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SOME PARAMETERS ESTIMATION METHODS FOR THE FLEXIBLE INVERTED POWER TOPP-LEONE DISTRIBUTION

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ABSTRACT

It is important to find new simple explicit forms for distributions instead of its old implicit forms which cause some problems in mathematical properties and generating random numbers. In this paper, a simple transformation is applied on the inverted power Topp-Leone (*IPTL*) distribution which gives more flexibility for the *IPTL* distribution properties and estimation methods. Some different methods of estimation are used and a simulation study is performed to compare between estimation methods.

KEYWORDS

The inverted power Topp-Leone distribution, maximum likelihood estimation method, maximum product space estimation method, least square estimation method, Bayesian estimation method, censored type II estimation method.

1. INTRODUCTION

Ahmed (2021) presented for the first time the inverted power Topp-Leone (*IPTL*) distribution with the following cumulative distribution function (*CDF*) and probability density function (*PDF)*

$$\underset{IPTL}{F}(x;\alpha,\beta) = 1-\left(x+1\right)^{-\alpha\beta}\left[2-\left(x+1\right)^{-\beta}\right]^{\alpha}; 0<x<\infty; \alpha,\beta>0; \alpha\beta\neq 1, \tag{1}$$

and

$$\underset{IPTL}{f}(x;\alpha,\beta) = 2\alpha\beta\left(x+1\right)^{-\alpha\beta-1}\left[1-\left(x+1\right)^{-\beta}\right]\left[2-\left(x+1\right)^{-\beta}\right]^{\alpha-1}, \tag{2}$$

when $\beta = 1$, the *IPTL* distribution reduces to the inverted Topp-Leone (*ITL)* distribution, some shapes of the density function for the *IPTL* distribution are illustrated in Figure 1.

The quantile function (*QF*) of the *IPTL* distribution has an implicit form which gives some problems in mathematical properties and generating random number, a simple transformation will be used in order to get rid of this problem to give an explicit form for the *QF*.

The main goal of this manuscript is to derive a simple form for the *QF*, *CDF* and *PDF* of the *IPTL* distribution to give more flexibility for the *IPTL* distribution and use it to derive some important estimation methods.

The rest of this paper is organized as follows: In section 2, a flexible form for the *IPTL* distribution is presented. In section 3, some non-Bayesian estimation methods are obtained. In section 4, some Bayesian estimation methods are applied. In section 5, censored type II estimation methods are performed. Finally, in Section 6, a simulation study is investigated between different methods of estimation for the *IPTL* distribution.

2. A FLEXIBLE FORM FOR THE *IPTL* DISTRIBUTION

A flexible form for the *IPTL* distribution can be given by using the complete square transformation as follows:

Since,

$$F(x;\alpha,\beta)=1-\left[2\left(x+1\right)^{-\beta}-\left(x+1\right)^{-2\beta}\right]^{\alpha},$$

then, adding +1 and -1 inside brackets gives

$$F(x;\alpha,\beta)=1-\left[1-1+2\left(x+1\right)^{-\beta}-\left(x+1\right)^{-2\beta}\right]^{\alpha},$$

hence,

$$F(x;\alpha,\beta)=1-\left\{1-\left[1-\left(x+1\right)^{-\beta}\right]^{2}\right\}^{\alpha};0<x<\infty;\alpha,\beta>0, \tag{3}$$

differentiating last equation with respect to x yields

$$f(x;\alpha,\beta)=2\alpha\beta\left(x+1\right)^{-\beta-1}\left[1-\left(x+1\right)^{-\beta}\right]\left\{1-\left[1-\left(x+1\right)^{-\beta}\right]^{2}\right\}^{\alpha-1}, \tag{4}$$

the *QF* of the *IPTL* distribution is given by

$$x_q=\frac{1}{\left\{1-\left[1-\left(1-q\right)^{\frac{1}{\alpha}}\right]^{\frac{1}{2}}\right\}^{\frac{1}{\beta}}}-1. \tag{5}$$

One can see, easily, the explicit forms of the *CDF*, *PDF* and *QF* of the *IPTL* distribution in (3), (4) and (5) which will help in using the following methods of parameters estimation.

