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A FINITE BUFFER REVERSE BALKING FEEDBACK MARKOVIAN QUEUING SYSTEM WITH RENEGING AND RETENTION OF IMPATIENT CUSTOMERS

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ABSTRACT

In this paper a single-server finite capacity feedback queuing system with buffer modified reverse balking and retention of impatient customers is studied. The steady-state probabilities of the model are derived and performance measures are studied iteratively. The numerical illustration of the model is also presented.

KEYWORDS

Queuing system, finite buffer, customer impatience, reneging, reverse balking, retention, feedback.

1. INTRODUCTION AND LITERATURE SURVEY

At times an arriving customer decides not to join the queue due to existing long queue in front of the system. The existing queue in front of the system demotivates the newly arriving customer to join the particular system for service. This behaviour is known as *balking* in queuing system and first studied by Haight (1959). There are other customer behaviour studied with respect to the arrival of customers, such as Jieyon and MacDonald (2009), associated h-transformation to the arriving customers and measured the increasing rate of the customers. However, in particular Jain et al., (2012) presented the concept of *reverse balking* and mentioned that, larger queues in front of the certain systems lead of higher trust. They quoted that, by looking at the longer existing queue an incoming customer may feel that the system is providing service of higher value. Hence, the instead of getting demotivated by looking at the queue, an arriving customer may feel motivated to join the particular system. The *reverse balking* is based on the fact that longer queues lead to higher trust in the system. For example; longer queue in front of ATM gives a surety that the machine is operational, longer queue in front of a restaurant may give an impression of better taste. The concept of *reverse balking* is gaining high visibility due to its practical application. Kumar and Som (2014) studied the concept of reverse balking along with impatient customers. Further, Kumar et al. (2015) studied an M/M/1/N feedback queuing system with reverse balking. The quoted that certain customers may not be satisfied with the service and return to the system as feedback customers. They studied the concept of feedback with reverse balking in steady-state. Som and Kumar (2017) studied a Markovian queuing system in heterogeneous service environment with reverse balking. Further, Som et al. (2020) studied heterogeneous multi-server infinite capacity feedback queuing system with reneging and retention. They derived an iterative solution of the model in steady-state. The performance measures of the model are studied. The numerical analysis of the model is also presented. They derived the iterative solution to the model and performed numerical analysis of the model. Som and Gupta (2017) made a comparative analysis of simulated and theoretical results of a single-server queuing system with reverse balking using c – programming. They found that there is no significant difference between the outputs of performance measures in theoretical and simulated environment. Bouchentouf and Messabihi (2017) also presented analysis of a heterogeneous server problem with reverse balking and reneging of impatient customers. Kumar and Som (2020) studied a multi-server queuing system with reverse balking with finite capacity. They have derived the solution of the model iteratively and obtained measures of performance. Saikia and Chaudhary (2019) added extension to the reverse balking probability function and presented the notion of buffer reverse balking. They studied the single-server queuing model with buffer reverse balking and reneging of impatient customers.

The concept of reneging customers is studied by Ancker and Gafarian (1963 and 1963a). They stated that a customer waiting in queue may get impatient after a threshold value of time and may decide to abandon the queue without completion of service. Kumar et al. (2012) quoted that a reneging customer may be retained with some probability. Retention of reneging customers is studied extensively. Kumar and Sharma (2013) extended their work to multi-server. They study a multi-server queuing system with retention of reneged customers. The authors derived an iterative solution of the model in steady-state. Kumar and Som (2015) studied a feedback queuing system with reverse balking, reverse reneging and retention of reneged customers. Som and Seth (2018) also studied the retention of reneged customers in the case of encouraged arrivals. Encouraged arrivals put the system under stress due to discounts and offers. They pondered upon that the encouraged arrivals may also cause a higher rate of reneging and therefore the retention of the customers become highly important. Kumar and Sharma (2018) studied the retention of impatient customers in transient-state for a single server queue. Further, Soodan and Kumar (2020) extended the study of impatient customers by studying a single-server queuing system with correlated reneging.

By reviewing the above literature it can be found that the reverse balking has a scope to be studied extensively. Hence in this paper we study a single-server feedback queuing system with buffer reverse balking and retention of impatient customers. The model is solved in steady-state iteratively. The necessary measures of performance are derived and the numerical analysis of the model is performed.

