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INFERENCE FOR LINEAR EXPONENTIAL DISTRIBUTION BASED ON EXTREME RANKED SET SAMPLING

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ABSTRACT

In this article, maximum likelihood estimation and Bayes estimators are derived for linear exponential distribution based on one- and m-cycle extreme ranked set sampling and simple random sample. These estimators are compared via their biases and mean squared error. This is done with respect to both symmetric and asymmetric loss functions. A real-life data set and simulation study are used to illustrate our procedures. This unequivocally shows that, in all circumstances taken into consideration for this study, ranked set sampling is more effective than both extreme ranked set sampling and simple random sample. Over a high number of cycles, a significant improvement is shown.

KEYWORDS

Extreme Ranked set sampling; Maximum likelihood estimation; Bayesian estimation; Mean squared error; Symmetric and asymmetric loss functions; Linear exponential distribution.

1. INTRODUCTION

To determine the mean pasture and forage yields, McIntyre (1952) initially developed ranked set sampling (RSS). It is a more affordable sampling technique than simple random sampling (SRS) for estimating the population mean in cases when visual ordering of sample units is straightforward but accurate measurement of the units is challenging and expensive. The RSS theory was put forward by Takahasi and Wakimoto (1968). They demonstrated that the mean of the RSS is a more accurate estimate of the population mean than the mean of the SRS. Dell and Clutter (1972) conducted research on the impact of ranking inaccuracy on RSS effectiveness. In biological and environmental sciences, the RSS has various statistical applications (see Barabesi and El-Sharaawi (2001)). In the RSS scheme, one first draws n^2 units at random from the population and partitions them into n sets of n units. Then units in each set are ranked without making actual measurements. From the first set of n units the first smallest-ranked unit is chosen. From the second set of n units, the second smallest ranked unit is measured. The process is continued until the unit ranked largest is measured from the nth set of n units. So, a RSS of size n is obtained. If this method is repeated m times, a RSS of size mn is obtained. However, ranking accuracy affects the efficiency of the estimator. When the set size is large, ranking error tends to occur. In order to reduce the error of ranking and keep optimality inherited in the original RSS procedure, Zamanzade and Mahdizadeh (2020) introduced the concept of varied set size RSS, which is coined as an extreme ranked set sampling (ERSS) in which only the largest or the smallest judgment ranked unit is chosen for quantification. In recent years, the ERSS design has received a lot of attention. Abu-Dayyeh and Al-Samawi (2009), Al-Saleh and Al-Ananbeh (2008), Al-Saleh and Samawi (2010) and Samawi and Al-Saleh (2013) took into account the issue of determining odds ratios based on ERSS. The procedure of ERSS is described as follows:

1. Draw n sets of simple random samples of size n.
2. Rank the elements of each set by visual inspection or by some other cheap method, without actual measurement of the characteristic of interest.
3. Measure accurately the minimum ordered observation from each set.
4. This process can be replicated m times to get m-cycle ERSS.

Clearly, it is not difficult to identify maximum or minimum units from each set, so ERSS is a very useful modification of RSS. It allows for an increase of set size without introducing too many ranking errors. In the sequel, the above sampling technique is called ERSS-type I. If, in step 3 of the sampling procedure, the maximum ordered unit is quantified from each sample, then the resulting design will be referred to as ERSS-type II. Recently, symmetric and asymmetric loss functions are considered in Bayesian inference, see Al-Hossain (2016), Al-Aboud (2009), Amin (2008), Hassan (2013), Howladera and Hossainb (2002), Kim and Song (2010), Kundua and Howladerb (2010), Ku and Kaya (2007), Martz and Waller (1982), Mohammadi and Pazira (2010), Soliman et al. ((2006), (2011)) and Zellner (1986).

The probability density function (pdf), the cumulative distribution function (cdf) and the survival function of the linear exponential distribution are given by

$$f(x;\theta,\mu) = (\theta+\mu x)\, exp\left(-\theta x-\frac{\mu}{2}x^2\right), x\geq 0, \theta,\mu>0, \tag{1}$$

$$F(x;\theta,\mu) = 1-exp\left(-\theta x-\frac{\mu}{2}x^2\right), x\geq 0, \theta,\mu>0, \tag{2}$$

and

$$\bar{F}(x;\theta,\mu) = exp\left(-\theta x-\frac{\mu}{2}x^2\right), x\geq 0, \theta,\mu>0, \tag{3}$$

respectively. The linear exponential distribution with parameters θ and μ will be denoted by LExp(θ, μ). Its reliability and hazard functions at mission time t are given respectively by

$$R(t;\theta,\mu) = 1-exp\left(-\theta x-\frac{\mu}{2}t^2\right) \text{ and } h(t,\theta,\mu) = \theta+\mu t, t>0, \tag{4}$$

In reliability analysis and applied statistics, this model has several uses. It is obvious that the $LExp(\theta,\mu)$ reduces to Rayleigh distribution if $a=0$. It was employed by Carbone et al. (1967) in their analysis of the lymphoma patient survival pattern. References and additional information on this model can be found in Al-Khedhairi (2008), Mahmoud et al. (2006) and Seo and Yum (1993).

In the next section, we present Bayes estimation to estimate the unknown parameters based on m-cycle ERSS and SRS. The Bayes estimates are obtained using both squared error loss function (SEL) and LINEX loss functions. Section 3, discusses the maximum likelihood estimation (MLE) of the unknown parameters based on m-cycle ERSS and SRS. Section 4 contains simulation studies. A summary and directions for our research are given in Section 5.

2. BAYES ESTIMATION

In this section, we obtain Bayes estimation based on ERSS and SRS of the unknown parameters of $LExp(\theta,\mu)$. This is done with respect to SEL and LINEX functions.

2.1 Bayes Estimation Based on m-Cycle ERSS

Consider $Y_{r_i}, i = 1,2,\dots,n$ be a one-cycle ERSS, (if $r = 1$, we can obtain ERSS-type I, and if $r = n$, we can obtain ERSS-type II). Then the density of Y_{r_i} is given by

$$g(y_{r_i}) = \frac{n!}{(r_i-1)!\,(n-r_i)!}\,[F(y_{r_i})]^{r_i-1}\,[\bar{F}(y_{r_i})]^{r_i-1} f(y_{r_i}).$$

The joint pdf of y_{r_i} is

$$f(y) = \prod_{i=1}^{n} g\left(y_{r_i}\right) \propto \prod_{i=1}^{n}\left[F\left(y_{r_i}\right)\right]^{r_i-1}\left[\bar{F}\left(y_{r_i}\right)\right]^{n-r_i} f\left(y_{r_i}\right). \tag{5}$$

Let $y_{lr_i}, l = 1,2,\dots,m, i = 1,2,\dots,n$ be m-cycle ERSS, then the likelihood function of m-cycle ERSS

$$L_1\left(\theta,\mu,\underline{y}\right) \propto \prod_{l=1}^{m}\prod_{i=1}^{n}\left(\theta+\mu\, y_{lr_i}\right)\left[1-\exp\left(-\theta\, y_{lr_i}-\frac{\mu}{2}\, y_{lr_i}^2\right)\right]^{r_i-1}$$
$$\times\, exp\left(-\sum_{l=1}^{n}\sum_{i=1}^{n}(n-r_i+1)\left(\theta\, y_{lr_i}+\frac{\mu}{2}\, y_{lr_i}^2\right)\right), \tag{6}$$

where $\underline{y} = (y_{1r_1},\dots,y_{1r_n}; y_{2r_1},\dots,y_{2r_n};\dots; y_{mr_1},\dots,y_{mr_n})$.