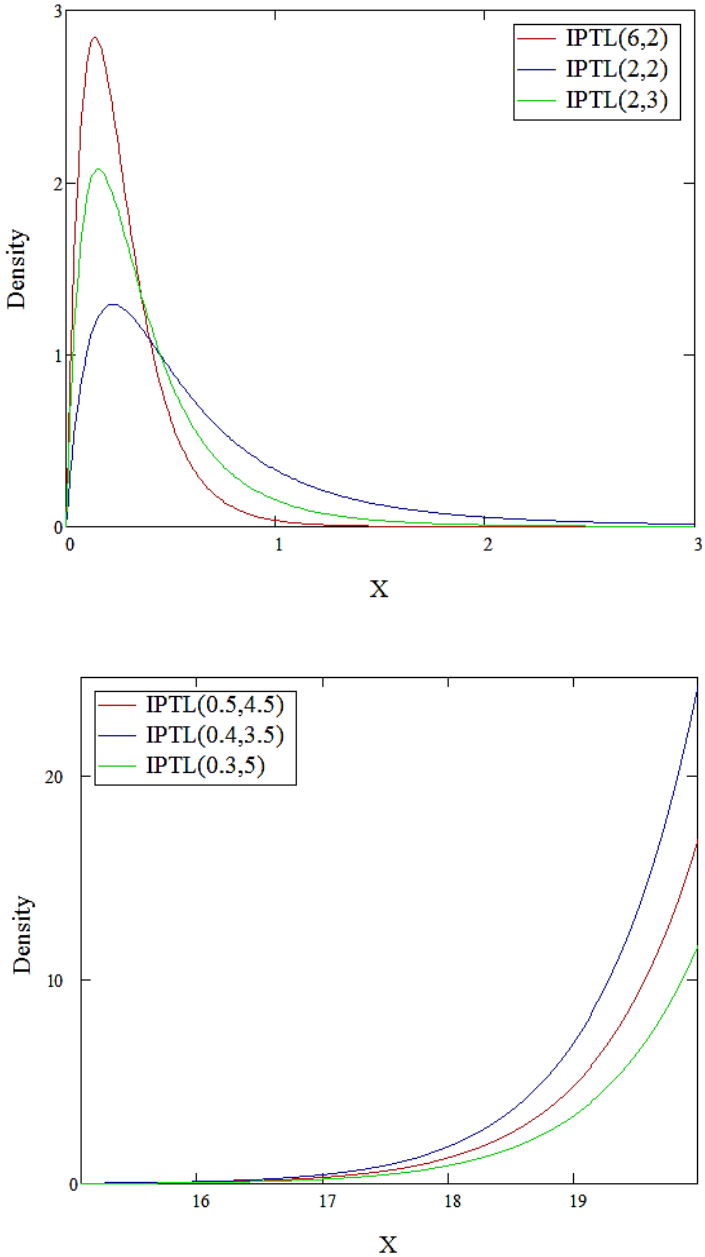

Figure 1: The *IPTL* density functions

3. SOME NON-BAYESIAN ESTIMATION METHODS

In this section, some classical estimation methods will be used.

3.1 Maximum Likelihood

Let $X_1, X_2, \ldots, X_n$ be *iid* random variables from the $IPTL(\alpha,\beta;x)$ distribution then the likelihood function for parameters α,β, Garthwaite *et al.*(2002), is given by

$$L(\alpha,\beta;x)=(2\alpha\beta)^n\prod_{i=1}^{n}(x_i+1)^{-\beta-1}\prod_{i=1}^{n}\left[1-(x_i+1)^{-\beta}\right]\prod_{i=1}^{n}\left\{1-\left[1-(x_i+1)^{-\beta}\right]^2\right\}^{\alpha-1},$$

the log likelihood function can be written as

$$\ell(\alpha,\beta;x)=n\log(2\alpha\beta)-(\beta+1)\sum_{i=1}^{n}\log(x_i+1)+\sum_{i=1}^{n}\log\left[1-(x_i+1)^{-\beta}\right]$$
$$+(\alpha-1)\sum_{i=1}^{n}\log\left\{1-\left[1-(x_i+1)^{-\beta}\right]^2\right\}.$$

The score functions for the parameters α and β are given by

$$\frac{\partial\ell}{\partial\alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n}\log\left\{1-\left[1-(x_i+1)^{-\beta}\right]^2\right\}, \tag{6}$$

and

$$\frac{\partial\ell}{\partial\beta}=\frac{n}{\beta}-\sum_{i=1}^{n}\log(x_i+1)+\sum_{i=1}^{n}\frac{(x_i+1)^{-\beta}\log(x_i+1)}{1-(x_i+1)^{-\beta}}$$
$$-2(\alpha-1)\sum_{i=1}^{n}\frac{\left[1-(x_i+1)^{-\beta}\right](x_i+1)^{-\beta}\log(x_i+1)}{1-\left[1-(x_i+1)^{-\beta}\right]^2}. \tag{7}$$

The unknown parameters of the maximum likelihood estimators *(MLEs)* are obtained by solving the nonlinear equations (6) and (7), numerically, using a suitable iterative technique such as the Newton–Raphson algorithm.

