\lambda-1}{\epsilon}(-1)^{\epsilon}}{\prod_{q=1}^{r}(\gamma_q+\epsilon+\alpha-1)}-\sum_{j=0}^{\lambda}\frac{(\alpha+\lambda)\binom{\lambda}{j}(-1)^{j}}{\prod_{q=1}^{r}(\gamma_q+j+\alpha-1)}\right]\\
&\qquad -m_{r-2}\left[\sum_{\epsilon=0}^{\lambda-1}\frac{\lambda\binom{\lambda-1}{\epsilon}(-1)^{\epsilon}}{\prod_{q=1}^{r-1}(\gamma_q+\epsilon+\alpha-1)}-\sum_{j=0}^{\lambda}\frac{(\alpha+\lambda)\binom{\lambda}{j}(-1)^{j}}{\prod_{q=1}^{r-1}(\gamma_q+j+\alpha-1)}\right]\\
&\quad= m_{r-1}\left[\sum_{\epsilon=0}^{\lambda-1}a_\epsilon(r)\lambda\binom{\lambda-1}{\epsilon}(-1)^{\epsilon}-\sum_{j=0}^{\lambda}a_j(r)(\alpha+\lambda)\binom{\lambda}{j}(-1)^{j}\right]\\
&\qquad -m_{r-2}\left[\sum_{\epsilon=0}^{\lambda-1}a_\epsilon(r)\,\lambda\binom{\lambda-1}{\epsilon}(-1)^{\epsilon}(\gamma_r+\epsilon+\alpha-1)\right.\\
&\qquad\left.-\sum_{j=0}^{\lambda-1}a_j(r)(\alpha+\lambda)\binom{\lambda-1}{j}(-1)^{j}(\gamma_r+j+\alpha-1)\right] \qquad (3.7)
\end{aligned}
$$

3.2 Case-I of $DGos$:

Based on Lai and Xie extension and case-I of $DGos$, we can write the pdf of $Z_{[r,i,w,l]}$ as

$$g_{d[r,i,w,l]}(z)=g_Z(z)+\delta R_d^*(r,i,w,l)\left[\lambda\sum_{i_1=0}^{\alpha-1}I_1g_{V_1}(z)-(\alpha+\lambda)\sum_{i_1=0}^{\alpha-1}I_2g_{V_2}(z)\right]. \qquad (3.8)$$

where $I_1, I_2, g_{V_1}(z), g_{V_2}(z), V_1$ and V_2 are defined in (3.2),(3.3) and (3.4)

Therefore, the moment generating function of $Z_{[r;i,w,l]}$ is

$$M_{[r,i,w,l]}(t)=M_Z(t)+\delta R_d^*(r,i,w,l)\left[\lambda\sum_{i_1=0}^{\alpha-1}I_1M_{V_1}(t)-(\alpha+\lambda)\sum_{i_1=0}^{\alpha-1}I_2M_{V_2}(t)\right] \qquad (3.9)$$

Theorem 3.2

Let $2\le r\le n$, and we get recurrence relation between moments of concomitants of case-I of $DGos$ from Lai and Xie extension as

$$
\begin{aligned}
M_{[r,i,w,l]}(t)-M_{[r-1,i,w,l]}(t)&=\delta m_{r-2}\left[\lambda\sum_{i_1=0}^{\alpha-1}I_1M_{V_1}(t)-(\alpha+\lambda)\sum_{i_1=0}^{\alpha-1}I_2M_{V_2}(t)\right]\\
&\times\sum_{\epsilon=0}^{\alpha-1}\binom{\alpha-1}{\epsilon}(-1)^{\epsilon}\left[\frac{\lambda(\epsilon+\lambda-1)}{\prod_{q=1}^{r}(\gamma_q+\epsilon-1+\lambda)}-\frac{\lambda(\epsilon+\lambda)(\alpha+\lambda)}{\prod_{q=1}^{r}(\gamma_q+\epsilon+\lambda)}\right]. \qquad (3.10)
\end{aligned}
$$

Proof:

By a similar way, we have the recurrence relation of case-I of $DGos$ by using

$$m_{r-2}=\frac{m_{r-1}}{\gamma_r},\ a_\epsilon(r)=\frac{1}{\prod_{q=1}^{r}\left(\gamma_q+\epsilon+\lambda\right)},(\gamma_r+\epsilon+\lambda)a_\epsilon(r)=a_\epsilon(r-1),a_{\epsilon-1}(r)$$
$$=\frac{1}{\prod_{q=1}^{r}\left(\gamma_q+\epsilon-1+\lambda\right)},(\gamma_r+\epsilon-1+\lambda)a_{\epsilon-1}(r)=a_{\epsilon-1}(r-1).$$

4. JOINT DISTRIBUTION OF TWO CONCOMITANTS

We derive the joint distribution of two concomitants of case-I of Gos and case-I of $DGos$. The following theorems deal with this matter:

4.1 Case-I of Gos:

When $Z_{[r,i,w,l]}$ and $Z_{[s,i,w,l]}$ be concomitants of the r-th and s-th for case-I of Gos, respectively. Then joint pdf of $Z_{[r,i,w,l]}$ and $Z_{[s,i,w,l]}$, $r<s$, is

$$\begin{aligned}
g_{(r,s,i,w,l)}(z_r,z_s)&=\int_{-\infty}^{\infty}\int_{-\infty}^{u_s} g_{Z|U}(z_r|u_r)g_{Z|U}(z_s|u_s)g_{(r,s,i,w,l)}(u_r,u_s)du_rdu_s\\
&=g_Z(z_r)g_Z(z_s)\int_{-\infty}^{\infty}\int_{-\infty}^{u_s}\{1+\delta^2A_{Z_r}A_{Z_s}[\lambda^2((1-G_U(u_r))(1-\\
&G_U(u_s))^{\alpha-1}(G_U(u_r)G_U(u_s))^{\lambda-1}\\
&-\lambda(\alpha+\lambda)((1-G_U(u_r))(1-G_U(u_s))^{\alpha-1}G_U(u_r)^{\lambda}G_U(u_s)^{\lambda-1}\\
&-\lambda(\alpha+\lambda)((1-G_U(u_r))(1-G_U(u_s))^{\alpha-1}G_U(u_r)^{\lambda-1}G_U(u_s)^{\lambda}\\
&+(\alpha+\lambda)^2((1-G_U(u_r))(1-G_U(u_s))^{\alpha-1}G_U(u_r)^{\lambda}G_U(u_s)^{\lambda}]\\
&+\delta A_{Z_r}[\lambda(1-G_U(u_r))^{\alpha-1}G_U(u_r)^{\lambda-1}-(\alpha+\lambda)(1-G_U(u_r))^{\alpha-1}G_U(u_r)^{\lambda}]\\
&+\delta A_{Z_s}[\lambda(1-G_U(u_s))^{\alpha-1}G_U(u_s)^{\lambda-1}-(\alpha+\lambda)(1-G_U(u_s))^{\alpha-1}G_U(u_s)^{\lambda}]\}\\
&\times\frac{m_{s-1}}{(-1+r)!\,(s-1-r)!\,(1+w)^{s-2}}\overline{G}_U(u_r)^w(1-\overline{G}_U(u_r)^{1+w}\\
&\times\left(\overline{G}_U(u_r)^{1+w}-\overline{G}_U(u_s)^{1+w}\right)^{s-1-r}\overline{G}_U(u_s)^{\gamma_s-1}g_U(u_r)g_U(u_s)du_rdu_s \qquad (4.1)
\end{aligned}$$

where

$$\begin{aligned}
A_{Z_r}&=(\lambda-(\alpha+\lambda)G_Z(z_r))(1-G_Z(z_r))^{\alpha-1}G_Z(z_r)^{\lambda-1},\\
A_{Z_s}&=(\lambda-(\alpha+\lambda)G_Z(z_s))(1-G_Z(z_s))^{\alpha-1}G_Z(z_s)^{\lambda-1}.
\end{aligned}\qquad (4.2)$$

Remark 4.1

To find the integration of (4.1) we use the following general forms of integrations, see Beg and Ahsanullah (2008),

$$\begin{aligned}
I_{p,q}&=\frac{m_{s-1}}{(-1+r)!\,(s-1-r)!\,(1+w)^{s-2}}\\
&\int_{-\infty}^{\infty}\int_{-\infty}^{u_s}\overline{G}_U(u_r)^p\overline{G}_U(u_r)^w(1-\overline{G}_U(u_r)^{1+w})^{-1+r}\\
&\times\left(\overline{G}_U(u_r)^{1+w}-\overline{G}_U(u_s)^{1+w}\right)^{s-1-r}\overline{G}_U(u_s)^{\gamma_s-1}\overline{G}_U(u_s)^q g_U(u_r)g_U(u_s)du_rdu_s\\
&=\frac{m_{s-1}}{(-1+r)!\,(s-1-r)!\,(1+w)^{s}}\beta\left(r,\frac{\gamma_r+p+q}{1+w}\right)\beta\left(s-r,\frac{\gamma_s+q}{1+w}\right). \qquad (4.3)
\end{aligned}$$