By using the following relations, see Balakrishnan (2008)

$$\prod_{l=1}^{m}\prod_{i=1}^{n}\left(\theta+\mu\, y_{lr_i}\right) = \sum_{u_1=0}^{n}\sum_{u_2=0}^{n}\cdots\sum_{u_m=0}^{n}\theta^{nm-V_{\underline{u}}}\,\mu^{V_{\underline{u}}}\left(\prod_{l=1}^{m}\varphi_{l,u_l}\left(\underline{y}\right)\right), \tag{7}$$

$$\varphi_{l,u_l}\left(\underline{y}\right) = \sum_{b_1=1}^{n-u_l+1} y_{b_1}\sum_{b_2=b_1+1}^{n-u_l+2} y_{b_2}\times\ldots\times\sum_{b_{u_l}=b_{u_l-1}+1}^{n} y_{b_{u_l}}\,, V_{\underline{u}} = \sum_{l=1}^{m} u_l. \tag{8}$$

By using Eqs (7) and (8), the likelihood function in Eq (6) is written as

$$L_1\left(\theta,\mu,\underline{y}\right) \propto \prod_{l=1}^{m} \sum_{\underline{t}_l} \sum_{u_l=0}^{n} W_{\underline{t}_l,u_l}\, \theta^{nm-V_{\underline{u}}}\ \mu^{V_{\underline{u}}} \times exp\left(-\sum_{l=1}^{m}\sum_{i=1}^{n}\left(n-r_i+t_{l,r_i}+1\right)\left(y_{lr_i}\theta+\frac{\mu}{2}\,y_{lr_i}^2\right)\right), \quad (9)$$

where

$$W_{\underline{t}_l,u_l} = \prod_{l=1}^{m}\varphi_{l,u_l}\left(\underline{y}\right)\prod_{i=1}^{n} r_i \binom{r_i-1}{t_{l,r_i}}\binom{n}{r_i}(-1)^{t_{l,r_i}},\quad \sum_{\underline{t}_l} = \sum_{t_{l,1}=0}^{0}\sum_{t_{l,2}=0}^{1}\cdots\sum_{t_{l,n}=0}^{n-1}.$$

For the Bayesian estimation setup, we need a suitable prior parameter distribution. Here, we consider the prior density, which suggested by Al-Khedhairi (2008) as

$$\pi\,(\theta,\mu;\ \delta) = \eta\rho exp\{-\theta\eta - \mu\rho\}, \theta\,,\mu > 0, \quad (10)$$

where δ is the vector of prior parameters and η,ρ are known positive constants. Using Eqs (9) and (10), the posterior density function is

$$\pi_1^*\left(\theta,\mu|\underline{y}\right) = K^{-1}\prod_{l=1}^{m}\sum_{\underline{t}_l}\sum_{u_l=0}^{n} W_{\underline{t}_l,u_l}\,\theta^{nm+V_{\underline{u}}}\,\mu^{V_{\underline{u}}} \times exp\left(-\theta\,(\sum_{l=1}^{m}\sum_{i=1}^{n} y_{lr_i}\left(n-r_i+t_{l,r_i}+1\right)+\eta)\right) \times exp\left(-\mu(\sum_{l=1}^{m}\sum_{i=1}^{n}\frac{y_{lr_i}^2}{2}\left(n-r_i+t_{l,r_i}+1\right)+\rho)\right), \quad (11)$$

where the normalized constant K is

$$K = \prod_{l=1}^{m}\sum_{\underline{t}_l}\sum_{u_l=0}^{n} W_{\underline{t}_l,u_l}\,\frac{\Gamma\left(nm+V_{\underline{u}}+1\right)}{\left(\sum_{l=1}^{m}\sum_{i=1}^{n} y_{lr_i}\left(n-r_i+t_{l,r_i}+1\right)+\eta\right)^{nm+V_{\underline{u}}+1}} \times\frac{\Gamma\left(V_{\underline{u}}+1\right)}{\left(\sum_{l=1}^{m}\sum_{i=1}^{n}\frac{y_{lr_i}^2}{2}\left(n-r_i+t_{l,r_i}+1\right)+\rho\right)^{V_{\underline{u}}+1}},$$

Bayesian estimation of θ and μ under a SEL function are

$$\tilde{\theta}_{BS} = E\left(\theta\middle|\underline{y}\right) = \int_0^\infty\int_0^\infty \theta\pi^*\left(\theta,\mu\middle|\underline{y}\right)d\theta d\mu = K^{-1}\prod_{l=1}^{m}\sum_{\underline{t}_l}\sum_{u_l=0}^{n} W_{\underline{t}_l,u_l}\,\frac{\Gamma\left(nm+V_{\underline{u}}+2\right)}{\left(\sum_{l=1}^{m}\sum_{i=1}^{n} y_{lr_i}\left(n-r_i+t_{l,r_i}+1\right)+\eta\right)^{nm+V_{\underline{u}}+2}} \times\frac{\Gamma\left(V_{\underline{u}}+1\right)}{\left(\sum_{l=1}^{m}\sum_{i=1}^{n}\frac{y_{lr_i}^2}{2}\left(n-r_i+t_{l,r_i}+1\right)+\rho\right)^{V_{\underline{u}}+1}}, \quad (12)$$

and

$$\tilde{\mu}_{BS} = K^{-1} \prod_{l=1}^{m} \sum_{\underline{t}_l} \sum_{u_l=0}^{n} W_{\underline{t}_l,u_l} \frac{\Gamma\left(nm + V_{\underline{u}} + 1\right)}{\left(\sum_{l=1}^{m}\sum_{i=1}^{n} y_{lr_i}\left(n - r_i + t_{l,r_i} + 1\right) + \eta\right)^{nm+V_{\underline{u}}+1}} \times \frac{\Gamma\left(V_{\underline{u}} + 2\right)}{\left(\sum_{l=1}^{m}\sum_{i=1}^{n} \frac{y_{lr_i}^2}{2}\left(n - r_i + t_{l,r_i} + 1\right) + \rho\right)^{V_{\underline{u}}+2}}, \tag{13}$$

