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SIMULTANEOUS CONFIDENCE INTERVALS WITH CONTINUITY CORRECTION FOR MULTINOMIAL DISTRIBUTIONS

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ABSTRACT

The paper extends seven methods for constructing confidence intervals that are developed for the estimation of binomial proportion with continuity correction to the estimation of multinomial proportion. The Bonferroni method is used to limit the overall confidence coefficient to $\propto$ %. The study considers each cell versus the remaining cells as a binomial estimate for the individual cell proportion (Cochran's, 1963). An analytical study is performed, for $n =$ 5, 10, 15, 20, 30 and 40; for number of categories limited to five, and for 5% and 1%. Results for the seven extended methods were checked for the invariance property and interval width. Results show that all methods satisfy invariance only for large n; the Blyth and Still method and the Wilson Score Method (Fleiss (1981) prove to do well in the estimation of multinomial proportion for all sample sizes; these two methods satisfy invariance property; however, the interval widths produced are not the narrowest. Hall Method gives the narrowest intervals, but it does not satisfy invariance. The Clopper-Pearson Exact Method satisfies invariance under some conditions when applied to $K > 1$, and the interval widths area bit higher than Method of Hall.

KEYWORDS

Binomial Probability; Bonferroni Inequality; Wald`s Intervals; Invariance property; Clopper and Pearson Exact limits; Fleiss limits; Wilson Score limits.

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1. INTRODUCTION

The multinomial distribution is a generalization of the binomial distribution; where, n independent trials that lead to a success in k categories, such that each category has a given fixed success probability P_k; the multinomial distribution gives the probability of any particular combination of numbers of successes for the various categories. When $k = 1$ and $n = 1$, then the distribution is the Bernoulli distribution, and when $k = 1$ and $n > 1$ then the distribution is the binomial distribution, and when $k > 1$ and $n > 2$ then it is the multinomial distribution. When k is a fixed infinite number, then we have k mutually exclusive outcomes, with probabilities $p_1, p_2, \dots . p_k$ where $p_i \geq 0$ and $\sum p_i = 1$, and n independent trials, then the vector $x_1, \dots . x_k$ follow a multinomial distribution, with parameters (n, p_i), where $i = 1, \dots k$; and since trials are independent then $\sum x_i = n$, and $x_i \in \{0,1, \dots . n\}$. The multinomial mass function takes the following form:

$$f(x_1 \dots x_k\, ;\, n, p_1 \dots . p_k) = \left\{\frac{n!}{x_1! \dots x_k!\,(n - \sum x)!}\, p_1^{x_1} \dots\, p_k^{x_k}\, \left(1 - \sum p_i\right)^{(n-\sum x)!} \dots\right\}$$

$$E(X_i) = np_i \text{ and } Var(X_i) = np_i\,(1 - p_i)$$

For a specified significance coefficient α. We wish to establish a set of intervals, S_i such that:

$$P\left[\bigcap_{i=1}^{k} \pi_i \ \in S_i\right] \leq (1 - \alpha) \tag{1}$$

where π_i is the true proportion in category i. Thus, it is required that the intersection probability for all intervals, S_i, contains π_i to be at most $= (1 - \alpha)$, and thus limiting the over-all confidence coefficient to α.

Cochran (1963) presented an approach to determine sample size in multinomial distributions; his approach is to consider each cell versus the remaining cells as a binomial estimate for the individual cell proportion; Cochran's underlying distribution is the standard normal distribution. The same course could be followed in establishing confidence limits for multinomial proportions; the difficulty arises from the inability to assess the value of the confidence coefficient for the entire set of intervals. To overcome this, Goodman (1965) gave a large sample confidence bounds for the multinomial proportion as $n \to \infty$ as:

$$LCL\,(\pi_i) \ \leq \ \pi_i \leq UCL(\pi_i) \tag{2}$$

where, the confidence limits is: $CL\,(\pi_i) = \hat{p}_i \pm\, \delta s_{\hat{p}_i}$, where $\hat{p}_i$ is the point estimate for the population proportion, π_i, $s_{\hat{p}_i}$ is the standard error for the estimator $\hat{p}_i$ and δ is the upper (α/k) of a chi-square distribution with one degree of freedom $(k = 1)$. The main difference between Goodman and Cochran approaches, not in the underlying distributions since degree of freedom for Goodman's is one, but in the upper and lower percentile points to be used; Cochran uses the standard normal curve (which is symmetrical) that cuts the total area to α at the tails, i.e. $z_{\propto/2}$, while Goodman`s uses the upper α/k % point of $\chi^2_{1,\propto}$ which is equivalent to $z\,^2_{\alpha/2}$. When the number of intervals is one (binomial: $k = 1$) then both Cochran and Goodman coincide, and the confidence coefficient is limited to α; but when $k > 1$ then Goodman's is the approach that satisfies equation [1]. Thus, when the $(1 - \alpha)$ is the desired overall coverage probability for the confidence intervals for k categories, ${\chi^2}_{(\alpha,k)}$ gives the upper $1 - \alpha$ quantile of the χ^2 distribution with k degrees of freedom. Using the simple Bonferroni-adjusted formula, Goodman performs well in most practical situations when the number of categories is greater than 2 and each cell count is greater than 5.

The procedure to be followed is to consider each cell versus the remaining cells as binomial (Cochran, 1965), but in order to satisfy equation [1], and employing Bonferroni inequality (Sokal and Rohlf, 1987), the $\left(1 - {}^{\alpha}/_{2k}\right)$ points of the standard normal distribution is used, where k is the number of intervals to be estimated.

Throughout this paper we denote c_k as the upper the $\left(1-{}^{\alpha}/_{2k}\right)$ points of the standard normal distribution. Thus, for $k = 1, \dots 5$, and for $\alpha = .05$ the c_k values are: 1.96, 2.24, 2.40, 2.51 and 2.58; and for $\alpha = .01$ the c_k values are 2.58, 2.81, 2.93, 3.00 and 3.90.

For each expression considered, the lower and upper CI limits are obtained for the estimation of the true population proportion $\pi_i(x)$, $i = 1,2, \dots k$, (k is limited to 5 in the study) and $x = 0,1, \dots n$; and for values of n = 5, 10, 15, 20, 30 and 40. Limits are checked for the invariance property as given by Blyth (1986) which states that the upper confidence limit of $\pi_{i(x)}$ equals to the complement of the lower limit of $1-\pi_{i(n-x)}$;

$$UCL(\pi_{i(x)}) = 1 - LCL(\pi_{i(n-x)} \tag{3}$$

The purpose of this paper is to estimate multinomial confidence limits when continuity correction factor applied by comparing six different approximate estimation procedures and the exact method of Clopper and Pearson (1934), these methods have been used for the binomial but extended to the multinomial using Cochran and Bonferroni approaches; comparisons are performed with regard to invariance property and the interval widths they produce.

Section 2, gives the multinomial version of Clopper and Pearson exact method for the estimation of the binomial proportion; Section 3, gives the different expressions used for the Binomial confidence interval limits as extended to the multinomial, and Section 4 gives the analytical results; conclusions and recommendations are given in Section 5.