Then, we can write

$$J_1 = \frac{m_{s-1}}{(-1+r)!(s-1-r)!(1+w)^{s-2}} \int_{-\infty}^{\infty}\int_{-\infty}^{u_s} (\overline{G}_U(u_r)\overline{G}_U(u_s))^{\alpha-1}(G_U(u_r)G_U(u_s))^{\lambda-1}$$
$$\times \overline{G}_U(u_r)^w(1-\overline{G}_U(u_r)^{1+w})^{-1+r}\left(\overline{G}_U(u_r)^{1+w}-\overline{G}_U(u_s)^{1+w}\right)^{s-1-r}\overline{G}_U(u_s)^{\gamma_s-1}$$
$$\times g_U(u_r)g_U(u_s)du_r du_s$$
$$= \frac{m_{s-1}}{(-1+r)!(s-1-r)!(1+w)^{s-2}} \int_{-\infty}^{\infty}\int_{-\infty}^{u_s} (\overline{G}_U(u_r)\overline{G}_U(u_s))^{\alpha-1}((1-\overline{G}_U(u_r))(1-\overline{G}_U(u_s)))^{\lambda-1}$$
$$\times \overline{G}_U(u_r)^w(1-\overline{G}_U(u_r)^{1+w})^{-1+r}\left(\overline{G}_U(u_r)^{1+w}-\overline{G}_U(u_s)^{1+w}\right)^{s-1-r}\overline{G}_U(u_s)^{\gamma_s-1}$$
$$\times g_U(u_r)g_U(u_s)du_r du_s$$
$$= \frac{m_{s-1}}{(-1+r)!(s-1-r)!(1+w)^{s-2}} \sum_{j_1=0}^{\lambda-1}\sum_{j_2=0}^{\lambda-1}\binom{\lambda-1}{j_1}\binom{\lambda-1}{j_2}(-1)^{j_1+j_2}$$
$$\times \int_{-\infty}^{\infty}\int_{-\infty}^{u_s} \overline{G}_U(u_r)^{j_1+\alpha-1}\overline{G}_U(u_s)^{j_2+\alpha-1}\overline{G}_U(u_r)^w g_U(u_r)(1-\overline{G}_U(u_r)^{1+w})^{-1+r}$$
$$\times \left(\overline{G}_U(u_r)^{1+w}-\overline{G}_U(u_s)^{1+w}\right)^{s-1-r}\overline{G}_U(u_s)^{\gamma_s-1}g_U(u_s)du_r du_s$$
$$= \frac{m_{s-1}}{(-1+r)!(s-1-r)!(1+w)^{s}} \sum_{j_1=0}^{\lambda-1}\sum_{j_2=0}^{\lambda-1}\binom{\lambda-1}{j_1}\binom{\lambda-1}{j_2}(-1)^{j_1+j_2}$$
$$\times \beta(r,\frac{\gamma_r+j_1+j_2+2\alpha-2}{1+w})\beta(s-r,\frac{\gamma_s+j_2+\alpha-1}{1+w}),$$

$$J_2 = \frac{m_{s-1}}{(-1+r)!(s-1-r)!(1+w)^{s-2}} \int_{-\infty}^{\infty}\int_{-\infty}^{u_s} (\overline{G}_U(u_r)\overline{G}_U(u_s))^{\alpha-1}G_U(u_r)^{\lambda}G_U(u_s)^{\lambda-1}$$
$$\times \overline{G}_U(u_r)^w(1-\overline{G}_U(u_r)^{1+w})^{-1+r}\left(\overline{G}_U(u_r)^{1+w}-\overline{G}_U(u_s)^{1+w}\right)^{s-1-r}\overline{G}_U(u_s)^{\gamma_s-1}$$
$$\times g_U(u_r)g_U(u_s)du_r du_s$$
$$= \frac{m_{s-1}}{(-1+r)!(s-1-r)!(1+w)^{s}} \sum_{j_1=0}^{\lambda}\sum_{j_2=0}^{\lambda-1}\binom{\lambda}{j_1}\binom{\lambda-1}{j_2}(-1)^{j_1+j_2}$$
$$\times \beta(r,\frac{\gamma_r+j_1+j_2+2\alpha-2}{1+w})\beta(s-r,\frac{\gamma_s+j_2+\alpha-1}{1+w}),$$

$$J_3 = \frac{m_{s-1}}{(-1+r)!(s-1-r)!(1+w)^{s-2}} \int_{-\infty}^{\infty}\int_{-\infty}^{u_s} (\overline{G}_U(u_r)\overline{G}_U(u_s))^{\alpha-1}G_U(u_r)^{\lambda-1}G_U(u_s)^{\lambda}$$
$$\times \overline{G}_U(u_r)^w(1-\overline{G}_U(u_r)^{1+w})^{-1+r}\left(\overline{G}_U(u_r)^{1+w}-\overline{G}_U(u_s)^{1+w}\right)^{s-1-r}\overline{G}_U(u_s)^{\gamma_s-1}$$
$$\times g_U(u_r)g_U(u_s)du_r du_s$$
$$= \frac{m_{s-1}}{(-1+r)!(s-1-r)!(1+w)^{s}} \sum_{j_1=0}^{\lambda-1}\sum_{j_2=0}^{\lambda}\binom{\lambda-1}{j_1}\binom{\lambda}{j_2}(-1)^{j_1+j_2}$$
$$\times \beta(r,\frac{\gamma_r+j_1+j_2+2\alpha-2}{1+w})\beta(s-r,\frac{\gamma_s+j_2+\alpha-1}{1+w}),$$

$$J_4 = \frac{m_{s-1}}{(-1+r)!(s-1-r)!(1+w)^{s-2}} \int_{-\infty}^{\infty}\int_{-\infty}^{u_s} (\overline{G}_U(u_r)\overline{G}_U(u_s))^{\alpha-1}G_U(u_r)^{\lambda}G_U(u_s)^{\lambda}$$
$$\times \overline{G}_U(u_r)^w(1-\overline{G}_U(u_r)^{1+w})^{-1+r}\left(\overline{G}_U(u_r)^{1+w}-\overline{G}_U(u_s)^{1+w}\right)^{s-1-r}\overline{G}_U(u_s)^{\gamma_s-1}$$
$$\times g_U(u_r)g_U(u_s)du_r du_s$$
$$= \frac{m_{s-1}}{(-1+r)!(s-1-r)!(1+w)^{s}} \sum_{j_1=0}^{\lambda}\sum_{j_2=0}^{\lambda}\binom{\lambda}{j_1}\binom{\lambda}{j_2}(-1)^{j_1+j_2}$$
$$\times \beta(r,\frac{\gamma_r+j_1+j_2+2\alpha-2}{1+w})\beta(s-r,\frac{\gamma_s+j_2+\alpha-1}{1+w}),$$

$$J_5 = \frac{m_{s-1}}{(-1+r)!(s-1-r)!(1+w)^{s-2}} \int_{-\infty}^{\infty}\int_{-\infty}^{u_s} \overline{G}_U(u_r)^{\alpha-1} G_U(u_r)^{\lambda-1} \overline{G}_U(u_r)^{w}$$
$$\times (1-\overline{G}_U(u_r)^{1+w})^{-1+r} \left(\overline{G}_U(u_r)^{1+w} - \overline{G}_U(u_s)^{1+w}\right)^{s-1-r} \overline{G}_U(u_s)^{\gamma_s-1}$$
$$\times g_U(u_r) g_U(u_s) du_r du_s$$
$$= \frac{m_{s-1}}{(-1+r)!(s-1-r)!(1+w)^{s}} \sum_{j_1=0}^{\lambda-1} \binom{\lambda-1}{j_1} (-1)^{j_1}$$
$$\times \beta(r, \frac{\gamma_r + j_1 + \alpha - 1}{1+w}) \beta(s-r, \frac{\gamma_s}{1+w}),$$

$$J_6 = \frac{m_{s-1}}{(-1+r)!(s-1-r)!(1+w)^{s-2}} \int_{-\infty}^{\infty}\int_{-\infty}^{u_s} \overline{G}_U(u_r)^{\alpha-1} G_U(u_r)^{\lambda} \overline{G}_U(u_r)^{w}$$
$$\times (1-\overline{G}_U(u_r)^{1+w})^{-1+r} \left(\overline{G}_U(u_r)^{1+w} - \overline{G}_U(u_s)^{1+w}\right)^{s-1-r} \overline{G}_U(u_s)^{\gamma_s-1}$$
$$\times g_U(u_r) g_U(u_s) du_r du_s$$
$$= \frac{m_{s-1}}{(-1+r)!(s-1-r)!(1+w)^{s}} \sum_{j_1=0}^{\lambda} \binom{\lambda}{j_1} (-1)^{j_1} \beta(r, \frac{\gamma_r + j_1 + \alpha - 1}{1+w}) \beta(s-r, \frac{\gamma_s}{1+w}),$$

$$J_7 = \frac{m_{s-1}}{(-1+r)!(s-1-r)!(1+w)^{s-2}} \int_{-\infty}^{\infty}\int_{-\infty}^{u_s} \overline{G}_U(u_s)^{\alpha-1} G_U(u_s)^{\lambda-1} \overline{G}_U(u_r)^{w}$$
$$\times (1-\overline{G}_U(u_r)^{1+w})^{-1+r} \left(\overline{G}_U(u_r)^{1+w} - \overline{G}_U(u_s)^{1+w}\right)^{s-1-r} \overline{G}_U(u_s)^{\gamma_s-1}$$
$$\times g_U(u_r) g_U(u_s) du_r du_s$$
$$= \frac{m_{s-1}}{(-1+r)!(s-1-r)!(1+w)^{s}} \sum_{j_2=0}^{\lambda-1} \binom{\lambda-1}{j_2} (-1)^{j_2}$$
$$\times \beta(r, \frac{\gamma_r + j_2 + \alpha - 1}{1+w}) \beta(s-r, \frac{\gamma_s + j_2 + \alpha - 1}{1+w}),$$

$$J_8 = \frac{m_{s-1}}{(-1+r)!(s-1-r)!(1+w)^{s-2}} \int_{-\infty}^{\infty}\int_{-\infty}^{u_s} \overline{G}_U(u_s)^{\alpha-1} G_U(u_s)^{\lambda} \overline{G}_U(u_r)^{w}$$
$$\times (1-\overline{G}_U(u_r)^{1+w})^{-1+r} \left(\overline{G}_U(u_r)^{1+w} - \overline{G}_U(u_s)^{1+w}\right)^{s-1-r} \overline{G}_U(u_s)^{\gamma_s-1}$$
$$\times g_U(u_r) g_U(u_s) du_r du_s$$
$$= \frac{m_{s-1}}{(-1+r)!(s-1-r)!(1+w)^{s}} \sum_{j_2=0}^{\lambda} \binom{\lambda}{j_2} (-1)^{j_2}$$
$$\times \beta(r, \frac{\gamma_r + j_2 + \alpha - 1}{1+w}) \beta(s-r, \frac{\gamma_s + j_2 + \alpha - 1}{1+w}).$$

