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K-TH RECORD MOMENTS AND CHARACTERIZATION OF INVERTED NADARAJAH-HAGHIGHI DISTRIBUTION

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ABSTRACT

In this paper, explicit forms of single and product moments with recurrence relations are obtained based on k-th lower record values from the inverted Nadarajah-Haghighi (N-H) distribution. We have also conducted a characterization of inverted N-H distribution using recurrence relations based on the same framework of lower k-th record values and truncated expectations.

KEYWORDS

k-th record values, single moments, product moments, recurrence relations, characterization.

1. INTRODUCTION

Record values are considered as those observations that exceed the previous ones in magnitude. There were first published in the work of Chandler (1952). Their significance is noted in a wide variety of practical situations such as industrial stress testing, meteorological analysis, hydrology, seismology, mining surveys, sports and athletic events. Various developments on record values and related topics are extensively studied in the literature. See for instance Ahsanullah (1995), Kamps (1995), Arnold *et al*. (1998), Ahsanullah and Nevzorov (2015). Dziubdziela and Kopcoinski (1976) have generalized the concept of record values by introducing k-th record values. They are defined in the following manner:

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identical distributed (*iid*) continuous random variables with *df* $F(x)$ and *pdf* $f(x)$. The *j*-th order statistic of a sample $X_1, X_2, ..., X_n$ is denoted by $X_{j:n}$. For a fixed positive integer *k*, we define the sequence $\{L_k(n), n \geq 1\}$ as *k*-th lower record times of $\{X_n, n \geq 1\}$ as

$$L_k(1) = 1,$$

$$L_k(n+1) = \min\left\{ j > L_k(n) : X_{k:L_k(n)+k-1} > X_{k:j+k-1} \right\}.$$

The sequence $\{Z_n^{(k)}, n \geq 1\}$ with $Z_n^{(k)} = X_{k:L_k(n)+k-1}$, $n = 1, 2, \ldots$ is called the sequence of *k*-th lower record values of $\{X_n, n \geq 1\}$. For convenience, we shall also take $Z_0^{(1)} = 0$. Note that for *k*=1 we have a sequence of ordinary lower record values $Z_n^{(1)} = X_{L(n)}$, $n \geq 1$.

The *pdf* of $Z_n^{(k)}$ and the joint *pdf* of $Z_m^{(k)}$ and $Z_n^{(k)}$ are as follows:

$$f_{Z_n^{(k)}}(x) = \frac{k^n}{\Gamma(n)}[-\ln F(x)]^{n-1}[F(x)]^{k-1} f(x),\ n \geq 1, \tag{1}$$

$$f_{Z_m^{(k)}, Z_n^{(k)}}(x, y) = \frac{k^n}{\Gamma(m)\Gamma(n-m)}[-\ln F(x)]^{m-1}\frac{f(x)}{F(x)} \times[\ln F(x) - \ln F(y)]^{n-m-1}[F(y)]^{k-1} f(y), \tag{2}$$

$y < x, 1 \leq m < n, n \geq 2$, respectively.

The conditional *pdf* of $Z_n^{(k)}$, given $Z_m^{(k)} = x$, is

$$f_{Z_n^{(k)}|Z_m^{(k)}}(x, y) = \frac{k^{n-m}}{\Gamma(n-m)}[\ln F(x) - \ln F(y)]^{n-m-1} \times\left[\frac{f(x)}{F(x)}\right]^{k-1}\frac{f(x)}{F(x)}, y < x. \tag{3}$$

Replacing $F(x)$ with $1 - F(x)$ we obtain expressions or upper k-th record values. The usefulness of record values on different inference problems may be found in e.g. Pawlas and Szynal (1998), Pawlas and Szynal (2000), Bieniek and Szynal (2002), Khan and Zia (2013) and Singh and Khan (2018), Vidovic (2019, 2021), Alam *et al.* (2021, 2022b).

Inverse transformations of distribution models are constantly introduced in statistical literature for the purpose of providing more flexible models with higher performance skills. There exist a significant literature of inverse (or inverted) distributions. Inverted distributions have a mayor impact in statistics due to their large modelling performances based on various shape styles of the density and hazard rate functions. Their applicability has a tremendous role in different real-life problems where non-inverted distributions have a noticeable lack of fit. For more details on inverted distributions we may refer to e.g. Sheikh *et al.* (1987), Lehmann and Shaffer (1988) and Ahmad (2007). In this paper we deal with an inverted N-H distribution which was introduced in Tahir *et al*. (2018). Its probability density function (pdf), is of the form

$$f(x) = \alpha\lambda x^{-2}\left(1+\lambda x^{-1}\right)^{\alpha-1}\exp\left\{1-\left(1+\lambda x^{-1}\right)^{\alpha}\right\},\ x > 0, \alpha > 0, \lambda > 0, \tag{4}$$

with the distribution function (*df*)

$$F(x) = \exp\left\{1-\left(1+\lambda x^{-1}\right)\right\}^{\alpha}\ x > 0, \alpha > 0, \lambda > 0. \tag{5}$$

This distribution is obtained by the transformation 1/Z, where Z follows the N-H distribution (see Nadarajah and Haghighi (2011)). Its fitting performances are improved over the N-H distribution by enabling the hazard rate function to exhibit a decreasing or bathtub form. By setting $\alpha = 1$, the inverted N-H distribution reduces to the inverted exponential distribution. With all this in mind, the inverted N-H distribution can be seen as a potential model for positive data with wide fitting abilities.