Under the LINEX loss function, the Bayesian estimates of θ and μ are given, respectively, by using Eq. (11)

$$\tilde{\theta}_{BL} = -\frac{1}{c} \log\left(E\left(exp(-c\theta)|\underline{y}\right)\right) = -\frac{1}{c}\log\left(\prod_{l=1}^{m} \sum_{\underline{t}_l} \sum_{u_l=0}^{n} W_{\underline{t}_l,u_l} \times \frac{\Gamma\left(nm + V_{\underline{u}} + 1\right)}{\left(\sum_{l=1}^{m}\sum_{i=1}^{n} y_{lr_i}\left(n - r_i + t_{l,r_i} + 1\right) + \eta + c\right)^{nm+V_{\underline{u}}+1}} \times \frac{\Gamma\left(V_{\underline{u}} + 1\right)}{\left(\sum_{l=1}^{m}\sum_{i=1}^{n} \frac{y_{lr_i}^2}{2}\left(n - r_i + t_{l,r_i} + 1\right) + \rho\right)^{V_{\underline{u}}+1}}\right), \tag{14}$$

and

$$\tilde{\mu}_{BL} = -\frac{1}{c}\log\left(\prod_{l=1}^{m} \sum_{\underline{t}_l} \sum_{u_l=0}^{n} W_{\underline{t}_l,u_l} \frac{\Gamma\left(nm + V_{\underline{u}} + 1\right)}{\left(\sum_{l=1}^{m}\sum_{i=1}^{n} y_{lr_i}\left(n - r_i + t_{l,r_i} + 1\right) + \eta\right)^{nm+V_{\underline{u}}+1}} \times \frac{\Gamma\left(V_{\underline{u}} + 1\right)}{\left(\sum_{l=1}^{m}\sum_{i=1}^{n} \frac{y_{lr_i}^2}{2}\left(n - r_i + t_{l,r_i} + 1\right) + \rho + c\right)^{V_{\underline{u}}+1}}\right). \tag{15}$$

2.2 Bayes Estimation based on m-Cycle SRS

Suppose that $x_{li}, i = 1, 2, \ldots, n, l = 1, 2, \ldots, m$ be m independent sets of order statistic each of size n from linear exponential distribution, then the joint density of $x_{li}, i = 1, 2, \ldots, n, l = 1, 2, \ldots, m$ is

$$f\left(\underline{x}|\theta, \mu\right) \propto \prod_{l=1}^{m}\prod_{i=1}^{n} f(x_{li}|\theta, \mu) \propto \left(\prod_{l=1}^{m}\prod_{i=1}^{n} (\theta + \mu x_{li})\right) exp\left(-\sum_{l=1}^{m}\sum_{i=1}^{n}\left(\theta x_{li} + \frac{\mu}{2} x_{li}^2\right)\right), \tag{16}$$

where $\underline{x} = (x_{11}, \ldots, x_{1n}; x_{21}, \ldots, x_{2n}; \ldots, x_{m1}, \ldots, x_{mn})$. Using Eq. (7), the likelihood function in Eq. (16) becomes

$$L_2(\theta,\mu,\underline{x}) = \prod_{l=1}^{m}\sum_{u_l=0}^{n}\varphi_{u_l}\,\theta^{nm-V_{\underline{u}}}\mu^{V_{\underline{u}}}\,exp\left(-\sum_{l=1}^{m}\sum_{i=1}^{n}\left(\theta x_{li}+\frac{\mu}{2}\,x_{li}^2\right)\right), \quad (17)$$

where

$$V_{\underline{u}} = \sum_{l=1}^{m}u_l\,, \text{and } \varphi_{u_l} = \sum_{b_1=1}^{n-u_l+1}x_{lb_1}\sum_{b_2=1}^{n-u_l+2}x_{lb_2}\times\ldots\times\sum_{b_{u_l}=b_{u_l-1}+1}^{n}x_{lb_{u_l}}\,.$$

From Eq. (10) and (17), the posterior density function is

$$\pi_2^*(\theta,\mu|\underline{x}) = A^{-1}\prod_{l=1}^{m}\sum_{u_l=0}^{n}\varphi_{u_l}\,\theta^{nm-V_{\underline{u}}}\mu^{V_{\underline{u}}}\;exp\left(-\theta\left(\eta+\sum_{l=1}^{m}\sum_{i=1}^{n}x_{li}\right)\right.$$
$$\left.-\;\mu\left(\rho+\sum_{l=1}^{m}\sum_{i=1}^{n}\frac{x_{li}^2}{2}\right)\right), \quad (18)$$

where

$$A = \prod_{l=1}^{m}\sum_{u_l=0}^{n}\varphi_{u_l}\;\frac{\Gamma\left(mn-V_{\underline{u}}+1\right)}{(\eta+\sum_{l=1}^{m}\sum_{i=1}^{n}x_{li})^{mn-d_{\underline{u}}+1}}\times\frac{\Gamma\left(V_{\underline{u}}+1\right)}{\left(\rho+\sum_{l=1}^{m}\sum_{i=1}^{n}\frac{x_{li}^2}{2}\right)^{V_{\underline{u}}+1}}.$$

Bayesian estimation of θ and μ under a SEL function are

$$\tilde{\theta}_{BS} = A^{-1}\prod_{l=1}^{m}\sum_{u_l=0}^{n}\varphi_{ul}\;\frac{\Gamma\left(mn-V_{\underline{u}}+2\right)}{(\eta+\sum_{l=1}^{m}\sum_{i=1}^{n}x_{li})^{mn-d_{\underline{u}}+2}}$$
$$\times\;\frac{\Gamma\left(V_{\underline{u}}+1\right)}{\left(\rho+\sum_{l=1}^{m}\sum_{i=1}^{n}\frac{x_{li}^2}{2}\right)^{V_{\underline{u}}+1}}, \quad (19)$$

and

$$\tilde{\mu}_{BS} = A^{-1}\prod_{l=1}^{m}\sum_{u_l=0}^{n}\varphi_{u_l}\;\frac{\Gamma\left(mn-V_{\underline{u}}+1\right)}{(\eta+\sum_{l=1}^{m}\sum_{i=1}^{n}x_{li})^{mn-d_{\underline{u}}+1}}\times\frac{\Gamma\left(V_{\underline{u}}+2\right)}{\left(\rho+\sum_{l=1}^{m}\sum_{i=1}^{n}\frac{x_{li}^2}{2}\right)^{V_{\underline{u}}+2}}. \quad (20)$$