The results of the paper can be of immense use for the organizations undergoing above mentioned customer behaviours. All the customers' behaviours pose a threat to the business as they result in the loss of the customers and goodwill of the organization. However, the retention adds value to the organizations at a certain cost. An organization may use the model developed in the paper to estimate the probabilities of interest and plan the business strategy well in advance in efficient manner.

Rest of the paper is arranged in 4 sections. Section 2 discusses the mathematical model formulation. Steady-state solution is drawn in section 3 of the model. 4th Section deals with the derivation of measures of performance, while numerical analysis is presented in section 5. The economic analysis of the model is performed in section 6. The conclusion of the paper is presented in section 7.

2. MATHEMATICAL MODEL FORMULATION

A single-server queuing model is formulated under following assumptions:

i) The arrivals occur one by one in accordance to Poisson process with parameter λ.

ii) Service times are exponentially distributed with parameter μ.

iii) Customers are serviced in the order of their arrival i.e. first come first served.

iv) Service is provided through a single channel.

v) The capacity of the system is finite say, N.

vi) Reneging times are exponentially distributed with parameter ξ.

vii) A customer a may reverse balk (refuses to join the system) with probability $(1-\frac{1}{\alpha-n\beta})$ and may not reverse balk (joins the system) with probability $\frac{1}{\alpha-n\beta}$ where α and β are constants to be chosen depending on the past or observed data such that $\alpha - n\beta > 0$ for n = 0, 1, 2… N-1.
(It is worthwhile to note here that as the number of customers in the system i.e. n will increase, the probability of customer joining the system $\frac{1}{\alpha-n\beta}$ will increase, which is the desired outcome in case of reverse balking phenomenon. It is for this reason such function is considered.)

viii) A satisfied customer leaves the system after service with probability p and joins the queue again as a feedback customer with probability $q = 1-p$.

ix) The probability of retention of a reneged customer is q' and the probability that customer is not retained is $p' = 1 - q'$.

In steady state with $P_n = \Pr\{n\ customers\ in\ the\ system\}$, equations governing the model are given by:

$$0 = -\frac{\lambda}{\alpha}P_0 + \mu p P_1 \tag{1}$$

$$0 = \frac{\lambda}{\alpha-(n-1)\beta}P_{n-1} - \left\{\frac{\lambda}{\alpha-n\beta} + \mu p + (n-1)\xi p'\right\}P_n + (\mu p + n\xi p')P_{n+1} \tag{2}$$

$$0 = \frac{\lambda}{\alpha-(N-1)\beta}P_{N-1} - \{\mu p + (N-1)\xi p'\}P_N \tag{3}$$

3. STEADY-STATE SOLUTION

On solving (1) - (3) iteratively, from (1) we get,

$$P_1 = \frac{\lambda}{\alpha\mu p}P_0 \tag{3.1}$$

For $n = 1$, from (2) we get,

$$\frac{\lambda}{\alpha}P_0 - \left\{\frac{\lambda}{\alpha-\beta} + \mu p\right\}P_1 + (\mu p + \xi p')P_2 = 0 \tag{3.2}$$

Simplifying (3.2) for P_2 and using value of P_1 from (3.1) we get,

$$\begin{aligned}(\mu p + \xi p')P_2 &= -\frac{\lambda}{\alpha}P_0 + \left\{\frac{\lambda}{\alpha-\beta} + \mu p\right\}\frac{\lambda}{\alpha\mu p}P_0 \\ &= -\frac{\lambda}{\alpha}P_0 + \left(\frac{\lambda}{\alpha-\beta}\right)\left(\frac{\lambda}{\alpha\mu p}\right)P_0 + \frac{\lambda}{\alpha}P_0\end{aligned} \tag{3.3}$$

Therefore,

$$P_2 = \left(\frac{\lambda}{\alpha-\beta}\right)\left(\frac{\lambda}{\alpha\mu p}\right)\frac{1}{(\mu p + \xi p')}P_0 = \prod_{i=1}^{2}\frac{\lambda}{[\alpha-(i-1)\beta][\mu p + (i-1)\xi p']}P_0 \tag{3.4}$$