Using the results of the previous then the pdf is

$$g_{(r,s,i,w,l)}(z_r, z_s)$$
$$= g_Z(z_r) g_Z(z_s) \{1 + \delta^2 A_{Z_r} A_{Z_s} [\lambda J_1 - \lambda(\alpha+\lambda) J_2 - \lambda(\alpha+\lambda) J_3 + (\alpha+\lambda)^2 J_4]$$
$$+ \delta A_{Z_r} [\lambda J_5 - (\alpha+\lambda) J_6] + \delta A_{Z_s} [\lambda J_7 - (\alpha+\lambda) J_8]\}$$
$$= g_Z(z_r) g_Z(z_s) + T_1 T_2 (\delta^2 [\lambda J_1 - \lambda(\alpha+\lambda)(J_2 + J_3) + (\alpha+\lambda)^2 J_4])$$
$$+ \delta T_1 g_Z(z_s) [\lambda J_5 - (\alpha+\lambda) J_6] + \delta T_2 g_Z(z_r) [\lambda J_7 - (\alpha+\lambda) J_8], \tag{4.4}$$

where

$$T_1 = \lambda \sum_{i_1=0}^{\alpha-1} I_1 g_{V_1}(z_r) - (\alpha+\lambda) \sum_{i_1=0}^{\alpha-1} I_2 g_{V_2}(z_r),$$
$$T_2 = \lambda \sum_{i_1=0}^{\alpha-1} I_1 g_{V_1}(z_s) - (\alpha+\lambda) \sum_{i_1=0}^{\alpha-1} I_2 g_{V_2}(z_s), \tag{4.5}$$

and $I_1, I_2, g_{V_1}(z), g_{V_2}(z), V_1$ and V_2 are defined in (3.2),(3.3) and (3.4).

The cdf concomitants $Z_{[r,i,w,l]}$ and $Z_{[s,i,w,l]}$, $r < s$, is

$$\begin{aligned} G_{(r,s,i,w,l)}(z_r, z_s) &= \int_{-\infty}^{\infty}\int_{-\infty}^{u_s} G_{Z|U}(z_r|u_r)G_{Z|U}(z_s|u_s)g_{(r,s,i,w,l)}(u_r,u_s)du_r du_s \\ &= G_Z(z_r)G_Z(z_s)\int_{-\infty}^{\infty}\int_{-\infty}^{u_s}\{1+\delta^2 A^*_{Z_r}A^*_{Z_s}((1-G_U(u_r)) \\ &\quad (1-G_U(u_s)))^{\alpha}(G_U(u_r)G_U(u_s))^{\lambda-1} \\ &\quad +\delta A^*_{Z_r}(1-G_U(u_r))^{\alpha}G_U(u_r)^{\lambda-1}+\delta A^*_{Z_s}(1-G_U(u_s))^{\alpha}G_U(u_s)^{\lambda-1}\} \\ &\quad \times \frac{m_{s-1}}{(-1+r)!(s-1-r)!(1+w)^{s-2}}\overline{G}_U(u_r)^w(1-\overline{G}_U(u_r)^{1+w})^{-1+r} \\ &\quad \times \left(\overline{G}_U(u_r)^{1+w}-\overline{G}_U(u_s)^{1+w}\right)^{s-1-r}\overline{G}_U(u_s)^{\gamma_s-1}g_U(u_r)g_U(u_s)du_r du_s, \end{aligned} \tag{4.6}$$

where

$$\begin{aligned} A^*_{Z_r} &= (1-G_Z(z_r))^{\alpha}G_Z(z_r)^{\lambda-1}, \\ A^*_{Z_s} &= (1-G_Z(z_s))^{\alpha}G_Z(z_s)^{\lambda-1}. \end{aligned} \tag{4.7}$$

$$\begin{aligned} J_1^* &= \frac{m_{s-1}}{(-1+r)!(s-1-r)!(1+w)^{s-2}}\int_{-\infty}^{\infty}\int_{-\infty}^{u_s}(\overline{G}_U(u_r)\overline{G}_U(u_s))^{\alpha}(G_U(u_r)G_U(u_s))^{\lambda-1} \\ &\quad \times \overline{G}_U(u_r)^w(1-\overline{G}_U(u_r)^{1+w})^{-1+r}\left(\overline{G}_U(u_r)^{1+w}-\overline{G}_U(u_s)^{1+w}\right)^{s-1-r}\overline{G}_U(u_s)^{\gamma_s-1} \\ &\quad \times g_U(u_r)g_U(u_s)du_r du_s \\ &= \frac{m_{s-1}}{(-1+r)!(s-1-r)!(1+w)^{s}}\sum_{j_1=0}^{\lambda-1}\sum_{j_2=0}^{\lambda-1}\binom{\lambda-1}{j_1}\binom{\lambda-1}{j_2}(-1)^{j_1+j_2} \\ &\quad \times \beta\left(r,\frac{\gamma_r+j_1+j_2+2\alpha}{w+1}\right)\beta\left(s-r,\frac{\gamma_s+j_2+\alpha}{w+1}\right), \end{aligned}$$

$$\begin{aligned} J_2^* &= \frac{m_{s-1}}{(-1+r)!(s-1-r)!(1+w)^{s-2}}\int_{-\infty}^{\infty}\int_{-\infty}^{u_s}\overline{G}_U(u_r)^{\alpha}G_U(u_r)^{\lambda-1}\overline{G}_U(u_r)^w \\ &\quad \times (1-\overline{G}_U(u_r)^{1+w})^{-1+r}\left(\overline{G}_U(u_r)^{1+w}-\overline{G}_U(u_s)^{1+w}\right)^{s-1-r}\overline{G}_U(u_s)^{\gamma_s-1} \\ &\quad \times g_U(u_r)g_U(u_s)du_r du_s \\ &= \frac{m_{s-1}}{(-1+r)!(s-1-r)!(1+w)^{s}}\sum_{j_1=0}^{\lambda-1}\binom{\lambda-1}{j_1}(-1)^{j_1}\beta\left(r,\frac{\gamma_r+j_1+\alpha}{w+1}\right)\beta\left(s-r,\frac{\gamma_s}{w+1}\right), \end{aligned}$$

$$\begin{aligned} J_3^* &= \frac{m_{s-1}}{(-1+r)!(s-1-r)!(1+w)^{s-2}}\int_{-\infty}^{\infty}\int_{-\infty}^{u_s}\overline{G}_U(u_s)^{\alpha}G_U(u_s)^{\lambda-1}\overline{G}_U(u_r)^w \\ &\quad \times (1-\overline{G}_U(u_r)^{1+w})^{-1+r}\left(\overline{G}_U(u_r)^{1+w}-\overline{G}_U(u_s)^{1+w}\right)^{s-1-r}\overline{G}_U(u_s)^{\gamma_s-1} \\ &\quad \times g_U(u_r)g_U(u_s)du_r du_s \\ &= \frac{m_{s-1}}{(-1+r)!(s-1-r)!(1+w)^{s}}\sum_{j_2=0}^{\lambda-1}\binom{\lambda-1}{j_2}(-1)^{j_2}\beta\left(r,\frac{\gamma_r+j_2+\alpha}{w+1}\right)\beta\left(s-r,\frac{\gamma_s+j_2+\alpha}{w+1}\right). \end{aligned}$$

Then we can write cdf as

$$G_{(r,s,i,w,l)}(z_r, z_s) = G_Z(z_r)G_Z(z_s)\{1+\delta^2 A^*_{Z_r}A^*_{Z_s}J_1^*+\delta A^*_{Z_r}J_2^*+\delta A^*_{Z_s}J_3^*\}. \tag{4.8}$$

The product moment of $Z_{[r,i,w,l]}, Z_{[s,i,w,l]}$ as $M_{[r,s,i,w,l]}(t_1,t_2)$ is simply acquired from (4.4) as

$$M_{[r,s,i,w,l]}(t_1,t_2) = M_{Z_r}(t_1)M_{Z_s}(t_2) + \delta^2 T_1^* T_2^*[\lambda J_1 - \lambda(\alpha+\lambda)(J_2+J_3) + (\alpha+\lambda)^2 J_4]) + \delta T_1^* M_{Z_s}(t_2)[\lambda J_5 - (\alpha+\lambda)J_6] + \delta T_2^* M_{Z_r}(t)[\lambda J_7 - (\alpha+\lambda)J_8], \quad (4.9)$$

where

$$T_1^* = \lambda \sum_{i_1=0}^{\alpha-1} I_1 M_{V_1 z_r}(t_1) - (\alpha+\lambda)\sum_{i_1=0}^{\alpha-1} I_2 M_{V_2 z_r}(t_1),$$
$$T_2^* = \lambda \sum_{i_1=0}^{\alpha-1} I_1 M_{V_1 z_s}(t_2) - (\alpha+\lambda)\sum_{i_1=0}^{\alpha-1} I_2 M_{V_2 z_s}(t_2), \quad (4.10)$$

where I_1, I_2, V_1 and V_2 are defined in (3.2) and (3.4).