Under the LINEX loss function, the Bayesian estimates of θ and μ are given, respectively, using Eq (18)

$$\tilde{\theta}_{BL} = -\frac{1}{c}\, log\left(\prod_{l=1}^{m}\sum_{u_l=0}^{n}\varphi_{u_l}\,\frac{\Gamma\left(mn-V_{\underline{u}}+1\right)}{\left(\eta+c+\sum_{l=1}^{m}\sum_{i=1}^{n}x_{li}\right)^{mn-\,d_{\underline{u}}+1}}\right.$$
$$\left.\times\,\frac{\Gamma\left(V_{\underline{u}}+1\right)}{\left(\rho+\sum_{l=1}^{m}\sum_{i=1}^{n}\frac{x_{li}^2}{2}\right)^{V_{\underline{u}}+1}}\right), \qquad (21)$$

$$\tilde{\mu}_{BL} = -\frac{1}{c}\, log\left(\prod_{l=1}^{m}\sum_{u_l=0}^{n}\varphi_{u_l}\,\frac{\Gamma\left(mn-V_{\underline{u}}+1\right)}{\left(\eta+\sum_{l=1}^{m}\sum_{i=1}^{n}x_{li}\right)^{mn-\,d_{\underline{u}}+1}}\right.$$
$$\left.\times\,\frac{\Gamma\left(V_{\underline{u}}+1\right)}{\left(\rho+c+\sum_{l=1}^{m}\sum_{i=1}^{n}\frac{x_{li}^2}{2}\right)^{V_{\underline{u}}+1}}\right). \qquad (22)$$

3. MAXIMUM LIKELIHOOD ESTIMATION

In this section, maximum likelihood estimation for the parameters of $LExp\ (\theta,\mu)$ based on RSS and SRS are derived.

3.1 Maximum Likelihood Estimation based on m-Cycle ERSS

Based on Eq. (6), we have

$$log\, L_1\ \left(\theta,\mu;\underline{y}\right) = \sum_{l=1}^{m}\sum_{i=1}^{n}\ (r_i-1)\log\left(1-\exp\left(-\theta\, y_{lr_i}-\frac{\mu}{2}\, y_{lr_i}^2\right)\right)$$
$$-\sum_{l=1}^{m}\sum_{i=1}^{n}(n-r_i+1)\left(\theta\, y_{lr_i}+\frac{\mu}{2}\, y_{lr_i}^2\right)$$
$$+\sum_{l=1}^{m}\sum_{i=1}^{n}\log\left(\theta+\mu\, y_{lr_i}\right). \qquad (23)$$

Under the assumption that both the parameters θ and μ are unknown, the $\tilde{\theta}_{ML}$ and $\tilde{\mu}_{ML}$, can be solved numerically from the following equations

$$\sum_{l=1}^{m}\sum_{i=1}^{n}\frac{-\,y_{lr_i}\,(\,r_i-1)\,exp\left(-\theta y_{lr_i}-\frac{\mu}{2}y_{lr_i}^2\right)}{1-exp\left(-\theta y_{lr_i}-\frac{\mu}{2}y_{lr_i}^2\right)}$$
$$+\sum_{l=1}^{m}\sum_{i=1}^{n}\frac{1}{\left(\theta+\mu y_{lr_i}\right)} = \sum_{l=1}^{m}\sum_{i=1}^{n}(n-r_i+1)\ \ y_{lr_i}\,, \qquad (24)$$

and

$$\sum_{l=1}^{m}\sum_{i=1}^{n}\frac{-0.5\, y_{lr_i}^2\,(r_i-1)\exp\left(-\theta y_{lr_i}-\frac{\mu}{2}y_{lr_i}^2\right)}{1-\exp\left(-\theta y_{lr_i}-\frac{\mu}{2}y_{lr_i}^2\right)} + \sum_{l=1}^{m}\sum_{i=1}^{n}\frac{y_{lr_i}}{(\theta+\mu y_{lr_i})} = \sum_{l=1}^{m}\sum_{i=1}^{n}(n-r_i+1)\frac{y_{lr_i}^2}{2}. \qquad (25)$$

3.2 Maximum Likelihood Estimation based on m-Cycle SRS

Based on Eq. (16), the log likelihood function is given by

$$log\, f(\underline{x}|\theta,\mu) = \sum_{l=1}^{m}\sum_{i=1}^{n}\left(log(\theta+\mu x_{li}) - \left(\theta x_{li}+\frac{\mu}{2}x_{li}^2\right)\right). \qquad (26)$$

The MLE of θ and μ can be solved numerically from the following equations

$$\sum_{l=1}^{m}\sum_{i=1}^{n}\frac{1}{\theta+\mu x_{li}} = m\overline{X} \text{ and } \sum_{l=1}^{m}\sum_{i=1}^{n}\frac{x_{li}}{\theta+\mu x_{li}} = \frac{m}{2}\overline{\overline{X}}, \qquad (27)$$

where

$$\overline{X} = \frac{1}{m}\sum_{l=1}^{m}\sum_{i=1}^{n}x_{li} \text{ and } \overline{\overline{X}} = \frac{1}{m}\sum_{l=1}^{m}\sum_{i=1}^{n}x_{li}^2 .$$

4. ILLUSTRATIVE EXAMPLES AND COMPARISONS

In this section, we conduct a Monte Carlo simulation study to verify how our methods work in practice and we consider one real data set for illustrative purposes.

4.1 Simulated Data

We use a simulated data set to illustrate the estimation techniques developed in previous sections. The following steps describe our methodology:

1. Choose values of the prior parameters η and ρ, and generate θ from Gamma (1, η) and μ from Gamma (1, ρ).
2. To examine the sensitivity of the estimation with respect to the hyper-parameters η and ρ, under the assumption that both hyper-parameters are known, we used 3 different choices of the hyper-parameters (η , ρ) = (0.5,0.3), (1.5,1.3), (2.5,2.3).
3. Based on those generated values of θ and μ in step (1), we generate n random samples, each of size n from the $LExp\,(\theta,\mu)$ model by using the transformation $X_i = \sqrt{\left(\frac{\theta}{\mu}\right)^2 - \frac{2}{\mu}\ln(1-U_i)} - \frac{\theta}{\mu}, i = 1,\dots,n$ where U_i from $U\,(0, 1)$. Then by using the procedure of one-cycle (m = 1) SRS, ERSS-type II, and RSS samples of size $n = 10, 15, 20$ obtained.
4. To obtain two-cycle SRS, ERSS-type I, ERSS-type II, and RSS, the previous step is replicated two times, so a sample size of $2n$ is obtained.
5. The different Bayes estimates $(.)_{\mathrm{BS}}$ and $(.)_{\mathrm{BL}}$ of θ and μ are computed through Eqs (12)-(15) and (19)-(22), as well as the ML estimate $(.)_{\mathrm{ML}}$ of θ and μ are calculated numerically from Eqs (24), (25) and (27).
6. Repeat Steps 1-5 for 5000 runs to obtain MSE and bias of all estimates for SRS,

ERSS-type I, ERSS-type II, and RSS of one- and two- cycle, respectively. So mean, MSE and bias are computed by, Mean $=\frac{1}{n^*}\sum_{i=1}^{n^*}\tilde{\vartheta}_i$, $\text{MSE}(\vartheta)=\frac{1}{n^*}\sum_{i=1}^{n^*}\left(\tilde{\vartheta}_i-\vartheta\right)^2$, $\tilde{\vartheta}_i$ is the estimator of (θ, μ) for the ith simulated data, n^* = 5000 and $\tilde{\vartheta}_{\text{Bias}}=\frac{1}{n^*}\sum_{i=1}^{n^*}\left(\tilde{\vartheta}_i-\vartheta\right)$. The bias and MSE of all the estimates are given in Tables 1-4, respectively.