One may realize that a functional relation between pdf and df can be expressed as

$$\left(x^2 + \lambda x) f(x)\right) = \alpha\lambda[1 - \ln F(x)][F(x)]. \tag{6}$$

The key role of this article is to present recurrence relations of single and double product moments of the inverted N-H distribution based on lower records. We use integration methods for obtaining such relations and we discussed potential applications of these results in those distribution arising as special cases of inverted N-H distribution in Section 2 and 3, with respect to single and product moments. The characterizations of inverted N-H distribution using recurrence relations and truncated moments are taken into account in Section 4, while Section 5 concludes this paper.

2. RELATIONS FOR SINGLE MOMENTS

Theorem 1

For $n \geq 1$ and $k \geq 1$, we have a representation of the record value moments from inverted N-H distribution as

$$E\left(Z_n^{(k)}\right)^j = \frac{(-1)^j \lambda^j}{\Gamma(n)} \sum_{p=0}^{\infty} a_p(j) \sum_{p=0}^{[p/\alpha]} \binom{[p/\alpha]}{q} \frac{\Gamma(q+n)}{k^q}, \tag{7}$$

where $[.]$ is a maximum integer.

Proof:

For the first step

$$E\left(Z_n^{(k)}\right)^j = \frac{k^n}{\Gamma(n)} \int_0^{\infty} x^j [-\ln F(x)]^{n-1} [F(x)]^{k-1} f(x) dx$$

$$E(Z_n^{(k)})^j = \frac{\lambda^j (-1)^j k^n}{\Gamma(n)} \sum_{p=0}^{\infty} a_p(j) \int_0^{\infty} (t+1)^{p/\alpha} t^{n-1} e^{-kt} dt$$

$$= \frac{\lambda^j (-1)^j k^n}{\Gamma(n)} \int_0^{\infty} \left[1 - \left\{-\ln F(x) + 1\right\}^{1/\alpha}\right]^{-j} [-\ln F(x)]^{n-1} [F(x)]^{k-1} f(x) dx.$$

Making the substitution $t = -\ln F(x)$, we find that

$$E\left(Z_n^{(k)}\right)^j = \frac{\lambda^j (-1)^j k^n}{\Gamma(n)} \int_0^{\infty} \left[1 - \{t+1\}^{1/\alpha}\right]^{-j} t^{n-1} e^{-kt} dt. \tag{8}$$

By using the binomial expansion found in Cordeiro and Andrade (2009), we can write

$$\left(1 - w^{1/\gamma}\right)^{-r} = 1 + \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i!} \prod_{j=0}^{i-1} (r+j) w^{i/\gamma} = \sum_{i=0}^{\infty} a_i(r) w^{i/\gamma}, \ |w| < 1, \tag{9}$$

From which we obtain

$$=\frac{\lambda^{j}(-1)^{j}k^{n}}{\Gamma(n)}\sum_{p=0}^{\infty}a_{p}(j)\sum_{p=0}^{[p/\alpha]}\binom{[p/\alpha]}{q}\int_{0}^{\infty}t^{q+n-1}e^{-kt}dt, \qquad (10)$$

where in the last step we used the gamma function and, hence, get the result.

Remark 1

i) Setting $\alpha=1$ in (7), we get the single moments of k-th lower record values from the inverted exponential distribution (see e.g. Dey *et al.* (2016)) as

$$E\left(Z_{n}^{(k)}\right)^{j}=\frac{(-1)^{j}\lambda^{j}}{\Gamma(n)}\sum_{p=0}^{\infty}a_{p}(j)\sum_{p=0}^{p}\binom{p}{q}\frac{\Gamma(q+n)}{k^{q}}.$$

Moreover for $k=1$ we get the single moments of lower record values from the inverted exponential distribution.

ii) Setting $k=1$ in (7), we get the single moments of lower record values from the inverted N-H distribution as

$$E\left(X_{L(n)}^{j}\right)=\frac{(-1)^{j}\lambda^{j}}{\Gamma(n)}\sum_{p=0}^{\infty}a_{p}(j)\sum_{p=0}^{[p/\alpha]}\binom{[p/\alpha]}{q}\Gamma(q+n).$$

iii) Setting $k=1$ and $n=1$ in (7), we get the *j*-th single moments from the inverted N-H distribution as

$$E(X^{j})=(-1)^{j}\lambda^{j}\,e\sum_{p=0}^{\infty}a_{p}(j)\Gamma(p/\alpha+1,1),$$

where

$$\sum_{p=0}^{[p/\alpha]}\binom{[p/\alpha]}{q}\Gamma(q+1)=e\,\Gamma(p/\alpha+1,1).$$

See also Tahir *et al.* (2018).

