Pak. J. Statist.
2023 Vol. 39(3), 259-270

RELATIONS FOR MOMENTS OF DUAL GENERALIZED ORDER STATISTICS FOR A TRANSMUTED INVERSE WEIBULL DISTRIBUTION AND CHARACTERIZATIONS

Saman Hanif Shahbaz[1] and **Muhammad Qaiser Shahbaz**[2]
Department of Statistics, King Abdulaziz University
Jeddah, Saudi Arabia
Email: [1]shmohamad2@kau.edu.sa
[2]mkmohamad@kau.edu.sa

ABSTRACT

Some relations for moments of dual generalized order statistics for a transmuted inverse Weibull distribution have been derived. These include relations for single, inverse, product and ratio moments. These relations are helpful to compute the higher order moments of dual generalized order statistics from the corresponding lower order moments when sample is available from a transmuted inverse Weibull distribution. The relations for moments of special cases of dual generalized order statistics are also presented. Some characterizations of the transmuted inverse Weibull distribution are also given on the basis of the moments of dual generalized order statistics.

KEY WORDS

Moments; Dual Generalized Order Statistics; Transmuted Inverse Weibull Distribution; Recurrence Relations.

AMS Subject Classification: 60E05, 62E10, 62G30.

1. INTRODUCTION

The dual generalized order statistics (*dgos*) is a classical method to study the properties of random variables which are arranged from highest to lowest. The *dgos* appears as opposite arrangements of the generalized order statistics, introduced by Kamps (1995). Burkschat et al. (2003) has given the joint distribution of *n dgos* as

$$f_{1,2,\ldots,n:n,m,k}\left(x_1,\ldots,x_n\right)=k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left[F\left(x_n\right)\right]^{k-1}f\left(x_n\right)\left[\prod_{i=1}^{n-1}\left\{F\left(x_i\right)\right\}^m f\left(x_i\right)\right], \quad (1)$$

where $\gamma_h=k+\left(n-h\right)\left(m+1\right)$ and $F\left(x_i\right)$ is cumulative distribution function of *i*th random variable. Burkschat et al. (2003) has further shown that the marginal distribution of a single *dgos* and joint distribution of two *dgos* are given as

$$f_{r:n,m,k}\left(x\right)=\frac{C_{r-1}}{\left(r-1\right)!}f\left(x\right)\left[F\left(x\right)\right]^{\gamma_r-1}g_m^{r-1}\left[F\left(x\right)\right], \quad (2)$$

and

$$f_{r,s:n,m,k}(x_1,x_2)=\frac{C_{s-1}}{(r-1)!(s-r-1)!}f(x_1)f(x_2)\left[F(x_1)\right]^m g_m^{r-1}\left[F(x_1)\right]$$
$$\times\left[F(x_2)\right]^{\gamma_s-1}\left[h_m\{F(x_1)\}-h_m\{F(x_2)\}\right]^{s-r-1}, \tag{3}$$

where

$$C_{r-1}=\prod_{h=1}^{r}\gamma_h$$

and

$$h_m(x)=\begin{cases}\dfrac{x^{m+1}}{m+1}; & m\neq-1\\ \ln x; & m=-1.\end{cases}\;;\; g_m(x)=\begin{cases}\dfrac{1}{m+1}\left(1-x^{m+1}\right); & m\neq-1\\ -\ln x; & m=-1.\end{cases}$$

Various models of random variables which are arranged in decreasing order appears as a special case of *dgos*. For example, the decreasing order statistics appears as special case of *dgos* when $m=0$ and $k=1$, Arnold et al. (2008). The lower record values, studied by Chandler (1952) and Dziubdziel and Kopociski (1976), emerges as special case for $m=-1$.

The studies on *dgos* mostly are focused on obtaining some methods for recursive computation of moments for specific choices of distributions in (2) and (3). Studies are also conducted to obtain some characterizations of distributions based upon the moments of *dgos*. The relations for moments of *dgos* for parent Inverse Weibull distribution have been obtained by Pawlas and Szynal (2001). The relations for moments of *dgos* for the power function distribution have been obtained by Athar and Faizan (2011). The recurrence relations for moments of ordered variables for transmuted distributions have not been explored much. Recently, Al–Sobhi et al. (2020) have obtained recurrence relations for moments of order statistics for a transmuted exponential distribution. More details on recurrence relations and characterizations of the distributions by using *dgos* and lower record values can be found in Ahsanullah and Nevzorov (2001) and in Shahbaz et al. (2016).