From the tabulated results, the following points are reported:

- In all scenarios taken into consideration, the MLE and various Bayes estimates based on ERSS have the minimum MSE in comparison to the MLE and Bayes estimates based on SRS, except ERSS-type I. This amply displays the effectiveness of ERSS-based inference.
- The MLE and different Bayes estimates based on RSS compete the MLE and Bayes estimates based on ERSS in all cases considered.
- The Bayes estimates relative to the LINEX loss function has the smallest MSE and bias as compared with the SEL loss function of Bayes estimates. This is due to the additional factor appeared in LINEX function.
- It is clear that the estimates based on both SRS, RSS and ERSS with two-cycle $(m=2)$ are better than the Bayes estimates with one-cycle $(m=1)$. In general, the better results are obtained by using a large number of cycles.
- As expected, all the estimators become better by getting closer to the population parameters value of θ and μ as n increase. It is also observed that all the estimators are getting better with decreasing η and ρ.

4.2 Application

In this subsection we present the analysis of a real-life data given by Cox and Lewis (1966). These data are taken from an experiment whose time intervals between successive pulses along a nerve fiber were measured in seconds. There data represent the 799 recorded waiting times. We first check whether the linear exponential distribution is appropriate for the data by using Kaplan-Meier estimator (KME) and the Kolmogorov-Smirnov (K-S) distance. For the given data, Figure 1 shows the P-P plot of the KME versus the fitted linear exponential survival function (SF). It suggests a fair match between the fitted SF and the empirical SF. We observe that the fitted linear exponential survival function's illustrated points are extremely close to the 45° line, indicating a very good fit. We also use the Kolmogorov Smirnov goodness-of-fit test to determine whether the linear exponential distribution is appropriate for describing these data set. The Kolmogorov Smirnov test statistic is 0.0368 with an associated p-value = 0.9994 $>$ 0.05, so linear exponential distribution is fitted to the above real data set. The one- and two- cycle RSS is in Table 5. Then the MLE and Bayes estimates of θ and μ are computed in Table 6.

5. CONCLUSION

In this paper, we have tackled the MLE and Bayesian inference, for the linear exponential model based on RSS, ERSS-type I, ERSS-type II and SRS. The Bayesian inference, using both symmetric and asymmetric loss functions are obtained using conjugate prior distribution. Also, comparisons are made between the different estimators based on a simulation study. The details have been explained using a real-life example. It is observed that the Bayesian estimators based on RSS have much smaller MSEs and biases than the corresponding Bayesian estimators based on SRS and ERSS. This in turn implies that RSS scheme is more efficient than SRS and ERSS schemes for estimating the model parameter for all cases considered in this study. Moreover, the Bayes estimators under the LINEX perform well when compared to their corresponding estimators under SEL.

Table 1
Mean, MSEs of the Estimators of 0 and μ based on one- and two-cycle RSS

m	n	η	ρ	Par	$(.)_{ML}$			$(.)_{SE}$			$(.)_{LENIX}$	
					Mean	**Bias**	**MSE**	**Mean**	**Bias**	**MSE**	**c=1**	**c= -0.1**
1	10	0.5	.03	θ	0.4700	0.1412	0.0309	0.4647	0.1120	0.0192	0.4633	0.4661
				μ	0.3845	0.1913	0.0664	0.4288	0.1701	0.0544	0.4049	0.4313
		1.5	1.3	θ	1.3000	0.3788	0.2264	1.6403	0.2242	0.0800	1.4538	1.4669
				μ	2.1149	1.2270	2.9561	1.2585	0.3887	0.2362	0.9358	1.3022
		2.5	2.3	θ	2.2019	0.5289	0.4372	2.4504	0.2490	0.0983	2.4426	2.4582
				μ	3.7165	2.2139	8.3522	0.6579	1.6417	2.7044	0.5146	0.6781
	15	0.5	.03	θ	0.4770	0.1046	0.0169	0.4732	0.0934	0.0135	0.4725	0.4740
				μ	0.3509	0.1382	0.0326	0.3744	0.1290	0.0297	0.3612	0.3757
		1.5	1.3	θ	1.3862	0.2707	0.1170	1.4877	0.1848	0.0544	1.4839	1.4916
				μ	1.7241	0.8462	1.3246	1.2457	0.4394	0.3154	0.9894	1.2766
		2.5	2.3	θ	2.3037	0.3966	0.2508	2.6269	0.2255	0.0811	2.6223	2.6316
				μ	3.3520	1.8098	5.7040	0.7205	1.5831	2.5304	0.5588	0.7431
	20	0.5	.03	θ	0.4859	0.0815	0.0104	0.4846	0.0766	0.0091	0.4841	0.4851
				μ	0.3289	0.1061	0.0188	0.3416	0.1017	0.0176	0.3334	0.3424
		1.5	1.3	θ	1.4337	0.2084	0.0693	1.5091	0.1596	0.0406	1.5065	1.5117
				μ	1.5597	0.6572	0.7467	1.2306	0.4364	03008	1.0309	1.2531
		2.5	2.3	θ	2.3707	0.3181	0.1311	2.6854	0.2261	0.0776	2.6821	2.6886
				μ	3.0271	1.4829	3.8335	0.8030	1.4978	2.2979	0.6191	0.8284
2	10	0.5	.03	θ	0.4871	0.1055	0.0174	0.4832	0.0938	0.0137	0.4824	0.4840
				μ	0.3397	0.1363	0.0315	0.3640	0.1253	0.0281	0.3514	0.3653
		1.5	1.3	θ	1.4029	0.2679	0.1158	1.4978	0.1877	0.0578	1.4938	1.5019
				μ	1.6945	0.8224	1.2281	1.2372	0.4406	0.3074	0.9872	1.2672
		2.5	2.3	θ	2.3234	0.4037	0.2593	2.6273	0.2261	0.0821	2.6223	2.6323
				μ	3.2878	1.7906	5.5809	0.7284	1.5740	2.5050	0.5644	0.7513
	15	0.5	.03	θ	0.4877	0.0734	0.0085	0.4875	0.0702	0.0078	1.4871	0.4879
				μ	0.3259	0.0966	0.0153	0.3355	0.0939	0.0147	0.3287	0.3362
		1.5	1.3	θ	1.4468	0.1927	0.0589	1.5117	0.1559	0.0382	1.5094	1.5140
				μ	1.4997	0.5902	0.5952	1.2223	0.4301	0.2873	1.0474	1.2415
		2.5	2.3	θ	2.3985	0.2840	0.1285	2.6931	0.2300	0.0748	2.6900	2.6961
				μ	2.8700	1.2909	2.8773	0.8541	1.4488	2.1732	0.6593	0.8804
	20	0.5	.03	θ	0.4946	0.0574	0.0052	0.4950	0.0563	0.0050	0.4947	0.4953
				μ	0.3135	0.0755	0.0090	0.3185	0.0745	0.0089	0.3143	0.1390
		1.5	1.3	θ	1.4674	0.1483	0.0350	1.5172	0.1322	0.274	1.5161	1.5193
				μ	1.4336	0.4655	0.3565	1.2331	0.3877	0.2303	1.1054	1.2473
		2.5	2.3	θ	2.4461	0.2239	0.0792	2.7025	0.2203	0.0688	2.7003	2.7048
				μ	2.6335	1.0236	1.7498	1.6001	1.3008	1.8323	0.7857	1.0341

Table 2

Mean, MSEs of the Estimators of θ and μ based on one- and two-cycle ERSS-type I.

