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# GENERALIZED MODIFIED MATRIX BESSEL DISTRIBUTION

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## ABSTRACT

This article presents the "Generalized Modified Matrix Bessel Distribution (GMMaB)", which is a generalization of the "Generalized Multivariate Modified Bessel Distribution" (GMMB). The features of the suggested distribution, including characteristic functions (C.f.), moments, marginal and conditional distributions, and the linear combination of Matrix variate in several forms have been investigated.

## KEYWORDS

Matrix normal distribution, Generalized Inverse, Gaussian distributions, Modified Bessel function of the third Kind.

## 1. INTRODUCTION

Numerous research in multivariate analysis focus on the independent selection of random samples from multivariate normal populations. In some instances, the data may be dependent yet uncorrelated, or the distribution of the data may belong to a family with larger tails than the normal distribution. To represent this kind of data, the multivariate t distribution is the preferred contaminated distribution.

Zellner (1976), Thabane and Drekie (2003) investigated the notion of the equality of vector means of two GMMB depending on Correlated samples.

The (GMMB) distribution is thought of as the generalization of multivariate normal distribution, t distribution, multivariate Bessel and modified Bessel distribution Thebane and Drekie (2004).

By combining the univariate or multivariate normal distribution with the inverse Gamma (IG) or the generalized inverse Gaussian (GIG) distribution for the scale (variance parameter in the univariate case to construct univariate t and univariate Generalized Modified Bessel (GMB) distribution, numerous studies have utilized distinct clear forms of the compound normal distribution. Drekie and Drebane (2003).

There is an additional mixing distribution utilized in Bayesian analysis. Utilizing an inverse Wishart (IW) conjugate prior distribution for the scale matrix in a multivariate normal distribution generates a matrix t distribution for the posterior and the predicted distribution.

 

In this research, the GMMB was extended to a (GMMaB) by combining the matrix normal distribution with the GIG distribution for the scale parameter. (GMMaB) density function was previously described in Section 2. Section 3 discusses various aspects of this distribution, including the characteristic function (cf) and some significant moments. In Section 4, marginal distributions of random variables were discussed. Although the conditional distribution is shown in Section 5. Section 6 presents the distribution of linear combinations of (GMMaB) variables in various situations.

## 2. MATERIAL AND METHODS

### 2.1 Probability Density Function of Generalized Modified Matrix Bessel Distribution

In this section, the mixed distribution has been used to generalize GMMB by defining the random matrix variables of order $(p \times n)$ where each column contains p dependent random variables and each row contains n dependent random variables, as a matrix normal distribution conditioned with scale variable w denoted by $X|w \sim N_{p,n}$ $(\mu, w\Sigma, w\,\Psi)$ with probability density function. given by:

$$f(X|w) = (2\pi)^{-\frac{np}{2}}\,|w\Sigma|^{-\frac{n}{2}}\,|\Psi|^{-\frac{p}{2}}.exp\left\{-\frac{1}{2w}\;tr[\Sigma^{-1}(X-\mu)\Psi^{-1}(X-\mu)'\,]\right\}\ldots \quad (1)$$

where $-\infty < X, \mu < \infty$ , $X$ is a random matrix of size $(p \times n)$, μ is a $(p \times n)$ matrix of location parameters, Σ is a positive definite matrix of size $(p \times p)$ which represents the variance - covariance among the rows of $X$ and Ψ is an $(n \times n)$ positive definite which represents the variances - covariances among columns of $X$. The random variable W∼ GIG $(\lambda, \delta, \nu)$ with p.d.f. defined as, Hormann and Leydold (2014)

$$f(w) = \frac{\left(\frac{\lambda}{\delta}\right)^{\frac{\nu}{2}}\,w^{\nu-1}}{2k_\nu\left(\sqrt{\lambda\delta}\right)}\;.exp\left\{-\frac{1}{2}\left(\frac{\delta}{w}+\lambda w\right)\right\}\ldots \quad (2)$$

when $(w, \lambda, \delta > 0)$ and zero otherwise, the parameters λ, δ represent scale parameters, ν is a shape parameter and $k_\nu(.)$ is a modified Bessel function of the third kind, so that See Kim and Genton, (2011)

$$k_\nu(u) = \frac{1}{2}\int_0^\infty z^{\nu-1}\exp\left\{-\frac{u}{2}\;(z+z^{-1}\,)\right\}dz\ldots \quad (3)$$