2. MULTINOMIAL VERSION OF CLOPPER AND PEARSON EXACT FOR THE BINOMIAL ESTIMATION

Exact upper and lower confidence limits for the binomial could be obtained by using an incomplete beta function or an F distribution (Hald, 1952; Brownlee, 1965), with degrees of freedom $(2(x+1), 2(n-x)$. Clopper and Pearson (1934) gave an exact approach for Binomial estimation that uses the F distribution (Hald, 1952; Brownlee, 1965); the method replaces the confidence level $(1-\propto)$ to $(\leq \frac{\propto}{2})$ beyond the lower limit and $\leq \frac{\propto}{2}$ above the upper limit (Blyth and Still, 1983), and thus different F values at each tail of the distribution. The method could be extended to the multinomial distribution $(k > 1)$ using Bonferroni approach and the tail probabilities would be $\frac{\propto}{2k}$. Defining x_i the total number of success $(x_i = 0, \dots n)$, the upper limit of Clopper and Pearson could be extended to the multinomial as:

$$CP - UCL(\pi_i) = \left(1 + \frac{n - x_i + 1}{x_i F_{\propto/2k(2x_i, 2(n-x_i+1)}}\right)^{-1} \tag{4}$$

where F has degrees of freedom $(2x_i, 2(n - x_i + 1)$. The lower limit is given by:

$$CP - LCL(\pi_i) = \left(1 + \frac{n - x_i}{(x_i + 1)F_{\left(1-\frac{\propto}{2k}\right),(2x_i+1),2(n-x_i)}}\right)^{-1} \tag{5}$$

With degrees of freedom $(2(x_i+1), 2(n-x_i)$. And when $x_i = 0$ the interval is maximum of $(0, (1-\left(\frac{\propto}{2}\right)^{\frac{1}{n}}$ and when $x_i = n$ the interval is minimum of $\left(\left(\frac{\propto}{2}\right)^{\frac{1}{n}}, 1\right)$.

3. MULTINOMIAL VERSIONS OF A STRAIGHTFORWARD NORMAL APPROXIMATION

Several estimation approaches were developed for the estimation of binomial proportions; the Wilson's Score (1927), Clopper-Pearson (1934), Fleiss (1981), Blyth and Still (1983), Pratt (1986) and a modification of Pratt (Vollset, 1993), Pires (2002), and Agresti (2003). These approaches were compared in the studies of Newcombe (1998), Rieczigel (2003), Tobi et al. (2005); and Borkowf (2006).

Paulson (1942) has shown that convergence exists between F_{υ_1,ν_2} distribution to the standard normal distribution as $\upsilon_1 \rightarrow \infty$ and $\nu_2 \rightarrow \infty$; Camp (1951) used this convergence to get a normal approximation to the binomial. In this section, six different methods that uses the normal approximation for the estimation of binomial confidence intervals are introduces and extended to the (MLN(n, π_i).

The Wald interval, used for large samples ($n\hat{p}$ and $n\hat{q} \geq 5$ and $.3 < \hat{p} < .7$) is the most popular used confidence interval for proportion (has been stated in all textbooks); it relies mainly on the normal approximation assumptions and takes the form:

$$W = \hat{p} \pm c_{\frac{\propto}{2}} \times s_{\hat{p}} \tag{6}$$

where $\hat{p} = \frac{x}{n}$ is the unbiased proportion with the attribute of interest in the sample, n is the sample size and $s_{\hat{p}} = \sqrt{\frac{\hat{p}(1-\hat{p}}{n}}$ is the standard error of $\hat{p}$, $c_{\frac{\propto}{2}}$ is the upper z value that makes the tail probability $= \frac{\propto}{2}$, i.e. $= z_{\propto/2}$ and W is asymptotically normal $N(0,1)$. Cochran (1963) recommends the use of t distribution with $(n-1)$ degrees of freedom for equation [6]; however, this correction has little effect for $n > 30$, and does not work well for $x = 0$ or $x = n$ (Cochran, 1977).

To correctly use normal approximation for continuity corrected expressions, some methods use $x \pm .5$ or $x \pm 1$ to the number of successes as a correction factor. Adjustments are also performed on the value of $\hat{p}$ and/or the standard error of $\hat{p}$. The corrected Wald test uses Yates (1934) continuity correction for $\hat{p} = \frac{x \pm .5}{n}$, and this affects the standard error by adding/subtracting a factor $\frac{1}{2n}$ to the standard error. The Wald corrected confidence limits ($WC - CL$) formula is extended to the multinomial with k categories as follows:

Method (A):

$$WC - CL = \frac{x \pm .5}{n} \pm \frac{c_k}{\sqrt{n}} \sqrt{\left[\frac{x \pm .5}{n}\left(1 - \frac{x \pm .5}{n}\right)\right]} \pm \frac{1}{2n} \tag{7}$$

where, $c_k = z_{\propto/2k}$. For the binomial case, the Wald and the corrected Wald behave poorlry interms of zero width interval at $x = 0$ or $x = n$ (Beal, 1987; Vollst, 1993); Newcombe, 1988; Pires, 2002 and Agresti, 2003). The Wald interval dominates for large samples, when the number of successes and the number of failures are greater than 5 ($n\hat{p} \geq 5$ and $n\hat{q} \geq 5$).

Based on WC [7] for the Binomial approximation, several alternative approaches have been developed. Molenaar (1973) gave a very accurate confidence bounds to the binomial; however, Molenaar`s limits were very complicated and do not satisfy the invariance property. Blyth and Still (1983, page 116)) have shown that equation [6] has failed to have asymptotic convergence as $\hat{p} = \frac{x}{n} \to 0$ and they gave a simple correction for equation [6] as the factor $\frac{c_k}{\sqrt{n}}$ is to be replaced by $\frac{c}{\sqrt{\{n-c^2-2c/\sqrt{n}-1/n\}}}$ where, $c = z_{\propto/2}$.

Extending Blyth and Still (1983) formula to the multinomial we get:

Method (B)

$$BS - CL(\pi_i) = \hat{p}_i \pm \frac{c}{\sqrt{\left\{n - c^2 - \frac{2c}{\sqrt{n}} - \frac{1}{n}\right\}}}\sqrt{[\,\hat{p}_i(1 - \hat{p}_i)]} + \frac{1}{2n} \tag{8}$$

Except lower limit = 0 for $x = 0$ and = 1 for $x = n$, $[c = z_{\propto/2k}]$.

Agresti and Coull (1998) have shown that the Wald interval is poor and the Clopper-Pearson interval is conservative; they proposed a straightforward adjustment to Wald. They suggested that by simply adding two successes and two failures and then use the Wald formula known as the Wilson Point estimator $\hat{p}_w$, defined as:

$$WPE(\pi_i) = p` \pm c \times \sqrt{\frac{p`(1-p`)}{n`}}$$

where, $p` = \frac{(2x+c^2)}{2n+c^2}$ and $n` = (n + c^2)$, and the Agresti Coull interval or the Wilson Point Estimator for the confidence interval as extended to the multinomial becomes:

Method (C):

$$WPE - CL(\pi_i) = \hat{p}_{w(i)} \pm c_k \sqrt{\frac{\hat{p}_{w(i)}\left(1 - \hat{p}_{w(i)}\right)}{(n + c_k^2)}} \tag{9}$$

where $c_k = z_{\propto/2k}$. Blyth and Still (1983) and Hald (1952) have derived an expression for the binomial (n, π), to be extended to the multinomial, as follows: let x_i denote a random variable which is multinomial (MLN(n, π_i), such as $0 \leq \pi_i \leq 1$. Let y_i has a normal distribution with the same mean and variance as x_i, then: $P(\,x_i \leq y_i \leq x_i\,) = \pi$. Then, for each integer a, the expression:

$$\frac{x_i - n\pi}{\sqrt{n\pi_i(1 - \pi)}} = c_k \sim N(0{,}1) \tag{10}$$

Let .5 be the correction continuity factor, then, $P(\,x_i - .5 \leq y_i \leq x_i + .5\,) = \pi_i$ (Hald, 1952), and equation [9] becomes:

$$\frac{x_i + .5 - n\pi}{\sqrt{n\pi_i(1 - \pi)}} = c_k ,$$

Squaring both sides, a quadratic form is obtained and solve to get the lower and upper limits to estimate single proportion ($k = 1$) for moderate and large n (Blyth and Still, 1983; Hald, 1952, p. 698). Hald formula is extended to the Multinomial as:

Method (D):

$$Hald - CL(\pi_i) = \frac{(x_i \pm .5) + \frac{c_k^2}{2} \pm c_k \sqrt{(x_i \pm .5) - \frac{(x_i \pm .5)^2}{n} + \frac{c_k^2}{4}}}{n + c_k^2} \quad (11)$$

$c_k = z_{\propto/2k}$ and Lower end $= 0$ at $x = 0$ and 1 for $x = n$.

