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QUARTIC TRANSMUTED FRECHET DISTRIBUTION: PROPERTIES AND APPLICATIONS

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ABSTRACT

In this article, I present the Quartic Transmuted Frechet distribution (QTFD), a new generalization of the Frechet distribution based Quartic on ranking transmutation map. Additionally, I have derived some properties of the proposed distribution including reliability and hazard functions, moments, moment generating function, quantile function and random number generation. The Maximum Likelihood technique was used to estimate the QTFD parameters. Finally, an application of the proposed distribution QTFD using two uncensored data is conducted to illustrate and compare with the base Frechet distribution, Transmuted Freshet distribution and Cubic Transmuted Freshet distribution. It has been observed that the proposed distribution QTFD provides a better fit for the two datasets as compared to the other distributions.

KEYWORDS

Cubic Transmuted, Frechet Distribution, Moments, Quartic Transmuted**,** Quantile function.

1. INTRODUCTION

The Frechet probability distribution, named after the French mathematician Maurice Frechet who developed it in 1925. It is a continuous probability distribution, also known as the inverse Weibull distribution by Khan et al. (2008). The Inverse Rayleigh and Inverse Exponential distributions can be considered as special cases of the Frechet distribution. The Frechet probability model is used to model maximum values in a data set. It is used to model a variety of phenomena, such as flood analysis, horse racing, human lifespans, maximum rainfall and river discharge in hydrology. The Frechet distribution is used particularly in the field of engineering reliability. It can also be used to model a variety of failure characteristics such as infant mortality, useful life, and wear-out periods. The cumulative distribution function (CDF) for the Frechet distribution and the probability density function (PDF) are defined as follows:

$$G(x) = e^{\left(-\left(\frac{x}{\mu}\right)^{-\sigma}\right)}, x > 0, \mu, \sigma > 0 \tag{1}$$

$$g(x) = \sigma\, \mu^{\sigma} x^{-(\sigma+1)} e^{\left(-\left(\frac{x}{\mu}\right)^{-\sigma}\right)}, x > 0, \mu, \sigma > 0 \tag{2}$$

Shaw and Buckley (2009), provided an intriguing technique for expanding an existing distribution with a new parameter for solving the problems related to financial mathematics and named the family as quadratic transmuted family (QRT) of distributions. The *cdf* of (QRT) map is:

$$F(x) = (1+\lambda)G(x) - \lambda G(x)^2, \text{where } |\lambda| \leq 1$$

where $G(x)$ is the cumulative distribution function (cdf) of the base distribution. Observe that at $\lambda = 0$, we get the baseline *cdf*. Mahmoud and Mandouh (2013) developed transmuted Frechet distribution. Abed Al-Kadim (2018) proposed generalized formula for transmuted distribution proposed by Shaw and Buckley (2009), cumulative distribution function of the Cubic Ranking transformation (CRT) map is:

$$F(x) = (1+\lambda_1)G(x) + (\lambda_2 - \lambda_1)G^2(x) - \lambda_2 G^3(x), |\lambda| \leq 1$$

where $\lambda_i \in [-1,1], i = 1,2$ and $-2 \leq \lambda_1 + \lambda_2 \leq 1$.

Rahman et al. (2018) developed a new generalized transmuted family of distributions; called *k*-transmuted families; which are defined as

$$F(x) = G(x) + [1 - G(x)] \sum_{i=1}^{k} \lambda_i [G(x)]^i, \quad (3)$$

with $\lambda_i \in [-1,1]$ for $i = 1,2,\ldots,k$ and $-k \leq \sum_{i=1}^{k} \lambda_i \leq 1$. The general transmuted family reduces to the base distribution for $\lambda_i = 0$ for $i = 1,2,\ldots,k$. The density function corresponding to Eq.(3) is

$$f(x) = g(x)\left[1 - \sum_{i=1}^{k} \lambda_i G^i(x)\left\{1 - \frac{i(1-G(x))}{G(x)}\right\}\right] \quad (4)$$

$G(x)$ and $g(x)$ is the cdf and pdf of the base distribution respectively.

In this paper, Quartic ranking transmutation map suggested by Rahman et al. (2018) is used to propose a new distribution that generalizes the Frechet distribution. This new version of the Frechet distribution is called Quartic Transmuted Frechet Distribution (QTFD). Some statistical properties are studied and the model parameters are estimated using the maximum likelihood method. Also, an application to two real data sets from rivers is illustrated and compared with the Frechet distribution (FD), transmuted Frechet distribution (TFD), cubic transmuted Frechet distribution (CTD) and quartic transmuted Frechet distribution (QTFD).

The rest of this paper is organized as follows: The new proposed distribution Quartic Transmuted Frechet (QTFD) is introduced in Section 2. We studied some statistical properties of QTFD such as reliability and hazard functions, moments, moment generating function, quantile function and random number generation in Section 3. Section 4 provides parameter estimation of the QTFD. An application of the QTFD to two uncensored data for the purpose of illustration is conducted in Section 5. Finally, Section 6 provides some concluding remarks.

2. QUARTIC TRANSMUTED FRECHET DISTRIBUTION

In this section, the new proposed distribution QTFD is discussed. This includes the cumulative distribution function (cdf), the probability density function (pdf), reliability and hazard function.

2.1 Cumulative and Density Functions for QTFD

In this section we have discussed the quartic transmuted family of distributions. The quartic transmuted map of distributions is obtained by setting $k = 3$ in Eq.(3) and is given as

$$F(x) = G(x) + \lambda_1 G(x)[1 - G(x)] + \lambda_2 G^2(x)[1 - G(x)] + \lambda_3 G^3(x)[1 - G(x)],$$

which can also be written as

$$F(x) = (1 + \lambda_1)G(x) + (\lambda_2 - \lambda_1)G^2(x) + (\lambda_3 - \lambda_2)G^3(x) - \lambda_3 G^4(x), x > 0 \tag{5}$$

where $\lambda_i \in [-1,1], i = 1,2,3$ and $-3 \leq \lambda_1 + \lambda_2 + \lambda_3 \leq 1$.