The recurrence relations for moments of *dgos* for a transmuted distribution are yet to be explored and, in this paper, we have obtained the recurrence relations for moments of *dgos* for transmuted inverse Weibull distribution, introduced by Khan and King (2014). The relations has been obtained for single, inverse, product and ratio moments. These relations are helpful in obtaining corresponding relations for the special cases. The paper also deals with some characterizations of the transmuted power function distribution based upon the single and product moments of *dgos*.

In the following we will give a brief about the transmuted inverse Weibull distribution.

2. TRANSMUTED INVERSE WEIBULL DISTRIBUTION

The inverse Weibull distribution appears as an inverse of the Weibull (1951) distribution. The density and distribution function of a random variable having inverse Weibull distribution are, respectively

$$g(x)=\frac{\theta\beta}{x^{\beta+1}}\exp\left(-\frac{\theta}{x^{\beta}}\right) \text{ and } G(x)=\exp\left(-\frac{\theta}{x^{\beta}}\right);\ x,\beta,\theta>0\ . \tag{4}$$

The distribution has been extensively studied by several authors. Some extensions of the distribution have also been given by using some families of distributions. Hanook et al. (2013) have used the beta–G family of distributions of Eugene et al. (2002) to propose the beta inverse Weibull distribution. Shahbaz et al. (2012) have proposed the Kumaraswamy inverse Weibull distribution by using (4) in the Kumaraswamy family of distributions by Cordeiro and Castro (2011). Khan and King (2014) have used the inverse Weibull distribution in the transmuted family of distributions, proposed by Shaw and Buckley (2007), to obtain the transmuted inverse Weibull distribution. The density and distribution function of the transmuted inverse Weibull distribution are

$$f(x)=\frac{\theta\beta}{x^{\beta+1}}\exp\left(-\frac{\theta}{x^{\beta}}\right)\left[1+\lambda-2\lambda\exp\left(-\frac{\theta}{x^{\beta}}\right)\right];\ x,\beta,\theta>0 \tag{5}$$

and

$$F(x)=\exp\left(-\frac{\theta}{x^{\beta}}\right)\left[1+\lambda-\exp\left(-\frac{\theta}{x^{\beta}}\right)\right];\ x,\beta,\theta>0\ , \tag{6}$$

where $-1\le\lambda\le1$ is the transmutation parameter. The transmuted inverse Weibull distribution reduces to the inverse Weibull distribution for $\lambda=0$. Also, for $\beta=2$, the transmuted inverse Rayleigh distribution, proposed by Ahmad et al. (2014), appears as a special case. It is easy to show that the density and distribution functions, given in (5) and (6), are related as

$$F(x)=\frac{x^{\beta+1}}{\theta\beta}f(x)+\lambda\sum_{j=0}^{\infty}\frac{(-1)^{j}}{j!}\theta^{j}x^{-\beta j}\ . \tag{7}$$

The relation (7) is very useful in obtaining the recurrence relations for moments of dual generalized order statistics for transmuted inverse Weibull distribution. The relation (7) is also useful in obtaining some characterizations for the transmuted inverse Weibull distribution in terms of moments of *dgos*. The recurrence relations and characterizations are given in the following sections.

3. RELATIONS FOR SINGLE AND INVERSE MOMENTS

In this section, we have obtained the recurrence relations for raw and inverse moments of *dgos* for transmuted inverse Weibull distribution. The relations are obtained by using following relation between moments of dual generalized order statistics for any baseline distribution which is given by Kotb et al. (2013) and Shahbaz et al. (2016)

$$\mu_{r:n,m,k}^{p}-\mu_{r-1:n,m,k}^{p}=-\frac{pC_{r-1}}{\gamma_r(r-1)!}\int_{-\infty}^{\infty}x^{p-1}\left[F(x)\right]^{\gamma_r}g_m^{r-1}\left[F(x)\right]dx$$

or

$$\mu_{r:n,m,k}^{p}-\mu_{r-1:n,m,k}^{p}=-\frac{pC_{r-1}}{\gamma_r(r-1)!}\int_{-\infty}^{\infty}x^{p-1}\left[F(x)\right]\left[F(x)\right]^{\gamma_r-1}g_m^{r-1}\left[F(x)\right]dx\,. \quad (8)$$

The recurrence relations for single moments are given in the following theorem.

