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# TOPP-LEONE INVERSE GOMPERTZ DISTRIBUTION: PROPERTIES AND DIFFERENT ESTIMATIONS TECHNIQUES AND APPLICATIONS

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## ABSTRACT

In this work, we proposed a new probability distribution based on the extension of the Gompertz distribution for modeling life time datasets known as the “Topp-Leone Inverse Gompertz Distribution (TLIG distribution)”. The TLIG distribution is derived using the logit of Topp-Leone generator and the Inverse Gompertz distribution (IGD) as the baseline distribution. Properties of TLIG distribution were examined. Five different estimation techniques namely maximum likelihood estimate (MLE), Weighted Least Squares Estimates (WLS), Ordinary Least Squares Estimates (OLS), Crammer-Von Miss Estimate (CVME) and Percentile Estimate (PCE) were considered to estimate the parameters of TLIG distribution. A Monte Carlo simulation technique was adopted to assess the consistency and efficiency of these parameter estimates. The usefulness of this new distribution is demonstrated with two real-life datasets whose results shows that the new TLIG distribution performs better than some familiar existing distribution.

## KEY WORDS

Inverse Gompertz, Topp-Leone distribution, Least Squares Estimates, Crammer-Von Miss Estimate, Monte Carlo simulation, Odd function

## 1. INTRODUCTION

The Gompertz distribution (GD) with two parameters was proposed by Gompertz (1825) and it has an application in modelling behavioral sciences and human mortality datasets. GD is a generalization of an exponential probability distribution (ED) and has a wide range of applications in actuarial studies and medical. The GD distribution has some relationship with some probability distributions such as double exponential distributions, Gumbel, Weibull, generalized logistic and exponential Willekens (2001). GD has an increasing failure rate, which reduces the capability and flexibility of the GD to model some phenomena in different areas. Authors such as Rasool et al., (2017), Read (1983), El-Gohary et al. (2015), Franses (1994), Khan et al., (2016), Makany (1991), El-Bassiouny et al., (2016), El-Bassiouny et al. (2017), Sanku et al., (2018), wu and Lee (1999) and Jafar et al., (2014) have tried to improve the properties of GD in other to improve the applicability of the model in different areas such as engineering, economics, medical sciences, actuarial, behavioral sciences, lifetime analysis, environmental or biological studies.

Consider a random variables (rvs) Y drawn from a GD with parameters $\lambda$ and $\beta$. The cumulative distribution function (CDF) is expressed as

$$G(y;\beta,\lambda) = 1 - e^{-\frac{\lambda}{\beta}(e^{\beta x}-1)}\ \lambda > 0, \beta > 0, y \geq 0 \tag{1.1}$$

and the probability density function (PDF) is expressed as

$$g(y;\beta,\lambda) = \alpha e^{\left(\beta x - \frac{\lambda}{\beta}(e^{\beta x}-1)\right)}\ \lambda > 0, \beta > 0, y \geq 0 \tag{1.2}$$

where $\lambda$ is a shape parameter and $\beta$ is a scale parameter

In recent times, researchers have shifted their attention to develop more flexible models using several techniques. One of the widely used techniques is known as Inverse Distribution (ID). Let $X$ be defined as non-negative rvs, then the probability distribution of a random variable $x = \frac{1}{y}$ is defined as ID. The ID have been widely adopted by several authors in modeling real-life probles in demography, biological sciences, actuarial studies, behavioral sciences etc. (See, Jiang et al., (1999), Drapella (1993), El-Morshedy et al., (2017), El-Gohary et al., (2015), Mudholkar and Kollia (1994), etc.). Eliwa et al., (2019), proposed a new distribution called inverse Gompertz distribution (IGD). Suppose $X$ is a rvs with parameters λ and β. The CDF of the ID is expressed as

$$G(x;\beta,\lambda) = e^{-\frac{\lambda}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}\ \lambda > 0, \beta > 0, x \geq 0 \tag{1.3}$$

while the PDF is expressed as

$$g(x;\beta,\lambda) = \frac{\lambda}{x^2} e^{-\frac{\lambda}{\beta}\left(e^{\frac{\beta}{x}}-1\right)+\frac{\beta}{x}}\ \lambda > 0, \beta > 0, x \geq 0 \tag{1.4}$$

where $\lambda$ and $\beta$ are shape and scale parameters.

Topp and Leone (1955) developed a new distribution known as Topp-Leone (the J-Shaped) distribution (TLD). This distribution have been widely used to model several real-life. Nadarajah and Kotz (2003) examined and reveal the importance of the TLD in analysing interval bounded data. In their study they discovered that the TLD exhibits bathtub failure rate functions and derived the closed form moment of the distribution, which disclosed many areas of its applicablity in reliability study. TLD have been studied by authors like Kotz and Seier (2007), Zhou et al., (2006), Ghitany et al., (2005), Nadarajah (2009), Zghoul (2011), amongst others. This work aimed in proposing a three-parameter distribution named Topp-Leone Inverse Gompertz Distribution (TLIG distribution) and examine some of its features. We employed parametric estimations such as using MLE, PCE, WLS, OLS and CVME and compare their estimates using simulations techniques. We analyzed two data set using the proposed distribution and observed that the proposed distribution provides a good fit for the considered datasets compared with three well-known distributions.

The remaining of this work is outlined as follows. In Section 2, we discussed the the theoretical framework of TLIG distribution which contains some of its properties. In Section 3, we discuss different estimation techniques for obtaining the estimate of the parameters of TLIG distribution. In Section 4, we presented the simulation studies of TLIG distribution and its application to two real-life datasets. Finally, in Section 5, discussion and conclusion based on the results obtained from the adopted estimators.

## 2. MATERIALS AND METHODS

### 2.1 The Topp-Leone Inverse Gompertz Distribution

Al-Shomrani et al., (2016) and Sangsanit and Bodhisuwan (2016) developed the Topp-Leone generated family of distribution stated as

$$F(x) = \left(1 - \overline{G}(x)^2\right)^{\alpha} \tag{2.1}$$

with the PDF known as

$$f(x) = 2\alpha g(x;\alpha)\overline{G}(x)\left(1 - \overline{G}(x)^2\right)^{\alpha-1} \tag{2.2}$$

where $\overline{G}(x) = 1 - G(x)$

where $G(x)$ is the CDF and $g(x)$ is the PDF of IGD stated in (1.3) and (1.4) respectively. The CDF of the proposed TLIG can be obtained by substituting (1.3) into (2.1)

$$F(x;\alpha,\beta,\gamma) = \left\{1 - \left(1 - e^{-\frac{\lambda}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}\right)^2\right\}^{\alpha} \quad \alpha > 0, \beta > 0, \lambda > 0, x > 0 \tag{2.3}$$

The expansion of the CDF can be expressed as

$$\sum_{j=0}^{\infty}\sum_{n=0}^{2i} \frac{(-1)^{i+m}\Gamma(\alpha+1)}{j!\,\Gamma(\alpha+1-j)}\binom{2i}{m} e^{-\frac{\lambda m}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}$$

The PDF of the TLIG distribution is expressed by substituting (1.3) and (1.4) into (2.2)

$$f(x;\alpha,\beta,\lambda) = 2\alpha\lambda x^{-2} e^{\frac{\beta}{x}} e^{-\frac{\lambda}{\beta}\left(e^{\frac{\beta}{x}}-1\right)} \left(1 - e^{-\frac{\lambda}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}\right)\left\{1 - \left(1 - e^{-\frac{\lambda}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}\right)^2\right\}^{\alpha-1}$$
$$\alpha > 0, \beta > 0, \lambda > 0, x > 0. \tag{2.4}$$

The expansion for the PDF can be expressed as

$$\sum_{j=0}^{\infty}\sum_{n=0}^{2i+n}\frac{(-1)^{i+m}2\Gamma(\alpha+1)}{j!\,\Gamma(\alpha+1-j)}\binom{2i-1}{n}2\alpha\lambda x^{-2}e^{\frac{\beta}{x}}e^{-\frac{\lambda}{\beta}\left(e^{\frac{\beta}{x}}-1\right)[1+m]}$$

The PDF and CDF Plots of TLIG distribution is shown in Figures 1 and 2 respectively. From Figure 1, we observed that the distribution is right skwed which means that the model can be used to model right skewed datasets. Also, from Figure 2, we can observation that the CDF falls within 0 and 1 which satisfy the probability condition.