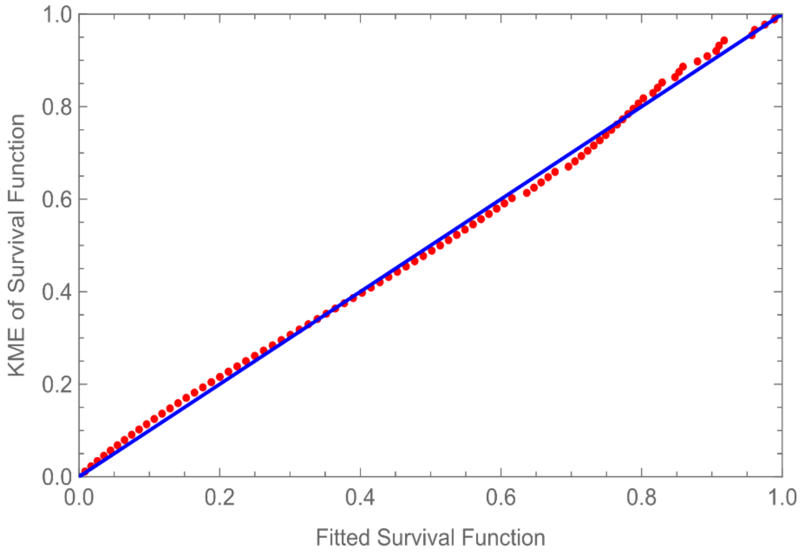
m	n	η	ρ	Par	$(.)_{ML}$			$(.)_{SE}$			$(.)_{LENIX}$	
					Mean	Bias	MSE	Mean	Bias	MSE	c=1	c= -0.1
1	10	0.5	.03	θ	0.3993	0.1875	0.0543	0.4615	0.1488	0.0381	0.4589	0.4640
				μ	1.9953	1.7389	8.0879	1.8420	1.5419	3.2981	1.1831	1.9625
		1.5	1.3	θ	1.2945	0.4497	0.3063	1.3078	0.3170	0.1434	1.2989	1.3169
				μ	4.4194	3.3500	17.8860	0.8513	0.4487	0.2037	0.6199	0.8882
		2.5	2.3	θ	2.2352	0.6554	0.6565	1.6089	0.8848	0.8636	1.5966	1.6251
				μ	4.7186	3.0179	13.9810	0.4587	1.8408	3.3888	0.3778	0.4965
	15	0.5	.03	θ	0.3998	0.1590	0.0379	0.4373	0.1301	0.0255	0.4356	0.4389
				μ	2.0820	1.8220	8.8893	1.9609	1.6610	3.7960	1.2539	2.0840
		1.5	1.3	θ	1.3230	0.3648	0.1976	1.3488	0.2781	0.1116	1.3425	1.3551
				μ	4.5251	3.4388	18.4780	0.8289	0.4699	0.2220	0.6062	0.8643
		2.5	2.3	θ	2.2971	0.5381	0.4451	1.8080	0.6982	0.5743	1.7974	1.8188
				μ	4.8351	3.1168	14.8860	0.4504	1.8532	3.4342	0.3720	0.4609
	20	0.5	.03	θ	0.4095	0.1386	0.0287	0.4310	0.1187	0.0205	0.4299	0.4321
				μ	1.9634	1.7020	6.1085	1.9800	1.6800	3.7310	1.2732	2.0986
		1.5	1.3	θ	1.3630	0.3073	0.1420	1.3849	0.2443	0.0885	1.3800	1.3899
				μ	4.5338	3.4524	18.6280	0.8149	0.4853	0.2362	0.5977	0.8494
		2.5	2.3	θ	2.3551	0.4688	0.3363	1.9451	0.5757	0.4133	1.9358	1.9545
				μ	4.8580	3.1184	14.769	0.4462	1.8546	3.4394	0.3690	0.4564
2	10	0.5	.03	θ	0.3974	0.1877	0.0524	0.4324	0.1583	0.0378	0.4310	0.4338
				μ	1.8547	1.5991	5.6800	1.7542	1.4542	3.3564	1.2329	1.8322
		1.5	1.3	θ	1.2921	0.4431	0.2933	1.3851	0.3221	0.1564	1.3797	1.3905
				μ	4.4512	3.3716	17.9170	0.8841	0.4179	0.1828	0.6403	0.9230
		2.5	2.3	θ	2.2731	0.6658	0.7027	1.9253	0.6369	0.5228	1.9159	1.9349
				μ	4.7832	3.0486	14.3340	0.4616	1.8408	3.3886	0.3798	0.4725
	15	0.5	.03	θ	0.4009	0.1575	0.0369	0.4217	0.1368	0.0278	0.4208	0.4225
				μ	1.9345	1.6773	5.9956	1.8599	1.5599	3.8275	1.3143	1.9378
		1.5	1.3	θ	1.3434	0.3575	0.1923	1.4200	0.2770	0.1177	1.4164	1.4237
				μ	4.5409	3.4574	18.6910	0.8508	0.4503	.2070	0.6197	0.8876
		2.5	2.3	θ	2.3128	0.5331	0.4420	2..0654	0.5213	0.3663	2..0582	2.0727
				μ	4.7775	3.0597	14.394	0.4523	1.8504	3.4242	0.3732	0.4627
	20	0.5	.03	θ	0.4113	0.1385	0.0288	0.4252	0.1236	0.0229	0.4246	0.4258
				μ	1.9570	1.6920	5.9638	1.9101	1.6099	4.0158	1.3505	1.9889
		1.5	1.3	θ	1.3616	0.3069	0.1425	1.4283	0.2460	0.0939	1.4256	1.4310
				μ	4.6399	3.5382	19.477	0.8337	0.4651	0.2189	0.6093	0.8694
		2.5	2.3	θ	2.3607	0.4686	0.3419	2.1619	0.4554	0.2881	2.1560	2.1679
				μ	4.8075	3.0746	14.461	0.4474	1.8532	3.4343	0.3699	0.4577