The probability density function of quartic transmuted map is

$$f(x) = g(x)[1 + \lambda_1 + 2(\lambda_2 - \lambda_1)G(x) + 3(\lambda_3 - \lambda_2)G^2(x) - 4\lambda_3 G^3(x)], x > 0, \tag{6}$$

where $g(x)$ is the pdf associated with baseline cdf $G(x)$.

Observe that for $\lambda_3 = 0$, the pdf of transmuted distribution reduces to the pdf of cubic transmuted distribution. And, observe that, at $\lambda_2 = \lambda_3 = 0$, reduces to the pdf of quadratic transmuted distribution. Also, observe that, at $\lambda_1 = \lambda_2 = \lambda_3 = 0$, we have the pdf of the baseline random variable, as it should be. The density function of quartic transmuted family of distributions given in Eq.(5) can be obtained by using $k = 3$ in the density function of general transmuted family of distributions given in Eq.(3).

The cdf and pdf of quartic transmuted Frechet are defined, respectively, as

$$F(x) = (1 + \lambda_1)e^{\left(-\left(\frac{x}{\mu}\right)^{-\sigma}\right)} + (\lambda_2 - \lambda_1)e^{\left(-2\left(\frac{x}{\mu}\right)^{-\sigma}\right)} + (\lambda_3 - \lambda_2)e^{\left(-3\left(\frac{x}{\mu}\right)^{-\sigma}\right)} - \lambda_3 e^{\left(-4\left(\frac{x}{\mu}\right)^{-\sigma}\right)} \tag{7}$$

which can also be written as

$$F(y) = (1 + \lambda_1)e^{-y} + (\lambda_2 - \lambda_1)e^{-2y} + (\lambda_3 - \lambda_2)e^{-3y} - \lambda_3 e^{-4y}, \quad \text{where } y = \left(\frac{x}{\mu}\right)^{-\sigma} \tag{8}$$

and the pdf

$$f(x) = \sigma\,\mu^{\sigma} x^{-(\sigma+1)} e^{\left(-\left(\frac{x}{\mu}\right)^{-\sigma}\right)} \left[(1 + \lambda_1) + 2(\lambda_2 - \lambda_1)e^{\left(-\left(\frac{x}{\mu}\right)^{-\sigma}\right)} + 3(\lambda_3 - \lambda_2)e^{\left(-2\left(\frac{x}{\mu}\right)^{-\sigma}\right)} - 4\lambda_3 e^{\left(-3\left(\frac{x}{\mu}\right)^{-\sigma}\right)}\right] \tag{9}$$

which can also be written *pdf* as

$$f(y) = \frac{\sigma}{\mu}\sqrt[\sigma]{y^{\sigma+1}}\, e^{-y}\left[1 + \lambda_1 + 2(\lambda_2 - \lambda_1)e^{-y} + 3(\lambda_3 - \lambda_2)e^{-2y} - 4\lambda_3 e^{-3y}\right] \tag{10}$$

where $\sigma, \mu, x > 0,\ \lambda_i \in [-1,1], i = 1,2,3$ and $-3 \leq \lambda_1 + \lambda_2 + \lambda_3 \leq 1$.

Now, by using Eq.10 can be expressed as

$$\int_{-\infty}^{\infty} f(x)dx = \int_{\infty}^{0} -\frac{\mu}{\sigma}(y^{\sigma+1})^{\frac{-1}{\sigma}}\frac{\sigma}{\mu}\sqrt[\sigma]{y^{\sigma+1}}\, e^{-y}\,[1 + \lambda_1 + 2(\lambda_2 - \lambda_1)e^{-y} + 3(\lambda_3 - \lambda_2)e^{-2y} - 4\lambda_3 e^{-3y}]dy$$

$$= \int_{0}^{\infty} e^{-y}\left[(1 + \lambda_1) + 2(\lambda_2 - \lambda_1)e^{-y} + 3(\lambda_3 - \lambda_2)e^{-2y} - 4\lambda_3 e^{-3y}\right] dy$$

$$= (1 + \lambda_1) + (\lambda_2 - \lambda_1) + (\lambda_3 - \lambda_2) - \lambda_3$$

$$= 1.$$

Figure 1 and Figure 2: illustrates some of possible shapes of the pdf and cdf for QTFD for selected values of λ_1, λ_2 and λ_3 where $\mu = 2$ and $\sigma = 3$.

From plot of pdf of Figure 1, the plots show that the QTF density can be skewed to the right, unimodal with varying degree of kurtosis depending on the values of the parameters.

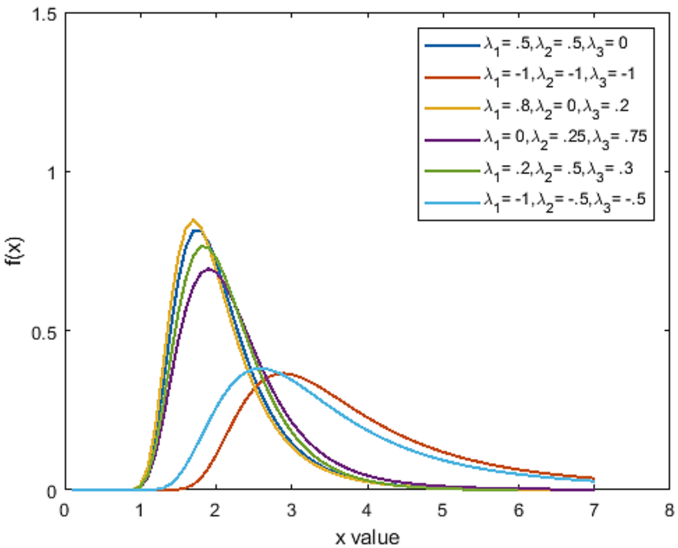

Figure 1: The pdf of QTFD for different Value of λ_1, λ_2 and λ_3 where $\mu = 2$ and $\sigma = 3$