Hall (1982, p. 649) showed that Hald approximation for the binomial [equation 10, $k = 1$] is underestimated; and gave a modified version that corrects for skewness and for Yates`s continuity factor and its multinomial version is:

Method (E):

$$Hall - CL(\pi_i) = \frac{x_i}{n} \pm \frac{c_k}{\sqrt{n}} \sqrt{\left(\frac{x_i}{n}\right)\left(1 - \frac{x_i}{n}\right)} + \frac{1}{6n} \times \left(1 - 2\left(\frac{x_i}{n}\right)(1 + 2c_k^2)\right). \quad (12)$$

Method E eliminates Yates's continuity correction, and $\frac{1}{6n} \times \left(1 - 2\left(\frac{x_i}{n}\right)(1 + 2c_k^2)\right)$ is a corrected factor for skewness, and $c_k = z_{\propto/2k}$.

Fleiss (1981) gave a confidence interval for a single proportion with continuity correction, called it Wilson Score (due to Wilson, 1927). Wilson Score corrected for continuity (Reed, 2007) could be extended to the Multinomial case as follows:

Method (F):

$$WS_{cc} - CL(\pi_i) = \frac{2n\hat{p_i} \pm 1 + {c_k}^2 \pm c\sqrt{{c_k}^2 \pm 2 - \frac{1}{n} + 4\hat{p_i}(n\hat{q}_i \mp 1)}}{2n + 2{c_k}^2} \quad (13)$$

where, $\hat{p}_i = \frac{x_i}{n}$, $\hat{q}_i = 1 - \hat{p}_i$ and $c_k = z_{\propto/2k}$.

Thus, Wilson score [12] adjusts the point estimate for proportion to become:

$\frac{x_i \pm .5 + \frac{c^2}{2}}{n + c^2}$ and the standard error to become:

$\frac{\sqrt{c^2 \pm \left(2 - \frac{1}{n}\right) + 4\hat{p_i}(n\hat{q}_i \mp 1)}}{2n + 2c^2}$ for the upper and lower limits.

Tobi et al. (2005) concluded that: the Wald method is very poor, and the Clopper-Pearson (CP) and Wilson Corrected Score (WS_{CC}) are the best choices to calculate confidence intervals for binomial proportion; Borkowf (2006) concluded that the Agresti-Coull (Point Wilson Estimator are substantially better than the Wald method.

4. ANALYTIC RESULTS

Confidence intervals are estimated at the 95% level of significance for methods A to F and the exact methods, for multinomial proportions $((n, P_i)$ for $n = 5, 10,15, 20, 30,$ and 40; $i = 1,2, \dots 5)$, and $x_i = 0, \dots n$. The $z_{\frac{.05}{2k}}$ values used for method A to F and for the exact methods is 1.96, 2.24, 2.4, 2.51 and 2.58 for $K = 1, 2, 3, 4, 5$ consecutively.

Given that invariance (Equation [3]); then:

$$UCL(\pi_{i(x)}) - LCL(\pi_{i(x)}) = UCL(\pi_{i(n-x)} - LCL(\pi_{i(n-x)}$$

i.e., the interval width for the estimation of $(\pi_{i(x)})$ = the interval width for the estimation of $\pi_{i(n-x)}$. The following results are obtained at the 5% level of significance.

4.1 95% Confidence Intervals for Multinomial Proportions

4.1.1 For $K = 1$

Interval widths are checked for Invariance as shown in Table (1). Widths were checked for invariance around the median value of n. The following is reached for each method: Thus, invariance is satisfied for Method A, D and F for all values of n, excluding at $x = 0$ and $x = n$. Method B, it is satisfied for $n = 10$, 15, and 20; and for $n = 30$ invariance is satisfied such that: $UCL(\pi_{i(x)}) = 1 - LCL(\pi_{i(n-x))}$ $1 \le x \le n - 1$. For $n = 40$, invariance is satisfied between $x = 0$ and $x = 39$ that is $UCL(\pi_{i(x)}) = 1 - LCL(\pi_{i(n-x-1))}$. Same is applied to method C, as shown in the Table. Method E does not satisfy invariance for all sample sizes considered except for $n = 40$, where the invariance equation becomes: $UCL(\pi_{i(x)}) = 1 - LCL(\pi_{i(n-x-4)}$ for $(3 \le x \le n - x - 4)$. Method E, though it does not satisfy invariance property, it gives negative lower limits for some of the lowest values of x, under all sample sizes, and it gives lower limits greater than one for values of $x = (n - 1)$ and $x = n$, and those values were treated as zeros. Method E, although does not satisfy invariance, yet it provides the narrowest interval width as compared to all other method widths considered.

Table 1
Analytical Results Reached for each Method, $K = 1, \alpha = 5\%$

Method / n	A	B	C	D	E	F	Exact
5	√	×	×	√	×	√	×
10	√	√	√	√	×	√	√
15	√	√	$0 \le X \le n-x-2$	√	×	√	√
20	√	√	$0 \le X \le n-x-1$	√	×	√	√
30	√	$0 \le X \le n-x-1$	$0 \le X \le n-x-2$	√	×	√	√
40	√	$0 \le X \le n-x-1$	$0 \le X \le n-x-2$	√	$3 \le X \le n-x-4$	√	√

√: satisfied ×: not satisfied;
$x \le X \le n - x - i$: satisfied only for these values of $x, i = 1,2, \dots$

The Exact method of CP, invariance is satisfied for all sample sizes considered except at $n = 5$. The Clopper and Pearson exact confidence interval widths satisfied invariance, for all values of n, except for $n = 5$. All methods widths increase up to $x =$ median then decrease except for Method E which talks a decreasing trend as x values increase; it also gives the narrowest interval widths as x increases.

Table (2) shows interval widths for the seven methods, for $n = 15$ and selected values of x and Figure gives 95% Confidence interval width for Methods A-F and the exact method, for $n = 40$.

Table 2
95% Confidence interval width for Methods A-F and the Exact Method, for $K = 1$ $n = 15$

n	A	B	C	D	E	F	EXACT
1	0.31	0.50	0.34	0.34	0.19	0.34	0.19
2	0.41	0.59	0.38	0.39	0.29	0.39	0.28
3	0.47	0.66	0.41	0.43	0.35	0.43	0.34
4	0.51	0.70	0.43	0.46	0.37	0.46	0.38
5	0.54	0.73	0.44	0.48	0.37	0.48	0.40
6	0.56	0.75	0.45	0.50	0.36	0.50	0.42
7	0.57	0.75	0.45	0.50	0.35	0.50	0.43
8	0.57	0.75	0.44	0.50	0.32	0.50	0.43
9	0.56	0.73	0.43	0.50	0.29	0.50	0.42
10	0.54	0.70	0.41	0.48	0.24	0.48	0.40
11	0.51	0.66	0.38	0.46	0.19	0.46	0.37
12	0.47	0.59	0.34	0.43	0.12	0.43	0.33
13	0.41	0.50	0.29	0.39	0.00	0.39	0.27
14	0.31	0.35	0.21	0.34	0.00	0.34	0.18