Theorem 1:

The single moments of dual generalized order statistics for the transmuted inverse Weibull distribution are related as

$$p\mu_{r:n,m,k}^{p+\beta}=\theta\beta\gamma_r\left(\mu_{r-1:n,m,k}^{p}-\mu_{r:n,m,k}^{p}\right)+\lambda\theta p\beta\left(\frac{\gamma_{r(k-1)}C_{r-1}}{\gamma_j C_{r-1(k-1)}}\right)\sum_{j=0}^{\infty}\frac{(-1)^j\theta^j}{j!(p-\beta j)} \quad (9)$$
$$\times\left(\mu_{r:n,m,k-1}^{p-\beta j}-\mu_{r-1:n,m,k-1}^{p-\beta j}\right),$$

where

$$\gamma_{r(k-1)}=(k-1)+(n-r)(m+1) \text{ and } C_{r(k-1)}=\prod_{h=1}^{r}\gamma_{h(k-1)}\,.$$

Proof:

Using (7) in (8), we have

$$\mu_{r:n,m,k}^{p}-\mu_{r-1:n,m,k}^{p}$$
$$=-\frac{pC_{r-1}}{\gamma_r(r-1)!}\int_0^{\infty}x^{p-1}\left\{\frac{x^{\beta+1}}{\theta\beta}f(x)+\lambda\sum_{j=0}^{\infty}(-1)^j\theta^j x^{-\beta j}\right\}\left[F(x)\right]^{\gamma_r-1}$$
$$\times g_m^{r-1}\left[F(x)\right]dx$$
$$=-\frac{pC_{r-1}}{\gamma_r\theta\beta(r-1)!}\int_0^{\infty}x^{p+\beta}f(x)\left[F(x)\right]^{\gamma_r-1}g_m^{r-1}\left[F(x)\right]dx$$
$$-\lambda\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\theta^j\frac{pC_{r-1}}{\gamma_r(r-1)!}\int_0^{\infty}x^{p-\beta j-1}\left[F(x)\right]^{\gamma_r-1}g_m^{r-1}\left[F(x)\right]dx$$

or

$$\mu_{r:n,m,k}^{p}-\mu_{r-1:n,m,k}^{p}=-\frac{p}{\theta\beta\gamma_r}\mu_{r:n,m,k}^{p+\beta}$$
$$-\lambda\sum_{j=0}^{\infty}(-1)^j\theta^j\frac{p(p-\beta j)\gamma_{r(k-1)}C_{r-1}C_{r-1(k-1)}}{\gamma_r\gamma_{r(k-1)}(p-\beta j)C_{r-1(k-1)}(r-1)!}$$
$$\times\int_0^{\infty}x^{p-\beta j-1}\left[F(x)\right]^{\gamma_r-1}g_m^{r-1}\left[F(x)\right]dx$$
$$=-\frac{p}{\theta\beta\gamma_r}\mu_{r:n,m,k}^{p+\beta}+\lambda\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\theta^j\frac{p\gamma_{r(k-1)}C_{r-1}}{(p-\beta j)\gamma_r C_{r-1(k-1)}}$$
$$\times\left(\mu_{r:n,m,k-1}^{p-\beta j}-\mu_{r-1:n,m,k-1}^{p-\beta j}\right).$$

Slight re–arrangement of above equation gives (9) and hence the theorem.

The recurrence relations for moments of *dgos* for transmuted inverse Weibull distribution can be readily obtained from (8) by using $\beta=2$. The relation (9) reduces to the relations for the moments of *dgos* for the inverse Weibull distribution for $\lambda=0$ as obtained by Pawlas and Szynal (2001).

Some corollaries can be immediately written from the above theorem and the same are given below.