3.2 Least Square

Let $X_1, X_2, \ldots, X_n$ be *iid* random variables from the *IPTL(α,β;x)* distribution then the summation of square for the error term, Singh *et al.*(2014) and Dey *et al.*(2017), is given by

$$\sum_{i=1}^{n}e_i^2=\sum_{i=1}^{n}\left\{F(x_i)-F_{Em}\right\}^2, \tag{8}$$

where, F_{EM} is the empirical *CDF* of the *IPTL(α,β;x)* distribution based on the mean rank, since,

$$F_{EM} = \frac{i}{n+1}, \tag{9}$$

then, substituting (3) and (9) into (8) gives

$$\sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} \left\{ 1 - \left\{ 1 - \left[1 - (x+1)^{-\beta} \right]^2 \right\}^{\alpha} - \frac{i}{n+1} \right\}^2.$$

The score functions for the parameters α and β are given by

$$\frac{\partial\left(\sum_{i=1}^{n} e_i^2\right)}{\partial\alpha} = -2\sum_{i=1}^{n} \left\{ 1 - \left\{ 1 - \left[1 - (x_i+1)^{-\beta} \right]^2 \right\}^{\alpha} - \frac{i}{n+1} \right\} \left\{ 1 - \left[1 - (x_i+1)^{-\beta} \right]^2 \right\}^{\alpha} \times \log\left\{ 1 - \left[1 - (x_i+1)^{-\beta} \right]^2 \right\}, \tag{10}$$

and

$$\frac{\partial\left(\sum_{i=1}^{n} e_i^2\right)}{\partial\beta} = 4\alpha\sum_{i=1}^{n} \left\{ 1 - \left\{ 1 - \left[1 - (x_i+1)^{-\beta} \right]^2 \right\}^{\alpha} - \frac{i}{n+1} \right\} \left\{ 1 - \left[1 - (x_i+1)^{-\beta} \right]^2 \right\}^{\alpha-1} \times \left[1 - (x_i+1)^{-\beta} \right] (x_i+1)^{-\beta} \log(x_i+1). \tag{11}$$

The unknown parameters of the least square estimators *(LSEs)* are obtained by solving the nonlinear equations (10) and (11), numerically, using a suitable iterative technique.

3.3 Maximum Product Space

Let $X_1, X_2, \ldots, X_n$ be *iid* random variables from the $IPTL(\alpha, \beta; x)$ distribution then the Geometric mean (*GM*) for parameters α, β, Singh *et al*. (2014) and Bhatti *et al.* (2021), is given by

$$GM = \sqrt[n+1]{\prod_{i=1}^{n+1} \left[F(x_i) - F(x_{i-1}) \right]};\ i = 1, 2, \ldots, n+1, \tag{12}$$

where, $F(\alpha, \beta; x_0) = 0$ and $F(\alpha, \beta; x_{n+1}) = 1$, then, taking logarithm of (12) yields

$$\log(GM) = \frac{1}{n+1} \sum_{i=1}^{n+1} \log\left\{ \left[F(x_i) - F(x_{i-1}) \right] \right\}, \tag{13}$$

one find that, the last equation can be rewritten as follows

$$\log(GM)=\frac{1}{n+1}\left\{\sum_{i=2}^{n}\log\left\{\left[F(x_i)-F(x_{i-1})\right]\right\}+\log\left[F(x_1)\right]+\log\left\{\left[1-F(x_n)\right]\right\}\right\},$$

substituting (3) into (13) leads to

$$\log(GM)=\frac{1}{n+1}\sum_{i=1}^{n+1}\log\left\{\left\{1-\left[1-\left(x_{i-1}+1\right)^{-\beta}\right]^2\right\}^{\alpha}-\left\{1-\left[1-\left(x_i+1\right)^{-\beta}\right]^2\right\}^{\alpha}\right\}.$$

The score functions for the parameters α and β are given by

$$\frac{\partial\log(GM)}{\partial\alpha}=\frac{1}{n+1}\sum_{i=1}^{n+1}\left\{\frac{\left\{1-\left[1-\left(x_{i-1}+1\right)^{-\beta}\right]^2\right\}^{\alpha}\log\left\{1-\left[1-\left(x_{i-1}+1\right)^{-\beta}\right]^2\right\}}{\left\{1-\left[1-\left(x_{i-1}+1\right)^{-\beta}\right]^2\right\}^{\alpha}-\left\{1-\left[1-\left(x_i+1\right)^{-\beta}\right]^2\right\}^{\alpha}}\right.$$
$$\left.-\frac{\left\{1-\left[1-\left(x_i+1\right)^{-\beta}\right]^2\right\}^{\alpha}\log\left\{1-\left[1-\left(x_i+1\right)^{-\beta}\right]^2\right\}}{\left\{1-\left[1-\left(x_{i-1}+1\right)^{-\beta}\right]^2\right\}^{\alpha}-\left\{1-\left[1-\left(x_i+1\right)^{-\beta}\right]^2\right\}^{\alpha}}\right\}, \tag{14}$$