Table 3
Mean, MSEs of the Estimators of θ and μ based on one- and two-cycle ERSS-type II

m	n	η	ρ	Par	$(.)_{ML}$			$(.)_{SE}$			$(.)_{LENIX}$	
					Mean	Bias	MSE	Mean	Bias	MSE	c=1	c= -0.1
1	10	0.5	.03	θ	0.4537	0.2053	0.0612	0.4774	0.1161	0.0190	0.4745	0.4803
				μ	0.3414	0.1420	0.0313	0.3486	0.0915	0.1327	0.3343	0.3500
		1.5	1.3	θ	1.2268	0.5379	0.4329	1.3698	0.2948	0.1265	1.3550	1.3846
				μ	1.7893	0.8843	1.2912	1.4351	0.3782	0.2103	1.1183	1.4727
		2.5	2.3	θ	1.950	0.8305	1.0753	2.6330	0.3003	0.1433	2.6190	2.6468
				μ	3.5827	1.9505	6.4166	0.9982	1.3013	1.7443	0.7308	1.0378
	15	0.5	.03	θ	0.4539	0.1878	0.0522	0.4709	0.1214	0.0215	0.4685	0.4734
				μ	0.3346	0.1215	0.0228	0.3362	0.0859	0.0109	0.3260	0.3372
		1.5	1.3	θ	1.2874	0.4816	0.3599	1.3780	0.3133	0.1431	1.3652	1.3908
				μ	1.6490	0.7256	0.8687	1.4592	0.4289	0.2775	1.1962	1.4882
		2.5	2.3	θ	2.0633	0.7452	0.8955	2.7726	0.3606	0.1896	2.5790	2.7860
				μ	3.3040	1.6687	4.7029	1.1832	1.1206	1.3844	0.8572	1.2302
	20	0.5	.03	θ	0.4542	0.1812	0.0493	0.4675	0.1328	0.0246	0.4654	0.4696
				μ	0.3298	0.1105	0.0186	0.3299	0.0837	0.0101	0.3220	0.3307
		1.5	1.3	θ	1.3335	0.4451	0.3093	1.3873	0.3268	0.1563	1.3764	1.3986
				μ	1.5519	0.6400	0.6595	1.4459	0.4501	0.3010	1.2299	1.4690
		2.5	2.3	θ	2.2040	0.6439	0.6798	2.8138	0.3972	0.2181	2.8004	2.8269
				μ	2.9534	1.3621	3.1587	1.3167	1.0028	1.1968	0.9644	1.3650
2	10	0.5	.03	θ	0.4584	0.2024	0.0594	0.4711	0.1515	0.0315	0.4692	0.0473
				μ	0.3403	0.1432	0.0314	0.3432	0.1125	0.0189	0.3338	0.3441
		1.5	1.3	θ	1.2605	0.5380	0.4371	1.3662	0.3751	0.2041	1.3561	1.3763
				μ	1.7375	0.8845	1.2970	1.4976	0.5575	0.4680	1.2504	1.5245
		2.5	2.3	θ	1.9849	0.8292	1.0761	2.6991	0.3999	0.2450	2.6869	2.7112
				μ	3.5737	1.9532	6.4251	1.2637	1.0509	1.0509	0.9203	1.3118
	15	0.5	.03	θ	0.4575	0.1872	0.0519	0.4655	0.1557	0.1557	0.4640	0.6470
				μ	0.3299	0.1216	0.0226	0.3310	0.1030	0.1030	0.3248	0.3316
		1.5	1.3	θ	1.2726	0.4916	0.3733	1.3391	0.3920	0.3920	1.3320	1.3483
				μ	1.6651	0.7478	0.9096	1.5350	0.5739	0.5739	1.3540	1.5539
		2.5	2.3	θ	2.0993	0.7308	0.8520	2.6988	0.4267	0.4267	2.6867	2.7106
				μ	3.2317	1.6201	4.3823	1.5753	0.9121	0.9121	1.1793	1.6257
	20	0.5	.03	θ	0.4550	0.1784	0.0478	0.4606	0.1563	0.1563	0.4594	0.4619
				μ	0.3305	0.1089	0.0184	0.3312	0.0964	0.0964	0.3264	0.3317
		1.5	1.3	θ	1.3244	0.4399	0.3067	1.3636	0.3845	0.3845	1.3572	1.3700
				μ	1.5774	0.6417	0.6716	1.5060	0.5467	0.5467	1.3664	1.5200
		2.5	2.3	θ	2.1752	0.5646	0.7113	2.6613	0.4324	0.4324	2.6501	2.6721
				μ	3.0212	1.3854	3.2798	1.7904	0.8961	0.8961	1.4041	1.8358