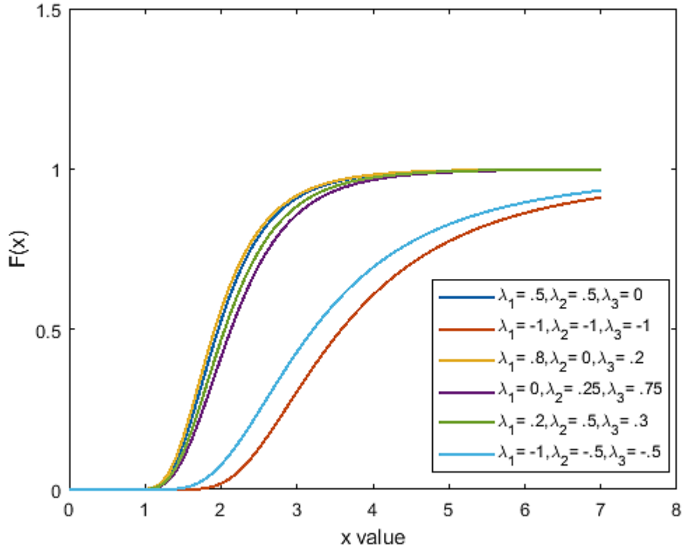

Figure 2: The cdf of QTFD for different Value of λ_1, λ_2 and λ_3 where $\mu = 2$ and $\sigma = 3$

2.2 Reliability and Hazard Function

The reliability function is defined as $R(x)$ = 1- $F(x)$ and for the QTFD is given as

$$R(y) = 1 - e^{-y}\left[(1+\lambda_1) + (\lambda_2 - \lambda_1)e^{-y} + (\lambda_3 - \lambda_2)e^{-2y} - \lambda_3 e^{-3y}\right]$$

The hazard function is defined as $h(x) = f(x)/R(x)$ and for the QTFD is given as

$$h(y) = \frac{\frac{\sigma}{\mu}\sqrt[\sigma]{y^{\sigma+1}}\, e^{-y}\left[(1+\lambda_1) + 2(\lambda_2 - \lambda_1)e^{-y} + 3(\lambda_3 - \lambda_2)e^{-2y} - 4\lambda_3 e^{-3y}\right]}{e^{y} - \left[(1+\lambda_1) + (\lambda_2 - \lambda_1)e^{-y} + (\lambda_3 - \lambda_2)e^{-2y} - \lambda_3 e^{-3y}\right]}$$

Figure 3 and Figure 4 illustrate a few potential shapes of the reliability and hazard functions for the QTFD using different combination of model parameters λ_1, λ_2 and λ_3 where $\mu = 2$ and $\sigma = 3$. The diagrams in Figures 3 and 4 indicate that the hazard rate function of the QTF distribution could be shaped as unimodal, bathtub, or upside-down bathtub. These properties suggest that the QTF distribution is suitable for modelling data sets with nonmonotonic hazard rate behaviour which are mostly encountered in practical situations.

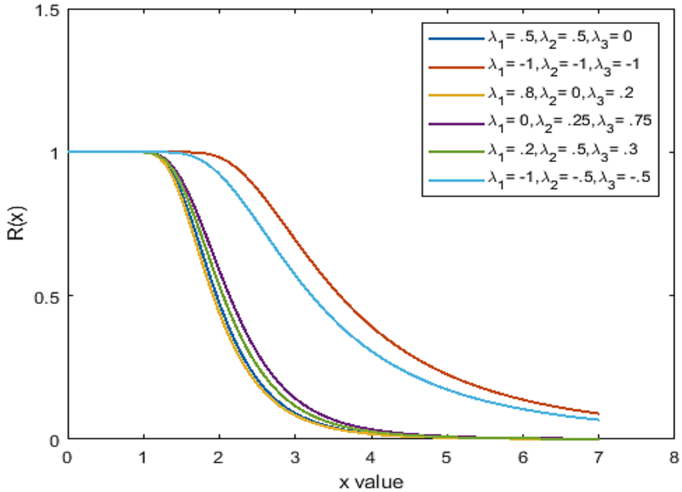

Figure 3: The Reliability Functions of QTFD for different Value of λ_1, λ_2 and λ_3 where $\mu = 2$ and $\sigma = 3$

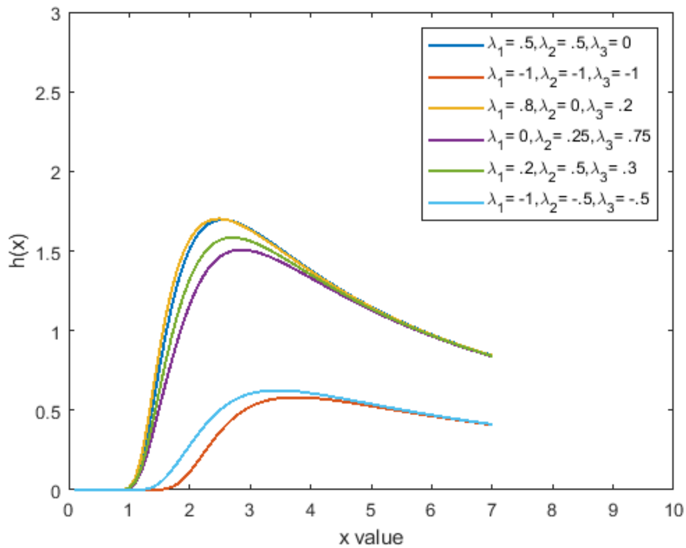

Figure 4: The Hazard Functions of QTFD for different value of λ_1, λ_2 and λ_3 where $\mu = 2$ and $\sigma = 3$

3. STATISTICAL PROPERTIES

In this section, some statistical properties for the proposed distribution, QTFD is demonstrated. These properties involve moments, moment generating function, quantile function and simulation the random sample.