and

$$\frac{\partial\log(GM)}{\partial\beta}=\frac{-2\alpha}{n+1}\sum_{i=1}^{n+1}\left\{\frac{\left\{1-\left[1-\left(x_{i-1}+1\right)^{-\beta}\right]^2\right\}^{\alpha-1}\left[1-\left(x_{i-1}+1\right)^{-\beta}\right]\left(x_{i-1}+1\right)^{-\beta}\log\left(x_{i-1}+1\right)}{\left\{1-\left[1-\left(x_{i-1}+1\right)^{-\beta}\right]^2\right\}^{\alpha}-\left\{1-\left[1-\left(x_i+1\right)^{-\beta}\right]^2\right\}^{\alpha}}\right.$$
$$\left.-\frac{\left\{1-\left[1-\left(x_i+1\right)^{-\beta}\right]^2\right\}^{\alpha-1}\left[1-\left(x_i+1\right)^{-\beta}\right]\left(x_i+1\right)^{-\beta}\log\left(x_i+1\right)}{\left\{1-\left[1-\left(x_{i-1}+1\right)^{-\beta}\right]^2\right\}^{\alpha}-\left\{1-\left[1-\left(x_i+1\right)^{-\beta}\right]^2\right\}^{\alpha}}\right\}. \tag{15}$$

The unknown parameters of the maximum product space estimators *(MPSEs)* are obtained by solving the nonlinear equations (14) and (15), numerically, using a suitable iterative technique.

4. SOME BAYESIAN ESTIMATION METHODS

Let $X_1, X_2, \ldots, X_n$ be *iid* random variables from the $IPTL(\alpha,\beta;x)$ distribution then the likelihood function for parameters α,β, Garthwaite *et al*. (2002), is given by

$$L(\alpha,\beta;x)=(2\alpha\beta)^{n}\prod_{i=1}^{n}(x_i+1)^{-\beta-1}\prod_{i=1}^{n}\left[1-(x_i+1)^{-\beta}\right]\prod_{i=1}^{n}\left\{1-\left[1-(x_i+1)^{-\beta}\right]^{2}\right\}^{\alpha-1},$$

for simplicity, non-informative prior distributions for parameters α and β will be used, respectively, Jeffreys (1998) and Chandra and Rathaur (2017), as follows

$$\pi(\alpha)=\frac{1}{\alpha};0<\alpha<a, \tag{16}$$

and

$$\pi(\beta)=\frac{1}{\beta};0<\beta<b, \tag{17}$$

then, the join posterior distribution is

$$\pi(\alpha,\beta;x)=\frac{L(\alpha,\beta;x)\pi(\alpha)\pi(\beta)}{\int_0^b\int_0^a L(\alpha,\beta;x)\pi(\alpha)\pi(\beta)\,d\alpha\,d\beta};0<\alpha<a;0<\beta<b;x>0,$$

last equation needs a numerical integration technique to be solved using a mathematical package.

The marginal posterior distribution of α and β can be given by, respectively,

$$\pi(\alpha;x)=\int_0^b\pi(\alpha,\beta;x)\,d\beta;0<\beta<b;x>0, \tag{18}$$

and

$$\pi(\beta;x)=\int_0^a\pi(\alpha,\beta;x)\,d\alpha;0<\alpha<a;x>0, \tag{19}$$

estimating α and β can be obtained using the squared error (*SE*) loss function or linear exponential (*LINEX*) loss function.

4.1 The *SE* Loss Function

In this subsection, estimation of the marginal posterior distributions will be performed using the *SE* loss function, or the quadratic loss function, which is a symmetric loss function for (18) and (19), Guure *et al*.(2012) as follows

$$E_{SE}(\alpha;x)=\int_0^a\alpha\,\pi(\alpha;x)\,d\alpha;0<\alpha<a;x>0, \tag{20}$$

and

$$E_{SE}(\beta;x)=\int_0^b\beta\,\pi(\beta;x)\,d\beta;0<\beta<b;x>0. \tag{21}$$

The unknown parameters of the Bayesian estimators are obtained by solving integrations in (20) and (21), numerically, using a suitable iterative technique.

4.2 The *LINEX* Loss Function

In this subsection, estimation of the marginal posterior distributions will be performed using the *LINEX* loss function which is an asymmetric loss function for (18) and (19), Guure *et al*.(2012), as follows

$$E_{LINEX}(\alpha;x) = -\frac{1}{h}\ln\left[\int_0^a e^{-h\alpha}\,\pi(\alpha;x)\,d\alpha\right];0<\alpha<a;x>0, \tag{22}$$

and

$$E_{LINEX}(\beta;x) = -\frac{1}{h}\ln\left[\int_0^b e^{-h\beta}\,\pi(\beta;x)\,d\beta\right];0<\beta<b;x>0. \tag{23}$$

On the other hand, h is the shape parameter for the *LINEX* function where the sign of h reflects the direction of asymmetry and its magnitude reflects the degree of asymmetry, when h closes to zero the *LINEX* loss is approximately *SE* loss.