Table 4
Mean, MSEs of the Estimators of θ and μ based on one- and two-cycle SRS

m	n	η	ρ	Par	$(.)_{ML}$			$(.)_{SE}$			$(.)_{LENIX}$	
					Mean	Bias	MSE	Mean	Bias	MSE	c=1	c= -0.1
1	10	0.5	.03	θ	0.4629	0.2211	0.0759	0.5184	0.1422	0.0354	0.5140	0.5229
				μ	0.4817	0.2890	0.2170	0.5576	0.2822	0.1827	0.4923	0.5652
		1.5	1.3	θ	1.2833	0.6079	0.5599	1.3427	0.3678	0.1986	1.3257	1.3601
				μ	2.5779	1.6593	5.7029	1.1658	0.2576	0.0953	0.8538	1.2113
		2.5	2.3	θ	2.1437	0.9107	1.2886	1.7016	0.7947	0.7322	1.6839	1.7196
				μ	3.9462	2.4148	9.7751	0.6707	1.6288	2.6637	0.5257	0.6909
	15	0.5	.03	θ	0.4566	0.1989	0.0595	0.4930	0.1360	0.0289	0.4899	0.4961
				μ	0.4405	0.2412	0.1233	0.4913	0.2214	0.1024	0.4510	0.4956
		1.5	1.3	θ	1.2956	0.5309	0.4312	0.3364	0.1723	0.1723	1.3713	1.3994
				μ	2.3045	1.4194	4.0935	1.2227	0.3104	0.1424	0.9060	1.2671
		2.5	2.3	θ	2.2056	0.7844	0.9651	1.9814	0.5486	0.3960	1..9649	1.9981
				μ	3.8518	2.3111	9.0939	0.6944	1.6092	2.6045	0.5408	0.7159
	20	0.5	.03	θ	0.4629	0.1829	0.0505	0.4865	0.1346	0.0278	0.4841	0.4890
				μ	0.4063	0.2051	0.0844	0.4485	0.1846	0.0712	0.4193	0.4515
		1.5	1.3	θ	1.3269	0.4691	0.3354	1.4166	0.3062	0.1467	1.4047	1.4285
				μ	2.1259	1.2491	3.1562	1.2505	0.3602	0.1988	0.9395	1.2926
		2.5	2.3	θ	2.2154	0.7122	0.7788	0.4961	0.1659	0.0439	0.4935	0.4988
				μ	0.4973	0.3025	0.2332	0.5285	0.2754	0.1857	0.4855	0.5332
2	10	0.5	.03	θ	0.4630	0.2208	0.0745	0.4961	0.1659	0.0439	0.4935	0.4988
				μ	0.4973	0.3025	0.2332	0.5285	0.2754	0.1857	0.4855	0.5332
		1.5	1.3	θ	1.2938	0.6010	0.5594	1.4382	0.4265	0.2872	1.2457	1.4509
				μ	2.5781	1.6934	5.8958	1.3210	0.4148	0.2715	0.9840	1.3675
		2.5	2.3	θ	2.1749	0.9077	1.2797	2.1135	0.5369	0.4130	2.0984	2.1288
				μ	3.9395	2.4014	9.6971	0.7526	1.5497	2.4417	0.5818	0.7765
	15	0.5	.03	θ	0.4627	0.1970	0.0585	0.4817	0.1602	0.0386	0.4799	0.4835
				μ	0.4304	0.2323	0.1146	0.4550	0.2104	0.0975	0.4313	0.4575
		1.5	1.3	θ	1.2910	0.5332	0.4295	1.4252	0.9318	0.2403	1.4156	1.4349
				μ	2.3322	1.4419	4.2528	1.4224	0.5170	0.4533	1.0929	1.4648
		2.5	2.3	θ	2.1844	0.7897	0.9572	2.3212	0.4408	0.2991	2.3085	2.3341
				μ	3.9480	2.3445	9.3430	0.8184	1.4854	2.2769	0.6261	0.8453
	20	0.5	.03	θ	0.4628	0.1810	0.0500	0.4769	0.1540	0.0357	0.4756	0.4783
				μ	0.4072	0.2054	0.0827	0.4257	0.1875	0.0716	0.4088	0.4274
		1.5	1.3	θ	1.3202	0.4877	0.3637	1.4319	0.3747	0.2215	1.4240	1.4398
				μ	2.1726	1.2913	3.3503	1.4711	0.5842	0.6095	1.1655	1.5083
		2.5	2.3	θ	2.2126	0.7123	0.7701	2.4304	0.4034	0.2601	2.4194	2.4415
				μ	3.8032	2.2104	8.3752	0.8771	1.4274	2.1361	0.6674	0.9063



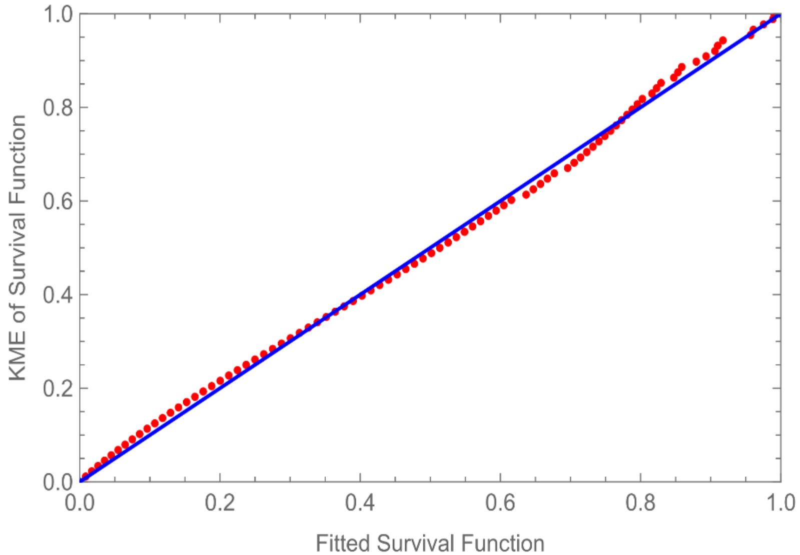


Figure 1: The Empirical and P-P plot of Kaplan-Meier Estimator of the Fitted Linear Exponential Model for Data Set

Table 5
A Ranked Set Sample Design with Sample Size n = 8 when m = 1,2

m	Samples							
1	0.02	0.10	0.11	0.14	0.2	0.34	0.56	0.59
	0.02	0.15	0.29	0.32	0.38	0.39	0.44	0.83
	0.02	0.04	0.06	0.06	0.07	0.09	0.14	0.24
	0.08	0.11	0.13	0.18	0.21	0.24	0.52	0.70
	0.01	0.02	0.08	0.16	0.3	0.40	0.55	0.90
	0.04	0.05	0.07	0.15	0.15	0.20	0.38	0.51
	0.01	0.08	0.08	0.13	0.15	0.35	0.68	1.35
	0.01	0.06	0.08	0.15	0.19	0.34	0.57	0.64
2	0.01	0.04	0.05	0.11	0.11	0.15	0.34	0.71
	0.01	0.05	0.13	0.15	0.17	0.32	0.39	0.45
	0.01	0.04	0.05	0.07	0.10	0.12	0.17	0.37
	0.01	0.16	0.21	0.30	0.30	0.31	0.32	0.39
	0.03	0.06	0.07	0.07	0.10	0.21	0.46	0.71
	0.03	0.04	0.05	0.07	0.14	0.3	0.42	0.55
	0.03	0.07	0.14	0.15	0.21	0.23	0.42	0.58
	0.05	0.05	0.14	0.18	0.22	0.36	0.39	0.90

Table 6
Bayesian Estimates and MLE based on one- and two-cycle SRS, RSS and ERSS for n= 8, where η = 1.5, ρ= 1.3.

m	*par*	RSS				ERSS – type I			
		$(.)_{ML}$	$(.)_{SE}$	$(.)_{LENIX}$		$(.)_{ML}$	$(.)_{SE}$	$(.)_{LENIX}$	
				c=-0.1	c=1			c=-0.1	c=1
1	θ	2.3067	2.9484	2.9641	2.9331	2.0276	1.8423	1.8623	1.8228
	μ	7.9952	1.0789	1.1370	0.7415	9.1424	0.8248	0.8605	0.6022
2	θ	3.3542	3.5169	3.5294	3.5047	2.1049	2.1116	5.1253	2.0980
	μ	3.9000	1.0728	1.1257	0.7500	8.9278	0.8329	0.8329	0.6069

m	*par*	ERSS – type II				SRS			
		$(.)_{ML}$	$(.)_{SE}$	$(.)_{LENIX}$		$(.)_{ML}$	$(.)_{SE}$	$(.)_{LENIX}$	
				c=-0.1	c=1			c=-0.1	c=1
1	θ	2.8992	3.8113	3.8374	3.7852	3.6769	2.4370	2.4776	2.3980
	μ	6.7016	1.3555	1.4416	0.8823	2.3807	0.9878	1.0355	0.6984
2	θ	1.3780	2.3274	2.3517	2.3028	2.3649	2.1938	2.2147	2.1732
	μ	5.5664	2.5526	2.6896	1.6306	2.3392	1.1081	1.1581	0.7870

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