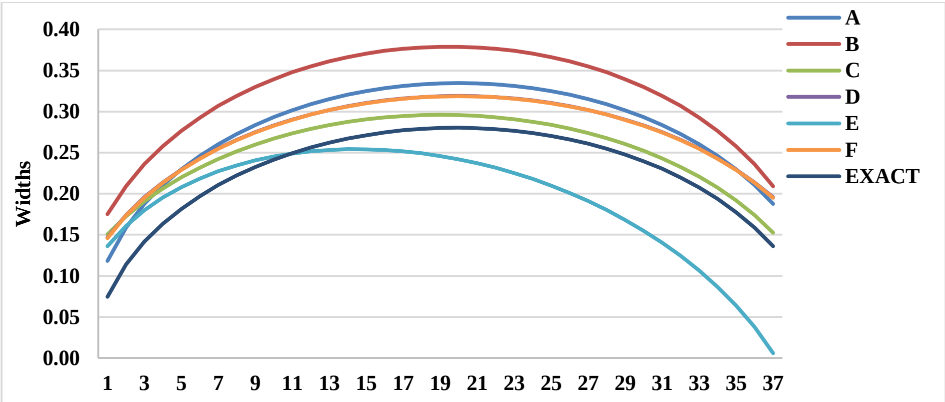

Figure 1: 95% Confidence Interval Widths for all Methods $k = 1$ $n = 40$

4.1.2 For $K = 2$

Table (3) shows that invariance is satisfied for Method A and D F for all values of n, excluding at $x = 0$ and $x = n$. It is satisfied for Method B, only for $n \geq 15$ where $UCL(\pi_{i(x)}) = 1 - LCL(\pi_{i(n-x-1))}$ $0 \leq x \leq n - x - 1$; for Method C, invariance is satisfied for $n \geq 20$, $UCL(\pi_{i(x)}) = 1 - LCL(\pi_{i(n-x-2))}$ and $1 \leq x \leq n - x - 2$ and for $n = 40, \mathrm{UCL}(\pi_{i(x)}) = 1 - \mathrm{LCL}(\pi_{i(n-x-2))}$ and $0 \leq x \leq n - x - 3$.

Method F, invariance is satisfied for $n = 5$ and $n = 40$ and is satisfied also for other values of n and for some x values only as shown in the Table (3). For the Exact method, invariance is satisfied for all sample sizes considered except at $n = 5$, and $UCL(\pi_{i(x)}) = 1 - LCL(\pi_{i(n-x))}$ and $1 \leq X \leq n - x$. Still Method E gives the narrowest interval widths for values around x=median, however, it gives negative lower limits for some of the lowest values of x, under all sample sizes, and it gives lower limits greater than one for values of $x = (n - 1)$ and $x = n$, and those values were treated as zeros. The exact method gives approximately equal width for x and $n - x$, widths for values of x less than median x are 1% higher than the values of x higher than the median x value. Method E, although does not satisfy invariance, yet it provides the narrowest interval width as compared to all other method widths considered.

Table (4) gives interval widths for all methods considered $K = 2$, $n = 20$, and Figure (2) gives 95% confidence interval widths for Methods A to F and for the Exact method for $K = 2$ and $n = 40$.

Figure (2) shows that as n increases all methods except Method D approaches symmetry, and thus invariance.

4.1.3 $k = 3$, 4 and 5

Tables 5, 6 and 7 gives summary tables for $K = 3$, 4 and 5 respectively. Interpretation of Tables 5, 6, and 7 is the same as the two previous tables. Methods A and D satisfy invariance as in Equation (3). Other Methods satisfy invariance only for some values of x and $(n - x - i), i = 0,1,2 \ldots .5$.

4.2 99% Confidence Intervals for Multinomial Proprtions

For the 1% level of significance, The $z_{\frac{.01}{2k}}$ used for all methods considered are 2.58, 2.81, 2.93, 3.0 and 3.9 for $K = 1$, 2, 3, 4 and 5 consecutively. Interval widths for $K = 5$ and $\propto = 5\%$ are the same as the widths for $K = 1$ and $\propto = 1\%$ for methods A to F but differ for the exact method and invariance is satisfied as in Table (7) above.

For $K = 2$, 3, 4, 5 and $\propto = 1\%$, and for all values of n; Method B and E produce lower intervals less than zero for small few values of x`s and produce upper limits more than 1 for high higher values of $x`s$ and do not satisfy invariance; and thus, will be excluded from the analysis. Tables (8), (9), (10) and (11) gives invariance check for Mehods A, C, D, F and the Exact methd For $k = 2$, 3, 4 and 5 respectively. Figures (6), (7), (8) and (9) show interval widths for $n = 40$ at $K = 2,3,4$ and $K = 5$. Figures (6), (7), (8) and (9) show interval widths for $n = 40$ at $K = 2$, 3, 4 and $K = 5$. It is clear from the graphs that Method E does not display invariance (not symmetrical) and it has the lowest interval widths among all other methods. As n increases the curves of A, D and F gets more smoother and approaches symmetry.

Table 3
Analytical Results Reached for each Method, $K = 2$, $\alpha = 5\%$

	A	B	C	D	E	F	Exact
5	√	×	×	√	×	√	×
10	√	×	×	√	×	$2 \leq X \leq n - x - 1$	√
15	√	$1 \leq X \leq n - x$	$4 \leq X \leq n - x - 2$	√	×	√	√
20	√	$1 \leq X \leq n - x$	√	√	×	$2 \leq X \leq n - x - 1$	$1 \leq X \leq n - x$
30	√	$1 \leq X \leq n - x$	$0 \leq X \leq n - x - 2$	√	$3 \leq X \leq n - x - 7$	$6 \leq X \leq n - x$	$3 \leq X \leq n - x$
40	√	$1 \leq X \leq n - x$	$0 \leq X \leq n - x - 3$	√	$2 \leq X \leq n - x - 6$	√	$1 \leq X \leq n - x$

Table 4
95% Confidence Interval Width for Methods A-F and the Exact Method, for $n = 20$

x	A	B	C	D	E	F	EXACT
1	0.26	0.42	0.32	0.30	0.16	0.30	0.16
2	0.35	0.50	0.36	0.34	0.24	0.35	0.25
3	0.41	0.56	0.39	0.38	0.31	0.39	0.31
4	0.45	0.60	0.41	0.41	0.34	0.42	0.34
5	0.48	0.64	0.42	0.43	0.36	0.44	0.38
6	0.51	0.66	0.43	0.45	0.37	0.47	0.40
7	0.53	0.68	0.44	0.47	0.37	0.48	0.41
8	0.54	0.69	0.45	0.48	0.36	0.48	0.43
9	0.55	0.70	0.45	0.48	0.35	0.49	0.43
10	0.55	0.70	0.47	0.48	0.33	0.49	0.43
11	0.55	0.69	0.44	0.48	0.31	0.48	0.43
12	0.54	0.68	0.43	0.48	0.29	0.48	0.42
13	0.53	0.66	0.41	0.47	0.26	0.47	0.41
14	0.51	0.64	0.40	0.45	0.22	0.45	0.39
15	0.48	0.60	0.37	0.43	0.17	0.43	0.37
16	0.45	0.56	0.34	0.41	0.12	0.40	0.34
17	0.41	0.50	0.30	0.38	0.06	0.37	0.30
18	0.35	0.42	0.25	0.34	0.00	0.33	0.24
19	0.26	0.29	0.18	0.30	0.00	0.28	0.15

Table 5

Invariance Results for Methods A – F and Exact Method for K=3, $\alpha = 5\%$

n	A	B	C	D	E	F	Exact
5	×	×	$0 \leq X \leq n - x - 3$	√	×	√	×
10	√	×	$0 \leq X \leq n - x - 3$	√	×	$2 \leq X \leq n - x - 1$	$3 \leq X \leq n - x$
15	√	$1 \leq X \leq n - x - 1$	$4 \leq X \leq n - x - 3$	√	×	$1 \leq X \leq n - x - 3$	$2 \leq X \leq n - x - 1$
20	√	$1 \leq X \leq n - x - 1$	$1 \leq X \leq n - x$	√	×	$0 \leq X \leq n - x - 3$	$0 \leq X \leq n - x - 3$
30	√	$1 \leq X \leq n - x - 1$	$1 \leq X \leq n - x - 1$	√	×	$0 \leq X \leq n - x - 3$	$1 \leq X \leq n - x$
40	√	$1 \leq X \leq n - x$	$1 \leq X \leq n - x - 2$	√	×	√	$1 \leq X \leq n - x$