3.1 The Moments

Theorem 3.1

Let X be a random variable has the QTFD, then the r[th] moment of X is given by

$$E(X^r) = \mu^r \Gamma\left(1 - \frac{r}{\sigma}\right)\left[(1 + \lambda_1) + \sqrt[\sigma]{2^r}\,(\lambda_2 - \lambda_1) + \sqrt[\sigma]{3^r}(\lambda_3 - \lambda_2) - \sqrt[\sigma]{4^r}\,\lambda_3\right], r < \sigma \quad (11)$$

$r = 0,1,2, \ldots$, where $\Gamma(1 - r/\sigma)$ is the gamma function.

Proof:

The r[th] moment is given by

$$E(X^r) = \int_0^\infty x^r f(x)dx$$

$$= \int_0^\infty x^r\ \sigma\mu^\sigma x^{-(\sigma+1)} e^{\left(-\left(\frac{x}{\mu}\right)^{-\sigma}\right)}\left[(1 + \lambda_1) + 2(\lambda_2 - \lambda_1)e^{\left(-\left(\frac{x}{\mu}\right)^{-\sigma}\right)} + 3(\lambda_3 - \lambda_2)e^{\left(-2\left(\frac{x}{\mu}\right)^{-\sigma}\right)} - 4\lambda_3 e^{\left(-3\left(\frac{x}{\mu}\right)^{-\sigma}\right)}\right] dx$$

Using the transformation $y = \left(\frac{\mu}{x}\right)^{\sigma}$, the expression reduces to

$$E(X^r) = \int_0^\infty \left(\frac{\mu}{\sqrt[\sigma]{y}}\right)^r e^{-y}\left[(1+\lambda_1) + 2(\lambda_2 - \lambda_1)e^{-y} + 3(\lambda_3 - \lambda_2)e^{-2y} - 4\lambda_3 e^{-3y}\right]dy$$

$$= \mu^r \int_0^\infty y^{-\frac{r}{\sigma}} e^{-y}\left[(1+\lambda_1) + 2(\lambda_2 - \lambda_1)e^{-y} + 3(\lambda_3 - \lambda_2)e^{-2y} - 4\lambda_3 e^{-3y}\right]dy$$

$$= \mu^r\left[(1+\lambda_1)\int_0^\infty y^{-\frac{r}{\sigma}} e^{-y}\,dy + 2(\lambda_2 - \lambda_1)\int_0^\infty y^{-\frac{r}{\sigma}} e^{-2y}\,dy + 3(\lambda_3 - \lambda_2)\int_0^\infty y^{-\frac{r}{\sigma}} e^{-3y}\,dy - 4\lambda_3\int_0^\infty y^{-\frac{r}{\sigma}} e^{-4y}\,dy\right]$$

Using the relation $\int_0^\infty t^b\, e^{-at}dt = \frac{\Gamma\,(1+b)}{a^{(1+b)}}$ we get

$$E(X^r) = \mu^r\,\Gamma\,(1 - r/\sigma)\left[(1 + \lambda_1) + \sqrt[\sigma]{2^r}\,(\lambda_2 - \lambda_1) + \sqrt[\sigma]{3^r}(\lambda_3 - \lambda_2) - \sqrt[\sigma]{4^r}\,\lambda_3\right]$$

Therefore, the theorem is proved.

By putting $r = 1,2$ respectively in Eq.(11), the mean and variance are given as follows:

$$E(X) = \mu\,\Gamma\left(1 - \frac{1}{\sigma}\right)\left[(1+\lambda_1) + \sqrt[\sigma]{2}\,(\lambda_2 - \lambda_1) + \sqrt[\sigma]{3}(\lambda_3 - \lambda_2) - \sqrt[\sigma]{4}\,\lambda_3\right], \sigma > 1$$

$$Var\,[X] = \mu^2\,\Gamma\left(1-\frac{2}{\sigma}\right)\left[(1+\lambda_1)+\sqrt[\sigma]{2^2}\,(\lambda_2-\lambda_1)+\sqrt[\sigma]{3^2}(\lambda_3-\lambda_2)-\sqrt[\sigma]{4^2}\,\lambda_3\right]$$
$$-\left\{\mu\,\Gamma\left(1-\frac{1}{\sigma}\right)\,[(1+\lambda_1)+\sqrt[\sigma]{2}\,(\lambda_2-\lambda_1)+\sqrt[\sigma]{3}(\lambda_3-\lambda_2)-\sqrt[\sigma]{4}\,\lambda_3]\right\}^2,\sigma>2.$$

Tables 1 provide the mean and variance of QTFD for various combinations of model parameters where M is the mean and Var is the variance.

Table 1
Mean and Variance of the QTFD for Various Combinations of the Parameters

Parameters		$\lambda_1=-1$, $\lambda_2=.5$, $\lambda_3=-.5$		$\lambda_1=-0.5$, $\lambda_2=0$, $\lambda_3=.5$		$\lambda_1=0$, $\lambda_2=-1$, $\lambda_3=1$	
		M	**Var**	**M**	**Var**	**M**	**Var**
$\sigma=3$	$\mu=1$	1.6809	1.3563	1.4318	0.8266	1.4045	0.8481
	$\mu=2$	3.3618	5.4249	2.8636	3.3066	2.8089	3.3929
	$\mu=4$	6.7236	21.6997	5.7273	13.2252	5.6178	13.5718
$\sigma=4$	$\mu=1$	1.4397	0.3897	1.2812	0.2606	1.2606	0.2718
	$\mu=2$	2.8793	1.5593	2.5624	1.0424	2.5212	1.0871
	$\mu=4$	5.7587	6.2361	5.1249	4.1686	5.0425	4.3472
$\sigma=6$	$\mu=1$	1.2560	0.1034	1.1646	0.0755	1.1508	0.0802
	$\mu=2$	2.5120	0.4135	2.3292	0.3021	2.3016	0.3204
	$\mu=4$	5.0241	1.6530	4.6585	1.2076	4.6032	1.2819