The following theorem gives the recurrence relations for single moments of k-th lower record values from (2).

Theorem 2

For $n\geq 1$ and $k\geq 1$, a recurrence relations based on the record value moments from inverted N-H distribution has the form

$$E\left(Z_{n}^{(k)}\right)^{j+2}=\frac{\alpha\lambda}{j+1}\left[k\,E\left(Z_{n-1}^{(k)}\right)^{j+1}-n\,E\left(Z_{n+1}^{(k)}\right)^{j+1}+\left(n-k-\frac{j+1}{\alpha}\right)E\left(Z_{n}^{(k)}\right)^{j+1}\right], \qquad (11)$$

For $j=0,1,\ldots$

Proof:

From (4) for $n \geq 1$ and $j = 0,1,...$ we have

$$E(Z_n^{(k)})^{j+2} - \lambda E(Z_n^{(k)})^{j+1} = \frac{\alpha\lambda k^n}{\Gamma(n)} \int_0^\infty x^j [1 - \ln F(x)][-\ln F(x)]^{n-1} [F(x)]^k \, dx.$$

$$= \frac{\alpha\lambda k^n}{\Gamma(n)} \int_0^\infty x^j [-\ln F(x)]^{n-1} [F(x)]^k \, dx + \frac{\alpha\lambda k^n}{\Gamma(n)} \int_0^\infty x^j [-\ln F(x)]^n [F(x)]^k \, dx$$

One can notice that (see Bieniek and Szynal, 2002)

$$E(Z_n^{(k)})^j - E(Z_{n-1}^{(k)})^j = -\frac{jk^{n-1}}{\Gamma(n)} \int_0^\infty x^{j-1} [-\ln F(x)]^{n-1} [F(x)]^k \, dx.$$

$$E(Z_n^{(k)})^{j+2} - \lambda E(Z_n^{(k)})^{j+1} = \alpha\lambda \left[\frac{k}{j+1} \left\{ E(Z_{n-1}^{(k)})^{j+1} - E(Z_n^{(k)})^{j+1} \right\} - \frac{n}{j+1} \left\{ E(Z_{n-1}^{(k)})^{j+1} - E(Z_n^{(k)})^{j+1} \right\} \right]. \quad (12)$$

Arranging (12) gives the results (11).

Remark 2

i) Setting $\alpha = 1$ in (11), we get the recurrence relation for single moments of k-th lower record values from the inverted exponential distribution as

$$E\left(Z_n^{(k)}\right)^{j+2} = \frac{\lambda}{j+1} \left[k E\left(Z_{n-1}^{(k)}\right)^{j+1} - n E\left(Z_{n+1}^{(k)}\right)^{j+1} + (n-k-j-1) E\left(Z_n^{(k)}\right)^{j+1} \right].$$

Moreover, for $k = 1$ we get the single moments of lower record values from the inverted exponential distribution.

ii) Setting $k = 1$ in (11), we get the recurrence relation for single moments of lower record values from the inverted N-H distribution as

$$E\left(X_{L(n)}^{j+2}\right) = \frac{\alpha\lambda}{j+1} \left[E\left(X_{L(n-1)}^{j+1}\right) - n E\left(X_{L(n+1)}^{j+1}\right) + \left(n - 1 - \frac{j+1}{\alpha}\right) E\left(X_{L(n)}^{j+1}\right) \right].$$

3. RELATIONS FOR PRODUCT MOMENTS

In this section, we derived the exact moment and recurrence relations for product moments of k-th lower record values from inverted N-H distribution (1).

Theorem 3

For $1 \leq m \leq n-1$ and $i, j = 0,1,...$, we have

$$E\left[(Z_m^{(k)})^i (Z_n^{(k)})^j\right] = \frac{(-1)^{r+s}\lambda^{r+s}}{\Gamma(m)\Gamma(n-m)} \sum_{p=0}^{n-m-1} (-1)^p \binom{n-m-1}{p} \sum_{q=0}^{\infty} a_q(i)$$
$$\times \sum_{r=0}^{[q/\alpha]} \binom{[q/\alpha]}{r} \sum_{s=0}^{\infty} a_t(j) \sum_{t=0}^{[s/\alpha]} \binom{[s/\alpha]}{t} \frac{\Gamma(t+n+r)}{(r+p+m)k^{t+r}} \qquad (13)$$

where [.] is a maximum integer.