Using Bayes theorem, the unconditional distribution of $X$ is given by

$$f(X) = \int_0^\infty f(X|w)\;f(w)dw\ldots \quad (4)$$

By substituting Eq (1) and Eq (2) into Eq (3) and summing the result, we get

$$f(X)=\frac{\left(\frac{\lambda}{\delta}\right)^{\frac{np}{4}}\ k_{\left(\frac{2\nu-np}{2}\right)}\left(\sqrt{\lambda\delta\left(1+\frac{tr[\Sigma^{-1}(X-\mu)\Psi^{-1}(X-\mu)'\,]}{\delta}\right)}\right)}{(2\pi)^{\frac{np}{2}}\,|\Sigma|^{\frac{n}{2}}\,|\Psi|^{\frac{p}{2}}k_{\nu}\left(\sqrt{\lambda\delta}\right)\left(1+\frac{tr[\Sigma^{-1}(X-\mu)\Psi^{-1}(X-\mu)'\,]}{\delta}\right)^{\left(\frac{2\nu-np}{4}\right)}}\ \ldots \tag{5}$$

The p.d.f. in Eq (5) is known as the GMMaB denoted by $X\sim\mathrm{GMMaB}_{\mathrm{p,n}}\ (\mu,\Sigma,\Psi,\lambda,\delta,\nu)$. When $n=1$, the $\mathrm{GMMaB}_{\mathrm{p,n}}$ reduces to $\mathrm{GMMB}_{\mathrm{p}}$, and when $n=p=1$, the distribution reduces to univariate generalized modified Bessel distribution GMB.

## 2.2 Characteristic Function and Some Moments

The characteristic function (c.f.) of a random matrix $X$ may be defined as follows:

$$\phi_X{}^{(T)}\ =\ Ee^{iT'X}\ \ldots \tag{6}$$

where $i=\sqrt{-1}$ , $T$ is a matrix of $(p\times n)$ order, and, the domain of $T$ is $-H<T<H$, and $H$ is a $(p\times n)$ matrix.

The expectation in Eq (4) can be obtained by using mixed distributions, so that

$$\phi_X{}^{(t)}\ =\ E_w\left(E_Xe^{it'X}\right)=E_w\left(\phi_{X|w}{}^{(T)}\right)\ldots \tag{7}$$

where $\phi_{X|w}{}^{(T)}$ is a c.f. of matrix normal defined by

$$\phi_{x|w}{}^{(T)}=e^{itr(T\prime\mu)}-\frac{w}{2}tr(\Sigma T\Psi T')\ \ldots \tag{8}$$

(See Kolloand Rosen, (2005)).

Substituting Equation (8) into Equation (6) gives the marginal c.f. of

$$X.\,\phi_X{}^{(T)}\ =\frac{\left(\frac{\lambda}{\delta}\right)^{\frac{2\nu}{2}}\ e^{itr(T\prime\mu)}k_{\nu}\left(\sqrt{\lambda\delta\left(1+\frac{tr(\Sigma T\Psi T')}{\lambda}\right)}\right)\left(1+\frac{tr(\Sigma T\Psi T')}{\lambda}\right)^{\frac{\nu}{2}}}{k_{\nu}\left(\sqrt{\lambda\delta}\right)}\ \ldots \tag{9}$$

To derive the first and second matrix moments of $X$, i.e., EX, EXX', the expression EX, EXX' may be used.

$$EX=E_w(E_x(X|w))\ and\ .EXX'=(E_x(XX'|w)).$$

$E(X|w)=\mu$ then $E(X)\ =\mu$ and $E(XX'|w)\ =\ w\ (\Sigma\otimes\Psi)$,

and therefore

$$E(XX'|w)\ =\int_0^{\infty}w\ (\Sigma\otimes\Psi)f(w)\ dw\ =(\Sigma\otimes\Psi)\frac{\left(\frac{\lambda}{\delta}\right)^{\frac{\nu}{2}}}{2k_{\nu}\left(\sqrt{\lambda\delta}\right)}$$

$$\int_0^{\infty}w^{\nu+1-1}\ e^{-\frac{1}{2}\left(\frac{1}{w}+\lambda w\right)}\ dw=\frac{(\Sigma\otimes\Psi)k_{(\nu+1)}\left(\sqrt{\lambda w}\right)\ \delta^{\frac{1}{2}}}{k_{\nu}\left(\sqrt{\lambda\delta}\right)\ \lambda^{\frac{1}{2}}}\ \ldots \tag{10}$$