Similarly, for $n = 2$, from (2) and using the value of P_1 and P_0 obtained above we get,

$$P_3 = \prod_{i=1}^{3}\frac{\lambda}{[\alpha-(i-1)\beta][\mu p + (i-1)\xi p']}P_0 \tag{3.5}$$

Generalizing we get,

$$\begin{aligned}P_n &= \Pr\{n\ customers\ in\ the\ system\} \\ &= \prod_{i=1}^{n}\frac{\lambda}{[\alpha-(i-1)\beta][\mu p + (i-1)\xi p']}P_0,\end{aligned} \tag{4}$$

Using condition of normality, $\sum_{n=0}^{N} P_n = 1$

$$\begin{aligned}P_0 &= Pr\{system\ is\ empty\} \\ &= \left\{1 + \sum_{n=1}^{N}\prod_{i=1}^{n}\frac{\lambda}{[\alpha-(i-1)\beta][\mu p + (i-1)\xi p']}\right\}^{-1}\end{aligned} \tag{5}$$

4. MEASURES OF PERFORMANCE

1. Expected System Size (L_s)

$$L_s = \sum_{n=1}^{N} nP_n = \sum_{n=1}^{N} n\left\{\prod_{i=1}^{n}\frac{\lambda}{[\alpha-(i-1)\beta][\mu p + (i-1)\xi p']}P_0\right\} \tag{6}$$

2. Expected queue length (L_q)

$$L_q = \sum_{n=1}^{N}(n-1)P_n = \sum_{n=1}^{N}(n-1)\left\{\prod_{i=1}^{n}\frac{\lambda}{[\alpha-(i-1)\beta][\mu p + (i-1)\xi p']}P_0\right\} \tag{7}$$

3. Average rate of reneging (R_r)

$$R_r = \sum_{n=1}^{N} (n-1)\xi p' P_n = \sum_{n=1}^{N} (n-1)\xi p' \left\{ \prod_{i=1}^{n} \frac{\lambda}{[\alpha-(i-1)\beta][\mu p+(i-1)\xi p']} P_0 \right\} \quad (8)$$

4. Average rate of retention (R_R)

$$R_R = \sum_{n=1}^{N} (n-1)\xi q' P_n = \sum_{n=1}^{N} (n-1)\xi q' \left\{ \prod_{i=1}^{n} \frac{\lambda}{[\alpha-(i-1)\beta][\mu p+(i-1)\xi p']} P_0 \right\} \quad (9)$$

5. Average rate of reverse balking (R_b)

$$R_b = \sum_{n=1}^{N} (1-\frac{\lambda}{\alpha-n\beta})\, P_n = \sum_{n=1}^{N} (1-\frac{\lambda}{\alpha-n\beta}) \left\{ \prod_{i=1}^{n} \frac{\lambda}{[\alpha-(i-1)\beta][\mu p+(i-1)\xi p']} P_0 \right\} \quad (10)$$

5. NUMERICAL ILLUSTRATION

In this section we present numerical illustration of the above model.

Table 1

Variation in $L_s, L_q,$ *and* R_r, R_R, R_b with respect to λ

We take, $N = 10, a = 20, b = 2, \mu = 2, \xi = 0.2, p' = 0.3, p = 0.4$

Average Rate of Arrival (λ)	Expected System Size (Ls)	Expected Queue Length (L_q)	Average Rate of Reneging (R_r)	Average Rate of Retention (R_R)	Average Rate of Reverse Balking (R_b)
6.0	0.75101	0.36018	0.02161	0.05042	0.23223
6.2	0.81969	0.41407	0.02484	0.05797	0.23092
6.4	0.89743	0.47676	0.02861	0.06675	0.22801
6.6	0.98552	0.54956	0.03297	0.07694	0.22324
6.8	1.08528	0.63379	0.03803	0.08873	0.21634
7.0	1.19798	0.73075	0.04384	0.10230	0.20703
7.2	1.32476	0.84164	0.05050	0.11783	0.19504
7.4	1.46655	0.96747	0.05805	0.13545	0.18011
7.6	1.62391	1.10892	0.06654	0.15525	0.16202
7.8	1.79698	1.26622	0.07597	0.17727	0.14063
8.0	1.98527	1.43908	0.08634	0.20147	0.11588
8.2	2.18771	1.62656	0.09759	0.22772	0.08782
8.4	2.40250	1.82706	0.10962	0.25579	0.05664
8.6	2.62722	2.03833	0.12230	0.28537	0.02264