Table 6

Invariance results for Methods A – F and Exact Method for $K = 4$, $\alpha = 5\%$

n	A	B	C	D	E	F	Exact
5	×	×	×	√	×	√	×
10	√	×	×	√	×	$2 \leq X \leq n - x - 1$	$2 \leq X \leq n - x$
15	√	×	$0 \leq X \leq n - x - 3$	√	×	$0 \leq X \leq n - x - 1$	$1 \leq X \leq n - x$
20	√	$0 \leq X \leq n - x$	$0 \leq X \leq n - x - 2$	$2 \leq X \leq n - x$	×	$0 \leq X \leq n - x - 1$	$1 \leq X \leq n - x$
30	√	√	$1 \leq X \leq n - x - 1$	$1 \leq X \leq n - x$	×	$0 \leq X \leq n - x - 1$	$1 \leq X \leq n - x$
40	√	$0 \leq X \leq n - x - 1$	$1 \leq X \leq n - x - 1$	$1 \leq X \leq n - x$	×	$0 \leq X \leq n - x - 1$	$1 \leq X \leq n - x$

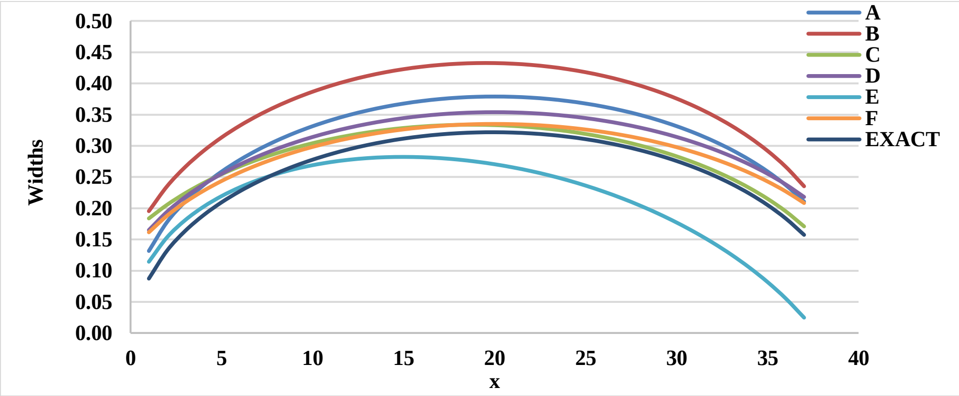

Figure 2: 95% Confidence Interval Widths for All Methods $k = 2$, $n = 40$

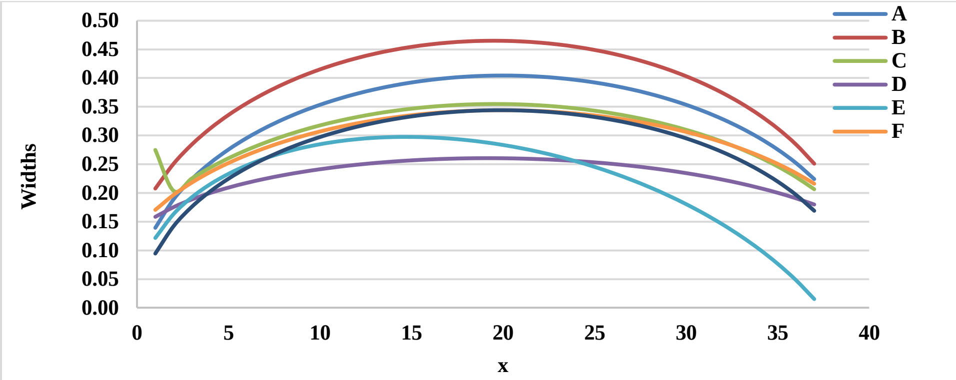

Figure 3: 95% Confidence Interval Widths for All Methods $k = 3, n = 40$

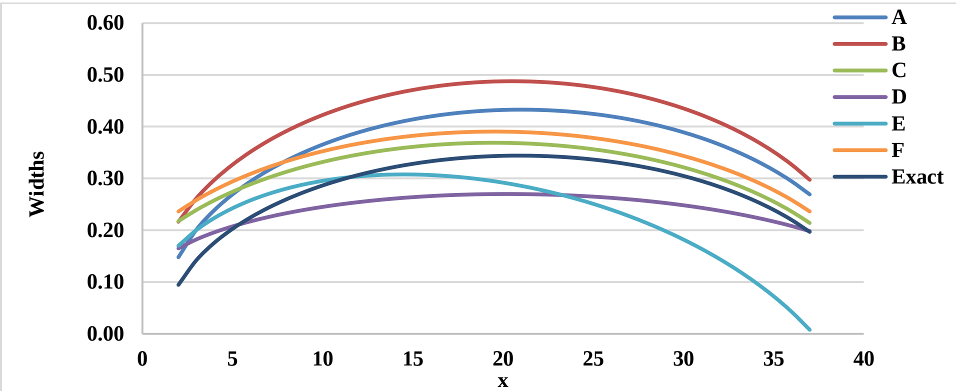

Figure (4): 95% Confidence Interval Widths for All Methods $k = 4, n = 40$

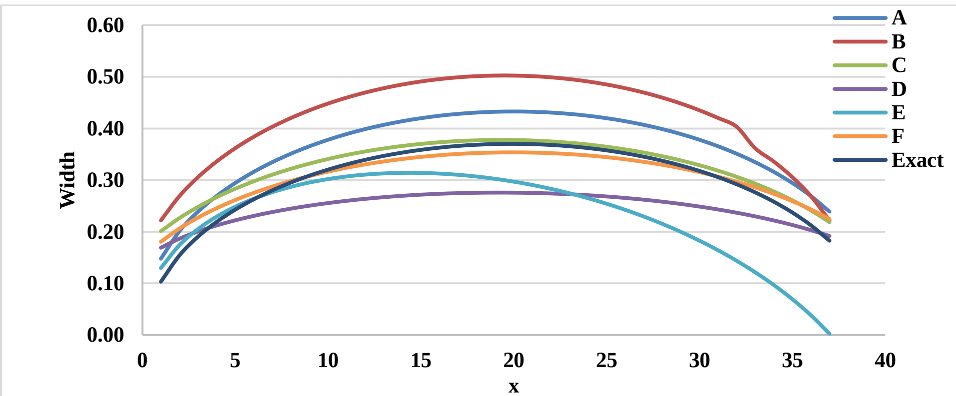

Figure (5): 95% Confidence Interval Widths for All Methods $k = 5, n = 40$

Table 7

Invariance results for Methods A – F and Exact Method for $K = 5, \alpha = 5\%$

n	A	B	C	D	E	F	Exact
5	×	×	×	√	×	√	×
10	√	×	×	√	×	√	$1 \le X \le n - x$
15	√	$4 \le X \le n - x - 1$	X	√	×	$0 \le X \le n - x - 1$	$1 \le X \le n - x$
20	√	$2 \le X \le n - x$	$0 \le X \le n - x - 2$	√	×	$0 \le X \le n - x - 1$	$1 \le X \le n - x$
30	√	$3 \le X \le n - x$	$0 \le X \le n - x - 4$	√	×	$0 \le X \le n - x - 1$	$1 \le X \le n - x$
40	√	$3 \le X \le n - x$	$1 \le X \le n - x - 3$	√	×	$0 \le X \le n - x$	$1 \le X \le n - x$

Table 8

Invariance results for Methods A – F and Exact Method for $K = 2$ and $\propto = 1\%$

n	A	B	C	D	E	F	Exact
5	×	×	×	√	×	√	×
10	√	×	×	√	×	√	$1 \le X \le n - x$
15	$4 \le X \le n - x - 1$	×	X	√	×	$1 \le X \le n - x$	$1 \le X \le n - x - 2$
20	$1 \le X \le n - x$	×	X	√	×	$1 \le X \le n - x$	$1 \le X \le n - x$
30	$3 \le X \le n - x$	×	$0 \le X \le n - x - 4$	√	×	$1 \le X \le n - x$	$1 \le X \le n - x$
40	$3 \le X \le n - x$	×	$0 \le X \le n - x - 4$	√	×	$1 \le X \le n - x$	$1 \le X \le n - x$