4.2 Case-I of $DGos$:

When $Z_{[r,i,w,l]}$ and $Z_{[s,i,w,l]}$ be concomitants of the r-th and s-th for case-I of $DGos$, respectively. Then joint pdf of $Z_{[r,i,w,l]}$ and $Z_{[s,i,w,l]}$, $r < s$, is:

$$\begin{aligned} g_{d(r,s,i,w,l)}(z_r,z_s) &= \int_{-\infty}^{\infty}\int_{-\infty}^{u_s} g_{Z|U}(z_r|u_r) g_{Z|U}(z_s|u_s) g_{d(r,s,i,w,l)}(u_r,u_s)\,du_r du_s \\ &= g_Z(z_r)g_Z(z_s)\int_{-\infty}^{\infty}\int_{-\infty}^{u_s}\{1+\delta^2 A_{Z_r}A_{Z_s} \\ &\quad [\lambda^2((1-G_U(u_r))(1-G_U(u_s)))^{\alpha-1}(G_U(u_r)G_U(u_s))^{\lambda-1} \\ &\quad -\lambda(\alpha+\lambda)((1-G_U(u_r))(1-G_U(u_s)))^{\alpha-1}G_U(u_r)^{\lambda}G_U(u_s)^{\lambda-1} \\ &\quad -\lambda(\alpha+\lambda)((1-G_U(u_r))(1-G_U(u_s)))^{\alpha-1}G_U(u_r)^{\lambda-1}G_U(u_s)^{\lambda} \\ &\quad +(\alpha+\lambda)^2((1-G_U(u_r))(1-G_U(u_s)))^{\alpha-1}(G_U(u_r)G_U(u_s))^{\lambda}] \\ &\quad +\delta A_{Z_r}[\lambda(1-G_U(u_r))^{\alpha-1}G_U(u_r)^{\lambda-1} - (\alpha+\lambda)(1-G_U(u_r))^{\alpha-1}G_U(u_r)^{\lambda}] \\ &\quad +\delta A_{Z_s}[\lambda(1-G_U(u_s))^{\alpha-1}G_U(u_s)^{\lambda-1} - (\alpha+\lambda)(1-G_U(u_s))^{\alpha-1}G_U(u_s)^{\lambda}]\} \\ &\quad \times \frac{m_{s-1}}{(-1+r)!(s-1-r)!(1+w)^{s-2}} G_U(u_r)^w(1-G_U(u_r)^{1+w})^{-1+r} \\ &\quad \times (G_U(u_r)^{1+w} - G_U(u_s)^{1+w})^{s-1-r}G_U(u_s)^{\gamma_s-1} \\ &\quad \times g_U(u_r)g_U(u_s)du_r du_s, \end{aligned} \quad (4.11)$$

where A_{Z_r} and A_{Z_s} are defined in (4.2).

Then, we can write

$$\begin{aligned} J_{d_1} &= \frac{m_{s-1}}{(-1+r)!(s-1-r)!(1+w)^{s-2}}\int_{-\infty}^{\infty}\int_{-\infty}^{u_s}(\overline{G}_U(u_r)\overline{G}_U(u_s))^{\alpha-1}(G_U(u_r)G_U(u_s))^{\lambda-1} \\ &\quad \times G_U(u_r)^w(1-G_U(u_r)^{1+w})^{-1+r}(G_U(u_r)^{1+w}-G_U(u_s)^{1+w})^{s-1-r}G_U(u_s)^{\gamma_s-1} \\ &\quad \times g_U(u_r)g_U(u_s)du_r du_s, \\ &= \frac{m_{s-1}}{(-1+r)!(s-1-r)!(1+w)^{s-2}} \\ &\quad \int_{-\infty}^{\infty}\int_{-\infty}^{u_s}((1-G_U(u_r))(1-G_U(u_s)))^{\alpha-1}(G_U(u_r)G_U(u_s))^{\lambda-1} \\ &\quad \times G_U(u_r)^w(1-G_U(u_r)^{1+w})^{-1+r}(G_U(u_r)^{1+w}-G_U(u_s)^{1+w})^{s-1-r}G_U(u_s)^{\gamma_s-1} \\ &\quad \times g_U(u_r)g_U(u_s)du_r du_s \\ &= \frac{m_{s-1}}{(-1+r)!(s-1-r)!(1+w)^{s-2}}\sum_{j_1=0}^{\alpha-1}\sum_{j_2=0}^{\alpha-1}\binom{\alpha-1}{j_1}\binom{\alpha-1}{j_2}(-1)^{j_1+j_2} \\ &\quad \times \int_{-\infty}^{\infty}\int_{-\infty}^{u_s} G_U(u_r)^{j_1+\lambda-1}G_U(u_s)^{j_2+\lambda-1}G_U(u_r)^w g_U(u_r)(1-G_U(u_r)^{1+w})^{-1+r} \\ &\quad \times (G_U(u_r)^{1+w}-G_U(u_s)^{1+w})^{s-1-r}G_U(u_s)^{\gamma_s-1}g_U(u_s)du_r du_s \\ &= \frac{m_{s-1}}{(-1+r)!(s-1-r)!(1+w)^{s}}\sum_{j_1=0}^{\alpha-1}\sum_{j_2=0}^{\alpha-1}\binom{\alpha-1}{j_1}\binom{\alpha-1}{j_2}(-1)^{j_1+j_2} \\ &\quad \times \beta(r, \tfrac{\gamma_r+j_1+j_2+2\lambda-2}{1+w})\beta(s-r, \tfrac{\gamma_s+j_2+\lambda-1}{1+w}), \end{aligned}$$

$$
\begin{aligned}
J_{d_2} &= \frac{m_{s-1}}{(-1+r)!(s-1-r)!(1+w)^{s-2}} \int_{-\infty}^{\infty}\int_{-\infty}^{u_s} (\overline{G}_U(u_r)\overline{G}_U(u_s))^{\alpha-1} G_U(u_r)^{\lambda} G_U(u_s)^{\lambda-1} \\
&\times G_U(u_r)^{w}(1-G_U(u_r)^{1+w})^{-1+r}(G_U(u_r)^{1+w}-G_U(u_s)^{1+w})^{s-1-r} G_U(u_s)^{\gamma_s-1} \\
&\times g_U(u_r) g_U(u_s) du_r du_s \\
&= \frac{m_{s-1}}{(-1+r)!(s-1-r)!(1+w)^{s}} \sum_{j_1=0}^{\alpha-1}\sum_{j_2=0}^{\alpha-1} \binom{\alpha-1}{j_1}\binom{\alpha-1}{j_2}(-1)^{j_1+j_2} \\
&\times \beta\left(r, \frac{\gamma_r+j_1+j_2+2\lambda-1}{1+w}\right)\beta\left(s-r, \frac{\gamma_s+j_2+\lambda-1}{1+w}\right),
\end{aligned}
$$

$$
\begin{aligned}
J_{d_3} &= \frac{m_{s-1}}{(-1+r)!(s-1-r)!(1+w)^{s-2}} \int_{-\infty}^{\infty}\int_{-\infty}^{u_s} (\overline{G}_U(u_r)\overline{G}_U(u_s))^{\alpha-1} G_U(u_r)^{\lambda-1} G_U(u_s)^{\lambda} \\
&\times G_U(u_r)^{w}(1-G_U(u_r)^{1+w})^{-1+r}(G_U(u_r)^{1+w}-G_U(u_s)^{1+w})^{s-1-r} G_U(u_s)^{\gamma_s-1} \\
&\times g_U(u_r) g_U(u_s) du_r du_s \\
&= \frac{m_{s-1}}{(-1+r)!(s-1-r)!(1+w)^{s}} \sum_{j_1=0}^{\alpha-1}\sum_{j_2=0}^{\alpha-1} \binom{\alpha-1}{j_1}\binom{\alpha-1}{j_2}(-1)^{j_1+j_2} \\
&\times \beta\left(r, \frac{\gamma_r+j_1+j_2+2\lambda-1}{1+w}\right)\beta\left(s-r, \frac{\gamma_s+j_2+\lambda}{1+w}\right),
\end{aligned}
$$

$$
\begin{aligned}
J_{d_4} &= \frac{m_{s-1}}{(-1+r)!(s-1-r)!(1+w)^{s-2}} \int_{-\infty}^{\infty}\int_{-\infty}^{u_s} (\overline{G}_U(u_r)\overline{G}_U(u_s))^{\alpha-1} (G_U(u_r) G_U(u_s))^{\lambda} \\
&\times G_U(u_r)^{w}(1-G_U(u_r)^{1+w})^{-1+r}(G_U(u_r)^{1+w}-G_U(u_s)^{1+w})^{s-1-r} G_U(u_s)^{\gamma_s-1} \\
&\times g_U(u_r) g_U(u_s) du_r du_s \\
&= \frac{m_{s-1}}{(-1+r)!(s-1-r)!(1+w)^{s}} \sum_{j_1=0}^{\alpha-1}\sum_{j_2=0}^{\alpha-1} \binom{\alpha-1}{j_1}\binom{\alpha-1}{j_2}(-1)^{j_1+j_2} \\
&\times \beta\left(r, \frac{\gamma_r+j_1+j_2+2\lambda}{1+w}\right)\beta\left(s-r, \frac{\gamma_s+j_2+\lambda}{1+w}\right),
\end{aligned}
$$

$$
\begin{aligned}
J_{d_5} &= \frac{m_{s-1}}{(-1+r)!(s-1-r)!(1+w)^{s-2}} \int_{-\infty}^{\infty}\int_{-\infty}^{u_s} \overline{G}_U(u_r)^{\alpha-1} G_U(u_r)^{\lambda-1} G_U(u_r)^{w} \\
&\times (1-G_U(u_r)^{1+w})^{-1+r}(G_U(u_r)^{1+w}-G_U(u_s)^{1+w})^{s-1-r} G_U(u_s)^{\gamma_s-1} \\
&\times g_U(u_r) g_U(u_s) du_r du_s \\
&= \frac{m_{s-1}}{(-1+r)!(s-1-r)!(1+w)^{s}} \sum_{j_1=0}^{\alpha-1} \binom{\alpha-1}{j_1}(-1)^{j_1} \\
&\times \beta\left(r, \frac{\gamma_r+j_1+\lambda-1}{1+w}\right)\beta\left(s-r, \frac{\gamma_s}{1+w}\right),
\end{aligned}
$$