### 2.3 Marginal Distributions

Marginal distributions of subsets of the random matrix $X$ are investigated, we partitioned the matrices $X, \mu, T, \Sigma$ and $\Psi$ as follows:

$X = (X_{11}\ X_{12}\ X_{21}\ X_{22}), \mu = (\mu_{11}\ \mu_{12}\ \mu_{21}\ \mu_{22}),$
$T = (T_{11}\ T_{12}\ T_{21}\ T_{22}), \Sigma = (\Sigma_{11}\ \Sigma_{12}\ \Sigma_{21}\ \Sigma_{22})$
and $\Psi = (\Psi_{11}\ \Psi_{12}\ \Psi_{21}\ \Psi_{22})$

where $(X_{11}, \mu_{11}, T_{11})$ are $(r \times s)$ matrices, $\Sigma_{11}$ and $\Psi_{11}$ are $(r \times r), (s \times s)$ respectively. $(X_{21}, \mu_{21}, T_{21})$ are $\big((p-r) \times s\big)$ matrices, $(X_{12}, \mu_{12}, T_{12})$ are $\big(r \times (n-s)\big)$ matrices, $(X_{22}, \mu_{22}, T_{22})$ are $\big((n-r) \times (n-r)\big)$ matrices, $\Sigma_{12}$ is a $\big(r \times (p-r)\big)$ matrix and $\Sigma_{22}$ is a $\big((p-r) \times (p-r)\big)$ matrix, $\Psi_{12}$ is a $\big(n \times (n-s)\big)$ matrix and $\Psi_{22}$ is an $\big((n-s) \times (n-s)\big)$ matrix.

Using the c.f. of $x$, the marginal distribution for each block of a random matrix is as follows:

The marginal distribution for every part of a random matrix by using c.f. of $x$ is as follows:

1. The c.f. of $X_{11}$ can obtained by substituting $T_{12}, T_{21}$ and $T_{22}$ to zero matrices in ${\phi_x}^{(T)}$ in order to get:

$$\phi_{X_{11}}{}^{(T_{11})} = \frac{\left(\frac{\lambda}{\delta}\right)^{\nu} e^{itr(T_{11}'\mu_{11})} k_\nu\left(\sqrt{\lambda\delta\left(1 + \frac{tr(\Sigma_{11}T_{11}\Psi_{11}T_{11}')}{\lambda}\right)}\right)}{\left(1 + \frac{tr(\Sigma_{11}T_{11}\Psi_{11}T_{11}')}{\lambda}\right)^{\frac{\nu}{2}} k_\nu\left(\sqrt{\lambda\delta}\right)} \quad \ldots \tag{11}$$

Eq (11) represents the c.f. of GMMaß with parameters $(\mu_{11}, \Sigma_{11}, \Psi_{11}, \lambda, \delta, \nu)$.

Thus $X_{11} \sim GMMaB_{r,s}\ (\mu_{11}, \Sigma_{11}, \Psi_{11}, \lambda, \delta, \nu)$ by the same way we can get the p.d.f of $X_{12}$ after substituting $T_{12}, T_{21}\ and\ T_{22}$ for zero matrices in ${\phi_X}^{(T)}$ to get:

$$\phi_{X_{12}}{}^{(T_{12})} = \frac{\left(\frac{\lambda}{\delta}\right)^{\nu} e^{itr(T_{12}'\mu_{12})} k_\nu\left(\sqrt{\lambda\delta\left(1 + \frac{tr(\Sigma_{11}T_{12}\Psi_{22}T_{12}')}{\lambda}\right)}\right)}{\left(1 + \frac{tr(\Sigma_{11}T_{12}\Psi_{22}T_{12}')}{\lambda}\right)^{\frac{\nu}{2}} k_\nu\left(\sqrt{\lambda\delta}\right)} \quad \ldots \tag{12}$$

Eq (12) is the c.f. of GMMaB with parameters $(\mu_{12}, \Sigma_{11}, \Psi_{22}, \lambda, \delta, \nu)$.

Thus $X_{12} \sim GMMaB_{r,n-s}\ (\mu_{12}, \Sigma_{11}, \Psi_{22}, \lambda, \delta, \nu)$

As well as in the marginal distributions for each $X_{21}, and\ X_{22}$ are as follows:

$X_{21} \sim GMMaB_{p-r,s}\ (\mu_{21}, \Sigma_{22}, \Psi_{11}, \lambda, \delta, \nu)$ and
$X_{22} \sim GMMaB_{p-r,n-s}\ (\mu_{22}, \Sigma_{22}, \Psi_{22}, \lambda, \delta, \nu)$ respectively.