Corollary 1:
The recurrence relations for the inverse moments of *dgos* for the transmuted inverse Weibull distribution is given as

$$p\mu_{r:n,m,k}^{\beta-p}=\theta\beta\gamma_r\left(\mu_{r:n,m,k}^{-p}-\mu_{r-1:n,m,k}^{-p}\right)-\lambda\theta p\beta\left(\frac{\gamma_{r(k-1)}C_{r-1}}{\gamma_r C_{r-1(k-1)}}\right)\sum_{j=0}^{\infty}\frac{(-1)^j\theta^j}{j!(\beta j+p)}$$
$$\times\left[\mu_{r:n,m,k-1}^{-(\beta j+p)}-\mu_{r-1:n,m,k-1}^{-(\beta j+p)}\right], \tag{10}$$

and can be easily obtained from (9) by replacing "*p*" with "*–p*"

Corollary 2:
The raw and inverse moments of lower record values for the transmuted inverse Weibull distribution are related as

$$p\mu_{K(r)}^{p+\beta}=\theta\beta k\left(\mu_{K(r-1)}^{p}-\mu_{K(r)}^{p}\right)$$
$$+\lambda\theta p\beta\left(\frac{k}{k-1}\right)^r\sum_{j=0}^{\infty}\frac{(-1)^j\theta^j}{j!(p-\beta j)}\left[\mu_{K-1(r)}^{p-\beta j}-\mu_{K-1(r-1)}^{p-\beta j}\right], \tag{11}$$

and

$$p\mu_{K(r)}^{\beta-p} = \theta\beta k\left(\mu_{K(r)}^{-p} - \mu_{K(r-1)}^{-p}\right)$$
$$-\lambda p\theta\beta\left(\frac{k}{k-1}\right)^r \sum_{j=0}^{\infty} \frac{(-1)^j \theta^j}{j!(\beta j+p)}\left[\mu_{K-1(r)}^{-(\beta j+p)} - \mu_{K-1(r-1)}^{-(\beta j+p)}\right] \quad (12)$$

and can be obtained by using $m = -1$ in (9) and (10), respectively.

Corollary 3:

The raw and inverse moments of reversed order statistics for the transmuted inverse Weibull distribution are related as

$$p\mu_{r:n}^{p+\beta} = \theta\beta(n-r+1)\left(\mu_{r-1:n}^{p} - \mu_{r:n}^{p}\right)$$
$$+\left(\frac{\lambda\theta p\beta n}{n-r+1}\right)\sum_{j=0}^{\infty} \frac{(-1)^j \theta^j}{j!(p-\beta j)}\left(\mu_{r:n}^{p-\beta j} - \mu_{r-1:n}^{p-\beta j}\right) \quad (13)$$

and

$$p\mu_{r:n}^{\beta-p} = \theta\beta(n-r+1)\left(\mu_{r:n}^{-p} - \mu_{r-1:n}^{-p}\right)$$
$$-\left(\frac{\lambda\theta p\beta n}{n-r+1}\right)\sum_{j=0}^{\infty} \frac{(-1)^j \theta^j}{j!(\beta j+p)}\left[\mu_{r:n}^{-(\beta j+p)} - \mu_{r-1:n}^{-(\beta j+p)}\right] \quad (14)$$

and can be obtained by using $m = 0$ and $k = 1$ in (9) and (10), respectively.

4. RELATIONS FOR PRODUCT AND RATIO MOMENTS

In this section, we have obtained the recurrence relations for product and ratio moments of *dgos* for transmuted inverse Weibull distribution. The relations are obtained by using following relation between moments of dual generalized order statistics for any baseline distribution which is given by Kotb et al. (2013) and Shahbaz et al. (2016)

$$\mu_{r,s:n,m,k}^{p,q} - \mu_{r,s-1:n,m,k}^{p,q} = -\frac{qC_{s-1}}{\gamma_s (r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{-\infty}^{x_1} x_1^p x_2^{q-1} f(x_1)\left[F(x_1)\right]^m$$
$$\times g_m^{r-1}\left[F(x_1)\right]\left[h_m(x_1)-h_m(x_2)\right]^{s-r-1}\left[F(x_2)\right]^{\gamma_s} dx_2 dx_1$$

or

$$\mu_{r,s:n,m,k}^{p,q} - \mu_{r,s-1:n,m,k}^{p,q} = -\frac{qC_{s-1}}{\gamma_s (r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{-\infty}^{x_1} x_1^p x_2^{q-1} f(x_1)\left[F(x_1)\right]^m$$
$$\times g_m^{r-1}\left[F(x_1)\right]\left[h_m(x_1)-h_m(x_2)\right]^{s-r-1}\left[F(x_2)\right]^{\gamma_s-1} \times\left[F(x_2)\right] dx_2 dx_1. \quad (15)$$

The recurrence relation is given in the following theorem.