The unknown parameters of the Bayesian estimators are obtained by solving integrations in (22) and (23), numerically, using a suitable iterative technique.

5. CENSORED TYPE II ESTIMATION METHODS

In this section, estimating parameters of the *IPTL* distribution will be performed using classical (non-Bayesian) censored type II and Bayesian censored type II

5.1 Non-Bayesian

Let $X_{(1)}, X_{(2)}, \ldots, X_{(r)}$ be *iid* random variables represents r failures for n components from the $IPTL(\alpha,\beta;x)$ distribution after a predetermined and fixed number of failures r then the censored type II likelihood function for parameters α,β, Ng *et al.* (2006), is given by

$$L_{CII}(\alpha,\beta;x) = \frac{n!}{(n-r)!}\left\{\frac{(2\alpha\beta)^r\prod_{i=1}^{r}(x_i+1)^{-\beta-1}\prod_{i=1}^{r}\left[1-(x_i+1)^{-\beta}\right]}{\prod_{i=1}^{r}\left\{1-\left[1-(x_i+1)^{-\beta}\right]^2\right\}^{\alpha-1}}\right\}$$
$$\times\left\{1-\left[1-(x_r+1)^{-\beta}\right]^2\right\}^{\alpha(n-r)},$$

the log likelihood function can be written as

$$\ell_{CII}(\alpha,\beta;x)=\log\left[\frac{n!}{(n-r)!}\right]+\left\{r\log(2\alpha\beta)\right.$$
$$+(-\beta-1)\sum_{i=1}^{r}\log(x_i+1)+\sum_{i=1}^{r}\log\left[1-(x_i+1)^{-\beta}\right]$$
$$\left.+(\alpha-1)\sum_{i=1}^{r}\log\left\{1-\left[1-(x_i+1)^{-\beta}\right]^2\right\}\right\}+\log\left\{1-\left[1-(x_r+1)^{-\beta}\right]^2\right\}^{\alpha(n-r)}.$$

The score functions for the parameters α and β are given by

$$\frac{\partial\ell_{CII}}{\partial\alpha}=\frac{r}{\alpha}+\sum_{i=1}^{r}\log\left\{1-\left[1-(x_i+1)^{-\beta}\right]^2\right\}+(n-r)\log\left\{1-\left[1-(x_r+1)^{-\beta}\right]^2\right\},\quad(24)$$

and

$$\frac{\partial\ell_{CII}}{\partial\beta}=\sum_{i=1}^{r}\frac{(x_i+1)^{-\beta}\log(x_i+1)}{1-(x_i+1)^{-\beta}}-2(\alpha-1)\sum_{i=1}^{r}\frac{\left[1-(x_i+1)^{-\beta}\right](x_i+1)^{-\beta}\log(x_i+1)}{1-\left[1-(x_i+1)^{-\beta}\right]^2}$$
$$+\frac{r}{\beta}-\sum_{i=1}^{r}\log(x_i+1)-2\alpha(n-r)\frac{\left[1-(x_r+1)^{-\beta}\right](x_r+1)^{-\beta}\log(x_r+1)}{1-\left[1-(x_r+1)^{-\beta}\right]^2}.\quad(25)$$

The unknown parameters of the censored type II maximum likelihood estimators *(CII-MLEs)* are obtained by solving the nonlinear equations (24) and (25), numerically, using a suitable iterative technique.

5.2 Bayesian

Let $X_{(1)},X_{(2)},\ldots,X_{(r)}$ be *iid* random variables represents r failures for n components from the $IPTL(\alpha,\beta;x)$ distribution after a predetermined and fixed number of failures r then the censored type II likelihood function for parameters α,β, Okasha (2014), is given by

$$L_{CII}(\alpha,\beta;x)=\frac{n!}{(n-r)!}\left\{(2\alpha\beta)^r\prod_{i=1}^{r}(x_i+1)^{-\beta-1}\prod_{i=1}^{r}\left[1-(x_i+1)^{-\beta}\right]\prod_{i=1}^{r}\left\{1-\left[1-(x_i+1)^{-\beta}\right]^2\right\}^{\alpha-1}\right\}$$
$$\times\left\{1-\left[1-(x_r+1)^{-\beta}\right]^2\right\}^{\alpha(n-r)},$$

non-informative prior distributions for parameters α and β will be used, respectively, from (16) and (17), then, the join posterior distribution is