Proof:

From (2), we have

$$E\left[(Z_m^{(k)})^i (Z_n^{(k)})^j\right] = \frac{k^n}{\Gamma(m)\Gamma(n-m)} \int_0^{\infty}\int_y^{\infty} x^i y^j [-\ln F(x)]^{m-1} \frac{f(x)}{F(x)}$$
$$\times [\ln F(x) - \ln F(y)]^{n-m-1} [F(y)]^{k-1} f(y)\, dx\, dy.$$
$$= \frac{k^n}{\Gamma(m)\Gamma(n-m)} \sum_{p=0}^{n-m-1} (-1)^p \binom{n-m-1}{p} \int_0^{\infty}\int_y^{\infty} x^i y^j [-\ln F(x)]^{p+m-1}$$
$$\times \frac{f(x)}{F(x)} [-\ln F(y)]^{n-m-p-1} [F(y)]^{k-1} f(y)\, dx\, dy.$$
$$= \frac{k^n}{\Gamma(m)\Gamma(n-m)} \sum_{p=0}^{n-m-1} (-1)^p \binom{n-m-1}{p}$$
$$\times \int_0^{\infty} y^j [-\ln F(y)]^{n-m-1} [F(y)]^{k-1} f(y)\, I(y) dy, \qquad (14)$$

where $I(y)$ has the form

$$I(y) = \int_y^{\infty} x^i [-\ln F(x)]^{p+m-1} \frac{f(x)}{F(x)} dx$$
$$= (-1)^i \lambda^i \int_y^{\infty} \left[1 - \{-\ln F(x) + 1\}^{1/\alpha}\right]^{-i} [-\ln F(x)]^{p+m-1} \frac{f(x)}{F(x)} dx.$$

By substituting $t = -\ln F(x)$, $I(y)$ simplifies to

$$I(y) = (-1)^i \lambda^i \int_0^{-\ln F(y)} [1 - \{t+1\}^{1/\alpha}]^{-i} t^{p+m-1} dt. \qquad (15)$$

Next, by using the binomial expansion as in (10), we can rewrite $I(y)$ as

$$I(y) = (-1)^i \lambda^i \sum_{q=0}^{\infty} a_q^{(i)} \int_0^{-\ln F(y)} (t+1)^{q/\alpha} t^{p+m-1} dt$$
$$= (-1)^i \lambda^i \sum_{q=0}^{\infty} a_q^{(i)} \sum_{r=0}^{[q/\alpha]} \binom{[q/\alpha]}{r} \int_0^{-\ln F(y)} t^{r+p+m-1} dt$$

$$= (-1)^i \lambda^i \sum_{q=0}^{\infty} a_q^{(i)} \sum_{r=0}^{[q/\alpha]} \binom{[q/\alpha]}{r} \frac{[-\ln F(x)]^{r+p+m}}{r+p+m}. \tag{16}$$

Inserting (16) in (14), we get

$$E\left[\left(Z_m^{(k)}\right)^i \left(Z_n^{(k)}\right)^j\right] = \frac{(-1)^i \lambda^i k^n}{\Gamma(m)\Gamma(n-m)} \sum_{p=0}^{n-m-1} (-1)^p \binom{n-m-1}{p} \sum_{q=0}^{\infty} a_q^{(i)}$$
$$\times \sum_{r=0}^{[q/\alpha]} \binom{[q/\alpha]}{r} \frac{1}{r+p+m} \int_0^{\infty} y^j [-\ln F(y)]^{r-n-1} [F(y)]^{k-1} \frac{f(y)}{F(y)} dy.$$

Making the substitution $z = -\ln F(y),$ we find that

$$E\left[\left(Z_m^{(k)}\right)^i \left(Z_n^{(k)}\right)^j\right] = \frac{(-1)^{i+j} \lambda^{i+j} k^n}{\Gamma(m)\Gamma(n-m)} \sum_{p=0}^{n-m-1} (-1)^p \binom{n-m-1}{p}$$
$$\times \sum_{q=0}^{\infty} a_q^{(i)} \sum_{r=0}^{[q/\alpha]} \binom{[q/\alpha]}{r} \frac{1}{r+p+m} \int_0^{\infty} \left[1 - \{z+1\}^{1/\alpha}\right]^{-j} z^{r-n-1} e^{-kz} dz.$$