We can observe that with increase in arrival rate the expected system size increases and so as expected length of queue and average rate of retention, while increase in rate of reneging means that with increase in arrival rate, more customers abandon the facility without completion of service. Also average rate of reverse balking decreases with increase in arrival rate, which means with more and more customers joining the system, number of customers refusing to join on arrival is decreasing.

Similarly, the numerical results are obtained by varying service rate.

Table 2
Variation in $L_s, L_q, and\ R_r$ with respect to μ
We take, $N = 10, a = 20, b = 2, \lambda = 7, \xi = 0.2, p' = 0.3, p = 0.4$

Average Rate of Service (μ)	Expected System Size (Ls)	Expected Queue Length (L_q)	Average Rate of Reneging (R_r)
2.0	1.9853	1.4391	0.0863
2.1	1.6941	1.1774	0.0706
2.2	1.4592	0.9691	0.0581
2.3	1.2698	0.8037	0.0482
2.4	1.1168	0.6724	0.0403
2.5	0.9925	0.5678	0.0341
2.6	0.8908	0.4840	0.0290
2.7	0.8068	0.4165	0.0250
2.8	0.7370	0.3618	0.0217
2.9	0.6782	0.3169	0.0190
3.0	0.6283	0.2799	0.0168
3.1	0.5855	0.2491	0.0149
3.2	0.5485	0.2232	0.0134
3.3	0.5162	0.2013	0.0121
3.4	0.4877	0.1826	0.0110

We can observe that with increase in service rate the expected system size decreases and so as expected length of queue, while decrease in rate of reneging means that more number of customers choose to wait rather than leaving the facility.

6. COST-MODEL

In this section we present the cost model by developing the functions of total expected cost (TEC), total expected revenue (TER) and total expected profit (TEP).

$$\boldsymbol{TEC} = C_s\mu + C_h L_s + C_b R_b + C_f q L_s + C_r R_r + C_R R_R + C_L P_N$$

$$= C_s\mu + [C_h + C_f q] \sum_{n=1}^{N} n \left\{ \prod_{i=1}^{n} \frac{\lambda}{[\alpha - (i-1)\beta][\mu p + (i-1)\xi p']} P_0 \right\}$$

$$+ C_b \sum_{n=1}^{N} (1$$

$$- \frac{\lambda}{\alpha - n\beta}) \left\{ \prod_{i=1}^{n} \frac{\lambda}{[\alpha - (i-1)\beta][\mu p + (i-1)\xi p']} P_0 \right\}$$

$$+ C_r \sum_{n=1}^{N} (n-1)\xi p' \left\{ \prod_{i=1}^{n} \frac{\lambda}{[\alpha - (i-1)\beta][\mu p + (i-1)\xi p']} P_0 \right\}$$

$$+ C_R \sum_{n=1}^{N} (n-1)\xi q' \left\{ \prod_{i=1}^{n} \frac{\lambda}{[\alpha - (i-1)\beta][\mu p + (i-1)\xi p']} P_0 \right\}$$

$$+ C_L \prod_{i=1}^{N} \frac{\lambda}{[\alpha - (i-1)\beta][\mu p + (i-1)\xi p']} P_0$$

$$\boldsymbol{TER} = R \times \mu(1 - P_0)$$

$$\boldsymbol{TEP} = TER - TEC$$

where, C_s= cost per service per unit time,

C_h= holding cost per unit per unit time,

C_b= cost associated to each reverse balked unit per unit time,

C_f = cost associated to each feedback customer per unit time,

R = revenue earned per unit per unit time,

C_r = cost associated with each reneged unit per unit time,

C_R = cost associated with each retained unit per unit time.

C_L = cost per lost unit per unit time

Any firm encountering the contemporary issues of reverse balking, feedback customers, reneging and retention of impatient customers operating under single server finite buffer environment can translate the cost model formulas in MS Excel taking firm specific values of various parameters involved to understand the economic aspect of their operations and can design effective strategies for smooth and efficient functioning of the system taking into consideration the performance measures as well as the cost factors involved.