2. Partition of $X, \mu$ and $T$ vertically into two submatrices, so that:

   $X = (X_{*1} : X_{*2})$ , where $X_{*1} = (X_{11}\ X_{21})$ and $X_{*2} = (X_{12}\ X_{22})$;
   $\mu = (\mu_{*1} : \mu_{*2})$, where $\mu_{*1} = (\mu_{11}\ \mu_{21})$ and $\mu_{*2} = (\mu_{12}\ \mu_{22})$;
   $T = (T_{*1} : T_{*2})$, where $T_{*1} = (T_{11}\ T_{21})$ and $T_{1*} = (T_{12}\ T_{22})$.

   The c.f. of $X$ defined in Eq (9) can be obtained as the joint c.f. of $X_{*1}$ and $X_{*2}$. The c.f. of $X_{*1}$ can be obtained by substituting $T_{*2}$ for a zero matrix in $\phi_X{}^{(T)}$ in order to get

$$\phi_{X_{*1}}{}^{(T_{*1})} = \frac{\left(\frac{\lambda}{\delta}\right)^{\nu} e^{itr(T_{*1}'\mu_{*1})} k_\nu\left(\sqrt{\lambda\delta\left(1+\frac{tr(\Sigma T_{*1}\Psi_{11}T_{*1}')}{\lambda}\right)}\right)}{\left(1+\frac{tr(\Sigma T_{*1}\Psi_{11}T_{*1}')}{\lambda}\right)^{\frac{\nu}{2}} k_\nu(\sqrt{\lambda\delta})} \quad \ldots \quad (13)$$

   which is the c.f. of $GMMaB_{p,s}(\mu_{*1}, \Sigma, \Psi_{11}, \lambda, \delta, \nu)$

   $\therefore X_{*1} \sim GMMaB_{p,s}(\mu_{*1}, \Sigma, \Psi_{11}, \lambda, \delta, \nu)$

   Similarly, the marginal distribution of $X_{*2}$ is:

   $X_{*2} \sim GMMaB_{p,(n-s)}(\mu_{*2}, \Sigma, \Psi_{22}, \lambda, \delta, \nu)$

3. Division of $X, \mu, T$ horizontally into two submatrices, so that

   $X = (X_{1*} \cdots\cdots X_{2*})$, where $X_{1*} = (X_{11}\ X_{21})$ and $X_{2*} = (X_{12}\ X_{22})$;
   $\mu = (\mu_{1*} \cdots\cdots \mu_{2*})$, where $\mu_{1*} = (\mu_{11}\ \mu_{21})$ and $\mu_{2*} = (\mu_{12}\ \mu_{22})$;
   $T = (T_{1*} \cdots\cdots T_{2*})$, where $T_{1*} = (T_{11}\ T_{21})$ and $T_{2*} = (T_{12}\ T_{22})$.

   The c.f. of $X$ defined in Eq (9) is the joint c.f. between $X_{1*}$ and $X_2 *$. The c.f. of $X_{1*}$ can be obtained by substituting $T_{2*}$ for a zero matrix

$$\phi_{X_{1*}}{}^{(T_{1*})} = \frac{\left(\frac{\lambda}{\delta}\right)^{\nu} e^{itr(T_{1*}'\mu_{1*})} k_\nu\left(\sqrt{\lambda\delta\left(1+\frac{tr(T_{1*}'\Sigma_{11}T_{1*}\Psi)}{\lambda}\right)}\right)}{\left(1+\frac{tr(T_{1*}'\Sigma_{11}T_{1*}\Psi)}{\lambda}\right)^{\frac{\nu}{2}} k_\nu(\sqrt{\lambda\delta})} \quad \ldots \quad (14)$$

   Thus $X_{1*} \sim GMMaB_{r,n}(\mu_{1*}, \Sigma_{11}, \Psi, \lambda, \delta, \nu)$ as the same way, the distribution of $X_{2*}$ is:

   $X_{2*} \sim GMMaB_{(p-r),n}(\mu_{2*}, \Sigma_{22}, \Psi, \lambda, \delta, \nu)$.