By using the binomial expansion (9), we can write

$$E\left[\left(Z_m^{(k)}\right)^i \left(Z_n^{(k)}\right)^j\right] = \frac{(-1)^{i+j} \lambda^{i+j} k^n}{\Gamma(m)\Gamma(n-m)} \sum_{p=0}^{n-m-1} (-1)^p \binom{n-m-1}{p}$$
$$\times \sum_{q=0}^{\infty} a_q^{(i)} \sum_{r=0}^{[q/\alpha]} \binom{[q/\alpha]}{r} \sum_{s=0}^{\infty} a_s^{(i)} \frac{1}{r+p+m} \int_0^{\infty} (z+1)^{s/\alpha} z^{r-n-1} e^{-kz} dz.$$
$$= \frac{(-1)^{i+j} \lambda^{i+j} k^n}{\Gamma(m)\Gamma(n-m)} \sum_{p=0}^{n-m-1} (-1)^p \binom{n-m-1}{p} \sum_{q=0}^{\infty} a_q^{(i)}$$
$$\times \sum_{r=0}^{[q/\alpha]} \binom{[q/\alpha]}{r} \sum_{s=0}^{\infty} a_s^{(i)} \sum_{t=0}^{[s/\alpha]} \binom{[s/\alpha]}{t} \frac{1}{r+p+m} \int_0^{\infty} z^{t+r+n-1} e^{-kz} dz. \tag{17}$$

where in the last expression we have used gamma function. This completes the proof.

Remark 3

i) Setting $\alpha = 1$ in (13), we obtain the product moments of k-th lower record values from the inverted exponential distribution

$$E\left[(Z_m^{(k)})^i (Z_n^{(k)})^j\right] = \frac{(-1)^{r+s} \lambda^{r+s}}{\Gamma(m)\Gamma(n-m)} \sum_{p=0}^{n-m-1} (-1)^p \binom{n-m-1}{p} \sum_{q=0}^{\infty} a_q^{(i)}$$
$$\times \sum_{r=0}^{q} \binom{q}{r} \sum_{s=0}^{\infty} a_t(j) \sum_{t=0}^{s} \binom{s}{t} \frac{\Gamma(t+n+r)}{(r+p+m) k^{t+r}}.$$

Moreover, for $k = 1$ we get the single moments of lower record values from the inverted exponential distribution.

ii) Setting $k=1$ in (13), we get the product moments of lower record values from the inverted N-H distribution as

$$E\left[(X_{L(m)}^{i})(X_{L(n)}^{j})\right]=\frac{(-1)^{r+s}\lambda^{r+s}}{\Gamma(m)\Gamma(n-m)}\sum_{p=0}^{n-m-1}(-1)^{p}\binom{n-m-1}{p}\sum_{q=0}^{\infty}a_q(i)$$
$$\times\sum_{r=0}^{[q/\alpha]}\binom{[q/\alpha]}{r}\sum_{s=0}^{\infty}a_t(j)\sum_{t=0}^{[s/\alpha]}\binom{[s/\alpha]}{t}\frac{\Gamma(t+n+r)}{(r+p+m)}.$$

iii) Setting $i=0$, we get a representation of the moments of single record as in (7).

Theorem 4

For $1\le m\le n-1$ and $i,j=0,1,\ldots$

$$E\left[(Z_m^{(k)})^{i+2}(Z_n^{(k)})^{j}\right]=\frac{\alpha\lambda}{i+1}\Bigg[kE\left[(Z_{m-1}^{(k)})^{i+1}(Z_n^{(k)})^{j}\right]$$
$$-mE\left[(Z_{m+1}^{(k)})^{i+1}(Z_n^{(k)})^{j}\right]+\left(m-k-\frac{i+1}{\alpha}\right)E\left[(Z_m^{(k)})^{i+1}(Z_n^{(k)})^{j}\right]\Bigg]. \quad (18)$$

For $m\ge 1$ and $i,j=0,1,\ldots$

$$E\left[(Z_m^{(k)})^{i+2}(Z_{m+1}^{(k)})^{j}\right]=\frac{\alpha\lambda}{i+1}\Bigg[kE\left[(Z_{m-1}^{(k)})^{i+1}(Z_{m+1}^{(k)})^{j}\right]$$
$$-(m+1)E\left[(Z_{m+1}^{(k)})^{i+j+1}\right]+\left(m-k-\frac{i+1}{\alpha}\right)E\left[(Z_m^{(k)})^{i+1}(Z_{m+1}^{(k)})^{j}\right]\Bigg]. \quad (19)$$

Proof:

From (2), we have

$$E\left[(Z_m^{(k)})^{i+2}(Z_n^{(k)})^{j}\right]+\lambda\left[E\left[(Z_m^{(k)})^{i+1}(Z_n^{(k)})^{j}\right]\right]=\frac{k^n}{\Gamma(m)\Gamma(n-m)}\int_0^{\infty}\int_y^{\infty}x^i y^j$$
$$(x^2+\lambda x)[-\ln F(x)]^{m-1}\frac{f(x)}{F(x)}\left[\ln F(x)-\ln F(y)\right]^{n-m-1}[F(y)]^{k-1}f(y)\,dx\,dy$$
$$=\frac{\alpha\lambda k^n}{\Gamma(m)\Gamma(n-m)}$$
$$\times\int_0^{\infty}\int_y^{\infty}x^i y^j[-\ln F(x)]^{m-1}[\ln F(x)-\ln F(y)]^{n-m-1}[F(y)]^{k-1}f(y)\,dx\,dy$$
$$+\frac{\alpha\lambda k^n}{\Gamma(m)\Gamma(n-m)}$$
$$\times\int_0^{\infty}\int_y^{\infty}x^i y^j[-\ln F(x)]^{m}[\ln F(x)-\ln F(y)]^{n-m-1}[F(y)]^{k-1}f(y)\,dx\,dy$$