Theorem 1:

The single moments of dual generalized order statistics for the transmuted inverse Weibull distribution are related as

$$q\mu_{r,s:n,m,k}^{p,q+\beta}=\theta\beta\gamma_s\left(\mu_{r,s-1:n,m,k}^{p,q}-\mu_{r,s:n,m,k}^{p,q}\right)$$
$$+\lambda\theta q\beta\left(\frac{\gamma_{s(k-1)}C_{s-1}}{\gamma_s C_{s(k-1)}}\right)\sum_{j=0}^{\infty}\frac{(-1)^j\,\theta^j}{j!(q-\beta j)}\times\left(\mu_{r,s:n,m,k-1}^{p,q-\beta j}-\mu_{r,s-1:n,m,k-1}^{p,q-\beta j}\right), \qquad (16)$$

where $\gamma_{r(k-1)}=(k-1)+(n-r)(m+1)$ and $C_{r(k-1)}=\prod_{h=1}^{r}\gamma_{h(k-1)}$.

Proof:

Using (7) in (15) we have

$$\mu_{r,s:n,m,k}^{p,q}-\mu_{r,s-1:n,m,k}^{p,q}=-\frac{qC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{-\infty}^{x_1}x_1^p x_2^{q-1}f(x_1)\left[F(x_1)\right]^m$$
$$\times g_m^{r-1}\left[F(x_1)\right]\left[h_m(x_1)-h_m(x_2)\right]^{s-r-1}\left[F(x_2)\right]^{\gamma_s-1}$$
$$\times\left[\frac{x^{\beta+1}}{\theta\beta}f(x)+\lambda\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\theta^j x^{-\beta j}\right]dx_2dx_1$$

or

$$\mu_{r,s:n,m,k}^{p,q}-\mu_{r,s-1:n,m,k}^{p,q}=-\frac{q}{\theta\beta\gamma_s}\mu_{r,s:n,m,k}^{p,q+\beta}-\lambda\sum_{j=0}^{\infty}\frac{(-1)^j\,\theta^j}{j!}$$
$$\times\frac{q(q-\beta j)\gamma_{s(k-1)}C_{s-1}C_{s(k-1)}}{\gamma_s(q-\beta j)\gamma_{s(k-1)}C_{s(k-1)}(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{-\infty}^{x_1}x_1^p x_2^{q-\beta j-1}f(x_1)$$
$$\times\left[F(x_1)\right]^m g_m^{r-1}\left[F(x_1)\right]\left[h_m(x_1)-h_m(x_2)\right]^{s-r-1}\left[F(x_2)\right]^{\gamma_s-1}dx_2dx_1$$

or

$$\mu_{r,s:n,m,k}^{p,q}-\mu_{r,s-1:n,m,k}^{p,q}=-\frac{q}{\theta\beta\gamma_s}\mu_{r,s:n,m,k}^{p,q+\beta}+\lambda\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\theta^j\frac{q\gamma_{s(k-1)}C_{s-1}}{\gamma_s(q-\beta j)C_{s(k-1)}}$$
$$\times\left(\mu_{r,s:n,m,k-1}^{p,q-\beta j}-\mu_{r,s-1:n,m,k-1}^{p,q-\beta j}\right).$$

Re–arranging above equation, we have (16) and hence the theorem.

Some corollaries that immediately follow from Theorem 2 are given below.

Corollary 4:

The recurrence relations for the ratio moments of *dgos* for the transmuted inverse Weibull distribution is given as

$$q\mu_{r,s:n,m,k}^{p,\beta-q} = \theta\beta\gamma_s\left(\mu_{r,s:n,m,k}^{p,-q} - \mu_{r,s-1:n,m,k}^{p,-q}\right)$$
$$-\lambda\theta q\beta\left(\frac{\gamma_{s(k-1)}C_{s-1}}{\gamma_s C_{s(k-1)}}\right)\sum_{j=0}^{\infty}\frac{(-1)^j\,\theta^j}{j!(\beta j+q)}\times\left[\mu_{r,s:n,m,k-1}^{p,-(\beta j+q)} - \mu_{r,s-1:n,m,k-1}^{p,-(\beta j+q)}\right], \tag{17}$$

and can be easily obtained from (16) by replacing "*p*" with "–*p*"