Table 1 (contd…)

Parameters		$\lambda_1=.2$, $\lambda_2=-.5$, $\lambda_3=0.2$		$\lambda_1=.5$, $\lambda_2=-.5$, $\lambda_3=0$		$\lambda_1=1$, $\lambda_2=1$, $\lambda_3=-1$	
		M	**Var**	**M**	**Var**	**M**	**Var**
$\sigma=3$	$\mu=1$	1.3679	0.9173	1.3016	0.8579	0.9518	0.0576
	$\mu=2$	2.7357	3.6700	2.6032	3.4316	1.9036	0.2305
	$\mu=4$	5.4714	14.6803	5.2063	13.7275	3.8072	0.9219
$\sigma=4$	$\mu=1$	1.2327	0.2928	1.1872	0.2776	0.9584	0.0314
	$\mu=2$	2.4655	1.1705	2.3744	1.1104	1.9167	0.1257
	$\mu=4$	4.9309	4.6832	4.7489	4.4406	3.8334	0.5027
$\sigma=6$	$\mu=1$	1.1321	0.0862	1.1040	0.0828	0.9686	0.0136
	$\mu=2$	2.2642	0.3448	2.2079	0.3315	1.9371	0.0548
	$\mu=4$	4.5285	1.3785	4.4158	1.3260	3.8742	0.2196

From Table 1 it is observed that, holding the shape and scale parameters σ, μ constants, as the transmuted parameters λ_1 and λ_2 increase and λ_3 decrease the mean of QTFD decrease. Whilst, for the scale parameter μ increase holding other parameters constants, the mean and variance increase. When shape parameters σ increase and other parameters constants the mean and variance decrease.

3.2 The Moment Generating Function

Theorem 3.2

Let X be a random variable has the QTFD, then the (MGF) of X is

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu^r \, \Gamma\left(1 - \frac{r}{\sigma}\right) \Big[(1 + \lambda_1) + \sqrt[\sigma]{2^r}\,(\lambda_2 - \lambda_1) + \sqrt[\sigma]{3^r}(\lambda_3 - \lambda_2) - \sqrt[\sigma]{4^r}\,\lambda_3\Big].$$

Proof:

We know that

$$M_X(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x) dx$$

Using series expansion of e^{tx},

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^{\infty} x^r f(x) dx = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(x^r)$$

Then

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu^r \, \Gamma\left(1 - \frac{r}{\sigma}\right) \Big[(1 + \lambda_1) + \sqrt[\sigma]{2^r}\,(\lambda_2 - \lambda_1) + \sqrt[\sigma]{3^r}(\lambda_3 - \lambda_2) - \sqrt[\sigma]{4^r}\,\lambda_3\Big].$$

3.3 Quantile Function

The quantile function theorem of the QTFD is stated as follow;

Theorem 3.3

Let X be random variable from the quartic rank transmuted Frechet probability distribution with parameters $\sigma > 0$, $\mu > 0, -3 \leq \lambda_1 + \lambda_2 + \lambda_3 \leq 1$. Then the quantile function of X, is given by

$$x_q = \mu \left(-\log\big(\mathrm{B}(q, \lambda_1, \lambda_2, \lambda_3)\big)\right)^{-\frac{1}{\sigma}} \tag{12}$$

Proof:

To compute the quantile function of the quartic rank transmuted Frechet probability distribution, we substitute x by x_q and $F(x)$ by q in Eq.(7) to get the equation

$$q = (1 + \lambda_1) e^{\left(-\left(\frac{x_q}{\mu}\right)^{-\sigma}\right)} + (\lambda_2 - \lambda_1) e^{\left(-2\left(\frac{x_q}{\mu}\right)^{-\sigma}\right)} + (\lambda_3 - \lambda_2) e^{\left(-3\left(\frac{x_q}{\mu}\right)^{-\sigma}\right)} - \lambda_3 e^{\left(-4\left(\frac{x_q}{\mu}\right)^{-\sigma}\right)} \tag{13}$$

Then, solve for x_q in Eq.(13) So, let $y = \mathrm{e}^{\left(-\left(\frac{x_q}{\mu}\right)^{-\sigma}\right)}$. Thus, Eq.13 becomes

$$q = (1+\lambda_1)\,\mathrm{y} + (\lambda_2 - \lambda_1)\,\mathrm{y}^2 + (\lambda_3 - \lambda_2)\,\mathrm{y}^3 - \lambda_3 y^4$$

and hence,

$$\lambda_3 y^4 + (\lambda_2 - \lambda_3)\,\mathrm{y}^3 + (\lambda_1 - \lambda_2)\,\mathrm{y}^2 + (-1-\lambda_1)\,\mathrm{y} + q = 0 \qquad (14)$$

Let $a_4 = \lambda_3$, $a_3 = (\lambda_2 - \lambda_3), a_2 = (\lambda_1 - \lambda_2), a_1 = (-1-\lambda_1)$ and $a_0 = q$ then the Eq.14 becomes $a_4 y^4 + a_3\mathrm{y}^3 + a_2\mathrm{y}^2 + a_1\mathrm{y} + a_0 = 0$, then Eq.14 solve from Nwosu et al. (2021).