Due to Alam *et al.* (2022a), it is noted that

$$E\left[(Z_{m+1}^{(k)})^i (Z_n^{(k)})^j\right] - E\left[(Z_m^{(k)})^i (Z_n^{(k)})^j\right] = -\frac{ik^n}{\Gamma(m+1)\Gamma(n-m)}$$
$$\times \int_\alpha^\beta \int_y^\beta x^{i-1} y^j [-\ln F(x)]^m [\ln F(x) - \ln F(y)]^{n-m-1} [F(y)]^{k-1} f(y)\,dx\,dy.$$

$$E\left[(Z_m^{(k)})^{i+2}(Z_n^{(k)})^j\right] + \lambda\left[E\left[(Z_m^{(k)})^{i+1}(Z_n^{(k)})^j\right]\right] = \frac{\alpha\lambda k}{i+1}\left[E\left[(Z_{m-1}^{(k)})^{i+1}(Z_n^{(k)})^j\right]\right]$$
$$-E\left[(Z_m^{(k)})^{i+1}(Z_n^{(k)})^j\right] + \frac{\alpha\lambda m}{i+1}\left[\left[E\left[(Z_m^{(k)})^{i+1}(Z_n^{(k)})^j\right]\right] - \left[E\left[(Z_{m+1}^{(k)})^{i+1}(Z_n^{(k)})^j\right]\right]\right].$$

By arranging the terms, we get the expression (18). Putting $n = m+1$ in (18) and noticing that $E\left[(Z_m^{(k)})^i (Z_m^{(k)})^j\right] = E\left[(Z_m^{(k)})^{i+j}\right]$, the recurrence relations (19) can be easily recognized.

Remark 4

i) Setting $\alpha = 1$ in (18), we get the recurrence relation for product moments of k-th lower record values from the inverted exponential distribution as

$$E[(Z_m^{(k)})^{i+2}(Z_n^{(k)})^j] = \frac{\lambda}{i+1}\left[kE\left[(Z_{m-1}^{(k)})^{i+1}(Z_n^{(k)})^j\right]\right.$$
$$\left.-mE\left[(Z_{m+1}^{(k)})^{i+1}(Z_n^{(k)})^j\right] + (m-k-i-1)E\left[(Z_m^{(k)})^{i+1}(Z_n^{(k)})^j\right]\right]$$

Moreover, for $k = 1$ we get the single moments of lower record values from the inverted exponential distribution.

ii) Setting $k = 1$ in (18), we get the recurrence relation for product moments of lower record values from the inverted N-H distribution as

$$E\left[(X_{L(m)}^{i+2})(X_{L(n)}^{j})\right] = \frac{\alpha\lambda}{i+1}\left[E\left[(X_{L(m-1)}^{i+1})(X_{L(n)}^{j})\right]\right.$$
$$\left.-mE\left[(X_{L(m+1)}^{i})(X_{L(n)}^{j})\right] + \left[m-1-\frac{i+1}{\alpha}\right]E\left[(X_{L(m)}^{i+1})(X_{L(n)}^{j})\right]\right]$$

iii) Setting $j = 0$ in (18), we get the recurrence relations of single moments as in (11).

4. CHARACTERIZATIONS

The characterization results can be used to know if the designed models fit the requirements probability distribution under study or not? There are several authors which are used the characterization theorem for order random variates such as order statistics and record values, we may refer to Athar and Akhtar (2015), Khan *et al.* (2015, 2016, 2017) and Alam *et al.* (2020).

In this part of the paper, we present a characterization of the inverted N-H distribution based on record values, using the following result found in Lin (1986):

Proposition 1

Let n_0 be any fixed non-negative integer, $-\infty \le a < b \le \infty$ and $g(x) \ge 0$ an absolutely continuous function with $g'(x) \ne 0$ a.e. on (a,b). Then the sequence of functions $\left\{(g(x))^n e^{-g(x)}, n \ge n_0\right\}$ is complete in $L(a,b)$ iff $g(x)$ is strictly monotone on (a,b).