Corollary 5:

The product and ratio moments of lower record values for the transmuted inverse Weibull distribution are related as

$$q\mu_{K(r,s)}^{p,q+\beta} = \theta\beta k\left(\mu_{K(r,s-1)}^{p,q} - \mu_{K(r,s)}^{p,q}\right)$$
$$+\lambda\theta q\beta\left(\frac{k}{k-1}\right)^r\sum_{j=0}^{\infty}\frac{(-1)^j\,\theta^j}{j!(q-\beta j)}\times\left[\mu_{K-1(r,s)}^{p,q-\beta j} - \mu_{K-1(r,s-1)}^{p,q-\beta j}\right], \tag{18}$$

and

$$q\mu_{K(r,s)}^{p,\beta-q} = \theta\beta k\left(\mu_{K(r,s)}^{p,-q} - \mu_{K(r,s-1)}^{p,-q}\right)$$
$$-\lambda\theta q\beta\left(\frac{k}{k-1}\right)^r\sum_{j=0}^{\infty}\frac{(-1)^j\,\theta^j}{j!(\beta j+q)}\times\left[\mu_{K-1(r,s)}^{p,-(\beta j+q)} - \mu_{K-1(r,s-1)}^{p,-(\beta j+q)}\right], \tag{19}$$

and can be obtained by using $m = -1$ in (16) and (17), respectively.

Corollary 6:

The raw and inverse moments of reversed order statistics for the transmuted inverse Weibull distribution are related as

$$q\mu_{r,s:n}^{p,q+\beta} = \theta\beta(n-s+1)\left(\mu_{r,s-1:n}^{p,q} - \mu_{r,s:n}^{p,q}\right)$$
$$+\left(\frac{\lambda\theta q\beta n}{n-s+1}\right)\sum_{j=0}^{\infty}\frac{(-1)^j\,\theta^j}{j!(q-\beta j)}\times\left(\mu_{r,s:n}^{p,q-\beta j} - \mu_{r,s-1:n}^{p,q-\beta j}\right), \tag{20}$$

and

$$q\mu_{r,s:n}^{p,\beta-q} = \theta\beta(n-s+1)\left(\mu_{r,s:n}^{p,-q} - \mu_{r,s-1:n}^{p,-q}\right)$$
$$-\left(\frac{\lambda\theta q\beta n}{n-s+1}\right)\sum_{j=0}^{\infty}\frac{(-1)^j\,\theta^j}{j!(\beta j+q)}\times\left[\mu_{r,s:n}^{p,-(\beta j+q)} - \mu_{r,s-1:n}^{p,-(\beta j+q)}\right], \tag{21}$$

and can be obtained by using $m = 0$ and $k = 1$ in (16) and (17), respectively.

We will, now, give some characterizations of the transmuted inverse Weibull distribution using single and product moments of *dgos*.

5. CHARACTERIZATIONS

In this section, we will give some characterizations of the transmuted inverse Weibull distribution on the basis of moments of *dgos*. These characterizations are given in the following Theorems.

Theorem 3:

The necessary and sufficient condition for a random variable X to have density and distribution functions (5) and (6) is that its moments are related as

$$p\mu_{r:n,m,k}^{p+\beta} = \theta\beta\gamma_r\left(\mu_{r-1:n,m,k}^{p} - \mu_{r:n,m,k}^{p}\right)$$
$$+\lambda\theta p\beta\left(\frac{\gamma_{r(k-1)}C_{r-1}}{\gamma_j C_{r-1(k-1)}}\right)\sum_{j=0}^{\infty}\frac{(-1)^j\theta^j}{j!(p-\beta j)}\left(\mu_{r:n,m,k-1}^{p-\beta j} - \mu_{r-1:n,m,k-1}^{p-\beta j}\right).$$

Proof:

The necessary condition immediately follows from Theorem 1. For sufficient condition, consider (7) and (8) and hence

$$-\frac{pC_{r-1}}{\gamma_r(r-1)!}\int_{-\infty}^{\infty}x^{p-1}\left[F(x)\right]^{\gamma_r} g_m^{r-1}\left[F(x)\right]dx$$
$$= -\frac{pC_{r-1}}{\gamma_r(r-1)!}\int_{-\infty}^{\infty}x^{p-1}\left[F(x)\right]^{\gamma_r-1}$$
$$\times\left\{\frac{x^{\beta+1}}{\theta\beta}f(x) + \lambda\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\theta^j x^{-\beta j}\right\}g_m^{r-1}\left[F(x)\right]dx$$

or

$$-\frac{pC_{r-1}}{\gamma_r(r-1)!}\int_{-\infty}^{\infty}x^{p-1}\left[F(x)\right]^{\gamma_r-1} g_m^{r-1}\left[F(x)\right]$$
$$\times\left[F(x) - \left\{\frac{x^{\beta+1}}{\theta\beta}f(x) + \lambda\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\theta^j x^{-\beta j}\right\}\right]dx = 0.$$

Using *Müntz–Száz* theorem; see Hwang and Lin (1984); to above equation we have

$$F(x) - \left\{\frac{x^{\beta+1}}{\theta\beta}f(x) + \lambda\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\theta^j x^{-\beta j}\right\} = 0$$
$$\Rightarrow F(x) = \frac{x^{\beta+1}}{\theta\beta}f(x) + \lambda\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\theta^j x^{-\beta j}$$

and hence the Theorem.

Theorem 4:

The necessary and sufficient condition for a random variable X to have density and distribution functions (5) and (6) is that its moments are related as

$$\mu_{r,s:n,m,k}^{p,q+\beta} = \frac{\theta\beta\gamma_s}{q}\left(\mu_{r,s-1:n,m,k}^{p,q} - \mu_{r,s:n,m,k}^{p,q}\right) + \lambda\theta\beta\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\theta^j\frac{\gamma_{s(k-1)}C_{s-1}}{(q-\beta j)C_{s(k-1)}} \times\left\{\mu_{r,s:n,m,k-1}^{p,q-\beta j} - \mu_{r-1,s:n,m,k-1}^{p,q-\beta j}\right\}.$$

Proof:

The necessary part immediately follows from Theorem 2. For sufficient part we consider (15) as

$$\mu_{r,s:n,m,k}^{p,q} - \mu_{r,s-1:n,m,k}^{p,q} = -\frac{qC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{-\infty}^{x_1} x_1^p x_2^{q-1} f(x_1)\left[F(x_1)\right]^m \times g_m^{r-1}\left[F(x_1)\right]\left[h_m(x_1)-h_m(x_2)\right]^{s-r-1}\left[F(x_2)\right]^{\gamma_s} dx_2 dx_1.$$

Using above equation with (7) we have

$$-\frac{qC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_0^c\int_0^{x_1} x_1^p x_2^{q-1} f(x_1)\left[F(x_1)\right]^m g_m^{r-1}\left[F(x_1)\right]\left[h_m(x_1)-h_m(x_2)\right]^{s-r-1}\times\left[F(x_2)\right]^{\gamma_s} dx_2 dx_1$$

$$= -\frac{qC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_0^c\int_0^{x_1} x_1^p x_2^{q-1} f(x_1)\left[F(x_1)\right]^m \times g_m^{r-1}\left[F(x_1)\right]\left[h_m(x_1)-h_m(x_2)\right]^{s-r-1}\left[F(x_2)\right]^{\gamma_s-1} \times\left\{\frac{x_2^{\beta+1}}{\theta\beta}f(x_2) + \lambda\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\theta^j x_2^{-\beta j}\right\}dx_2 dx_1$$

or

$$-\frac{qC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_0^c\int_0^{x_1} x_1^p x_2^{q-1} f(x_1)\left[F(x_1)\right]^m g_m^{r-1}\left[F(x_1)\right]\left[h_m(x_1)-h_m(x_2)\right]^{s-r-1} \times\left[F(x_2)\right]^{\gamma_s-1}\left[F(x_2) - \left\{\frac{x_2^{\beta+1}}{\theta\beta}f(x_2) + \lambda\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\theta^j x_2^{-\beta j}\right\}\right]dx_2 dx_1 = 0$$

Using *Müntz-Száz* theorem; see Hwang and Lin (1984); to above equation we have

$$F(x_2)-\left\{\frac{x_2^{\beta+1}}{\theta\beta}f(x_2)+\lambda\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\theta^j x_2^{-\beta j}\right\}$$

$$=0\Rightarrow F(x_2)=\frac{x_2^{\beta+1}}{\theta\beta}f(x_2)+\lambda\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\theta^j x_2^{-\beta j}$$

and hence the Theorem.