$$y_{k,j} = \frac{(-1)^k v^2 + (-1)^j\sqrt{v^4 - 2v\left(v(v^2 + {p^*}_{4,2}) + (-1)^k {p^*}_{4,1}\right)}}{2v} - \frac{a_3}{4a_4}, \quad k = 1,2;\ j = 1,2$$

where

$${p^*}_{n,k} = \sum_{j=0}^{n-k} \frac{(-1)^j}{a_n}\binom{k+j}{j} a_{k+j}\left(\frac{a_{n-1}}{na_n}\right)^j,\ v^2 = z$$

and

$$z = \left(-\frac{{p^*}_{3,0}}{2} + (-1)^j\sqrt{\left(\frac{{p^*}_{3,0}}{2}\right)^2 + \left(\frac{{p^*}_{3,1}}{3}\right)^3}\right)^{\frac{1}{3}} - \frac{{p^*}_{3,1}}{2}\left(-\frac{{p^*}_{3,0}}{2} + (-1)^j\sqrt{\left(\frac{{p^*}_{3,0}}{2}\right)^2 + \left(\frac{{p^*}_{3,1}}{3}\right)^3}\right)^{-\frac{1}{3}}$$

Let $\mathrm{B}(\mathrm{q},\lambda_1,\lambda_2,\lambda_3)$ be defined by

$$\mathrm{B}(\mathrm{q},\lambda_1,\lambda_2,\lambda_3) = \frac{(-1)^k v^2 + (-1)^j\sqrt{v^4 - 2v\left(v(v^2 + {p^*}_{4,2}) + (-1)^k {p^*}_{4,1}\right)}}{2v} - \frac{a_3}{4a_4}$$

Hence,

$$y = \mathrm{e}^{\left(-\left(\frac{x_q}{\mu}\right)^{-\sigma}\right)} = \mathrm{B}(\mathrm{q},\lambda_1,\lambda_2,\lambda_3)$$

Take natural Logarithm to both sides to get $-\left(\frac{x_q}{\mu}\right)^{-\sigma} = log\mathrm{B}(\mathrm{q},\lambda_1,\lambda_2,\lambda_3)$

Then, we have the equation

$$x_q = \frac{\mu}{\sqrt[\sigma]{-\log\left(\mathrm{B}(\mathrm{q},\lambda_1,\lambda_2,\lambda_3)\right)}}.$$

4. PARAMETERS ESTIMATION

The discussion of maximum likelihood estimate (MLE) for QTFD parameter is covered in this section. Let $X_1, X_2, \dots, X_n$ be a random sample of size n from QTFD. Then the likelihood function is given by

$$L(\mu, \sigma, \lambda_1, \lambda_2, \lambda_3) = \prod_{i=1}^{n} f(x_i)$$

$$= \prod_{i=1}^{n} \frac{\sigma}{\mu} \sqrt[\sigma]{y_i^{\sigma+1}}\, e^{-y_i} \left[1 + \lambda_1 + 2(\lambda_2 - \lambda_1)e^{-y_i} + 3(\lambda_3 - \lambda_2)e^{-2y_i} - 4\lambda_3 e^{-3y_i} \right]$$

$$= \left(\frac{\sigma}{\mu}\right)^n e^{-(\sum_{i=1}^{n} y_i)} \sqrt[\sigma]{\prod_{i=1}^{n} y_i^{\sigma+1}} \prod_{i=1}^{n} \left[1 + \lambda_1 + 2(\lambda_2 - \lambda_1)e^{-y_i} + 3(\lambda_3 - \lambda_2)e^{-2y_i} - 4\lambda_3 e^{-3y_i} \right]$$

Then, the Log likelihood function of a vector of parameters given as,

$$l = lnL = nln\frac{\sigma}{\mu} - \sum_{i=1}^{n} [y_i - \frac{\sigma+1}{\sigma} ln y_i - ln[1 + \lambda_1 + 2(\lambda_2 - \lambda_1)e^{-y_i} + 3(\lambda_3 - \lambda_2)e^{-2y_i} - 4\lambda_3 e^{-3y_i}] \quad (15)$$

Differentiate Eq.(15) in relation to the unknown parameters μ, σ, λ_1, λ_2 and λ_3 we have

$$\frac{\partial l}{\partial \mu} = \frac{\sigma n}{\mu} - \sum_{i=1}^{n} y_i + \frac{2(\lambda_2 - \lambda_1)e^{-y_i} + 6(\lambda_3 - \lambda_2)e^{-2y_i} - 12\lambda_3 e^{-3y_i}}{1 + \lambda_1 + 2(\lambda_2 - \lambda_1)e^{-y_i} + 3(\lambda_3 - \lambda_2)e^{-2y_i} - 4\lambda_3 e^{-3y_i}} \quad (16)$$

$$\frac{\partial l}{\partial \sigma} = 2n\frac{1+\sigma}{\sigma^2} - \frac{1}{\sigma}\sum_{i=1}^{n} \frac{y_i(2(\lambda_2 - \lambda_1)e^{-y_i} + 6(\lambda_3 - \lambda_2)e^{-2y_i} - 12\lambda_3 e^{-3y_i})}{1 + \lambda_1 + 2(\lambda_2 - \lambda_1)e^{-y_i} + 3(\lambda_3 - \lambda_2)e^{-2y_i} - 4\lambda_3 e^{-3y_i}} \quad (17)$$

$$\frac{\partial l}{\partial \lambda_1} = \sum_{i=1}^{n} \frac{1 - 2e^{-y_i}}{1 + \lambda_1 + 2(\lambda_2 - \lambda_1)e^{-y_i} + 3(\lambda_3 - \lambda_2)e^{-2y_i} - 4\lambda_3 e^{-3y_i}} \quad (18)$$

$$\frac{\partial l}{\partial \lambda_2} = \sum_{i=1}^{n} \frac{2e^{-y_i} - 3e^{-2y_i}}{1 + \lambda_1 + 2(\lambda_2 - \lambda_1)e^{-y_i} + 3(\lambda_3 - \lambda_2)e^{-2y_i} - 4\lambda_3 e^{-3y_i}} \quad (19)$$

$$\frac{\partial l}{\partial \lambda_3} = \sum_{i=1}^{n} \frac{3e^{-2y_i} - 4e^{-3y_i}}{1 + \lambda_1 + 2(\lambda_2 - \lambda_1)e^{-y_i} + 3(\lambda_3 - \lambda_2)e^{-2y_i} - 4\lambda_3 e^{-3y_i}} \quad (20)$$

Putting Eq.(16), Eq.(17), Eq.(18), Eq.(19), Eq.(20) equals zero while concurrently resolving them. In order to acquire the numerical solution to the nonlinear equations, statistical software can be used. We can compute the maximum likelihood estimators (MLEs) of parameters (μ, σ, λ_1, λ_2 and λ_3) using quasi-Newton procedure, or computer packages/ soft wares such as R, SAS, Ox, MATLAB and MATHEMATICA.