Theorem 4

Let $k \ge 1$ and $j \ge 0$ be integers. A necessary and sufficient condition for a random variable X to have a pdf (4) is

$$E(Z_n^{(k)})^{j+2} = \frac{\alpha\lambda}{j+1}\left[k\,E(Z_{n-1}^{(k)})^{j+1} - n\,E(Z_{n+1}^{(k)})^{j+1} + \left(n-k-\frac{j+1}{\alpha}\right)E(Z_n^{(k)})^{j+1}\right], \quad (20)$$

For $n = 1, 2, \ldots$

Proof:

The necessary part follows from (11). On the other hand if the recurrence relation (20) is satisfied, then by rearranging we have

$$E(Z_n^{(k)})^{j+2} + \lambda\,E(Z_n^{(k)})^{j+1} = \frac{\alpha\lambda k}{j+1}\left\{E(Z_{n-1}^{(k)})^{j+1} - E(Z_n^{(k)})^{j+1}\right\}$$
$$-\frac{\alpha\lambda n}{j+1}\left\{E(Z_{n-1}^{(k)})^{j+1} - E(Z_n^{(k)})^{j+1}\right\}. \quad (21)$$

Reasoning as in Bieniek and Szynal (2002), we deduce that

$$\frac{k^n}{\Gamma(n)}\int_0^\infty x^j(x^2+\lambda x)[-\ln F(x)]^{n-1}[F(x)]^{k-1} f(x)dx$$
$$= \frac{\alpha\lambda k^n}{\Gamma(n)}\int_0^\infty x^j[-\ln F(x)]^{n-1}[F(x)]^k\,dx + \frac{\alpha\lambda k^n}{\Gamma(n)}\int_0^\infty x^j[-\ln F(x)]^n[F(x)]^k\,dx$$

$$\int_0^\infty x^j[-\ln F(x)]^{n-1}[F(x)]^{k-1} f(x)$$
$$\times\left\{(x^2+\lambda x) - \frac{\alpha\lambda[F(x)]}{f(x)} - \frac{\alpha\lambda[-\ln F(x)][F(x)]}{f(x)}\right\}dx = 0.$$

It now follow from the above proposition with

$$g(x) = -\ln F(x)$$

that

$$(x^2+\lambda x)f(x) = \alpha\lambda[1-\ln F(x)][F(x)].$$

which proves that $f(x)$ has the form as given in (4).

Theorem 6

Let X be a non-negative random variable having an absolutely continuous *df* $F(x)$, $f(0)=0$ and $0 \le F(x) \le 1$ for all $x > 0$, then

$$E\left[F(Z_n^{(k)})(Z_l^{(k)})=x\right] = g_{n|l}(x) = \exp\left\{1-(1+\lambda x^{-1})^{\alpha}\right\}\left(\frac{k}{k+1}\right)^{n-l},$$

$$l = m, m+1, \tag{22}$$

If and only if

$$F(x) = \exp\left\{1-(1+\lambda x^{-1})^{\alpha}\right\} \quad x>0, \alpha>0, \lambda>0.$$

Proof:

From (3), we have

$$E\left[F(Z_n^{(k)})(Z_m^{(k)})=x\right] = \frac{k^{n-m}}{\Gamma(n-m)}\int_0^x \exp\left\{1-(1+\lambda y^{-1})^{\alpha}\right\}$$

$$\times\left[\ln F(x)-\ln F(y)\right]^{n-m-1}\left[\frac{f(x)}{F(x)}\right]^{k-1}\frac{f(x)}{F(x)}dy. \tag{23}$$

By setting $u = \dfrac{F(y)}{F(x)} = \dfrac{\exp\left\{1-(1+\lambda y^{-1})^{\alpha}\right\}}{\exp\left\{1-(1+\lambda x^{-1})^{\alpha}\right\}}$, from (5) and (23), we have

$$E\left[F(Z_n^{(k)})(Z_m^{(k)})=x\right] = \frac{k^{n-m}}{\Gamma(n-m)}\exp\left\{1-(1+\lambda x^{-1})^{\alpha}\right\}\int_0^1 u^k[-\ln u]^{n-m-1}du.$$

Using the identity found in (Gradshteyn and Ryzhik, 2007, p. 551):

$$\int_0^1 x^{\upsilon-1}[-\ln x]^{\mu-1}dx = \frac{\Gamma(\mu)}{\upsilon^{\mu}}.$$

We obtain the relation (22).

To prove the sufficient part, we reason as

$$\frac{k^{n-m}}{\Gamma(n-m)}\int_0^x \exp\left\{1-(1+\lambda y^{-1})^{\alpha}\right\}\left[\ln F(x)-\ln F(y)\right]^{n-m-1}$$

$$[F(x)]^{k-1} f(x)dx = [F(x)]^k\, g_{n|m}(x), \tag{25}$$

where

$$g_{n|m}(x) = \exp\left\{1-(1+\lambda x^{-1})^{\alpha}\right\}\left(\frac{k}{k+1}\right)^{n-m}. \tag{26}$$

Differentiating (25) both sides with respect to x, we get

$$\frac{k^{n-m} f(x)}{\Gamma(n-m) F(x)} \int_0^x \exp\left\{1-(1+\lambda y^{-1})^{\alpha}\right\} \left[\ln F(x) - \ln F(y)\right]^{n-m-2}$$

$$[F(x)]^{k-1} f(y)dy = g'_{n|m}(x)[F(x)]^k + k\, g_{n|m}(x)[F(x)]^{k-1} f(x),$$

i.e.