$$\pi_{CII}(\alpha,\beta;x)=\frac{L_{CII}\left(\alpha,\beta;x\right)\pi(\alpha)\pi(\beta)}{\int_0^b\int_0^a L_{CII}\left(\alpha,\beta;x\right)\pi(\alpha)\pi(\beta)\,d\alpha\,d\beta};0<\alpha<a;0<\beta<b;x>0,$$

last equation needs a numerical integration technique to be solved using a mathematical package. The marginal posterior distribution of α and β can be given by, respectively,

$$\pi_{CII}(\alpha;x)=\int_0^b \pi_{CII}(\alpha,\beta;x)\,d\beta;0<\beta<b;x>0, \tag{26}$$

and

$$\pi_{CII}(\beta;x)=\int_0^a \pi_{CII}(\alpha,\beta;x)\,d\alpha;0<\alpha<a;x>0, \tag{27}$$

estimation of the marginal posterior distributions will be performed using the *SE* loss function, or the quadratic loss function, which is a symmetric loss function for (26) and (27) as follows

$$E_{CII}(\alpha;x)=\int_0^a \alpha\,\pi_{CII}(\alpha;x)\,d\alpha;0<\alpha<a;x>0, \tag{28}$$

and

$$E_{CII}(\beta;x)=\int_0^b \beta\;\pi_{CII}(\beta;x)\,d\beta;0<\beta<b;x>0. \tag{29}$$

The unknown parameters of the Bayesian estimators are obtained by solving integrations in (28) and (29), numerically, using a suitable iterative technique.

6. A SIMULATION STUDY

In this section, some simulation studies will be performed in order to investigate between estimators' behaviors of estimated methods.

In this study, some estimation methods for parameters of the *IPTL* distribution are given using random numbers with fixed seeds to compare between estimators behavior, obtaining parameters estimates algorithm is mentioned in the following steps:

Step (1): Generating a random sample $X_1, X_2, \ldots, X_n$ of sizes $n=$ *(10, 20, 30, 50, 100, 300)* using the flexible *IPTL* distribution with fixed seeds of random numbers.

Step (2): Using a set values of parameters as: $(\alpha=2,\ \beta=3)$.

Step (3): Solving normal equations of estimators for estimation methods in (6),(7); (10),(11); (14),(15); (20),(21); (22),(23); (24),(25) and (28),(29) by iteration to estimate distribution parameters for every method.

Step (4): Calculating biases, *MLEs* and *RMSE* (the root of mean squared error) of the *IPTL* distribution for every method.

Step (5): Repeating step (1) to step (4), 10000 times.

In this study, random numbers with fixed seeds are generated via Mathcad package v15 where the conjugate gradient iteration method is performed. All results are included in tables and graphs which are indicated in the appendix I and II.

From study results, included in appendices, for all methods; as sample size increases, biases and *RMSEs* decrease, moreover, when sample size increases, the distribution estimators can be more consistent.

In classical (non-Bayesian) estimation methods, one can see that, the best efficient estimation method, according to biases and *RMSEs,* is the *MPS* method.

In Bayesian estimation methods, it is clear that, the best efficient estimation method, according to biases and *RMSEs*, is the Bayesian estimation using *LINEX loss* function. On the other hand, Bayesian estimation methods give better efficiency than classical methods, according to biases and *RMSEs.*

In censored type II estimation methods, obviously, Bayesian estimation method yields better efficiency than classical method, according to biases and *RMSEs.*

7. CONCLUSION

Using the complete square transformation on the *IPTL* distribution gives big flexibility for the distribution, especially, in mathematical properties and generating random numbers which helps to use different parameters estimation methods. The *MPS* method is very efficient estimation method having a good performance with small biases and *RMSEs.* Bayesian estimation methods have a better performance with the smallest biases and *RMSEs* when it is compared with classical estimation methods in complete and censored samples. Author encourages researchers to study more about *MPS* and Bayesian estimation methods.

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APPENDIX I

Estimation Methods Tables

MLE_Method

Sample Size	Parameters	Mean of Estimators	Biases	Total Bias	RMSE	Total RMSE
10	$\alpha=2$	209.825	207.825	207.97	533.227	533.46
	$\beta=3$	10.74	7.74		15.79	
20	$\alpha=2$	92.98	90.98	91.059	326.469	326.606
	$\beta=3$	6.781	3.781		9.473	
30	$\alpha=2$	47.032	45.032	45.087	210.606	210.698
	$\beta=3$	5.231	2.231		6.218	
50	$\alpha=2$	16.149	14.149	14.192	100.882	100.951
	$\beta=3$	4.1	1.1		3.715	
100	$\alpha=2$	4.087	2.087	2.14	14.142	14.274
	$\beta=3$	3.475	0.475		1.936	
300	$\alpha=2$	2.285	0.285	0.32	1.453	1.738
	$\beta=3$	3.144	0.144		0.953	