Table 9

Invariance Results for Methods A – F and Exact Method for $K = 3$ and $\propto = 1\%$

n	A	B	C	D	E	F	Exact
5	×	×	×	√	×	√	×
10	X	×	×	√	×	$1 \le X \le n - x - 2$	$2 \le X \le n - x$
15	$6 \le X \le n - x$	×	$3 \le X \le n - x - 2$	√	×	$1 \le X \le n - x - 3$	$1 \le X \le n - x$
20	$1 \le X \le n - x$	×	$1 \le X \le n - x$	√	×	$1 \le X \le n - x - 3$	$1 \le X \le n - x$
30	√	×	$1 \le X \le n - x - 1$	√	×	$1 \le X \le n - x - 3$	$1 \le X \le n - x$
40	√	$4 \le X \le n - x$	$1 \le X \le n - x - 1$	√	×	$1 \le X \le n - x - 2$	$1 \le X \le n - x$

Table 10
Invariance Results for Methods A – F and Exact Method for $K = 4$ and $\propto = 1\%$

n	A	B	C	D	E	F	Exact
5	×	×	×	√	×	√	×
10	$0 \leq X \leq n - x - 2$	×	×	√	×	$1 \leq X \leq n - x$	$1 \leq X \leq n - x$
15	$1 \leq X \leq n - x - 2$	×	$0 \leq X \leq n - x - 3$	√	×	$2 \leq X \leq n - x - 1$	$2 \leq X \leq n - x$
20	$1 \leq X \leq n - x - 2$	×	$1 \leq X \leq n - x - 3$	√	×	$1 \leq X \leq n - x - 1$	$1 \leq X \leq n - x$
30	$0 \leq X \leq n - x - 4$	$8 \leq X \leq n - x$	$0 \leq X \leq n - x - 3$	√	×	$1 \leq X \leq n - x - 3$	$1 \leq X \leq n - x$
40	$0 \leq X \leq n - x - 4$	$10 \leq X \leq n - x - 1$	$0 \leq X \leq n - x - 3$	√	×	$0 \leq X \leq n - x - 3$	$1 \leq X \leq n - x$

Table 11
Invariance Results for Methods A – F and Exact Method for $K = 5$ and $\propto = 1\%$

n	A	B	C	D	E	F	Exact
5	×	×	×	√	×	√	×
10	×	×	$1 \leq X \leq n - x - 3$	√	×	√	$3 \leq X \leq n - x$
15	×	×	$2 \leq X \leq n - x - 3$	√	×	√	$1 \leq X \leq n - x$
20	×	×	$2 \leq X \leq n - x - 1$	√	×	√	$1 \leq X \leq n - x$
30	$3 \leq X \leq n - x$	$4 \leq X \leq n - x$	$1 \leq X \leq n - x$	√	×	√	$1 \leq X \leq n - x$
40	$3 \leq X \leq n - x$	$2 \leq X \leq n - x - 1$	$1 \leq X \leq n - x$	√	×	√	$1 \leq X \leq n - x$

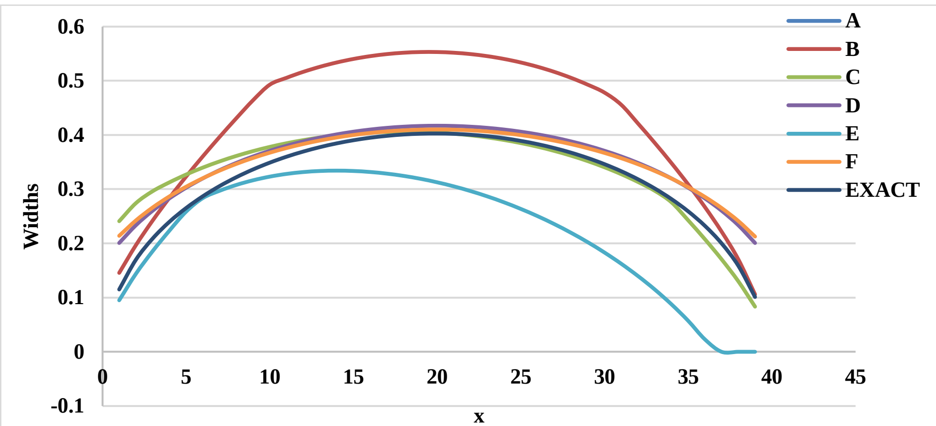

Figure 6: 99% Confidence Interval Widths for All Methods $k = 2, n = 40$

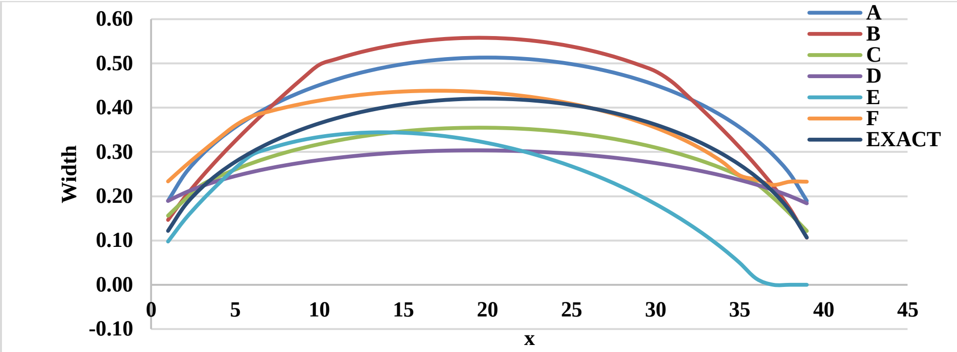

Figure 7: 99% Confidence Interval Widths for All Methods $k = 3, n = 40$

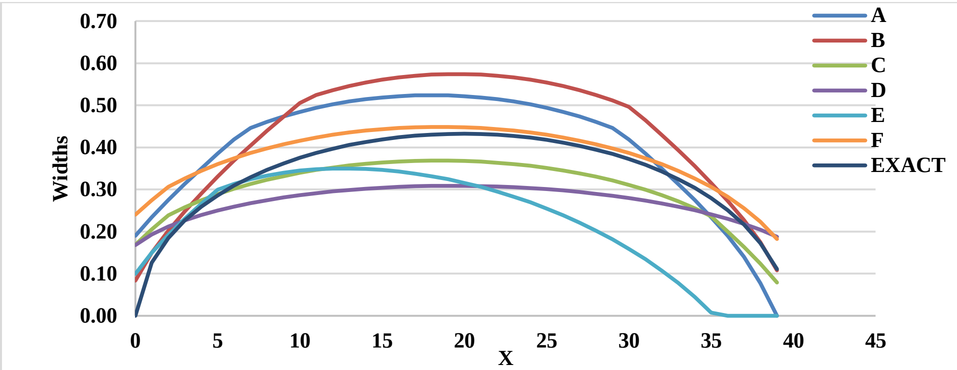

Figure 8: 99% Confidence Interval Widths for All Methods $k = 4, n = 40$

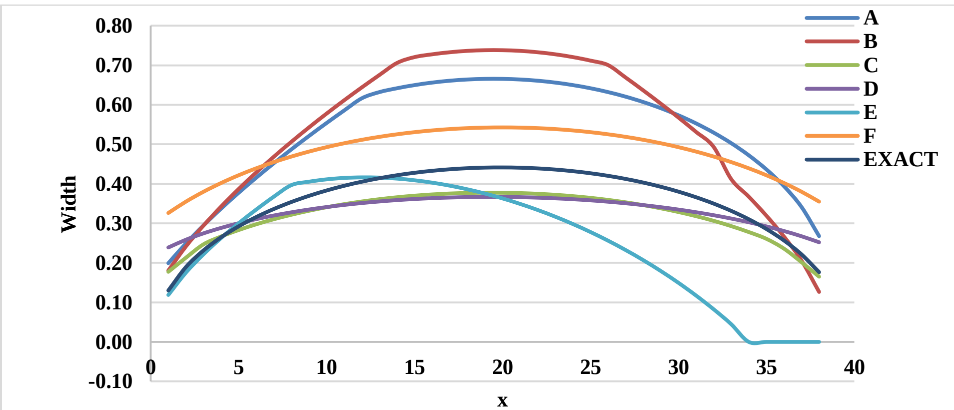

Figure 9: 99% Confidence Interval Widths for All Methods $k = 5, n = 40$

4.3 Maximum Interval Widths

All methods involved in the analysis produce widths that are increasing form $x = 0$ or 1 to some point (usually the median x) and then decrease to the last $x = n - 1$ or $x = n$. Widths for all Methods and for $k = 1$ to 5, and $\alpha = 5\%$ and 1% = were checked to find out the point where the curve starts to change from increasing to decreasing. It is noted that the maximum width for all Methods studied lie at one paricular value of x, usually around the median value, as shown in Table (12).