## 3. STATISTICAL PROPERTIES

In this section, we looked at some of the properties of $TLIG(\alpha,\beta,\lambda)$ such s reversed hazard function, hazard function, survival function, the odds function and so on.

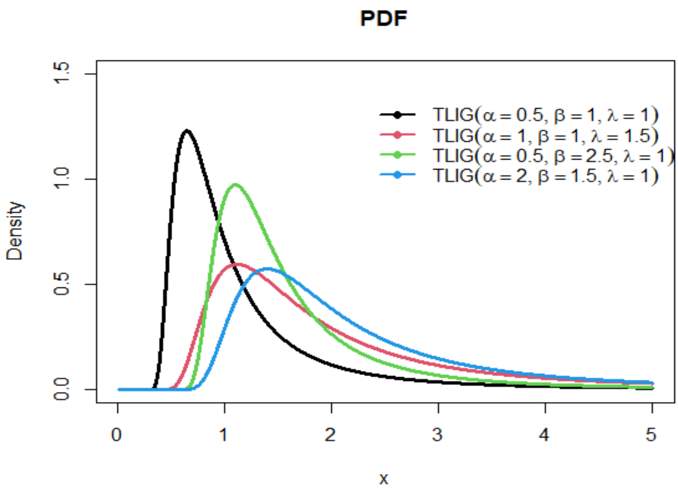


**Figure 1: PDF Plot of TLIG**

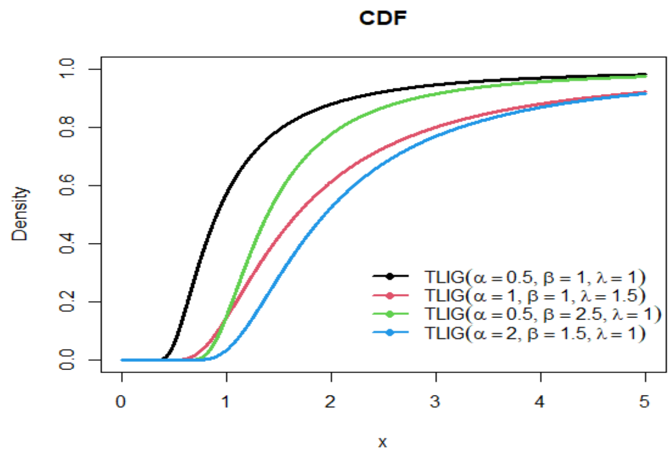


**Figure 2: CDF Plot of TLIG**

**3.1 Survival Function**

The mathematical expression for obtaining the survival function is stated as

$$S(x) = 1 - F(x;\alpha,\beta,\lambda) \tag{3.1}$$

Therefore the survival function of TLIG distribution is derived by substituting (2.3) into (3.1) which resulted to

$$S(x) = 1 - \left\{1 - \left(1 - e^{-\frac{\lambda}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}\right)^2\right\}^{\alpha} \quad \alpha > 0, \beta > 0, \lambda > 0, x > 0 \tag{3.2}$$

The survival plots at different parameter values for the proposed distribution is given in Figure 3.

**3.2 Hazard Function**

The mathematical expression for obtaining the hazard function is stated as

$$h(x) = \frac{f(x)}{1 - F(x)} \tag{3.3}$$

The hazard function of TLIG distribution is derived by substituting (2.3) and (2.4) into (3.3) and it resulted into

$$h(x) = \frac{2\alpha\lambda x^{-2} e^{\frac{\beta}{x}} e^{-\frac{\lambda}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}\left(1 - e^{-\frac{\lambda}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}\right)\left\{1 - \left(1 - e^{-\frac{\lambda}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}\right)^2\right\}^{\alpha-1}}{1 - \left\{1 - \left(1 - e^{-\frac{\lambda}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}\right)^2\right\}^{\alpha}}$$
$$\alpha > 0, \beta > 0, \lambda > 0, x > 0. \tag{3.4}$$

The hazard plots at different parameter values for the proposed distribution is given in Figure 4.

**3.3 Reversed Hazard Function**

The mathematical expression for obtaining the reversed hazard function is stated as

$$\tau(x) = \frac{f(x)}{F(x)} \tag{3.5}$$

Hence, we obtain the reversed hazard function by substituting (2.3) and (2.4) in (3.5)

$$\tau(x) = \frac{2\alpha\lambda x^{-2} e^{\frac{\beta}{x}} e^{-\frac{\lambda}{\beta}\left(e^{\frac{\beta}{x}}-1\right)} \left(1 - e^{-\frac{\lambda}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}\right) \left\{1 - \left(1 - e^{-\frac{\lambda}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}\right)^2\right\}^{\alpha-1}}{\left\{1 - \left(1 - e^{-\frac{\lambda}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}\right)^2\right\}^{\alpha}}$$

$$\alpha > 0, \beta > 0, \lambda > 0, x > 0 \qquad (3.6)$$

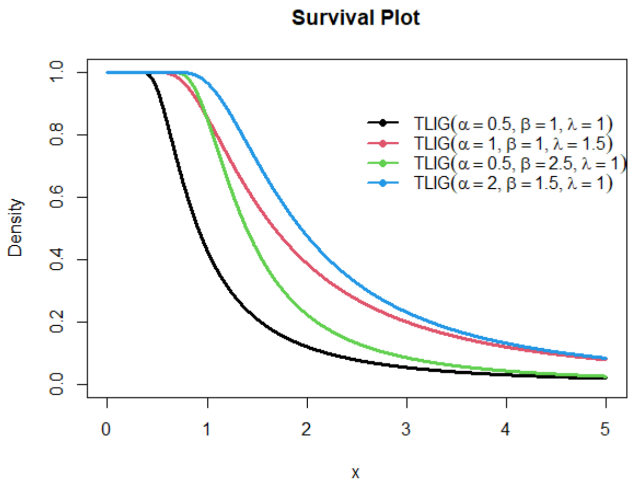


**Figure 3: Survival for TLIG Distribution**

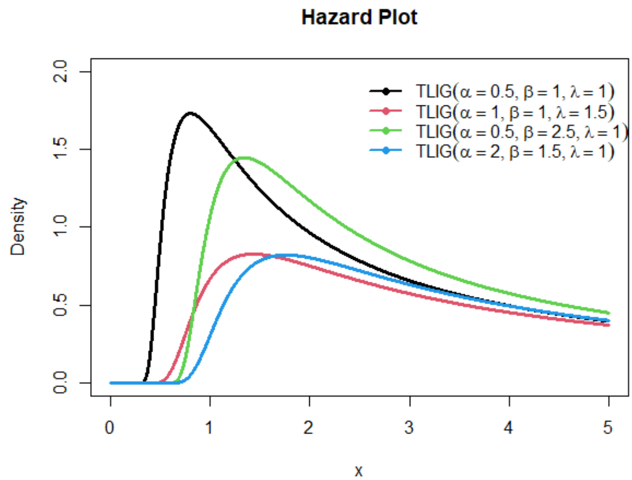


**Figure 4: Hazard Plot for TLIG distribution**

**3.4 Odd Function**

The odd function of any distribution is stated as

$$O(x) = \frac{F(x)}{S(x)} \tag{3.7}$$

The odd function for TLIG distribution is obtained by substituting (2.3) and (2.4) into (3.7)

$$O(x) = \frac{\left\{1 - \left(1 - e^{-\frac{\lambda}{\beta}\left(e^{\frac{\beta}{x}} - 1\right)}\right)^2\right\}^\alpha}{1 - \left\{1 - \left(1 - e^{-\frac{\lambda}{\beta}\left(e^{\frac{\beta}{x}} - 1\right)}\right)^2\right\}^\alpha} \quad \alpha > 0, \beta > 0, \lambda > 0, x > 0 \tag{3.8}$$