## 2.4 Stochastic Independence

The c.f. of $X|w$ defined in Eq (6), is the product of the c.f's of $X_{11}, X_{12}, X_{21}, X_{22}$ . conditioned on $w$ when ( $\Sigma_{12}, \Sigma_{21}, \Psi_{12}, \Psi_{21}$) are zero matrices i.e.:

$$\Phi_{X|w}{}^{(t)} = e^{itr(T_{11}'\mu_{11})} + \frac{w}{2} tr(\Sigma_{11} T_{11} \Psi_{11} T_{11}{}'). e^{itr(T_{22}'\mu_{22})} + \frac{w}{2} tr(\Sigma_{22} T_{22} \Psi_{22} T_{22}{}') .$$

$$e^{itr(T_{12}'\mu_{12})} - \frac{w}{2} tr(\Sigma_{11} T_{12} \Psi_{22} T_{12}{}'). e^{itr(T_{21}'\mu_{21})} - \frac{w}{2} tr(\Sigma_{22} T_{21} \Psi_{11} T_{21}{}')$$

$$= \prod_{i,j}^{2} \Phi_{X_{ij}|w}{}^{(T_{ij})} \; \ldots \qquad (15)$$

The unconditional c.f. of $x$ defined as:

$$\phi_x{}^{(T)} = \prod_{i,j}^{2} \int_0^\infty \Phi_{X_{ij}|w}{}^{(T_{ij})} f(w) dw \; \ldots \qquad (16)$$

Substituting Eq's (2, 15) into Eq (16), it is concluded that

$$\phi_x{}^{(T)} = \prod_{i,j}^{2} \Phi_{X_{ij}|w}{}^{(T_{ij})} \ldots \qquad (17)$$

Thus $X_{11}, X_{12}, X_{21}$ *and* $X_{22}$ are independent if $(\Sigma_{12}, \Sigma_{21}, \Psi_{12}$ *and* $\Psi_{21})$ are zero matrices.

## 2.5 Conditional Distributions

Each conditional distribution for the three types of $X$ partition matrices stated in (4) is shown below:

For the partitioned matrices defined in (4,2), the conditional p.d.f. for each $X_{*1}, X_{*2}$ can be obtained as follows:

1. The p.d.f. of a random matrix $X|w$ defined in eq (1) is a $N_{pn}(vec(\mu), (\Psi \otimes \Sigma))$

$$f(w) = f(w) = (2\pi)^{-\frac{np}{2}} (w)^{-\frac{np}{2}} |\Sigma|^{-\frac{n}{2}} |\Psi|^{-\frac{p}{2}} . exp \left\{ -\frac{1}{2w} [vec'(X-\mu)(\Psi \otimes \Sigma)^{-1} vec(X-\mu)'] \right\} \ldots \qquad (18)$$

where $(vec)$ is a vector operator which transforms a matrix to column vector and $\otimes$ is a kroneker product, and

$vec\, X = (vec(X_{*1})\; vec(X_{*2}))$, $Vec\, \mu = (vec(\mu_{*1})\; vec(\mu_{*2}))$, *and* $\Psi \otimes \Sigma = (\Psi_{11}\; \Psi_{12}\; \Psi_{21}\; \Psi_{22}) \otimes \Sigma$

Since $\Psi$ is a patitioned (Block) matrix and $\Sigma$ is not partitioned thus

$$(\Psi \otimes \Sigma) = (\Psi_{11} \otimes \Sigma\; \Psi_{12} \otimes \Sigma\; \Psi_{21} \otimes \Sigma\; \Psi_{22} \otimes \Sigma)$$

It is well known that the conditional distribution of

$vecX_{*1}|vecX_{*2}, w \sim N_{pn}(vec\delta^*_{1.2}, w\Sigma^*_{1.2})$ where

$$vec\delta_{1.2}^{*} = vec\mu_{*1} + (\Psi_{12}\otimes\Sigma)(\Psi_{22}\otimes\Sigma)^{-1}(vecX_{*2} - vec\mu_{*2}) \ldots \qquad (19)$$

and

$$\Sigma_{1.2}^{*} = (\Psi_{11}\otimes\Sigma) - (\Psi_{12}\otimes\Sigma)(\Psi_{22}\otimes\Sigma)^{-1}(\Psi_{21}\otimes\Sigma) \ldots \qquad (20)$$