$$
\begin{aligned}
J_{d_6} &= \frac{m_{s-1}}{(-1+r)!(s-1-r)!(1+w)^{s-2}} \int_{-\infty}^{\infty}\int_{-\infty}^{u_s} \overline{G}_U(u_r)^{\alpha-1} G_U(u_r)^{\lambda} G_U(u_r)^{w} \\
&\times (1-G_U(u_r)^{1+w})^{-1+r}(G_U(u_r)^{1+w}-G_U(u_s)^{1+w})^{s-1-r} G_U(u_s)^{\gamma_s-1} \\
&\times g_U(u_r) g_U(u_s) du_r du_s \\
&= \frac{m_{s-1}}{(-1+r)!(s-1-r)!(1+w)^{s}} \sum_{j_1=0}^{\alpha-1} \binom{\alpha-1}{j_1}(-1)^{j_1} \beta\left(r, \frac{\gamma_r+j_1+\lambda}{1+w}\right)\beta\left(s-r, \frac{\gamma_s}{1+w}\right),
\end{aligned}
$$

$$
\begin{aligned}
J_{d_7} &= \frac{m_{s-1}}{(-1+r)!(s-1-r)!(1+w)^{s-2}} \int_{-\infty}^{\infty}\int_{-\infty}^{u_s} \overline{G}_U(u_s)^{\alpha-1} G_U(u_s)^{\lambda-1} G_U(u_r)^{w} \\
&\times (1-G_U(u_r)^{1+w})^{-1+r}(G_U(u_r)^{1+w}-G_U(u_s)^{1+w})^{s-1-r} G_U(u_s)^{\gamma_s-1} \\
&\times g_U(u_r) g_U(u_s) du_r du_s \\
&= \frac{m_{s-1}}{(-1+r)!(s-1-r)!(1+w)^{s}} \sum_{j_2=0}^{\alpha-1} \binom{\alpha-1}{j_2}(-1)^{j_2} \beta\left(r, \frac{\gamma_r+j_2+\lambda-1}{1+w}\right)\beta\left(s-r, \frac{\gamma_s+j_2+\lambda-1}{1+w}\right),
\end{aligned}
$$

$$\begin{aligned} J_{d_8} &= \frac{m_{s-1}}{(-1+r)!(s-1-r)!(1+w)^{s-2}} \int_{-\infty}^{\infty}\int_{-\infty}^{u_s} \overline{G}_U(u_s)^{\alpha-1} G_U(u_s)^{\lambda} G_U(u_r)^{w} \\ &\times (1-G_U(u_r)^{1+w})^{-1+r}(G_U(u_r)^{1+w}-G_U(u_s)^{1+w})^{s-1-r} G_U(u_s)^{\gamma_s-1} \\ &\times g_U(u_r) g_U(u_s) du_r du_s \\ &= \frac{m_{s-1}}{(-1+r)!(s-1-r)!(1+w)^{s}} \sum_{j_2=0}^{\alpha-1} \binom{\alpha-1}{j_2} (-1)^{j_2} \beta(r, \tfrac{\gamma_r+j_2+\lambda}{1+w}) \beta(s-r, \tfrac{\gamma_s+j_2+\lambda}{1+w}). \end{aligned}$$

Then we can write pdf as

$$\begin{aligned} & g_{d(r,s,i,w,l)}(Z_r, Z_s) \\ &= g_Z(z_r) g_Z(z_s)\{1+\delta^2 A_{Z_r} A_{Z_s}[\lambda J_{d_1} - \lambda(\alpha+\lambda) J_{d_2} - \lambda(\alpha+\lambda) J_{d_3} + (\alpha+\lambda)^2 J_{d_4}] \\ &+ \delta A_{Z_r}[\lambda J_{d_5} - (\alpha+\lambda) J_{d_6}] + \delta A_{Z_s}[\lambda J_{d_7} - (\alpha+\lambda) J_{d_8}]\} \\ &= g_Z(z_r) g_Z(z_s) + T_1 T_2 (\delta^2[\lambda J_{d_1} - \lambda(\alpha+\lambda)(J_{d_2}+J_{d_3}) + (\alpha+\lambda)^2 J_{d_4}]) \\ &+ \delta T_1 g_Z(z_s)[\lambda J_{d_5} - (\alpha+\lambda) J_{d_6}] + \delta T_2 g_Z(z_r)[\lambda J_{d_7} - (\alpha+\lambda) J_{d_8}], \end{aligned} \tag{4.12}$$

where T_1, T_2 are defined in (4.5).

The cdf of concomitants $Z_{[r,i,w,l]}$ and $Z_{[s,i,w,l]}$, $r < s$, is

$$\begin{aligned} & G_{d(r,s,i,w,l)}(Z_r, Z_s) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{u_s} G_{Z|U}(z_r|u_r) G_{Z|U}(z_s|u_s) g_{d(r,s,i,w,l)}(u_r, u_s) du_r du_s \\ &= G_Z(z_r) G_Z(z_s) \int_{-\infty}^{\infty}\int_{-\infty}^{u_s} \{1+\delta^2 A^*_{Z_r} A^*_{Z_s}((1-G_U(u_r)) \\ &\quad (1-G_U(u_s)))^{\alpha}(G_U(u_r) G_U(u_s))^{\lambda-1} \\ &+ \delta A^*_{Z_r}(1-G_U(u_r))^{\alpha} G_U(u_r)^{\lambda-1} + \delta A^*_{Z_s}(1-G_U(u_s))^{\alpha} G_U(u_s)^{\lambda-1}\} \\ &\times \frac{m_{s-1}}{(-1+r)!\,(s-1-r)!\,(1+w)^{s-2}} G_U(u_r)^{w}(1-G_U(u_r)^{1+w})^{-1+r} \\ &\times (G_U(u_r)^{w+1} - G_U(u_s)^{w+1})^{s-1-r} G_U(u_s)^{\gamma_s-1} g_U(u_r) g_U(u_s) du_r du_s, \end{aligned} \tag{4.13}$$

where $A^*_{Z_r}$ and $A^*_{Z_s}$ are defined in (4.7).

$$\begin{aligned} J^*_{d_1} &= \frac{m_{s-1}}{(-1+r)!(s-1-r)!(1+w)^{s-2}} \int_{-\infty}^{\infty}\int_{-\infty}^{u_s} (\overline{G}_U(u_r)\overline{G}_U(u_s))^{\alpha}(G_U(u_r) G_U(u_s))^{\lambda-1} \\ &\times G_U(u_r)^{w}(1-G_U(u_r)^{1+w})^{-1+r}(G_U(u_r)^{1+w}-G_U(u_s)^{1+w})^{s-1-r} G_U(u_s)^{\gamma_s-1} \\ &\times g_U(u_r) g_U(u_s) du_r du_s \\ &= \frac{m_{s-1}}{(-1+r)!(s-1-r)!(1+w)^{s}} \sum_{j_1=0}^{\alpha}\sum_{j_2=0}^{\alpha} \binom{\alpha}{j_1}\binom{\alpha}{j_2} (-1)^{j_1+j_2} \\ &\times \beta(r, \tfrac{\gamma_r+j_1+j_2+2\lambda-2}{1+w}) \beta(s-r, \tfrac{\gamma_s+j_2+\lambda-1}{1+w}), \end{aligned}$$

$$\begin{aligned} J^*_{d_2} &= \frac{m_{s-1}}{(-1+r)!(s-1-r)!(1+w)^{s-2}} \int_{-\infty}^{\infty}\int_{-\infty}^{u_s} \overline{G}_U(u_r)^{\alpha} G_U(u_r)^{\lambda-1} \\ &\times G_U(u_r)^{w}(1-G_U(u_r)^{1+w})^{-1+r}(G_U(u_r)^{1+w}-G_U(u_s)^{1+w})^{s-1-r} G_U(u_s)^{\gamma_s-1} \\ &\times g_U(u_r) g_U(u_s) du_r du_s \\ &= \frac{m_{s-1}}{(-1+r)!(s-1-r)!(1+w)^{s}} \sum_{j_1=0}^{\alpha} \binom{\alpha}{j_1} (-1)^{j_1} \beta(r, \tfrac{\gamma_r+j_1+\lambda-1}{1+w}) \beta(s-r, \tfrac{\gamma_s}{1+w}), \end{aligned}$$

$$J_{d_3}^* = \frac{m_{s-1}}{(-1+r)!(s-1-r)!(1+w)^{s-2}} \int_{-\infty}^{\infty} \int_{-\infty}^{u_s} G_U(u_s)^{\lambda-1} \overline{G}_U(u_s)^{\alpha}$$
$$\times G_U(u_r)^w (1 - G_U(u_r)^{1+w})^{-1+r} (G_U(u_r)^{1+w} - G_U(u_s)^{1+w})^{s-1-r} G_U(u_s)^{\gamma_s - 1}$$
$$\times g_U(u_r) g_U(u_s) du_r du_s$$
$$= \frac{m_{s-1}}{(-1+r)!(s-1-r)!(1+w)^s} \sum_{j_2=0}^{\alpha} \binom{\alpha}{j_2} (-1)^{j_2} \beta(r, \frac{\gamma_r + j_2 + \lambda - 1}{1+w}) \beta(s - r, \frac{\gamma_s + j_2 + \lambda - 1}{1+w}).$$

Then we can write cdf as

$$G_{d(r,s,i,w,l)}(Z_r, Z_s) = G_Z(z_r) G_Z(z_s) \{1 + \delta^2 A_{Z_r}^* A_{Z_s}^* J_{d_1}^* + \delta A_{Z_r}^* J_{d_2}^* + \delta A_{Z_s}^* J_{d_3}^*\}. \quad (4.14)$$

The product moment of $Z_{[r,i,w,l]}, Z_{[s,i,w,l]}$ as $M_{[r,s,i,w,l]}(t_1, t_2)$ is simply acquired from (4.12) as

$$M_{[r,s,i,w,l]}(t_1, t_2)$$
$$= M_{Z_r}(t_1) M_{Z_s}(t_2) + \delta^2 T_1^* T_2^* [\lambda J_{d_1} - \lambda(\alpha + \lambda)(J_{d_2} + J_{d_3}) + (\alpha + \lambda)^2 J_{d_4}]$$
$$+ \delta T_1^* M_{Z_s}(t_2) [\lambda J_{d_5} - (\alpha + \lambda) J_{d_6}] + \delta T_2^* M_{Z_r}(t_1) [\alpha J_{d_7} - (\alpha + \lambda) J_{d_8}], \quad (4.15)$$

where T_1^*, T_2^* are defined in (4.10).