**3.5 Cumulative Hazard Function**

The mathematical expression for obtaining the cummulative hazard function is stated as

$$H(x) = -log(S(x)) \tag{3.9}$$

The cumulative hazard function for TLIG distribution is obtained by substituting (2.3) and (2.4) into (3.9)

$$H(x) = -log\left(1 - \left\{1 - \left(1 - e^{-\frac{\lambda}{\beta}\left(e^{\frac{\beta}{x}} - 1\right)}\right)^2\right\}^\alpha\right) \quad \alpha > 0, \beta > 0, \lambda > 0, x > 0 \tag{3.10}$$

**3.6 The Quantile, Skewness, Median, Kurtosis and Mode**

The quantile function of a distribution can be obtained using the mathematical expression

$$Q(u) = F^{-1}(u) \tag{3.11}$$

Therefore, the quantile function of the TLIG distribution can be expressed as

$$Q(u) = \frac{\beta}{\log\left(1 - \frac{\beta}{\lambda}\log\left(1 - \sqrt{1 - u^{1/\alpha}}\right)\right)} \tag{3.12}$$

where, u is uniformly distributed with the range (0,1). When u = 0.5 in (3.12), then the median of the TLIG distribution can be expressed as

$$Med(x) = \frac{\beta}{\log\left(1 - \frac{\beta}{\lambda}\log\left(1 - \sqrt{1 - 0.5^{1/\alpha}}\right)\right)} \tag{3.13}$$

Kenney and Keeping (1962) and Moors (1988) developed the skewness (Sk) and kurtosis (Ku) of any given distribution as

$$Sk = \frac{Q\left(\frac{3}{4}\right) + Q\left(\frac{1}{4}\right) - 2Q\left(\frac{1}{2}\right)}{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right)} \tag{3.14}$$

and

$$Ku = \frac{Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right) + Q\left(\frac{3}{8}\right) - Q\left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{2}{8}\right)} \tag{3.15}$$

One of the properties about Sk and Ku is that both properties are less affected by outliers and can be easily adopted for any distributions whose moments does not exist.

### 3.7 Moments

The moment can be used to examine many properties of a distribution such as dispersion, tendency, kurtosis and skewness,. If $X \sim TLIGD(\alpha, \beta, \lambda)$, then the $r^{th}$ moment expression of TLIG distribution can be expressed as

$$\mu_r' = \int_0^\infty x^r f(x) dx$$

So, therefore the r$^{th}$ moment of the TLIG distribution is stated as

$$\mu_r' = \int_0^\infty x^r 2\alpha\lambda x^{-2} e^{\frac{\beta}{x}} e^{-\frac{\lambda}{\beta}\left(e^{\frac{\beta}{x}}-1\right)} \left(1 - e^{-\frac{\lambda}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}\right) \left\{1 - \left(1 - e^{-\frac{\lambda}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}\right)^2\right\}^{\alpha-1} dx \tag{3.16}$$

Using the binomial and exponential expansions, Eqn (3.16) can be expressed as

$$\mu_r' = \sum_{i,j,k,l=0}^{\infty} (-1)^{i+j+k+l} \binom{\alpha-1}{i} \binom{2i-1}{j} \binom{k}{l} \frac{2\alpha\lambda^{k+1}(j+1)^k \beta^{r-k-1}(l-1)^r}{k!\,\Gamma(r-1)} \tag{3.17}$$

### 3.8 Mode

The mode of the TLIG distribution can be derived by differentiating the logarithm of the PDF (2.4), $x_0$, with respect to $x$, and equating the expression to zero. After some manipulations, the mode is derived by solving the non-linear equation below:

$$x_0 = -\frac{2n}{x} - \frac{\beta n}{x^2} + \frac{\lambda n \mathrm{e}^{\frac{\beta}{x}}}{x^2} - \frac{n\lambda \mathrm{e}^{\frac{\beta}{x}} \mathrm{e}^{-\frac{\lambda\left(\mathrm{e}^{\frac{\beta}{x}}-1\right)}{\beta}}}{x^2\left(1-\mathrm{e}^{-\frac{\lambda\left(\mathrm{e}^{\frac{\beta}{x}}-1\right)}{\beta}}\right)} + \frac{2(\alpha-1)n\left(1-\mathrm{e}^{-\frac{\lambda\left(\mathrm{e}^{\frac{\beta}{x}}-1\right)}{\beta}}\right)\lambda \mathrm{e}^{\frac{\beta}{x}} \mathrm{e}^{-\frac{\lambda\left(\mathrm{e}^{\frac{\beta}{x}}-1\right)}{\beta}}}{x^2\left(1-\left(1-\mathrm{e}^{-\frac{\lambda\left(\mathrm{e}^{\frac{\beta}{x}}-1\right)}{\beta}}\right)^2\right)} \tag{3.18}$$

The expression (3.18) cannot be obtained analytically, hence the mode can be obtained numerically.

### 3.9 Distribution of Order Statistics of TLIG Distribution

Given that $X_{1,}X_2, X_3, \cdots, X_n$ is a random sample from the Topp-Leone Inverse Gompertz (TLIG) distribution with the corresponding order statistics $X_{1:n}, X_{2:n}$, $X_{3:n}, \cdots, X_{n:n}$, then the probability density function of the $r^{th}$ order statistics $X = X_{r:n}$, $r = 1,2,3, \ldots, n$ denoted by $f_{r:n,}(x)$, for $1 \leq x \leq n$ is defined as

$$f_{r:n,}(x) = \frac{n!}{(r-1)!\,(n-r)!} f(x) F^{r-1}(x)[1-F(x)]^{n-r} \tag{3.19}$$

where $f(x)$ and $F(x)$ are respectively the pdf and the cdf of the random variable $X$. We derive the density function of the order statistics for Topp-Leone Inverse Gompertz (TLIG) distribution as

$$f_{r:n}(x) = \frac{2\alpha\lambda n!}{(r-1)!\,(n-r)!} x^{-2} e^{\frac{\beta}{x}} e^{\frac{\lambda}{\beta}\left(1-e^{\frac{\beta}{x}}\right)}\left(1-e^{\frac{\lambda}{\beta}\left(1-e^{\frac{\beta}{x}}\right)}\right) \left[1-\left(1-e^{\frac{\lambda}{\beta}\left(1-e^{\frac{\beta}{x}}\right)}\right)^2\right]^{\alpha-1}\left[1-\left(1-e^{\frac{\lambda}{\beta}\left(1-e^{\frac{\beta}{x}}\right)}\right)^2\right]^{\alpha(r-1)} \times\left\{1-\left[1-\left(1-e^{\frac{\lambda}{\beta}\left(1-e^{\frac{\beta}{x}}\right)}\right)^2\right]^{\alpha}\right\}^{n-r}$$