The p.d.f. of $vecX_{*1}|vecX_{*2}$ by using Bayes theorem is:

$$f(vecX_{*1}|vecX_{*2}) = \int_0^\infty f(vecX_{*1}|vecX_{*2}, w)f(w)dw$$

$$= \int_0^\infty \frac{(w)^{-\frac{ps}{2}} exp\left\{-\frac{1}{2w}\left[vec(X_{*1} - \delta_{1.2}^{*})'\Sigma_{1.2}^{*\,-1}vec(X_{*1} - \delta_{1.2}^{*})\right]\right\}}{(2\pi)^{\frac{ps}{2}}|\Sigma_{1.2}^{*}|^{\frac{1}{2}}}$$

$$.\frac{\left(\frac{\lambda}{\delta}\right)^{\frac{\nu}{2}} w^{\nu-1}}{2k_\nu\left(\sqrt{\lambda\delta}\right)} . \exp\left\{-\frac{1}{2}\left(\frac{\delta}{w} + \lambda w\right)\right\}dw \ldots \qquad (21)$$

$$f(vecX_{*1}|vecX_{*2}) = \frac{\left(\frac{\lambda}{\delta}\right)^{\frac{ps}{4}} k_{\left(\frac{2\nu-ps}{4}\right)}\left(\sqrt{\lambda\delta\left(1+\frac{tr\Sigma_{1.2}^{*\,-1}vec(X_{*1}-\delta_{*1})vec'(X_{*1}-\delta_{*1})}{\delta}\right)}\right)}{(2\pi)^{\frac{ps}{2}}|\Sigma_{1.2}^{*}|^{\frac{1}{2}}k_\nu(\sqrt{\lambda\delta})\left(1+\frac{tr\Sigma_{1.2}^{*\,-1}vec(X_{*1}-\delta_{*1})vec'(X_{*1}-\delta_{*1})}{\delta}\right)^{\left(\frac{2\nu+ps}{4}\right)}} \cdots \qquad (22)$$

But $\Sigma_{1.2}^{*} = (\Psi_{11} - \Psi_{12}\Psi_{22}^{-1}\Psi_{21})\otimes\Sigma = \Psi_{1.2}\otimes\Sigma$

$$f(X_{*1}|X_{*2}) = \frac{\left(\frac{\lambda}{\delta}\right)^{\frac{ps}{4}} k_{\left(\frac{2\nu-ps}{4}\right)}\left(\sqrt{\lambda\delta\left(1+\frac{tr[\Sigma^{-1}(X_{*1}-\delta_{*1})\Psi_{1.2}^{-1}(X_{*1}-\delta_{*1})']}{\delta}\right)}\right)}{(2\pi)^{\frac{ps}{2}}|\Sigma|^{\frac{s}{2}}|\Psi_{1.2}|^{\frac{p}{2}}k_\nu\left(\sqrt{\lambda\delta}\right)\left(1+\frac{tr[\Sigma^{-1}(X_{*1}-\delta_{*1})\Psi_{1.2}^{-1}(X_{*1}-\delta_{*1})']}{\delta}\right)^{\left(\frac{2\nu+ps}{4}\right)}} \ldots \qquad (23)$$

Thus $X_{*1}|X_{*2}\sim GMMaB_{p,s}(\delta_{*1}, \Sigma, \Psi_{1.2}, \lambda, \delta, \nu)$ where $\delta_{*1}$ can be found by writing $vec\delta_{*1}$ defined in Eq (10) in matrix form;

$$\delta_{*1} = \mu_{*1} + (X_{*2} - \mu_{*2})\Psi_{22}^{-1}\Psi_{21} \ldots \qquad (24)$$

In the same way, the conditional distribution of $X_{*2}|X_{*1}$ can be obtained and denoted by:

$$X_{*2}|X_{*1}\sim GMMaB_{p,n-s}(\delta_{*2}, \Sigma, \Psi_{2.1}, \lambda, \delta, \nu),$$

where

$$\delta_{*2} = \mu_{*2} + (X_{*1} - \mu_{*1})\Psi_{11}^{-1}\Psi_{12} \ \ldots \qquad (25)$$

2. Conditional distributions for each block of (2x2) Block random matrix $X$. The conditional distributions of $X_{11}|X_{21},X_{*2}$ and $X_{21}|X_{11},X_{*2}$ depending on the conditional p.d.f. of $X_{*1}|X_{*2},w$, $X_{*2}|X_{*1},w$ respectively.