5. THE SHANNON ENTROPY

For concomitants of case-I of Gos from Lai and Xie extensions, obtaining an explicit form of Shannon entropy is simple.

Theorem 5.1

Let $Z_{[r,i,w,l]}$ is a concomitants of rth case-I of Gos form Lai and xie extensions in (2.1), then from (1.13), an exlicit form of shannon entropy of $Z_{[r,i,w,l]}$, is given by:

$$H(Z_{[r,i,w,l]}) = H(Z) - \delta R_{[r,i,w,l]}^* [\lambda \Phi_{g_1}(z) - (\alpha + \lambda) \Phi_{g_2}(z)] - W(r, \delta, i, w, l) \quad (5.1)$$

where the Shannon entropy for Z is

$$H(Z) = -\int_{-\infty}^{\infty} g_Z(z) \ln g_Z(z) dz, \quad (5.2)$$

$$\Phi_{g_1}(z) = \int_{-\infty}^{\infty} g_Z(z)(1 - G_Z(z))^{\alpha-1} G_Z^{\lambda-1}(z) \ln g_Z(z) dz, \quad (5.3)$$

$$\Phi_{g_2}(z) = \int_{-\infty}^{\infty} g_Z(z)(1 - G_Z(z))^{\alpha-1} G_Z^{\lambda}(z) \ln g_Z(z) dz, \quad (5.4)$$

and

$$W(r, \delta, i, w, l) = -\sum_{j=0}^{\infty} (-1)^j (\delta R_{[r,i,w,l]}^*)^{j+1} [\delta R_{[r,i,w,l]}^* \sum_{p=0}^{\alpha-1} \binom{\alpha - 1}{p} (-1)^p$$
$$[(\lambda - 1)(\frac{\lambda}{\lambda+p} I_1^* - \frac{\alpha+\lambda}{\lambda+p+1} I_2^*) - (\alpha - 1)(\frac{\lambda}{\lambda+p} I_3^* - \frac{\alpha+\lambda}{\lambda+p+1} I_4^*) - (\alpha + \lambda)(\frac{\lambda}{\lambda+p} I_5^*$$
$$- \frac{\alpha+\lambda}{\lambda+p+1} I_6^*)] + (\lambda - 1) I_7^* - (\alpha - 1) I_8^* - (\alpha + \lambda) I_9^*]. \quad (5.5)$$

Proof:

From (2.1) and (1.13) then we have:

$$H(Z_{[r,i,w,l]})$$
$$= -\int_{-\infty}^{\infty} g_Z(z)(1+\delta R^*_{[r,i,w,l]}(\lambda-(\alpha+\lambda)G_Z(z))(1-G_Z(z))^{\alpha-1}G_Z^{\lambda-1}(z))$$
$$\times \ln(g_Z(z)(1+\delta R^*_{[r,i,w,l]}(\lambda-(\alpha+\lambda)G_Z(z))(1-G_Z(z))^{\alpha-1}G_Z^{\lambda-1}(z)))\mathrm{d}z$$
$$= H(Z)-\delta R^*_{[r,i,w,l]}[\lambda\Phi_{g_1}(z)-(\alpha+\lambda)\Phi_{g_2}(z)]-W(r,\delta,i,w,l), \qquad (5.6)$$

where

$$W(r,\delta,i,w,l)$$
$$= \int_{-\infty}^{\infty} g_Z(z)(1+\delta R^*_{[r,i,w,l]}(\lambda-(\alpha+\lambda)G_Z(z))(1-G_Z(z))^{\alpha-1}G_Z^{\lambda-1}(z))$$
$$\ln(1+\delta R^*_{[r,i,w,l]}(\lambda-(\alpha+\lambda)G_Z(z))(1-G_Z(z))^{\alpha-1}G_Z^{\lambda-1}(z))\mathrm{d}z, \qquad (5.7)$$

we used integration by part, let

$$u = \ln(1+\delta R^*_{[r,i,w,l]}(\lambda-(\alpha+\lambda)G_Z(z))(1-G_Z(z))^{\alpha-1}G_Z^{\lambda-1}(z)), \qquad (5.8)$$

then

$$du = \sum_{j=0}^{\infty} (-1)^j(\delta R^*_{[r,i,w,l]})^{j+1}(\lambda-(\alpha+\lambda)G_Z(z))^j(1-G_Z(z))^{j(\alpha-1)}G_Z^{j(\lambda-1)}(z)$$
$$\times g_Z(z)[(\lambda-1)(\lambda-(\alpha+\lambda)G_Z(z))(1-G_Z(z))^{\alpha-1}G_Z^{\lambda-2}(z)$$
$$-(\alpha-1)(\lambda-(\alpha+\lambda)G_Z(z))(1-G_Z(z))^{\alpha-2}G_Z^{\lambda-1}(z)$$
$$-(\alpha+\lambda)(1-G_Z(z))^{\alpha-1}G_Z^{\lambda-1}(z)]\mathrm{d}z, \qquad (5.9)$$

let, we have

$$dv = \int g_Z(z)(1+\delta R^*_{[r,i,w,l]}g_Z(z)(\lambda-(\alpha+\lambda)G_Z(z))(1-G_Z(z))^{\alpha-1}G_Z^{\lambda-1}(z))\mathrm{d}z, \qquad (5.10)$$

then

$$\mathrm{V} = G_Z(z)+\delta R^*_{[r,i,w,l]}\sum_{p=0}^{\alpha-1}\binom{\alpha-1}{p}(-1)^p[\frac{\lambda G_Z^{\lambda+p}(z)}{\lambda+p}-\frac{(\alpha+\lambda)G_Z^{\lambda+p+1}(z)}{\lambda+p+1}]. \qquad (5.11)$$

Then we have

$$W(r,\delta,i,w,l) = -\int_{-\infty}^{\infty} vdu =$$
$$-\sum_{j=0}^{\infty}(-1)^j(\delta R^*_{[r,i,w,l]})^{j+1}[\delta R^*_{[r,i,w,l]}\sum_{p=0}^{\alpha-1}\binom{\alpha-1}{p}(-1)^p$$
$$[(\lambda-1)(\frac{\lambda}{\lambda+p}I_1^*-\frac{\alpha+\lambda}{\lambda+p+1}I_2^*)-(\alpha-1)(\frac{\lambda}{\lambda+p}I_3^*-\frac{\alpha+\lambda}{\lambda+p+1}I_4^*)$$
$$-(\alpha+\lambda)(\frac{\lambda}{\lambda+p}I_5^*-\frac{\alpha+\lambda}{\lambda+p+1}I_6^*)]+(\lambda-1)I_7^*-(\alpha-1)I_8^*-(\alpha+\lambda)I_9^*], \qquad (5.12)$$

where

$$I_1^* = \int_{-\infty}^{\infty} g_Z(z)(\lambda-(\alpha+\lambda)G_Z(z))^{j+1}(1-G_Z(z))^{(\alpha-1)(j+1)}G_Z^{2\lambda+p+j(\lambda-1)-2}(z)\mathrm{d}z$$
$$= \sum_{h=0}^{j+1}\binom{j+1}{h}(-(\alpha+\lambda))^h\lambda^{j+1-h}$$
$$\int_{-\infty}^{\infty} g_Z(z)(1-G_Z(z))^{(\alpha-1)(j+1)}G_Z^{2\lambda+p+j(\lambda-1)+h-2}(z)\mathrm{d}z$$

by putting $x = 1 - G_Z(z)$. Therefore, we get

$$I_1^* = \sum_{h=0}^{j+1} \binom{j+1}{h} (-(\alpha+\lambda))^h \lambda^{j+1-h} \int_0^1 x^{(\alpha-1)(j+1)}(1-x)^{2\lambda+p+j(\lambda-1)+h-2} \mathrm{d}x$$
$$= \sum_{h=0}^{j+1} \binom{j+1}{h} (-(\alpha+\lambda))^h \lambda^{j+1-h}$$
$$\times \beta((\alpha-1)(j+1)+1, 2\lambda+p+h+j(\lambda-1)-1)$$

$$I_2^* = \int_{-\infty}^{\infty} g_Z(z)(\lambda-(\alpha+\lambda)G_Z(z))^{j+1}(1-G_Z(z))^{(\alpha-1)(j+1)} G_Z^{2\lambda+p+j(\lambda-1)-1}(z)\mathrm{d}z$$
$$= \sum_{h=0}^{j+1} \binom{j+1}{h} (-(\alpha+\lambda))^h \lambda^{j+1-h} \beta((\alpha-1)(j+1)+1, 2\lambda+p+h+j(\lambda-1))$$