5. APPLICATION OF QTFD

In this section the (QTF) distribution applied to two data as follows**.** The first data in Table 2 is represents the remission times (in months) of 128 bladder cancer Patients was introduced by Okasha et al. (2021). The second data in Table 3 the total milk production proportion in the first birth of 107 cows living in the Carnauba farm in Brazil given by Saraçoğlu and Tanış (2018).

Table 2
The Remission Times (in months) of 128 Bladder Cancer Patients

0.08	0.20	0.40	0. 0.50	0.51	0.81	0.90	1.05	1.19	1.26
1.35	1.40	1.46	1.76	2.02	2.02	2.07	2.09	2.23	2.26
2.46	2.54	2.62	2.64	2.69	2.69	2.75	2.83	2.87	3.02
3.70	3.82	3.25	3.31	3.36	3.36	3.48	3.52	3.57	3.64
4.51	4.87	3.88	4.18	4.23	4.26	4.33	4.34	4.40	4.50
5.41	5.49	4.98	5.06	5.09	5.17	5.32	5.32	5.34	5.41
6.97	7.09	5.62	5.71	5.85	6.25	6.54	6.76	6.93	6.94
7.87	7.93	7.26	7.28	7.32	7.39	7.59	7.62	7.63	7.66
9.74	10.06	8.26	9.74	10.06	8.26	9.74	10.06	8.26	9.74
12.03	12.07	10.34	10.66	10.75	11.25	11.64	11.79	11.98	12.02
12.63	13.11	13.29	13.80	14.24	14.76	14.77	14.83	15.96	16.62
17.12	17.14	17.36	18.10	19.13	20.28	21.73	22.69	23.63	25.74
25.82	26.31	32.15	34.26	36.66	43.01	46.12	79.05		

Table 3
Milk Production Data

0.4365	0.4260	0.5140	0.6907	0.7471	0.2605	0.6196	0.8781	0.4990	0.6058	0.6891
0.5770	0.5394	0.1479	0.2356	0.6012	0.1525	0.5483	0.6927	0.7261	0.3323	0.0671
0.2361	0.4800	0.5707	0.7131	0.5853	0.6768	0.5350	0.4151	0.6789	0.4576	0.3259
0.2303	0.7687	0.4371	0.3383	0.6114	0.3480	0.4564	0.7804	0.3406	0.4823	0.5912
0.5744	0.5481	0.1131	0.7290	0.0168	0.5529	0.4530	0.3891	0.4752	0.3134	0.3175
0.5285	0.5232	0.6465	0.0650	0.8492	0.8147	0.3627	0.3906	0.4438	0.4612	0.3188
0.2160	0.6707	0.6220	0.5629	0.4675	0.6844	0.3413	0.4332	0.0854	0.3821	0.4694
0.3635	0.4111	0.5349	0.3751	0.1546	0.4517	0.2681	0.4049	0.5553	0.5878	0.4741
0.1167	0.6750	0.5113	0.5447	0.4143	0.5627	0.5150	0.0776	0.3945	0.4553	0.4470
0.3598	0.7629	0.5941	0.6174	0.6860	0.0609	0.6488	0.2747			

Summary statistics of the two data sets is demonstrated in Table 4. In order to compare and utilize these data sets, four alternative distributions: (QTFD) model, the Frechet distribution (FD), transmuted Frechet distribution (TFD), Cubic Transmuted Frechet (CTFD) have been compared. Tables 5 and 7 show the estimated values of the model parameters along with the corresponding standard errors for a few models that were chosen using the MLE method. In Tables 6 and 8, the goodness of fit of the (QTFD) model, (CTFD), (TFD) and (FD) has been introduced using different comparison measures we consider some criteria like (-2$\mathfrak{L}(\theta)$): where is $\mathfrak{L}(\theta)$ the maximum value of log-likelihood function, AIC (Akaike Information Criterion), BIC (Bayesian Information Criterion), and HQIC (Hanan-Quinn information criterion), for the data set. In general the better fit of the distribution corresponds to the smaller value of the statistics (-2$\mathfrak{L}(\theta)$), AIC, CAIC, BIC, and HQIC, by using the following relations:

$$AIC = -2\mathfrak{L}(\theta) + 2K,\ BIC = -2\mathfrak{L}(\theta) + Klog(n)$$

and $HQIC = -2\mathfrak{L}(\theta) + 2Klog(log(n))$

where K is the number of parameters estimated and n is the sample size. Furthermore, the Kolmogorov-Smirnov $(K - S)$ statistics and associated p-values were obtained along with the Cramer-von Mises (w*) and Andersen-Darling (A*) statistics. A very good fit of the model to the data is shown by reduced values for all three goodness-of-fit indicators. Large p-values also indicate a good fit for the model, which is another benefit.

Plots of the empirical and theoretical cdfs and pdfs for fitted distributions are given in Figure 5, and Figure 6. These Figures shows that: the curve of the pdf and cdf QTFD is closer to the curve of the sample of data than the curve of the pdf and cdf of CTFD, TFD and FD. So, the QTFD is a superior model than one based on the CTFD, TFD and FD.