 The conditional p.d.f. of $X_{*1}|X_{*2},w$ is:

$$X_{*1}|X_{*2},w \sim N_{p,s}(\delta_{*1},w\Sigma,\Psi_{1.2})$$

 but $X_{*1}=(X_{11}\ X_{21}\ ),\delta_{*1}=(\delta_{*11}\ \delta_{*12}\ )$ and $\Sigma=(\Sigma_{11}\ \Sigma_{12}\ \Sigma_{21}\ \Sigma_{22}\ )$ also $\Sigma\otimes\Psi_{1.2}$ (by sing the properties of Kronecker product (Al-Zhour and Kilicman, 2007).

$$\Sigma\otimes\Psi_{1.2}=(\Sigma_{11}\otimes\Psi_{1.2}\ \Sigma_{12}\otimes\Psi_{1.2}\ \Sigma_{21}\otimes\Psi_{1.2}\ \Sigma_{22}\otimes\Psi_{1.2}\ )=V$$
$$X_{11}|X_{*2},X_{21},w \sim N_{r,s}(\delta_{1.2},w\Sigma_{1.2},\Psi_{1.2})$$

$$f(vecX_{11}|vecX_{*2},vecX_{21},w) = \frac{exp\left\{-\frac{1}{2w}\left[vec'(X_{11}-\delta_{*1.2})V_{1.2}{}^{-1}vec(X_{11}-\delta_{*1.2})\ \right]\right\}}{(2\pi)^{\frac{rs}{2}}(w)^{\frac{rs}{2}}|V_{1.2}|^{\frac{1}{2}}}\ldots \quad (26)$$

 where

$$V_{1.2}=(\Sigma_{11}\otimes\Psi_{1.2})-(\Sigma_{12}\otimes\Psi_{1.2})(\Sigma_{22}\otimes\Psi_{1.2})^{-1}(\Sigma_{21}\otimes\Psi_{1.2}) = \Sigma_{1.2}\otimes\Psi_{1.2}\ldots \quad (27)$$

 and

$$vec\delta_{*1.2}=vec\delta_{*11}+(\Sigma_{12}\otimes\Psi_{1.2})(\Sigma_{22}\otimes\Psi_{1.2})^{-1}vec(X_{*12}-\delta_{*12}) = vec\delta_{*11}+(\Sigma_{12}\Sigma_{22}^{-1}\otimes I_s)vec(X_{21}-\delta_{*12})\ldots \quad (28)$$

 By using property $vec(AB)=(I\otimes A)vecB=(B'\otimes I)vecA$
 then $EX_{11}|X_{*2},X_{21}=\delta_{*11}+\Sigma_{12}\Sigma_{22}^{-1}(X_{21}-\delta_{*12})$

 As the same way the conditional distribution of

$$X_{21}|X_{*2},X_{11} \sim GMMaB_{p-r,s}(\delta_{*2.1},\Sigma_{2.1},\Psi_{1.2},\lambda,\delta,\nu)$$

 where

$$\delta_{*2.1}=\delta_{*21}+\ \Sigma_{21}\Sigma_{11}^{-1}(X_{11}-\delta_{*11})\ldots \quad (29)$$

 Depending on the conditional distribution of $X_{*2}|X_{*1},w$, the conditional distributions of $X_{12}|X_{*1},X_{22},w$, and $X_{22}|X_{*1},X_{12},w$ are respectively defined as:

$$vecX_{12}|vecX_{*1},vecX_{22},w \sim N_{r(,n-s)}(vec\delta_{*2.1},w\Sigma_{1.2},\Psi_{2.1}),$$

 where

$$vec\delta_{*2.1}=vec\delta_{*21}+\left(\left(\Sigma_{12}\Sigma_{22}^{-1}\otimes I_{(n-s)}\right)vec(X_{22}-\delta_{*22})\right)$$

 and

$$X_{12}|X_{*1},X_{22} \sim GMMaB_{r,(n-s)}(\delta_{*2.1},\Sigma_{1.2},\Psi_{2.1},\lambda,\delta,\nu),$$

 where

$$\delta_{*2.1}=\delta_{*21}+\ \Sigma_{12}\Sigma_{22}^{-1}(X_{22}-\delta_{*22})\ldots \quad (30)$$

 In the same way

$$X_{22}|X_{*1},X_{11} \sim GMMaB_{(p-r),(n-s)}(\delta_{*2.2},\Sigma_{1.2},\Psi_{2.1},\lambda,\delta,\nu),$$

 where

$$\delta_{*2.2}=\delta_{*22}+\ \Sigma_{21}\Sigma_{11}^{-1}(X_{21}-\delta_{*21})\ldots \quad (31)$$