$$I_3^* = \int_{-\infty}^{\infty} g_Z(z)(\lambda-(\alpha+\lambda)G_Z(z))^{j+1}(1-G_Z(z))^{(\alpha-2)+j(\alpha-1)} G_Z^{2\lambda+p+j(\lambda-1)-1}(z)\mathrm{d}z$$
$$= \sum_{h=0}^{j+1} \binom{j+1}{h} (-(\alpha+\lambda))^h \lambda^{j+1-h}$$
$$\times \beta((\alpha-2)+j(\alpha-1)+1, 2\lambda+p+h+j(\lambda-1))$$

$$I_4^* = \int_{-\infty}^{\infty} g_Z(z)(\lambda-(\alpha+\lambda)G_Z(z))^{j+1}(1-G_Z(z))^{(\alpha-2)+j(\alpha-1)} G_Z^{2\lambda+p+j(\lambda-1)}(z)\mathrm{d}z$$
$$= \sum_{h=0}^{j+1} \binom{j+1}{h} (-(\alpha+\lambda))^h \lambda^{j+1-h}$$
$$\times \beta((\alpha-2)+j(\alpha-1)+1, 2\lambda+p+h+j(\lambda-1)+1)$$

$$I_5^* = \int_{-\infty}^{\infty} g_Z(z)(\lambda-(\alpha+\lambda)G_Z(z))^{j}(1-G_Z(z))^{(\alpha-1)(j+1)} G_Z^{\lambda+p+(j+1)(\lambda-1)}(z)\mathrm{d}z$$
$$= \sum_{h=0}^{j} \binom{j}{h} (-(\alpha+\lambda))^h \lambda^{j-h}$$
$$\times \beta((\alpha-1)(j+1)+1, \lambda+p+h+(j+1)(\lambda-1)+1)$$

$$I_6^* = \int_{-\infty}^{\infty} g_Z(z)(\lambda-(\alpha+\lambda)G_Z(z))^{j}(1-G_Z(z))^{(\alpha-1)(j+1)} G_Z^{\lambda+p+(j+1)(\lambda-1)+1}(z)\mathrm{d}z$$
$$= \sum_{h=0}^{j} \binom{j}{h} (-(\alpha+\lambda))^h \lambda^{j-h}$$
$$\times \beta((\alpha-1)(j+1)+1, \lambda+p+h+(j+1)(\lambda-1)+2)$$

$$I_7^* = \int_{-\infty}^{\infty} g_Z(z)(\lambda-(\alpha+\lambda)G_Z(z))^{j+1}(1-G_Z(z))^{(j+1)(\alpha-1)} G_Z^{(j+1)(\lambda-1)}(z)\mathrm{d}z$$
$$= \sum_{h=0}^{j+1} \binom{j+1}{h} (-(\alpha+\lambda))^h \lambda^{j+1-h}$$
$$\times \beta((j+1)(\alpha-1)+1, (j+1)(\lambda-1)+h+1)$$

$$I_8^* = \int_{-\infty}^{\infty} g_Z(z)(\lambda-(\alpha+\lambda)G_Z(z))^{j+1}(1-G_Z(z))^{(\alpha-2)+j(\alpha-1)} G_Z^{\lambda+j(\lambda-1)}(z)\mathrm{d}z$$
$$= \sum_{h=0}^{j+1} \binom{j+1}{h} (-(\alpha+\lambda))^h \lambda^{j+1-h}$$
$$\times \beta((\alpha-2)+j(\alpha-1)+1, \lambda+h+j(\lambda-1)+1)$$

$$I_9^* = \int_{-\infty}^{\infty} g_Z(z)(\lambda-(\alpha+\lambda)G_Z(z))^{j}(1-G_Z(z))^{(j+1)(\alpha-1)} G_Z^{\lambda+j(\lambda-1)}(z)\mathrm{d}z$$
$$= \sum_{h=0}^{j} \binom{j}{h} (-(\alpha+\lambda))^h \lambda^{j-h} \beta((j+1)(\alpha-1)+1, \lambda+h+j(\lambda-1)+1)$$

In the next subsection, we will apply the Shannon entropy of concomitants case-I of Gos of Lai and Xie extension on exponential distribution.

5.1 Exponential Distribution

Exponential distribution's pdf and cdf are provided by, respectively:

$$g(z) = e^{-z}, \tag{5.13}$$

$$G(Z) = 1 - e^{-z}, 0 \leq z < \infty. \tag{5.14}$$

Theorem 5.2

Let $Z_{[r,i,w,l]}$ is the concomitant of r-th case-I of Gos for Exponential distribution from (5.13) and (5.14) then, from (5.6), then the Shannon entropy of $Z_{[r,i,w,l]}$ is given by:

$$\begin{aligned} H\left(Z_{[r,i,w,l]}\right) = 1 - W(r,\delta,i,w,l) - \delta R^*_{[r,i,w,l]} \\ \times\ [\lambda\beta(\alpha,\lambda)(\psi(\alpha) - \psi(\alpha+\lambda)) - (\alpha+\lambda) \\ \beta(\alpha,\lambda+1)(\psi(\alpha) - \psi(\alpha+\lambda+1))] \end{aligned} \tag{5.15}$$

where the Euler's constant is $\nu = -\Gamma'(1)$, the digamma function is $\psi(.)$ and $W(r,\alpha,i,w,k)$ is defined by (5.5).

Proof:

From (5.6), (5.13) and (5.14), we have:

$$\begin{aligned} H\left(Z_{[r,i,w,l]}\right) &= -\int_0^\infty e^{-z}\ln(e^{-z})\,\mathrm{d}z - W(r,\delta,i,w,l) - \delta R^*_{[r,i,w,l]} \\ &\times [\lambda \int_0^\infty (e^{-z})^\alpha (1-e^{-z})^{\lambda-1}\ln(e^{-z})\mathrm{d}z \\ &-(\alpha+\lambda)\int_0^\infty (e^{-z})^\alpha(1-e^{-z})^\lambda \ln(e^{-z})\mathrm{d}z] \\ &= 1 - W(r,\delta,i,w,l) - \delta R^*_{[r,i,w,l]}[\lambda\Phi_{g_1}(z) - (\alpha+\lambda)\Phi_{g_2}(z)]. \end{aligned}$$

In order to find

$\Phi_{g_1}(z) \quad = \int_0^\infty e^{-z}(e^{-z})^{\alpha-1}(1-e^{-z})^{\lambda-1}\ln(e^{-z})\mathrm{d}z$, we note that

$$\begin{aligned} \eta(t) &= \int_0^\infty [g(z)]^t (e^{-z})^{\alpha-1}(1-e^{-z})^{\lambda-1}\mathrm{d}z \\ &= \int_0^\infty [e^{-z}]^t(e^{-z})^{\alpha-1}(1-e^{-z})^{\lambda-1}\mathrm{d}z \\ &= \int_0^1 x^{t-1}x^{\alpha-1}(1-x)^{\lambda-1}\mathrm{d}x = \beta(t+\alpha-1,\lambda). \end{aligned}$$

By putting $x = e^{-z}$.

Therefore, we get $\eta`(t) = \beta(t+\alpha-1,\lambda)[\psi(t+\alpha-1) - \psi(t+\alpha+\lambda-1)]$, then $\eta`(1) = \Phi_{g_1}(z) = \beta(\alpha,\lambda)[\psi(\alpha) - \psi(\alpha+\lambda)]$.

By the same manner, we can obtain the

$$\Phi_{g_2}(z) = \beta(\alpha,\lambda+1)[\psi(\alpha) - \psi(\alpha+\lambda+1)].$$

6. FISHER INFORMATION

The following theorem introduces the Fisher information for concomitants of case-I of Gos of Lai and Xie extension as:

Theorem 6.1

Let $Z_{[r,i,w,l]}$ is the concomitant of r-th case-I of Gos from (2.1), then from (1.14), the Fisher information of $Z_{[r,i,m,l]}$ is given by:

$$I(Z_{[r,i,w,l]}) = \int_{-\infty}^{\infty}\left[\frac{\partial \ln g_{[r,i,w,l]}(z)}{\partial z}\right]^2 g_{[r,i,w,l]}(z) = I + 2\delta R^*_{[r,i,w,l]} II + (\delta R^*_{[r,i,w,l]})^2 III, \quad (6.1)$$

where

$$I = \int_{-\infty}^{\infty}\left[\frac{\partial \ln g_Z(z)}{\partial z}\right]^2 g_{[r,i,w,l]}(z)\mathrm{d}z. \quad (6.2)$$

$$II = \int_{-\infty}^{\infty} g_Z(z)`g_Z(z)[(\lambda-1)(\lambda-(\alpha+\lambda)G_Z(z))(1-G_Z(z))^{\alpha-1}G_Z^{\lambda-2}(z) -(\alpha-1)(\lambda-(\alpha+\lambda)G_Z(z))(1-G_Z(z))^{\alpha-2}G_Z^{\lambda-1}(z) -(\alpha+\lambda)\big(1-G_Z(z)\big)^{\alpha-1}G_Z^{\lambda-1}(z)\big]\mathrm{d}z. \quad (6.3)$$

$$III = \int_{-\infty}^{\infty}\frac{g_Z^3(z)}{1+\delta R^*_{[r,i,w,l]}(\lambda-(\alpha+\lambda)G_Z(z))(1-G_Z(z))^{\alpha-1}G_Z^{\lambda-1}(z)} [(\lambda-1)(\lambda-(\alpha+\lambda)G_Z(z))(1-G_Z(z))^{\alpha-1}G_Z^{\lambda-2}(z) -(\alpha-1)(\lambda-(\alpha+\lambda)G_Z(z))(1-G_Z(z))^{\alpha-2}G_Z^{\lambda-1}(z) -(\alpha+\lambda)(1-G_Z(z))^{\alpha-1}G_Z^{\lambda-1}(z)]^2\mathrm{d}z. \quad (6.4)$$

This is shift-invariant Fisher information, which is Fisher information for the location parameter. Noting that it differs from what BuHamra and Ahsanullah (2005) introduced. We will apply the last theorem for the exponential distribution in the following subsections.