LS_Method

Sample Size	Parameters	Mean of Estimators	Biases	Total Bias	RMSE	Total RMSE
10	$\alpha=2$	20.842	18.842	25.065	94.362	97.297
	$\beta=3$	19.53	16.53		23.717	
20	$\alpha=2$	19.287	17.287	18.935	89.395	90.393
	$\beta=3$	10.727	7.727		13.395	
30	$\alpha=2$	13.206	11.206	12.23	57.689	58.481
	$\beta=3$	7.9	4.9		9.592	
50	$\alpha=2$	7.879	5.879	6.354	38.832	39.202
	$\beta=3$	5.41	2.41		5.371	
100	$\alpha=2$	4.284	2.284	2.536	14.679	14.94
	$\beta=3$	4.103	1.103		2.78	
300	$\alpha=2$	2.255	0.255	0.427	1.609	2.04
	$\beta=3$	3.343	0.343		1.255	

MPS_Method

Sample Size	Parameters	Mean of Estimators	Biases	Total Bias	RMSE	Total RMSE
10	$\alpha=2$	15.23	13.23	24.272	70.568	75.651
	$\beta=3$	23.349	20.349		27.263	
20	$\alpha=2$	9.417	7.417	12.537	49.662	52.16
	$\beta=3$	13.107	10.107		15.949	
30	$\alpha=2$	5.253	3.253	7.084	28.626	30.594
	$\beta=3$	9.293	6.293		10.794	
50	$\alpha=2$	2.928	0.928	3.265	9.191	10.915
	$\beta=3$	6.131	3.131		5.888	
100	$\alpha=2$	2.2	0.2	1.395	4.301	5.058
	$\beta=3$	4.381	1.381		2.661	
300	$\alpha=2$	1.933	-0.067	0.468	1.097	1.557
	$\beta=3$	3.463	0.463		1.105	

Bayesian_Method - SE Loss Function

Sample Size	Parameters	Mean of Estimators	Biases	Total Bias	RMSE	Total RMSE
10	$\alpha=2$	4.414	2.414	3.992	2.807	4.691
	$\beta=3$	6.18	3.18		3.759	
20	$\alpha=2$	3.541	1.541	2.499	1.697	2.798
	$\beta=3$	4.968	1.968		2.224	
30	$\alpha=2$	3.44	1.44	2.32	1.539	2.511
	$\beta=3$	4.818	1.818		1.984	
50	$\alpha=2$	3.45	1.45	2.312	1.547	2.495
	$\beta=3$	4.801	1.801		1.958	
100	$\alpha=2$	3.482	1.482	2.298	1.598	2.483
	$\beta=3$	4.757	1.757		1.9	
300	$\alpha=2$	3.405	1.405	2.257	1.495	2.436
	$\beta=3$	4.766	1.766		1.923	

Bayesian_Method-*LINEX* Loss Function (h=-1)

Sample Size	Parameters	Mean of Estimators	Biases	Total Bias	RMSE	Total RMSE
10	$\alpha=2$	3.867	1.867	2.625	1.927	2.709
	$\beta=3$	4.845	1.845		1.904	
20	$\alpha=2$	3.705	1.705	2.395	1.724	2.422
	$\beta=3$	4.682	1.682		1.701	
30	$\alpha=2$	3.393	1.393	2.169	1.698	2.385
	$\beta=3$	4.663	1.663		1.674	
50	$\alpha=2$	3.3	1.3	2.104	1.701	2.384
	$\beta=3$	4.655	1.655		1.671	
100	$\alpha=2$	3.287	1.287	2.102	1.709	2.384
	$\beta=3$	4.663	1.663		1.662	
300	$\alpha=2$	3.179	1.179	2.032	1.688	2.37
	$\beta=3$	4.655	1.655		1.664	

Censored II_Method

Sample Size	Parameters	R	Mean of Estimators	Biases	Total Bias	RMSE	Total RMSE
10	*α=2* *β=3*	5	186.674 18.333	184.674 15.333	185.31	379.212 34.504	380.778
20	*α=2* *β=3*	15	119.262 9.398	117.262 6.398	117.437	313.009 16.11	313.424
30	*α=2* *β=3*	25	72.212 6.255	70.212 3.255	70.287	228.746 9.053	228.925
50	*α=2* *β=3*	45	23.565 4.424	21.565 1.424	21.612	101.918 4.719	102.027
100	*α=2* *β=3*	90	7.614 3.611	5.614 0.611	5.647	35.379 2.367	35.458
150	*α=2* *β=3*	140	3.709 3.34	1.709 0.34	1.743	11.346 1.675	11.469