Table (12) showed that:

a) Maximum widths get smaller as sample size increases.

b) The x value at maximum width is the median value for most n values under all values of k, however for $\alpha = 1\%$ the x values are shifter left as compared to $\alpha = 5\%$.

c) The E method produces the minimum maximum width for for $k = 1$ and 2; however Method D gives the smallest maximum width for $n = 30$ and 40 for $k = 3$, 4 and 5 for $\alpha = 5\%$ and $\alpha = 1\%$.

d) At $\alpha = 5\%$, the exact method gives the second smallest width for $k = 1$ and 2; Methdo D gives the second smallest for $k = 3$, 4 and 5.

e) At $\alpha = 1\%$, the exact method or the D Method gives the second smallest width for $k = 1$ and 2; however Method E gives the second smallest maximum width for $n = 30$ and 40 for $k = 3$, 4 and 5.

f) Widths for Method B reaches 1 for $n = 10$ and 15 for $\alpha = 5\%$ and for $n \leq 20$ for $\alpha = 1\%$.

5. CONCLUSIONS

Six methods that rely on the normal sampling distribution for the estimation of Binomial proportion and the exact method of Clopper Pearson that relies on the F distribution for the binomial estimation were extended to the multinomial distribution variate using the Bonferroni approach when the number of categories is 1 to 5, for $n = 5$, 10, 15, 20, 30 and 40 and at significance level 5% and 1%. Methods were compared according to invariance property as given by Blyth (1986) and also compared according to the width of intervals produced.

1) It is found that the invariance property could be measures as equal width at x and at $(n - x)$. It is found that:

At $\alpha = 5\%$,

a) The Corrected Wald Method (Method A) dominates for $k = 1$ and 2; for $k = 3$ it fails at $n = 5$, for $n = 10$, 15 and 20 the method satisfy invariance for values between 1 and $(n - 1)$, and is satisfies for $n > 20$. For $K = 4$, invariance is satisfies for $n > 5$, and for $K = 5$ invariance is satisfies for $n > 10$.

b) The Blyth and Still Method (Method B) does satisfy invariance property for k=1 $(n > 5)$; $k = 2$ and 3 for $n > 10$; $k = 4$ and $k = 5$ for $n > 15$.

c) The Wilson Point Estimator Method (Method C) satisfies invariance property for $k = 1$ and $k = 3$ $(n > 5)$; $k = 2$, 4 and 5 for $n > 10$.
d) The Blyth and Still Method 2 (Method D) does satisfy invariance property for $k = 1, .., 5$ and for all values of $n =$ (5, 10, 15, 20, 30 and 40.
e) The Hall Method (Method E), does not satisfy invariance for $k = 1$ and $k = 2$ except for large $n(> 20)$; for $k = 33$, 5 and 5, invariance is satisfied for $n > 30$, under some conditions on x and $(n - x)$.
f) The Wilson Score Method (Method F) satisfies invariance property for $k = 1$ for all values of n; $K = 2$, 3, 4 and 5 invariance is satisfied under some conditions on x and $(n - x)$.
g) The Clopper-Pearson Method (The exact method) for $k = 1$, invariance is not satisfied at $n = 5$; for $K = 2$ inviance is satisfied for $n = 5$ and 10, and under some conditions on x and $n - x$ for $n > 10$. For $K = 3$, invariance is not satisfies for $n = 5$; for $n = 10$, 15 and 20, invariance is satisfied under some conditions on x and $(n - x)$; for $n > 20$ ivariance is satisfied (excluding $x = 0$ and $x = n$). For $K = 4$ and $k = 5$, invariance is not satisfies for $n = 5$; for 10 invariance is satisfied under some conditions on x and $(n - x)$; for $n > 10$ invariance is satisfied (excluding $x = 0$ and $x = n$).

At $\alpha = 1\%$,

a) The Corrected Wald Method (Method A) does not satisfy invariance for $k = 1$, $n = 5$ and 10; for $k = 2$, it fails at $n < 20$; for $k = 3$, 4 and 5, it fails at $n < 15$ and $k = 5$ the method satisfy invariance $n = 30$ and 40 under some conditions on x and $(n - x)$.
b) The Blyth and Still Method (Method B) does not satisfy invariance property for $k = 1$ or $k = 2$ (all n`s); invariance is satisfied for $k = 3$ and $k = 4$ $(n > 30)$; $K = 5$ $(n > 20)$ under some conditions on x and $(n - x)$.
c) The Wilson Point Estimator Method (Method C) does not satisfiy invariance property for $k = 1$; for $k = 2$ $(n > 20)$; $k = 3$, 4 and 5 $(n > 10)$, under some conditions on x and $(n - x)$.
d) The Blyth and Still Method 2(Method D) does satisfy invariance property for $k = 1, \dots, 5$ and for all values of $n =$ (5, 10, 15, 20, 30 and 40.
e) The Hall Method (Method E) does not satisfy invariance for all $k = 1 \ldots 5$ and all values of n.
f) The Wilson Score Method (Method F) satisfies invariance property for $k = 1$ for all values of n; $K = 2$, 3, 4 and 5 $(n > 5)$ invariance are satisfied under some conditions on x and $(n - x)$.
g) The Clopper-Pearson Method (The exact method), for $k = 1$, invariance is satisfied for all n values; for $K = 2$ $(n = 5, 10, 15, 20)$ under some conditions on x and $(n - x)$, and satisfied for $n > 20)$; satisfied for $k = 3$ and 4 $(n = 5)$, and for $n = 10$ and 15 satisfied under some conditions and satisfied for $n > 20)$; $K = 4$ and 5, inviance is satisfied for $n < 20$, under some conditions on x and $(n - x)$; for $n > 20$ invariance is satisfied (excluding $x = 0$ and $x = n$). For $K = 5$, invariance is satisfies for $n = 5$; for 10 invariance is satisfied under some conditions on x and $(n - x)$; for $n > 10$ invariance is satisfied (excluding $x = 0$ and $x = n$).

2) Interval widths were also compared, it is found that Method E (Hall Method) produces the smallest maximum width, and thus narrower confidence interval width, and thus minimum marginal error at both levels of significance, for the different values of x and n; followed by either the Exact method or the Blyth and Still Method 2 (Method D).

RECOMMENATIONS

Method 2 of Blyth and Still (Method D) and Wilson Score Method (Method F) prove to do well in the estimation of multinomial proportion; these two methods satisfy invariance property; however, the interval widths produced are not the narrowest. The Clopper-Pearson Exact Method satisfies invariance under some conditions when applied to $K > 1$, and the interval widths area bit higher than Method E (Hall method).