6.1 Exponential Distribution

Theorem 6.2

Let $Z_{[r,i,w,l]}$ is the concomitant of r-th case-I of Gos for exponential distribution function from (5.13) and (5.14) then from (6.1), the Fisher information of $Z_{[r,i,w,l]}$ is given by:

$$I(Z_{[r,i,w,l]}) = 1 + \sum_{j=0}^{\infty}(-1)^j(\delta R^*_{[r,i,w,l]})^{j+2}[(\lambda-1)^2 I_1 - 2(\lambda-1)(\alpha-1)I_2 -2(\lambda-1)(\alpha+\lambda)I_3 + (\alpha-1)^2 I_4 + (\alpha+\lambda)^2 I_5 - 2(\alpha-1)(\alpha+\lambda)I_6]. \quad (6.5)$$

Proof:

From (5.13), (5.14) and (6.1), we have:

$$I\big(Z_{[r,i,w,l]}\big) = \int_{-\infty}^{\infty}\left[\frac{\partial \ln e^{-z}}{\partial z}\right]^2 g_{[r,i,w,l]}(z)\mathrm{d}z + 2\delta R^*_{[r,i,w,l]} \int_{-\infty}^{\infty}[-(\lambda-1)(\lambda-(\alpha+\lambda)(1-e^{-z}))(e^{-z})^{\alpha+1}(1-e^{-z})^{\lambda-2} -(\alpha-1)(\lambda-(\alpha+\lambda)(1-e^{-z}))(e^{-z})^{\alpha}(1-e^{-z})^{\lambda-1} +(\alpha+\lambda)(e^{-z})^{\alpha+1}(1-e^{-z})^{\lambda-1}]\mathrm{d}z + (\delta R^*_{[r,i,w,l]})^2 \int_{-\infty}^{\infty}\frac{e^{-3z}}{1+\delta R^*_{[r,i,w,l]}(1-e^{-z})^{\lambda-2}(z)e^{-(\alpha-1)z}(\lambda-(\alpha+\lambda)(1-e^{-z}))} [(\lambda-1)(\lambda-(\alpha+\lambda)(1-e^{-z}))(e^{-z})^{\alpha-1}(1-e^{-z})^{\lambda-2} +(\alpha-1)(\lambda-(\alpha+\lambda)(1-e^{-z}))(e^{-z})^{\alpha-2}(1-e^{-z})^{\lambda-1} -(\alpha+\lambda)(e^{-z})^{\alpha-1}(1-e^{-z})^{\lambda-1}]^2\mathrm{d}z = I + 2\delta R^*_{[r,i,w,l]} II + (\delta R^*_{[r,i,w,l]})^2 III,$$

where

$$I=\int_{-\infty}^{\infty}\left[\frac{\partial \ln e^{-z}}{\partial z}\right]^2 g_{[r,i,w,l]}(z)\mathrm{d}z=1. \tag{6.6}$$

$$II=\int_{-\infty}^{\infty}[-(\lambda-1)(\lambda-(\alpha+\lambda)(1-e^{-z}))(e^{-z})^{\alpha+1}(1-e^{-z})^{\lambda-2}$$
$$-(\alpha-1)(\lambda-(\alpha+\lambda)(1-e^{-z}))(e^{-z})^{\alpha}(1-e^{-z})^{\lambda-1}$$
$$+(\alpha+\lambda)(e^{-z})^{\alpha+1}(1-e^{-z})^{\lambda-1}]\mathrm{d}z,$$ by putting $x=1-\exp(-z)$.

Therefore, we get

$$\int_0^1 [(\alpha+\lambda)(\lambda-1)(1-x)^{\alpha}x^{\lambda-1}-\lambda(\lambda-1)(1-z)^{\alpha}x^{\lambda-2}]$$
$$+[(\alpha+\lambda)(\alpha-1)(1-x)^{\alpha-1}x^{\lambda}-\lambda(\alpha-1)(1-x)^{\alpha-1}x^{\lambda-1}]$$
$$+(\alpha+\lambda)(1-x)^{\alpha}x^{\lambda-1}\mathrm{d}x$$
$$=zero. \tag{6.7}$$

$$III=\int_{-\infty}^{\infty}\frac{(e^{-z})^3}{1+\delta R^*_{[r,i,w,l]}(\lambda-(\alpha+\lambda)(1-e^{-z}))(e^{-z})^{\alpha-1}(1-e^{-z})^{\lambda-1}}$$
$$[(\lambda-1)(\lambda-(\alpha+\lambda)(1-e^{-z}))(e^{-z})^{\alpha-1}$$
$$(1-e^{-z})^{\lambda-2}+(\alpha-1)(\lambda-(\alpha+\lambda)(1-e^{-z}))(e^{-z})^{\alpha-2}(1-e^{-z})^{\lambda-1}$$
$$-(\alpha+\lambda)(e^{-z})^{\alpha-1}(1-e^{-z})^{\lambda-1}]^2\mathrm{d}z \quad =\int_{-\infty}^{\infty}\sum_{j=0}^{\infty}(-1)^j(\delta R^*_{[r,i,w,l]})^j$$
$$(\lambda-(\alpha+\lambda)(1-e^{-z}))^j(e^{-z})^{3+j(\alpha-1)}(1-e^{-z})^{j(\lambda-1)}$$
$$[(\lambda-1)(\lambda-(\alpha+\lambda)(1-e^{-z}))(e^{-z})^{\alpha-1}(1-e^{-z})^{\lambda-2}$$
$$+(\alpha-1)(\lambda-(\alpha+\lambda)(1-e^{-z}))(e^{-z})^{\alpha-2}(1-e^{-z})^{\lambda-1}$$
$$-(\alpha+\lambda)(e^{-z})^{\alpha-1}(1-e^{-z})^{\lambda-1}]^2\mathrm{d}z$$
$$=\sum_{j=0}^{\infty}(-1)^j(\delta R^*_{[r,i,w,l]})^j[(\lambda-1)^2 I_1`-2(\lambda-1)(\alpha-1)I_2`$$
$$-2(\lambda-1)(\alpha+\lambda)I_3`+(\alpha-1)^2 I_4`+(\alpha+\lambda)^2 I_5-2(\alpha-1)(\alpha+\lambda)I_6`]. \tag{6.8}$$

By putting $x=1-\exp(-z)$. Therefore, we get

$$I_1`=\int_0^1(\lambda-(\alpha+\lambda)x)^{j+2}(1-x)^{2\alpha+j(\alpha-1)}x^{2(\lambda-2)+j(\lambda-1)}\mathrm{d}x$$
$$=\sum_{k=0}^{j+2}\lambda^{j+2-k}(-(\alpha+\lambda))^k\binom{j+2}{k}$$
$$\times\beta(2\lambda-3+j(\lambda-1)+k,2\alpha+j(\alpha-1)+1)$$

$$I_2`=\int_0^1(\lambda-(\alpha+\lambda)x)^{j+2}(1-x)^{2\alpha-1+j(\alpha-1)}x^{2\lambda-3+j(\lambda-1)}\mathrm{d}x$$
$$=\sum_{k=0}^{j+2}\lambda^{j+2-k}(-(\alpha+\lambda))^k\binom{j+2}{k}\beta(2\lambda-2+j(\lambda-1)+k,2\alpha+j(\alpha-1))$$

$$I_3`=\int_0^1(\lambda-(\alpha+\lambda)x)^{j+1}(1-x)^{2\alpha+j(\alpha-1)}x^{2\lambda-3+j(\lambda-1)}\mathrm{d}x$$
$$=\sum_{k=0}^{j+1}\lambda^{j+1-k}(-(\alpha+\lambda))^k\binom{j+1}{k}\beta((j+2)(\lambda-1)+k,2\alpha+j(\alpha-1)+1)$$

$$I_4`=\int_0^1(\lambda-(\alpha+\lambda)x)^{j+2}(1-x)^{(j+2)(\alpha-1)}x^{(j+2)(\lambda-1)}\mathrm{d}x$$
$$=\sum_{k=0}^{j+2}\lambda^{j+2-k}(-(\alpha+\lambda))^k\binom{j+2}{k}$$
$$\times\beta((j+2)(\lambda-1)+1+k,(j+2)(\alpha-1)+1)$$

$$I_5` = \int_0^1 (\lambda-(\alpha+\lambda)x)^j(1-x)^{2\alpha+j(\alpha-1)}x^{(j+2)(\lambda-1)}\mathrm{d}x$$
$$= \sum_{k=0}^{j} \lambda^{j-k}(-(\alpha+\lambda))^k \binom{j}{k} \beta((j+2)(\lambda-1)+1+k, 2\alpha+j(\alpha-1)+1)$$

$$I_6` = \int_0^1 (\lambda-(\alpha+\lambda)x)^{j+1}(1-x)^{2\alpha-1+j(\alpha-1)}x^{(j+2)(\lambda-1)}\mathrm{d}x$$
$$= \sum_{k=0}^{j+1} \lambda^{j+1-k}(-(\alpha+\lambda))^k \binom{j+1}{k} \beta((j+2)(\lambda-1)+1+k, 2\alpha+j(\alpha-1)).$$

7. CONCLUSIONS

In this article, we consider a new extensions of Morgenstern family is Lai and Xie extensions and discuss their concomitants for case-I of generalized order statistics and case-I of dual generalized order statistics. Additionally, recurrence relation between moments is found for the recommended models. We have also derived the expression for the joint distribution of concomitants for generalized order statistics and its dual. We established the Shannon entropy and Fisher information with exponential distribution as an example from Lai and Xie extensions.

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