$$f_{p:n}(x)=\frac{2\alpha\lambda n!}{(r-1)!\,(n-r)!}x^{-2}e^{\frac{\beta}{x}}e^{\frac{\lambda}{\beta}\left(1-e^{\frac{\beta}{x}}\right)}\left(1-e^{\frac{\lambda}{\beta}\left(1-e^{\frac{\beta}{x}}\right)}\right)$$

$$\left[1-\left(1-e^{\frac{\lambda}{\beta}\left(1-e^{\frac{\beta}{x}}\right)}\right)^{2}\right]^{\alpha r-1}\left\{1-\left[1-\left(1-e^{\frac{\lambda}{\beta}\left(1-e^{\frac{\beta}{x}}\right)}\right)^{2}\right]^{\alpha}\right\}^{n-r}$$

$$=\frac{2\alpha\lambda n!}{(r-1)!\,(n-r)!}\Psi_{(i,j,k,m,p,q)}x^{-q-2} \tag{3.20}$$

where

$$\Psi_{(i,j,k,m,p,q)}=\frac{2\alpha\lambda n!}{(r-1)!\,(n-r)!}\sum_{i,j,k,m,p,q=0}\binom{n-r}{i}\binom{\alpha r+i-1}{j}$$

$$\binom{2j+1}{k}\binom{m}{p}(-1)^{i+j+k+p}\frac{\lambda^{m}\beta^{q-m}(k+1)^{m}(p+1)^{q}}{m!\,q!}x^{-q-2}$$

## 4. DIFFERENT ESTIMATION METHODS

### 4.1 Maximum Likelihood Estimation (MLE)

Let $x_1, x_2, \ldots x_n$ be random variables drawn from $TLIGD(\alpha,\beta,\lambda)$. The log-likelihood function of the distribution can be stated as

$$\log L(\alpha,\beta,\lambda|x)=n\log(2\alpha\lambda)-2\log(x)+\beta\sum_{i=1}^{n}\frac{1}{x}-\frac{\lambda}{\beta}\sum_{i=1}^{n}\left(e^{\frac{\beta}{x}}-1\right)$$

$$+\sum_{i=1}^{n}\left(1-e^{\left(e^{\frac{\beta}{x}}-1\right)}\right)+(\alpha-1)\sum_{i=1}^{n}\log\left(1-\left(1-e^{-\frac{\lambda}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}\right)^{2}\right) \tag{4.1}$$

To maximizing the log-likelihood function $logL(\alpha,\beta,\lambda)$, we differentiate eqn (4.1) w.r.t $\alpha$, $\beta$ and $\lambda$ which resulted to

$$\frac{\partial l}{\partial\alpha}=\frac{n}{\alpha}+\sum_{i=0}^{n}\log(\varpi) \tag{4.2}$$

$$\frac{\partial l}{\partial\beta}=\sum_{i=0}^{n}\frac{1}{x}+\frac{\lambda}{\beta^{2}}\sum_{i=0}^{n}\gamma-\frac{\lambda}{\beta}\sum_{i=0}^{n}\frac{e^{\frac{\beta}{x}}}{x}-\sum_{i=0}^{n}\left(\frac{\lambda}{\beta^{2}}\gamma-\frac{\lambda}{\beta}\frac{e^{\frac{\beta}{x}}}{x}\right)\xi$$

$$+2(\alpha-1)\sum_{i=1}^{n}\frac{\varphi}{\varpi}\left(\frac{\lambda}{\beta^{2}}\gamma-\frac{\lambda}{\beta x}e^{\frac{\beta}{x}}\right)\xi \tag{4.3}$$

$$\frac{\partial l}{\partial\lambda}=\frac{n}{\lambda}-\frac{1}{\beta}\sum_{i=1}^{n}\gamma+\frac{1}{\beta}\sum_{i=1}^{n}\gamma\xi-\frac{2(\alpha-1)}{\beta}\sum_{i=1}^{n}\frac{\varphi\gamma\xi}{\beta\varpi} \tag{4.4}$$

where

$$\varpi = 1-\left(1-e^{-\frac{\lambda}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}\right)^2, \ \ \varphi = 1-e^{-\frac{\lambda}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}, \ \ \xi = e^{-\frac{\lambda}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}$$

and $\gamma = e^{\frac{\beta}{x}}-1$

To obtain the MLE estimates of parameters $\hat{\alpha}_{MLE}$, $\hat{\beta}_{MLE}$ and $\hat{\lambda}_{MLE}$ we equate the expression (4.2), (4.3) and (4.4) to zero and solve the system of nonlinear equation. It was deduced that the solution of these system of equations can not be obtained analytically, so we adopted a numerical techniques which is known as Newton Raphson techniques. Many authors have adopted this approximation techniques for obtaining the estimate of parameters for some lifetime distributions; see among others, Obisesan et al. (2015), Adegoke et al. (2018), Sule et al. (2020), Sule et al. (2021), Dibal et al. (2019), Adegoke et al. (2019); Ogunsanya et al. (2021).

The asymptotic distribution for the TLIG distribution contain $3 \times 3$ element observed information matrix which is stated as

$$\sqrt{n}(\hat{\theta}-\theta) \sim N_3(0, \Lambda^{-1})$$

where $\Lambda$ is the expected information matrix. Thus, the expected information matrix is expressed as

$$\Lambda^{-1} = -E\begin{bmatrix} \frac{\partial^2 l}{\partial\alpha^2} & \frac{\partial^2 l}{\partial\alpha\partial\beta} & \frac{\partial^2 l}{\partial\alpha\partial\lambda} \\ \frac{\partial^2 l}{\partial\alpha\partial\beta} & \frac{\partial^2 l}{\partial\beta^2} & \frac{\partial^2 l}{\partial\lambda\partial\beta} \\ \frac{\partial^2 l}{\partial\alpha\partial\lambda} & \frac{\partial^2 l}{\partial\lambda\partial\beta} & \frac{\partial^2 l}{\partial\lambda^2} \end{bmatrix}$$

$$\frac{\partial^2 l}{\partial\alpha^2} = -\frac{n}{\alpha^2}, \frac{\partial^2 l}{\partial\alpha\partial\beta} = \sum\left(\frac{2\varphi\xi\rho}{\varpi}\right), \frac{\partial^2 l}{\partial\alpha\partial\lambda} = -\sum\left(\frac{2\varphi\gamma\xi}{\beta\varpi}\right),$$

$$\frac{\partial^2 l}{\partial\beta^2} = -\sum\left(2\frac{\lambda}{\beta^3}\gamma\right) + \sum\left(\frac{2\lambda e^{\frac{\beta}{x}}}{\beta^2 x}\right) - \sum\left(\frac{\lambda e^{\frac{\beta}{x}}}{\beta x}\right) - \sum\left(\frac{\xi\mu}{\varphi}\right) - \sum\left(\frac{\xi\rho^2}{\varphi}\right)$$
$$-\sum\left(\frac{\xi^2\rho^2}{\varphi^2}\right) - \sum\left(\frac{2(\alpha-1)\xi^2\rho^2}{\varpi}\right) + \sum\left(\frac{2(\alpha-1)\varphi\gamma^2\xi}{\beta^2\varpi}\right)$$