A numerical illustration by considering some specific parameter values.

Table 3
Variation in TEC, TER, TEP with respect to λ
We take, $N = 10, a = 20, b = 2, \mu = 3, \xi = 0.2, p' = 0.3, p = 0.4,$
$C_s = 15, C_h = 3, C_b = 4, C_f = 3, C_r = 3, C_R = 3, C_L = 15, R = 200$

λ	TEC	TER	TEP
6.0	84.8641	158.1518	73.2877
6.2	85.2609	164.7028	79.4419
6.4	85.7115	171.5391	85.8276
6.6	86.2243	178.6980	92.4737
6.8	86.8080	186.2150	99.4070
7.0	87.4712	194.1215	106.6503
7.2	88.2218	202.4410	114.2192
7.4	89.0666	211.1859	122.1193
7.6	90.0104	220.3537	130.3433
7.8	91.0552	229.9230	138.8678
8.0	92.1996	239.8513	147.6517
8.2	93.4380	250.0733	156.6353
8.4	94.7607	260.5019	165.7412
8.6	96.1538	271.0308	174.8770
8.8	97.5995	281.5399	183.9405
9.0	99.0774	291.9022	192.8248

From the above table, it can be observed that as the arrival rate increases, values of TEC, TER and TEP all increases.

We can also observe the variation in TEC, TER and TEP w.r.t. service rate as below:

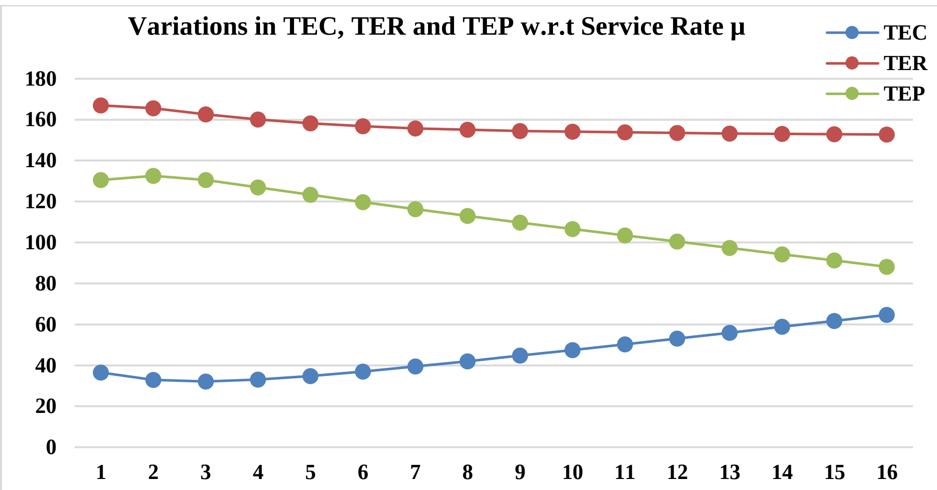

Figure 1: We take, $N = 10, a = 20, b = 2, \lambda = 6, \xi = 0.2, p' = 0.3, p = 0.4,$
$C_s = 15, C_h = 3, C_b = 4, C_f = 3, C_r = 3, C_R = 3, C_L = 15, R = 200$

From the graph above, it can be observed that TEC decreases for initial few values in which arrival rate dominates service rate and then starts increasing with further increase in service rate. TER decreases with increase in service rate whereas TEP increases for initial arrival rate dominating values and then starts decreasing as service rate increases.

7. CONCLUSIONS AND FUTURE SCOPE

In this paper a queuing system with modified reverse balking, feedback, and retention of impatient customers is developed. The model is solved in steady-state. The steady-state probability and necessary measures of performance is derived. The numerical illustration of the model is also presented.

The results of the paper can be utilised by the various manufacturing and service organizations in order to design effective business strategies. The theoretical scope of the model is very wide as the model can be studied under different set of conditions such as multi-sever, heterogeneous service and transient state. The simulation of the model can also be performance and the theoretical results of the model under different set of conditions can be compared. The scope of application of the model is universal with suitable customization.

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