Table 12
The x Values that Produce Maximum Interval Width

k	n	Maximum Interval Widths $\alpha = 5\%$								Maximum Interval Widths $\alpha = 1\%$							
		x	***A***	***B***	***C***	***D***	***E***	***F***	***Exact***	***X***	***A***	***B***	***C***	***D***	***E***	***F***	***Exact***
1	10	5	.72	1.0	.52	.60	.36	.60	.50	5	.90	1.0	.61	.66	.37	.69	.65
	15	8	.57	.75	.44	.50	.32	.50	.43	7	.73	1.0	.55	.58	.39	.60	.65
	20	11	.49	.60	.39	.44	.29	.44	.29	10	.63	.84	.49	.52	.36	.54	.50
	30	15	.39	.46	.34	.37	.27	.37	.32	14	.50	.61	.43	.44	.33	.45	.42
	40	20	.33	.38	.30	.32	.24	.32	.28	18	.43	.50	.38	.39	.31	.40	.37
2	10	5	.80	1.0	.57	.63	.37	.64	.58	5	.43	1.0	.64	.67	.36	.72	.70
	15	7	.64	.89	.50	.54	.37	.55	.49	8	.44	1.0	.57	.60	.35	.63	.60
	20	9	.55	.70	.47	.48	.35	.49	.43	10	.54	.95	.52	.55	.37	.57	.54
	30	15	.44	.52	.38	.40	.30	.41	.37	14	.45	.67	.46	.47	.35	.48	.45
	40	20	.38	.43	.33	.35	.27	.34	.32	17	.40	.55	.41	.41	.33	.41	.40
3	10	5	.86	1.0	.59	.54	.37	.67	.61	5	1.0	1.0	.59	.57	.35	.72	.72
	15	7	.68	.99	.52	.44	.38	.57	.52	7	.89	1.0	.52	.49	.40	.60	.63
	20	10	.59	.76	.47	.38	.34	.51	.46	9	.75	.91	.47	.43	.40	.56	.56
	30	15	.47	.56	.40	.31	.31	.43	.39	14	.60	.68	.40	.35	.36	.49	.47
	40	20	.40	.46	.35	.26	.28	.34	.34	19	.51	.56	.35	.30	.33	.44	.42
4	10	5	.91	1.0	.60	.55	.37	.68	.61	5	.97	1.0	.60	.57	.35	.86	.74
	15	7	.73	1.0	.54	.45	.39	.59	.52	7	.90	1.0	.54	.49	.44	.65	.64
	20	10	.63	. 80	.48	.39	.33	.52	.46	9	.77	.92	.49	.43	.40	.59	.57
	30	14	.50	.59	.42	.32	.33	.44	.39	14	.61	.70	.42	.36	.35	.51	.49
	40	19	.43	.48	.37	.27	.30	.39	.34	17-20	.52	.57	.37	.31	.32	.45	.43
5	10	5	.91	1.00	.61	.55	.37	.69	.75	5	1.0	1.0	.61	.59	.22	.81	.75
	15	7	.73	1.00	.55	.46	.39	.53	.64	7	1.0	1.0	.55	.53	.38	.74	.65
	20	10	.63	.84	.50	.40	.36	.54	.59	10	.97	1.0	.50	.48	.37	.69	.59
	30	15	.50	.61	.43	.32	.32	.45	.50	14	.78	.90	.43	.41	.40	.60	.50
	40	20	.43	.50	.38	.28	.30	.35	.44	19	.67	.74	.38	.37	.38	.54	.44

REFERENCES

1. Agresti, A. and Coull, B.A. (1998). Approximate is better than exact for interval estimation of binomial proportions. *The American Statistician*, 52, 119-126.
2. Anderson, T. and Burstein, H. (1967). Approximating the upper binomial confidence limit. *J. Amer. Statist*., 62, 857-62.
3. Blyth, C.R. (1986). Approximate Binomial Confidence Limits. *Journal of the American Statistical Association*, 81, 843-855.
4. Blyth, C.R. and Still, H.A. (1983). Binomial confidence intervals. *Journal of the American Statistical Association*, 78(381), 108-116.
5. Böhning, D. (1994). Better Approximate Confidence Intervals for a Binomial Parameter. *Statistical Society of Canada*, 22(2), 207-218.
6. Borkowf, C.B. (2006). Constructing binomial confidence intervals with near nominal coverage by adding a single imaginary failure or success. *Statistics in Medicine*, 25(21), 3679-3695.
7. Brownlee, K.A. (1965). *Statistical Theory and Methodology in Science and Engineering*. 2nd ed., John Wiley, New York.
8. Camp, B.H. (1951). Approximate of the Point Binomial. *Annals of Mathematical Statistics*, 22, 130-131.
9. Clopper, C.J. and Pearson, E.S. (1943). The Use of Confidence or Fiducial Limits Illustrated in the Case of the Binomial. *Biometrika*, 26, 404-413.
10. Cochran, W.G. (1963). *Sampling Technique* (2nd Ed.), John Wiley & Sons, New York.
11. Cochran, W.G. (1977). *Sampling Technique* (3rd Ed.), John Wiley & Sons, New York.
12. Colin, C. and Pearson, E. (1934). Approximate Binomial Confidence Limits. *Biometrika*, 26, 404-413.
13. Fleiss, J.H. (1981). *Statistical Methods for Rate Proportions* (2nd Ed.). John Wiley & Sons, New York.
14. Gebhardt, F. (1969). Some numerical comparisons of several approximations to the binomial distribution. *J. Amer. Statist*., 84, 1638-46.
15. Goodman, L.A. (1965). On simultaneous Confidence Interval for multinomial Proportions. *Technometrics*, 7, 247-254.
16. Hald, A. (1952). *Statistical Theory with Engineering Applications*, John Wiley, New York.
17. Hall, P. (1982). Improving the normal approximation when constructing one-sided confidence intervals for binomial or Poisson parameters. *Biometrika*, 69(3), 647-652.
18. Molenaar, W. (1970). *Approximations to the poisson, binomial and hypergeometric distribution functions. MC Tracts*. Centrum Voor Wiskunde en Informatica.
19. Molenaar, W. (1973). Simple Approximations to the Poisson, Binomial, and Hypergeometric Distributions. *Biometrics*, 29(2), 403-407.
20. Newcombe, R.G. (1989). Two Sided Confidence Intervals for the Single Proportion: Comparison of Seven Methods. *Statistical Methods Medical Research*, 17, 857-872.
21. Peizer, D.B. and Pratt, J.W. (1968). A normal approximation for binomial, F, beta, and other common, related tail probabilities. *Journal of the American Statistical Association*, 63(324), 1416-1456.
22. Pires, A.M. (2002). Confidence Intervals for Binomial Proportion: Comparison of Methods and Software Evaluation. *Proceeding of the Conference ComStat* 2002.

23. Pratt, J.W. (1968). A Normal Approximation for Binomial, F, Beta, and other common, related tail Probabilities. *J. Amer. Statist.*, 63, 1457-1483.
24. Raff, M.S. (1956). On approximating the Point Binomial. *J. Amer. Statist.*, 61, 293-303.
25. Reed III, J.F. (2007). Better binomial confidence intervals. *Journal of Modern Applied Statistical Methods*, 6(1), 153-161.
26. Reiczigel, J. (2003). Confidence Intervals for Binomial Parameter: Some New Considerations. *Statistical Methods Medical Research*, 22, 611-621.
27. Sokal, R. and Rohif, F. (1987). *Introduction to Biostatistics*, 2nd Edition, W.H. Freeman and Company, New York.
28. Tobi, H. van den Berg, P.B. and delong-van den Berg, L.T.W. (2005). Small Proportions: What to report for confidence intervals. *Pharmacoepidemiology and Drug Safety*, 14, 239-247.
29. Vollset, S.E. (1993). Confidence Intervals for Binomial Proportion. *Statistical Methods Medical Research*, 12, 809-824.
30. Wilson, E.B. (1927). Probable Inference, the Law of Succession, and Statistical Inference. *Journal of the American Statistical Association*, 22, 209-212.
31. Wise, M.E. (1954). A quickly convergent expansion for cumulative hypergeometric probabilities, direct and inverse. *Biometrika*, 41, 317-329.