$$\frac{\partial^2 l}{\partial\lambda\partial\beta} = \sum\left(\frac{n\gamma}{\beta^2}\right) - \sum\left(\frac{e^{\frac{\beta}{x}}}{\beta x}\right) - \sum\left(\frac{\xi\psi}{\varphi}\right)\sum\left(\frac{\gamma\xi\rho}{\beta\varphi}\right) + \sum\left(\frac{\gamma\xi\rho}{\beta\varphi^2}\right)$$
$$-\sum\left(\frac{2(\alpha-1)\gamma\xi^2\rho}{\beta\varpi}\right) + \sum\left(\frac{2(\alpha-1)\varphi\xi\psi}{\varpi}\right)$$
$$-\sum\left(\frac{2(\alpha-1)\varphi\gamma\xi\rho}{\beta\varpi}\right) + \sum\left(\frac{4(\alpha-1)\varphi^2\gamma\xi\rho}{\beta\varpi^2}\right)$$

$$\frac{\partial^2 l}{\partial \lambda^2} = -\frac{n}{\lambda} - \sum\left(\frac{\gamma^2\xi}{\beta^2\varphi}\right) - \sum\left(\frac{\gamma^2\xi^2}{\beta^2\varphi^2}\right) - \sum\left(\frac{2(\alpha-1)\gamma^2\xi^2}{\beta^2\varpi}\right) + \sum\left(\frac{2(\alpha-1)\varphi\gamma^2\xi}{\beta^2\varpi}\right) - \sum\left(\frac{4(\alpha-1)\varphi^2\xi^2}{\beta^2\varpi^2}\right)$$

where

$$\mu = -\frac{2\lambda\left(e^{\frac{\beta}{x}}-1\right)}{\beta^3} + \frac{2\lambda e^{\frac{\beta}{x}}}{\beta^2 x} - \frac{\lambda e^{\frac{\beta}{x}}}{\beta x^2}, \rho = \frac{\lambda\left(e^{\frac{\beta}{x}}-1\right)}{\beta^2} - \frac{\lambda e^{\frac{\beta}{x}}}{\beta x}, \psi = \frac{e^{\frac{\beta}{x}}-1}{\beta^2} - \frac{e^{\frac{\beta}{x}}}{\beta x}$$

**4.2 Least Squares Method (LSM)**

This techniques was developed by Swain et al. (1988) to obtain the estimates of Beta distribution parametrs. Supposed $F(X_{(i)})$ represents the order statistics (OS) distribution function from a random sample $x = (x_1, \dots, x_n)$. It was discovered that $F(X_{(i)}) \sim Beta(i, n-i+1)$. So, therefore the mean and variance can be expressed as

$$E(F(X_{(i)})) = \frac{i}{n+1}\ V(F(X_{(i)})) = \frac{i(n-i+1)}{(n+1)^2(n+2)}$$

with these expectations (means) and variances, we obtain two variants of the LSM.

**4.2.1 Ordinary Least Squares (OLS)**

Supposed $x = (x_1, \dots, x_n)$ is a random samples generated from TLIG distribution in increasing order. Let the related OS be $X_{1:n}, X_{2:n}, \dots, X_{n:n}$. Then, the OLS estimates of the parameters $\hat{\alpha}_{OLS}$, $\hat{\beta}_{OLS}$ and $\hat{\lambda}_{OLS}$ can be gotten by differentiating the eqn. (4.5) with respect to $\alpha$, $\beta$ and $\lambda$ and equate to system of equations to zero.

$$S(\alpha,\beta,\lambda|x) = \sum_{i=1}^{n}\left[F(x_{(i)}|\alpha,\beta,\lambda) - \frac{i}{n+1}\right]^2 \tag{4.5}$$

$$S(\alpha,\beta,\lambda|x) = \sum_{i=1}^{n}\left[\left\{1-\left(1-e^{-\frac{\lambda}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}\right)^2\right\}^{\alpha} - \frac{i}{n+1}\right]^2 \tag{4.6}$$

**4.2.2 Weighted Least Squares (WLS)**

Supposed $x = (x_1, \dots, x_n)$ is a random samples generated from TLIG distribution in increasing order. Let the related OS be $X_{1:n}, X_{2:n}, \dots, X_{n:n}$. The WLS estimates of $\hat{\alpha}_{WLS}$, $\hat{\beta}_{WLS}$ and $\hat{\lambda}_{WLS}$ can be obtained by minimizing the eqn. (4.7) (i.e. differentiating the eqn. (4.7) with respect to $\alpha$, $\beta$ and $\lambda$ and equate to system of equations to zero)

$$W(\alpha,\beta,\lambda|x) = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(x_{(i)}|\alpha,\beta,\lambda) - \frac{i}{n+1}\right]^2 \tag{4.7}$$

$$W(\alpha,\beta,\lambda|x)=\frac{(n+1)^2(n+2)}{i(n-i+1)}\sum_{i=1}^{n}\left[\left\{1-\left(1-e^{-\frac{\lambda}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}\right)^2\right\}^{\alpha}-\frac{i}{n+1}\right]^2 \tag{4.8}$$

**4.3 Crammer-von Mises (CVM)**

Supposed $x=(x_1,\ldots,x_n)$ is a random samples generated from TLIG distribution in increasing order. Let the related OS be $X_{1:n},X_{2:n},\ldots,X_{n:n}$. Then, the CVM function is given by

$$C(\alpha,\beta,\lambda|x)=\frac{1}{12n}+\left[F(x_{(i)}|\alpha,\beta,\lambda)-\frac{2i-1}{2n}\right]^2 \tag{4.9}$$

Then, the CVM estimates of the parameters $\hat{\alpha}_{CVM}$, $\hat{\beta}_{CVM}$ and $\hat{\lambda}_{CVM}$ can be gotten by differentiating the eqn. (4.9) with respect to $\alpha$, $\beta$ and $\lambda$ and equate to system of equations to zero.

$$C(\alpha,\beta,\lambda)=\frac{1}{12n}+\left[\left\{1-\left(1-e^{-\frac{\lambda}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}\right)^2\right\}^{\alpha}-\frac{2i-1}{2n}\right]^2 \tag{4.10}$$

**4.5 Method of Percentiles**

The percentile techniques was proposed by Kao(1958), Kao(1959). Let $p_i=\frac{i}{n+1}$ be an estimate of the CDF $F(x_i;\alpha,\beta,\lambda)$, then the percentile estimate for $\hat{\alpha}_{pce}$, $\hat{\beta}_{pce}$ and $\hat{\lambda}_{pce}$ can be gotten by differentiating eqn (4.11) with respect to $\alpha,\beta$ and $\lambda$ and solving numerically since the equations are intractable.

$$Z=\sum_{i=1}^{n}\left(x_i-\frac{\beta}{\log\left(1-\frac{\beta}{\lambda}\log\left(1-\sqrt{1-p^{1/\alpha}}\right)\right)}\right)^2. \tag{4.11}$$

## 5. SIMULATION AND DATA ANALYSIS

**5.1 Simulation**

This section, we focused on Monte Carlo simulation in order to study the behaviour of the estimated values for the five different techniques (MLEs, LSEs, WLSEs, CVMEs and Percentiles) with respect to different sample sizes $n$ and parameter values. We adopted R software to generate samples for different sample sizes having different values of parameter. 1000 samples were drawn for different sample size $n=20$, 50, 100, and 200 from $TLIGD(\alpha,\beta,\lambda)=[(1.5,1.0,1.0),(0.5,1.0,0.5),(1.5,1.0,1.5)]$ respectively. The average biases (BIAS) and mean-squared errors (MSEs) were computed using

$$Bias = \frac{1}{1000}\sum_{i=1}^{1000}\left(\frac{\widehat{\Theta}-\Theta}{\Theta}\right) \tag{5.1}$$

$$MSE = \frac{1}{1000}\sum_{i=1}^{1000}\left(\widehat{\Theta}-\Theta\right)^2 \tag{5.2}$$

Table 1 shows the result obtained from the simulated values which includes the Bias and the Mean Square errors (MSE) for the parameters of the TLIG distribution using five (5) different estimation methods. From the above results, we deduced that for all estimation techniques and parameter values, as $n$ increases both the Bias and MSE decreases which confirms their consistency. It was also discovered that in most cases, the MLE estimates is said to be the best estimator as it contains smallest values of the Bias and MSE, while the percentile estimator tends to be the weakest estimator since it has the largest values of Bias and MSE. In this study Fitdristrplus and MaxLik packages in R programming were adopted to perform the analysis.