Censored II_Bayesian Method-*SE* Loss Function

Sample Size	Parameters	r	Mean of Estimators	Biases	Total Bias	RMSE	Total RMSE
10	*α=2* *β=3*	5	0.669 1.195	-1.331 -1.805	2.243	1.752 2.138	2.764
20	*α=2* *β=3*	15	0.78 1.581	-1.22 -1.419	1.871	1.682 1.806	2.468
30	*α=2* *β=3*	25	2.826 2.683	0.826 -0.317	0.884	1.642 0.753	1.807
50	*α=2* *β=3*	45	2.657 3.081	0.657 0.081	0.662	1.473 0.426	1.533
100	*α=2* *β=3*	90	2.553 3.102	0.553 0.102	0.562	1.379 0.268	1.405
150	*α=2* *β=3*	140	2.498 2.927	0.498 -0.073	0.503	1.268 0.544	1.379

APPENDIX II

Estimation Methods Graphs

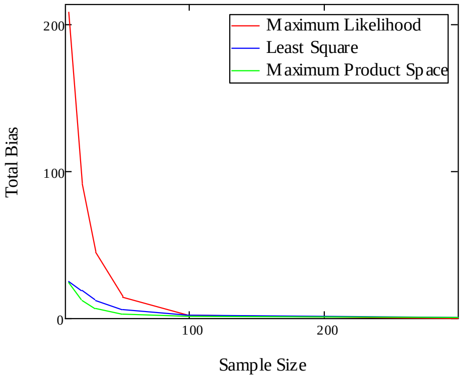

Figure 2: Classical Methods Biases

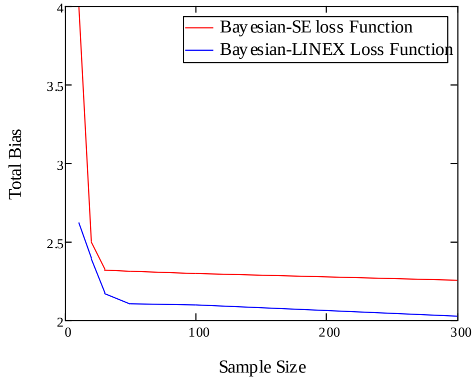

Figure 3: Bayesian Methods Biases

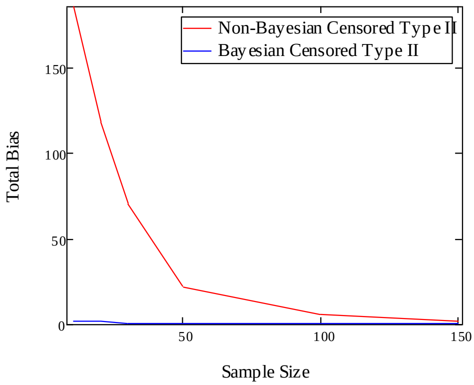

Figure 4: Censored Type II Methods Biases

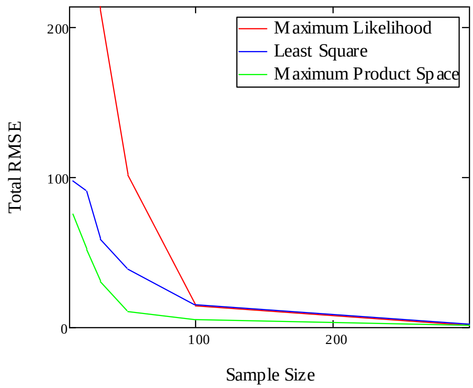

Figure 5: Classical Methods RMSE

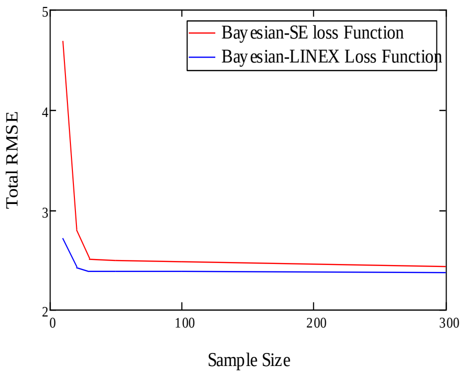

Figure 6: Bayesian Methods RMSE

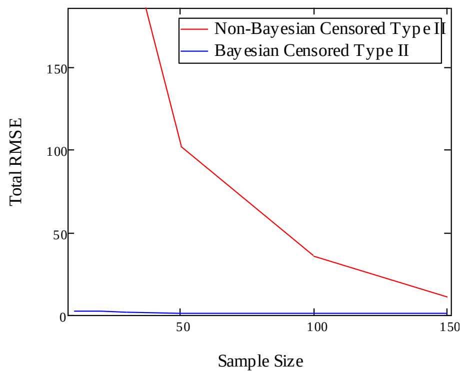

Figure 7: Censored Type II Methods RMSE