$$kg'_{n|m+1}(x)[F(x)]^{k-1} f(x) = g'_{n|m}(x)[F(x)]^k + kg_{n|m}(x)[F(x)]^{k-1} f(x).$$

Therefore, we have the differential equation

$$\frac{f(x)}{F(x)} = \frac{g'_{n|m}(x)}{k\left[g_{n|m+1}(x) - g_{n|m}(x)\right]}.$$

where

$$g'_{n|m}(x) = \alpha\lambda x^{-2}(1+\lambda x^{-1})^{\alpha-1} \exp\left\{1-(1+\lambda x^{-1})^{\alpha}\right\}\left(\frac{k}{k+1}\right)^{n-m}$$

and

$$g'_{n|m+1}(x) - g_{n|m}(x) = \exp\left\{1-(1+\lambda x^{-1})^{\alpha}\right\}\frac{1}{k}\left(\frac{k}{k+1}\right)^{n-m}.$$

Hence, we obtain the functional relation

$$\frac{f(x)}{F(x)} = \alpha\lambda x^{-2}(1+\lambda x^{-1})^{\alpha-1}$$

Which implies that $F(x) = \exp\left\{1-(1+\lambda x^{-1})^{\alpha}\right\}$. This conclude the sufficiency part.

Theorem 7

Suppose an absolutely continuous (with respect to Lebesgue measure) random variable X has the *df* $F(x)$ and the *pdf* $f(x)$ for $0 < x < \infty$, such that $f'(x)$ and $E(X \mid X \le x)$ exist for all x, then

$$E(X \mid X \le x) = g(x)\,\eta(x), \tag{27}$$

where

$$\eta(x) = \frac{f(x)}{F(x)}$$

and

$$g(x) = \frac{x}{\alpha\lambda\, x^{-2}\,(1+\lambda x^{-1})^{\alpha-1}} - \frac{1}{f(x)}\int_0^x \exp\left\{1-(1+\lambda u^{-1})^{\alpha}\right\} du$$

if and only if

$$f(x) = \alpha\lambda x^{-2}(1+\lambda x^{-1})^{\alpha-1} \exp\left\{1-(1+\lambda x^{-1})^{\alpha}\right\},\ x > 0, \alpha > 0, \lambda > 0$$

Proof:

From (4), we have

$$E(X \mid X \le x) = \frac{\alpha\lambda}{F(x)} \int_0^x u^{-1}(1+\lambda u^{-1})^{\alpha-1} \exp\left\{1-(1+\lambda u^{-1})^{\alpha}\right\} du. \tag{28}$$

Integrating (28) by parts treating $'\alpha\lambda u^{-2}(1+\lambda u^{-1})^{\alpha-1} \exp\left\{1-(1+\lambda u^{-1})^{\alpha}\right\}'$ for integration and rest of the integrand for differentiation, we get

$$E(X \mid X \le x) = \frac{1}{F(x)}\left[x \exp\left\{1-(1+\lambda x^{-1})^{\alpha}\right\} - \int_0^x \exp\left\{1-(1+\lambda u^{-1})^{\alpha}\right\} du \right]. \tag{29}$$

After multiplying and dividing by $f(x)$ in (29), we get the result given in (27).

For sufficient, from (27) we have that

$$\int_0^x u f(u) dt = g(x)\, f(x). \tag{30}$$

Differentiating (30) on both the sides with respect to x, we find that

$$xf(x) = g'(x) f(x) + g(x) f'(x),$$

from which we have

$$\frac{f'(x)}{f(x)} = \frac{x - g'(x)}{g(x)} = \alpha\lambda\, x^{-2}(1+\lambda\, x^{-1})^{\alpha-1}.$$

Hence

$$g'(x) = x - g(x)\left[\alpha\lambda\, x^{-2}(1+\lambda\, x^{-1})^{\alpha-1}\right]. \tag{31}$$

Integrating both the sides in (31) with respect to x, we get

$$f(x) = C\alpha\lambda x^{-2}(1+\lambda x^{-1})^{\alpha-1} \exp\left\{1-(1+\lambda x^{-1})^{\alpha}\right\}.$$

Now, using the condition $\int_0^\infty f(x)\, dx = 1$, we obtain

$$f(x) = \alpha\lambda x^{-2}(1+\lambda x^{-1})^{\alpha-1} \exp\left\{1-(1+\lambda x^{-1})^{\alpha}\right\},\ x>0, \alpha>0, \lambda>0.$$

5. CONCLUSION

This article describes the recurrence relations of the single and product moments of k-th lower record values in Sections 2 and 3, respectively. These results are interesting for researchers since they provide the ability of evaluating higher order moments based on the first several known moments. These results are of interest since they may potentially reduce time and costs. In Section 4, characterization of inverted N-H distribution is given through recurrence relations and using truncated moments.

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