### 5.2 Data Analysis

This section focused on application of TLIG distribution on two real life datasets to illustrate that the TLIG distribution can be a good lifetime model. We compared TLIG distribution with seven probability distributions such as the with inverse Gompertz distribution (IGD), exponential distribution (EXP), alpha power ILo (APILo), inverse Lomax (ILo), inverse exponential (IEXP), alpha power inverted Topp Leone (APITL) and inverted Topp Leone (ITL). The fitted models were compared using different measures such as the Hannan-Quinn information criterion (HQIC), Akaike information criterion (AIC), corrected AIC (CAIC) and Kolmogorov-Smirnov (KS) statistic along with their $P$-value for all fitted models for two datasets.

**Table 1**
**The Bias Values of the Estimates and their MSE**

| n | Parameters | MLE | | OLS | | WLS | | CRA | | PER | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Bias | MSE | Bias | MSE | Bias | MSE | Bias | MSE | Bias | MSE |
| 20 | $\alpha = 1.5$ | 0.14385 | 0.02069 | 0.25784 | 0.06648 | 0.21459 | 0.04605 | 0.16961 | 0.02877 | -2.26540 | 1.60130 |
| | $\beta = 1.0$ | -0.23050 | 0.05313 | -0.55609 | 0.30924 | -0.32352 | 0.10466 | -0.20370 | 0.41492 | 0.62490 | 0.39050 |
| | $\lambda = 1.0$ | -0.23713 | 0.05623 | 0.09118 | 0.00831 | 0.08929 | 0.00797 | -0.39028 | 0.15232 | 1.28860 | 1.66060 |
| | $\alpha = 0.5$ | 0.04366 | 0.00191 | 0.12539 | 0.01572 | 0.06814 | 0.00464 | 0.05661 | 0.00320 | -0.36610 | 0.13400 |
| | $\beta = 1.0$ | -0.10510 | 0.01104 | -0.15068 | 0.02276 | -0.11891 | 0.01414 | -0.10884 | 0.01185 | 0.22600 | 0.05110 |
| | $\lambda = 0.5$ | -0.11360 | 0.01291 | -0.14764 | 0.02180 | -0.06385 | 0.00407 | -0.15448 | 0.02386 | 0.34440 | 0.11860 |
| | $\alpha = 1.5$ | 0.13849 | 0.01918 | 0.34630 | 0.11992 | 0.24992 | 0.06246 | 0.17179 | 0.29511 | -2.21180 | 1.46840 |
| | $\beta = 1.0$ | -0.34095 | 0.11025 | -0.83027 | 0.68934 | -0.48515 | 0.23537 | -0.33228 | 0.11041 | 0.69840 | 0.48770 |
| | $\lambda = 1.5$ | -0.24448 | 0.05977 | 0.06162 | 0.00380 | 0.05204 | 0.00271 | -0.36124 | 0.13050 | 1.63105 | 2.66034 |
| 50 | $\alpha = 1.5$ | 0.07816 | 0.00611 | 0.14456 | 0.02090 | 0.13287 | 0.01766 | 0.15201 | 0.02311 | -1.05419 | 1.11132 |
| | $\beta = 1.0$ | -0.10111 | 0.01022 | -0.35519 | 0.12616 | -0.25251 | 0.06376 | 0.05650 | 0.00319 | 0.20710 | 0.04289 |
| | $\lambda = 1.0$ | -0.12496 | 0.01562 | 0.05284 | 0.00279 | -0.06086 | 0.00370 | -0.31136 | 0.09695 | 1.12210 | 1.25916 |
| | $\alpha = 0.5$ | -0.00689 | 0.00472 | 0.08430 | 0.00711 | 0.05131 | 0.00263 | 0.04699 | 0.00221 | -0.19479 | 0.03794 |
| | $\beta = 1.0$ | -0.09759 | 0.00953 | -0.11610 | 0.01348 | -0.08594 | 0.00739 | -0.10669 | 0.01138 | 0.08942 | 0.00634 |
| | $\lambda = 0.5$ | -0.01179 | 0.00014 | -0.06435 | 0.00414 | 0.04992 | 0.00249 | -0.11521 | 0.01327 | 0.22704 | 0.05155 |
| | $\alpha = 1.5$ | 0.06278 | 0.00394 | 0.22307 | 0.04976 | 0.16472 | 0.02713 | 0.12565 | 0.01579 | -1.03709 | 1.07555 |
| | $\beta = 1.0$ | -0.23400 | 0.05474 | -0.42500 | 0.18062 | -0.26419 | 0.06980 | -0.27994 | 0.07837 | 0.24473 | 0.05999 |
| | $\lambda = 1.5$ | -0.23275 | 0.05417 | 0.04174 | 0.00174 | -0.04188 | 0.00175 | -0.18143 | 0.03292 | 1.57260 | 2.47300 |

| n | Parameters | MLE | | OLS | | WLS | | CRA | | PER | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Bias | MSE | Bias | MSE | Bias | MSE | Bias | MSE | Bias | MSE |
| 100 | α = 1.5 | 0.02905 | 0.00084 | 0.13424 | 0.01802 | 0.12588 | 0.01585 | 0.28905 | 0.08355 | -0.64560 | 0.41680 |
| | β =1.0 | -0.15831 | 0.02506 | -0.25314 | 0.06408 | -0.13305 | 0.01770 | 0.05974 | 0.00357 | 0.17260 | 0.02980 |
| | λ =1.0 | 0.02216 | 0.00049 | 0.02162 | 0.00047 | 0.01878 | 0.00035 | 0.19211 | 0.03691 | 0.34200 | 0.11700 |
| | α = 0.5 | -0.06872 | 0.00005 | 0.04576 | 0.00209 | 0.04152 | 0.00172 | 0.01484 | 0.02204 | -0.16960 | 0.02876 |
| | β =1.0 | -0.06536 | 0.00427 | -0.05913 | 0.00350 | -0.06733 | 0.00453 | -0.08085 | 0.00654 | 0.07963 | 0.00028 |
| | λ =0.5 | 0.02294 | 0.00053 | -0.03426 | 0.00117 | 0.00970 | 0.00009 | -0.03480 | 0.00121 | 0.11450 | 0.01315 |
| | α = 1.5 | 0.04772 | 0.00228 | 0.16700 | 0.02789 | 0.11780 | 0.01388 | 0.07473 | 0.00558 | -0.46940 | 0.22056 |
| | β =1.0 | -0.16938 | 0.02869 | -0.33673 | 0.11388 | -0.18202 | 0.03313 | -0.21858 | 0.04778 | 0.13422 | 0.01802 |
| | λ =1.5 | 0.01703 | 0.00029 | 0.02819 | 0.00079 | 0.03509 | 0.00123 | -0.05186 | 0.00269 | 0.32336 | 0.10456 |
| 200 | α = 1.5 | 0.01572 | 0.00025 | 0.20599 | 0.04243 | 0.06715 | 0.00451 | 0.09487 | 0.00900 | 0.19170 | 0.03673 |
| | β =1.0 | -0.13401 | 0.01796 | -0.26517 | 0.07032 | -0.03020 | 0.00091 | -0.04511 | 0.00204 | -0.00247 | 0.00001 |
| | λ =1.0 | -0.03112 | 0.00097 | -0.01222 | 0.00015 | 0.01257 | 0.00016 | -0.08302 | 0.00689 | -0.09370 | 0.00878 |
| | α = 0.5 | 0.00303 | 0.00001 | 0.05018 | 0.00252 | 0.01930 | 0.00037 | 0.04612 | 0.00213 | -0.15053 | 0.02266 |
| | β =1.0 | -0.00671 | 0.00004 | -0.08345 | 0.00696 | -0.06367 | 0.00405 | -0.05950 | 0.00354 | 0.01687 | 0.00800 |
| | λ =0.5 | -0.04921 | 0.00242 | -0.02713 | 0.00074 | -0.00065 | 0.00000 | -0.02963 | 0.00088 | 0.08224 | 0.00676 |
| | α = 1.5 | 0.02335 | 0.00055 | 0.16585 | 0.02751 | 0.74892 | 0.00561 | 0.07341 | 0.00539 | 0.11880 | 0.01411 |
| | β =1.0 | -0.16512 | 0.02726 | -0.25742 | 0.06627 | -0.03736 | 0.00140 | -0.13283 | 0.01764 | 0.01442 | 0.00021 |
| | λ =1.5 | 0.00823 | 0.00007 | -0.01534 | 0.00024 | 0.01084 | 0.00012 | 0.04867 | 0.00237 | -0.10826 | 0.01172 |