6. CONCLUSIONS

In this paper we have obtained the recurrence relations for single, inverse, product and ratio moments of *dgos* for transmuted inverse Weibull distribution. We have also obtained the recurrence relations for moments of the special cases of *dgos*. We have also given some characterizations for the transmuted inverse Weibull distribution on the basis of single and product moments of *dgos*. These relations are useful to recursively compute the higher order moments from the lower order moments.

REFERENCES

1. Ahmad, A., Ahmad, S.P. and Ahmed, A. (2014). Transmuted inverse Rayleigh distribution: a generalization of the inverse Rayleigh distribution. *Math. Theory and Modeling*, 4, 90-98.
2. Ahsanullah, M. and Nevzorov, V.B. (2001). *Ordered Random Variables*, Nova Science Publishers, USA.
3. Al-Sobhi, M., Hanif Shahbaz, S. and Shahbaz, M.Q. (2020). Recursive Computation for Moments of Order Statistics for Transmuted Exponential Distribution. *Advances and Applications in Statistics*, 63, 175-189.
4. Arnold, B.C., Balakrishnan, N. and Nagaraja, H.N. (2008). *A First Course in Order Statistics*, SIAM, USA.
5. Athar, H. and Faizan, M. (2011). Moments of lower generalized order statistics for Power Function distribution and a characterization. *Inter. J. of Stat. Sci.*, 11, 125-134.
6. Burkschat, M., Cramer, E. and Kamps, U. (2003). Dual generalized order statistics. *Metron*, 61(01), 13-26.
7. Chandler, K.N. (1952). The distribution and frequency of record values. *J. Royal Stat. Soc. B*, 14(02), 220-228.
8. Cordeiro, G.M. and de Castro, M. (2011). A new family of generalized distributions. *J. Stat. Comp. Simul.*, 81, 883-898.
9. Dziubdziel, W.A. and Kopociski, B. (1976). Limiting properties of the k-th record value, *App. Math.*, 15(02), 187-190.
10. Eugene, N., Lee, C. and Famoye, F. (2002). Beta-normal distribution and its applications. *Comm. Statist. Theor. Meth.*, 31, 497-512.
11. Hanook, S., Shahbaz, M.Q., Mohsin, M. and Kibria, B.M.G. (2013). A note of beta-inverse Weibull distribution. *Comm. Stat. Theo. Meth.*, 42, 320-335.
12. Hwang, J.S. and Lin, G.D. (1984). Extensions of *Müntz-Száz* theorems and application. *Analysis*, 4(12), 143-160.
13. Kamps, U. (1995). A concept of generalized order statistics, *J. Stat. Plann. and Infer.*, 48(01), 1-23.

14. Khan, M.S. and King, R. (2014). A new class of transmuted inverse Weibull distribution for reliability analysis. *American J. Math. & Manag. Sci.*, 33, 261-286.
15. Kotb, M.S., El-Din, M.M.M. and Newer, H.A. (2013). Recurrence relations for single and product moments of lower generalized order statistics from a general class of distributions and its characterizations. *Int. J. of Pure and App. Math.*, 87(02), 229-247.
16. Pawlas, P. and Szynal, D. (2001). Recurrence relations for single and product moment of lower generalized order statistics from Inverse Weibull distribution. *Demonstratio Mathematica*, 34(2), 353-358
17. Shahbaz, M.Q., Ahsanullah M., Hanif Shahbaz, S. and Al-Zahrani B. (2016) *Ordered Random Variables: Theory and Applications*, Atlantis Press and Springer.
18. Shahbaz, M.Q., Shahbaz, S. and Butt, N.S. (2012). The Kumaraswamy Inverse Weibull distribution. *Pak. J. Stat. & OR.*, 8, 479-489.
19. Shaw, W.T. and Buckley, I.R. (2009). The alchemy of probability distributions: beyond Gram-Charlier expansions, and a skew-kurtotic-normal distribution from a rank transmutation map. *arXiv preprint arXiv:0901.0434*.
20. Weibull, W. (1951) A statistical distribution of wide applicability. *J. App. Mech.*, 18, 293-297.