Table 4
Descriptive Statistics of Data Set 1, 2

Data	Mean	Median	Skewness	Kurtosis
Set 1	9.366	72	3.326	16.154
Set 2	0.46885	0.4741	-0.34008	2.81599

Table 5
MLE's of the Parameters and respective SE's for Various Distributions for Data Set 1

Distribution	Parameter	Estimate	SE
QTFD	μ	8.87222	1.07551 i
	σ	0.561994	0.0294738
	λ_1	0.999999	0.70089
	λ_2	0.999998	2.33143
	λ_3	-0.999997	2.18569
CTFD	μ	4.62696	0.547042 i
	σ	0.650616	0.0287267
	λ_1	$1.55917*10^{-6}$	0.110986
	λ_2	0.999998	0.577835
TFD	μ	7.51061	1.14396 i
	σ	0.599552	0.00649901
	λ	1	0.321967 i
FD	μ	3.25822	0.407438
	σ	0.75208	0.0424239

Table 6
Goodness-of Fit Statistics using the Selection Criteria Values for Data Set 1

Model	$-2\mathfrak{L}(\theta)$	AIC	BIC	HQIC	W*	A*	K-S	P-value
QTFD	849.708	859.709	873.97	865.50	0.3153	2.0347	0.10092	0.1474
CTFD	856.31	864.31	875.72	868.95	0.3798	2.4243	0.10753	0.1036
TFD	862.382	868.382	876.94	871.86	0.6017	3.7276	0.3368	$4.89*10^{-13}$
FD	888.002	892.002	897.71	894.32	0.7443	4.5464	0.1408	0.0125

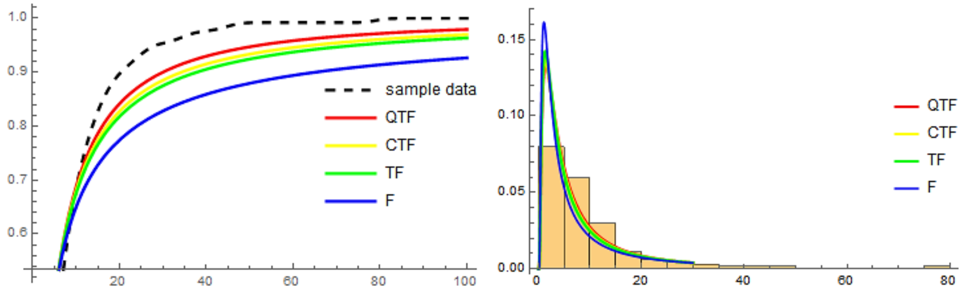

Figure 5: Fitted cdf Plots and the Empirical Distribution and Fitted pdf for of the Data Set 1

Table 7
MLE's of the Parameters and respective SE's
for Various Distributions for Data Set 2

Distribution	Parameter	Estimate	SE
QTFD	μ	0.549065	0.0168784 i
	σ	0.82464	0.0425318
	λ_1	1	0.708479
	λ_2	1	2.04513
	λ_3	-1	1.94668
CTFD	μ	0.355729	0.0255606 i
	σ	0.934335	0.0445717
	λ_1	$3.13137*10^{-8}$	0.170492 i
	λ_2	1	0.508379
TFD	μ	0.500906	0.0538594 i
	σ	0.85529	0.0326237
	λ	1	0.405734 i
FD	μ	0.279539	0.0278422
	σ	1.03497	0.0605856

Table 8
Goodness-of Fit Statistics using the Selection Criteria Values for Data Set 2

Model	$-2\mathfrak{L}(\theta)$	AIC	BIC	HQIC	W*	A*	K-S	P-value
QTFD	51.499	61.499	74.86	66.92	1.63562	8.96831	0.22516	$3.88*10^{-5}$
CTFD	60.529	68.526	79.22	72.86	1.74329	9.47750	0.22732	$3.15*10^{-5}$
TFD	68.206	74.206	82.22	77.46	2.08476	11.1197	0.46757	$2.2*10^{-16}$
FD	97.448	101.448	106.79	103.62	2.23607	11.8209	0.26414	$6.55*10^{-7}$

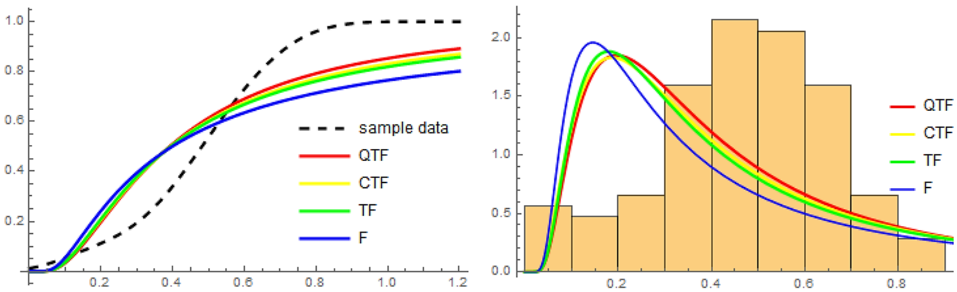

Figure 6: Fitted cdf Plots and the Empirical Distribution
and Fitted pdf for of the Data Set 2

In Table 6 and 8, we compare the QTF model with the CTF, TF and F distributions. It is noted that the proposed model has the lowest values for the AIC, BIC, HQIC, W*, A* and K-S tests statistics among all fitted models, as well as the highest p value. So, the QTFD can be chosen as the best model among the competing distributions studied in this article. Figure 5 and 6 indicate that the QTF distribution provides a better fit than others models considered for both data.

6. CONCLUDING REMARKS

In this article, a new generalization of the Frechet distribution called the Quartic Transmuted Frechet (QTFD) distribution is suggested. Furthermore, some properties of the QTFD including survival and hazard functions, mean, variance, quantile function and random number generation are derived. The estimation of the distribution parameters is performed using maximum Likelihood method. In order to test a goodness of fit for QTFD, the distribution is fitted to a two uncensored data and compared with some related distributions. It is observed that QTFD works better than these distributions.

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