### 5.2.1 Data Set 1

In this section, we considered the waiting times (in secs), between 65 successive eruptions of the Kiama Blowhole. A digital watch was used to recorded the values on 12th of July, 1998 by Jim Irish and has been widely used by different authors such as Pinho et al. (2015) Aryal et al. (2017). The data inclues:

83, 51, 87, 60, 28, 95, 8, 27, 15, 10, 18, 16, 29, 54, 91, 8, 17, 55, 10, 35, 47, 77, 36, 17, 21, 36, 18, 40, 10, 7, 34, 27, 28, 56, 8, 25, 68, 146, 89, 18, 73, 69, 9, 37, 10, 82, 29, 8, 60, 61, 61, 18, 169, 25, 8, 26, 11, 83, 11, 42, 17, 14, 9, 12.

Table 2 displayed the MLE estimates and the goodness of fit. Figure 5 provided estimated PDF, CDF, PP-plot, and QQ-plot of TLIG distribution for data 1.

**Table 2**
**The MLEs, and the Goodness of Fit for Data 1**

| Model | $\alpha$ | $\beta$ | $\lambda$ | KS | P – Value | AIC | CAIC | HQIC |
|---|---|---|---|---|---|---|---|---|
| TLIG | 0.2032 | 3.9343 | 96.3525 | 0.062 | 0.7643 | 593.5388 | 593.0903 | 593.9355 |
| IGD | - | 14.214 | 8.215729 | 0.1134 | 0.5583 | 597.1796 | 597.3763 | 599.7311 |
| IEXP | - | - | 20.4135 | 0.1621 | 0.2058 | 600.3507 | 600.415 | 602.9022 |
| EXP | - | - | 0.0251 | 0.1664 | 0.4575 | 601.6254 | 601.6899 | 604.1769 |

### 5.3.2 Data Set 2

In the second dataset, the active repair times (hr) for an airborne communication transceiver was considered Jorjensen (2012). The data are as follows:

0.50, 0.60, 0.60, 0.70, 0.70, 0.70, 0.80, 0.80, 1.00, 1.00, 1.00, 1.00, 1.10, 1.30, 1.50, 1.50, 1.50, 1.50, 2.00, 2.00, 2.20, 2.50, 2.70, 3.00, 3.00, 3.30, 4.00, 4.00, 4.50, 4.70, 5.00, 5.40, 5.40, 7.00, 7.50, 8.80, 9.00, 10.20, 22.00 and 24.50.

These data have been analyzed in Ibrahim et al. (2021). Table 3 displayed the MLE estimates and the goodness of fit. Figure 6 provided estimated PDF, CDF, PP-plot, and QQ-plot of TLIG distribution for data 2.