3. For the partitioned matrices defined in (4.3), the conditional p.d.f. for each $X_{1*}$ and $X_{2*}$ can be obtained as follows:

   In Eq (15) we have

   $vec\ X = (vec(X_{1*})\ vec(X_{2*})\ )\ , Vec\ \mu = (vec(\mu_{1*})\ vec(\mu_{2*})\ ),$
   and $\Psi \otimes \Sigma = \Psi \otimes (\Sigma_{11}\ \Sigma_{12}\ \Sigma_{21}\ \Sigma_{22}\ )$

   $\Psi \otimes \Sigma = (\Psi \otimes \Sigma_{11}\ \Psi \otimes \Sigma_{12}\ \Psi \otimes \Sigma_{21}\ \Psi \otimes \Sigma_{22}\ )$

   The conditional distribution of

   $X_{1*}|X_{2*}, w\ is\ N_{r,n}(\theta_{1*}, \Psi \otimes w\Sigma_{1.2})$

   where
   $vec\theta_{1*} = vec\mu_{1*} + (\Psi \otimes \Sigma_{12})(\Psi \otimes \Sigma_{22})^{-1} vec(X_{2*} - \mu_{2*})$
   $= vec\mu_{1*} + (I_{11} \otimes \Sigma_{12}\Sigma_{22}^{-1}) vec(X_{2*} - \mu_{2*})$
   $\therefore \theta_{1*} = \mu_{1*} + \Sigma_{12}\Sigma_{22}^{-1}(X_{2*} - \mu_{2*})$
   and $X_{1*}|X_{2*} \sim GMMaB_{r,(n-s)}(\theta_{1*}, \Sigma_{1.2}, \Psi, \lambda, \delta, \nu),$

   In the same way the conditional distribution of

   $X_{2*}|X_{1*} \sim GMMaB_{r,(n-s)}(\theta_{2*}, \Sigma_{2.1}, \Psi, \lambda, \delta, \nu)$

   where $\theta_{2*} = \mu_{2*} + \Sigma_{21}\Sigma_{11}^{-1}(X_{1*} - \mu_{1*})$.

## 2.6 Distributions of Linear Combinations

1. If $X \sim GMMaB_{p,n}(\mu, \Sigma, \Psi, \lambda, \delta, \nu)$ and if there is a random matrix $Z = DX$ where D is an $(m \times p)$ matrix of constants then the p.d.f. of $Z$ can be obtained as follows:

   It is known that

   $vecX|w \sim N_{p,n}\ (vec\mu, \Psi \otimes w\Sigma)$
   and $vecZ = vec(DX) = (I_n \otimes D) vecX$

   Thus $vecZ|w \sim N_{mn}\ ((I_n \otimes D)vec\mu, (I_n \otimes D)(\Psi \otimes w\Sigma)(I_n \otimes D'))$

   After simplification:

   $Z|w \sim N_{m,n}\ (D\mu, wD\Sigma D', \Psi)$

   and the unconditional distribution of $Z$ is:

   $Z \sim GMMaB_{m,n}(D\mu, wD\Sigma D', \Psi, \lambda, \delta, \nu)$

2. If there is a random matrix $Z = DXB$ where $D$ is an $(m \times p)$ matrix of constants and B is also an $(n \times e)$ matrix of constants then

   $Z|w \sim N_{m,n}\ (D\mu B, wD\Sigma D', B\Psi B')$

   and the unconditional distribution of $Z$ is:

   $Z \sim GMMaB_{m,n}(D\mu B, D\Sigma D', B\Psi B', \lambda, \delta, \nu).$

## 3. CONCLUSIONS

The generalized multivariate modified Bessel distribution might be extended to a matrix variate by making use of a combination of the generalized inverse Gaussian distribution and the matrix normal distribution. The use of the combination was successful in achieving this goal. The functioning of (GMMaB) from a theoretical standpoint has been looked at. In addition, we get to the conclusion that the marginal distributions of each block in a random matrix $X$ are also a. (GMMaB). The blocks of a random matrix are dependent on one another, with the exception of the case in which the covariance matrices linking the rows of $X$ are zero matrices. $\Sigma_{12}$ and $\Sigma_{21}$ are both zero matrices, and if the covariance matrices between columns of $X$ $\Psi_{12}$ and $\Psi_{21}$ are zero matrices, then the distributions of linear functions of (GMMaB) are also a (GMMaB).

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