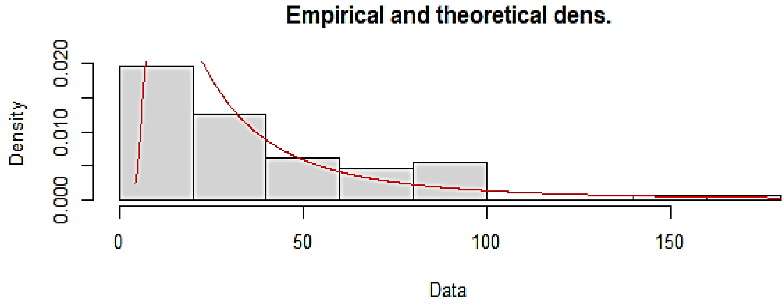


**Figure 5(a): Empirical and Theoretical Density of Fitted Model TLIG Distribution on Data 1**

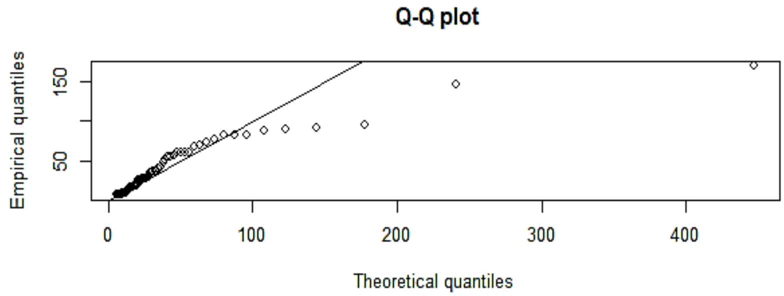


**Figure 5(b): Q-Q Plot of Fitted Model TLIG Distribution on Data 1**

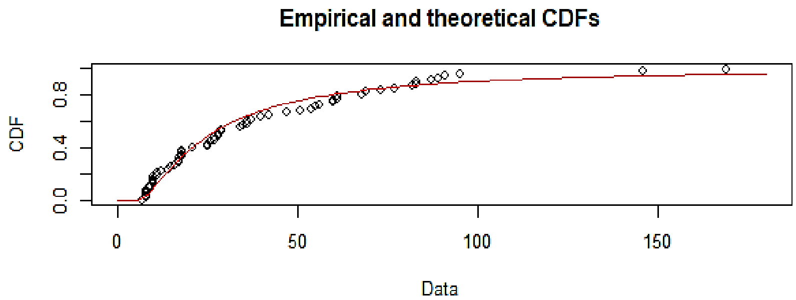


**Figure 5(c): Emperical and Theoretical CDFs of Fitted Model TLIG Distribution on Data 1**

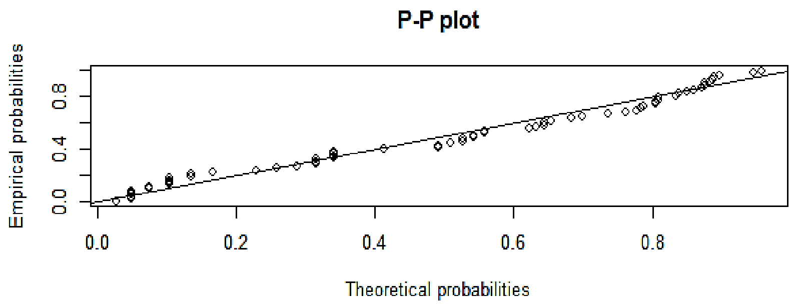


**Figure 5(d): P-P Plot of Fitted Model TLIG Distribution on Data 1**

**Table 3**
**The MLEs, and the Goodness of Fit for Data 2**

| Model | $\alpha$ | $\beta$ | $\lambda$ | KS | P-Value | AIC | CAIC | HQIC |
|---|---|---|---|---|---|---|---|---|
| TLIG | 0.1131 | *0.2918* | 12.5177 | 0.0974 | 0.9733 | 183.333 | 183.999 | 185.165 |
| ITL | | 1.2533 | | 0.1479 | 0.3450 | 188.950 | 189.055 | 189.5613 |
| APITL | 6.9168 | 1.8963 | | 0.1090 | 0.7278 | 187.442 | 187.766 | 188.6638 |
| ILo | | 3.7381 | 0.4900 | 0.1106 | 0.7092 | 189.016 | 189.340 | 190.2378 |
| APILo | 10.919 | 23.999 | 0.0293 | 0.1123 | 0.6945 | 191.596 | 192.262 | 193.4282 |

**Table 4**
**The OLS, WLS. Percentile and Crammer-von Mises Estimates for the Two Reallife Datasets**

| | Data 1 | | | Data 2 | | |
|---|---|---|---|---|---|---|
| | $\alpha$ | $\beta$ | $\lambda$ | $\alpha$ | $\beta$ | $\lambda$ |
| **OLS** | 0.2267 | 3.8670 | 100.169 | 4.185 | 0.56170 | 0.4988 |
| **WLS** | 0.2223 | 3.9192 | 101.648 | 4.3819 | 1.1678 | 0.5023 |
| **Percentile** | 0.2425 | 2.6862 | 102.117 | 4.1377 | 1.0215 | 0.2989 |
| **CRA** | 0.2664 | 3.5833 | 101.348 | 4.0161 | 0.5128 | 0.6031 |

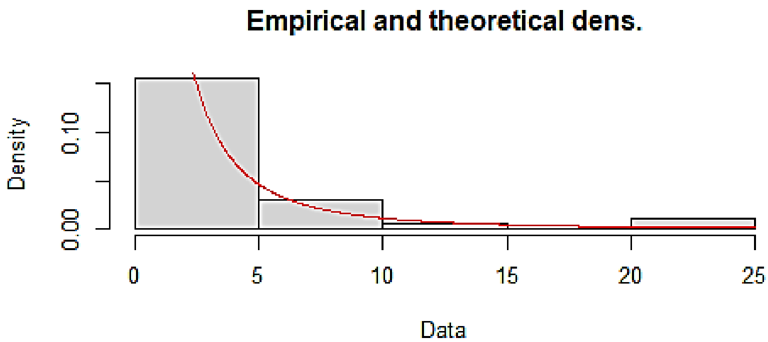


**Figure 6(a): Empirical and Theoretical Density of Fitted Model TLIG Distribution on Data 2**

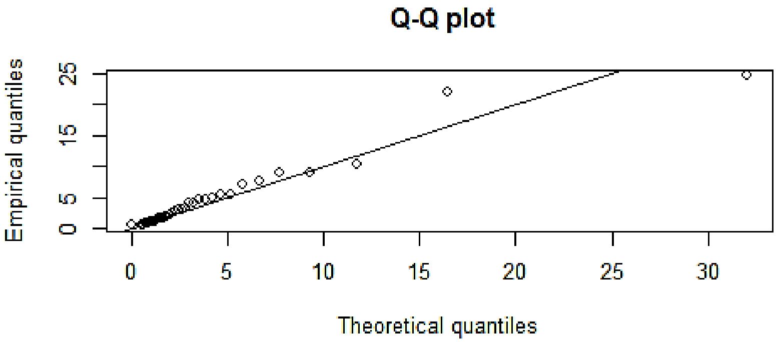


**Figure 6(b): Q-Q Plot of Fitted Model TLIG Distribution on Data 2**

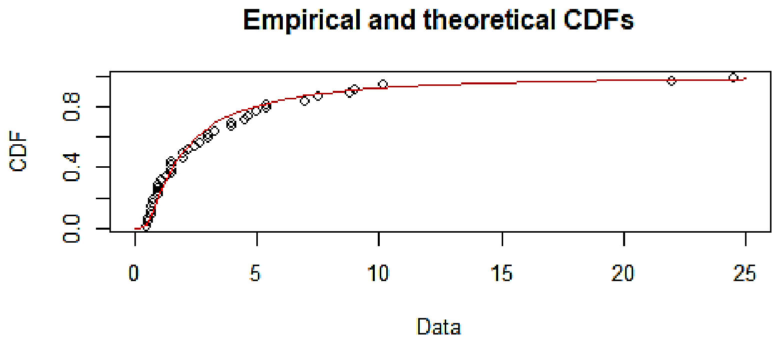


**Figure 6(c): Emperical and Theoretical CDFs and of Fitted Model TLIG Distribution on Data 2**

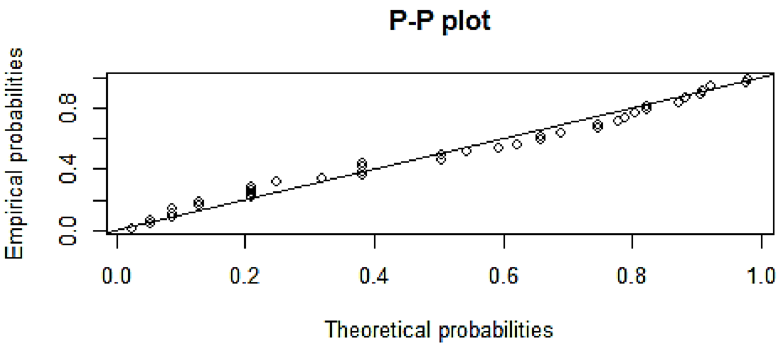


**Figure 6(d): P-P Plot of Fitted Model TLIG Distribution on Data 2**

## 6. CONCLUSION

In this study, we proposed a new distribution known as TLIG distribution. Some properties of this distribution were derived five different techniques of estimation we examined. Table 1 shows the result obtained from the simulated studies which includes the Bias and the MSEs for the Top-Leone Inverse Gompertz (TLIG) parameters using five (5) different estimation techniques. The result shows, that for all estimation techniques and parameter values, as n increases both the Bias and MSE decreases which proves their consistency. It was also noted that the MLE techniques performs better than any other estimators. We observed that all techniques provide small bias for all predetermined values of the parameter. The five estimators are said to be consistent because the MSEs decrease as n increases for all studied cases, but overall the MLE estimator performs better than other estimators.

Tables 2 and 3 shows the values of MLE, AIC, CAIC, HQIC, and KS statistic along with its P-value for all fitted models for the two real data. We compared the models with IGD, EXP, IEXP, ILo, APILo, APITL and ITL and the result is displayed in Tables 2 and 3. The empirical and theoretical density, Q-Q plot., emperical and theoretical CDFs and P-P plot of the fitted model TLIG distribution is shown in Figures 5 and 6. These figures provide indication that the TLIG distribution has the smallest values of AIC, CAIC, HQIC, KS and the largest P-value among all fitted models for the two datasets.

Generally, from Table 1, we can conclude that the maximum likelihood estimates are preferable when compared with other estimates for the given datasets. We showed empirically that the proposed TLIG distribution reveals its superiority over other competitive distribution for the two real data.

## ACKNOWLEDGEMENTS

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