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SOME MODIFIED CLASSES OF ESTIMATORS FOR POPULATION VARIANCE USING AUXILIARY ATTRIBUTE

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ABSTRACT

This study introduces some modified classes of estimators of population variance in simple random sampling utilizing information on auxiliary attribute which cover traditional mean, ratio and regression estimators and Kadilar and Cingi (2004) type estimators adapted by Singh and Malik (2014) for particular values of characterizing scalars. The expressions of bias and mean square error of the proposed classes of estimators are tabulated up to the first order of approximation. The proposed classes of estimators are theoretically compared with the contemporary estimators existing till date and the efficiency conditions are obtained. Subsequently, the theoretical results are numerically enhanced by using a real data set.

KEYWORDS

Bias, Mean square error, Efficiency, Auxiliary attribute.

1. INTRODUCTION

In sampling theory, it is long familiar that the utilization of auxiliary information helps to improve the efficiency of the estimators of population parameters of choice such as mean or total and variance of the variable under study. In literature of survey sampling, several authors have introduced a wide range of estimators of population parameters based on information about the population parameters of the auxiliary variable. Some recent relevant studies in this direction like, Zaman and Kadilar (2020), Bhushan and Kumar (2020a, b), Bhushan et al. (2020a, b, c, d, e), Bhushan et al. (2021), Zaman and Kadilar (2021a, b) can be viewed. However, in many practical situations, instead of an auxiliary variable *x* there exist an auxiliary attribute (say, ϕ) which is highly correlated with the study variable y. For example:

i) Amount of milk produce (y) and a particular breed of cow (ϕ),
ii) Amount of production of paddy crop (y) and a particular variety of paddy (ϕ),
iii) Height of persons (y) and sex (ϕ) etc.

In these types of situations, considering the advantage of point bi-serial correlation (ρ) into practice various renowned statisticians like, Naik and Gupta (1996), Jhajj et al. (2006), Singh et al. (2008), Abd-Elfattah et al. (2010), Grover and Kaur (2011), Singh and Solanki (2012), Koyuncu (2012), Haq and Shabbir (2014), Zaman (2018, 2020, 2021), Zaman and Kadilar (2019) and Bhushan and Gupta (2020) suggested various improved estimators for the estimation of population mean consist of information on auxiliary attribute. In most of the cases, one may be interested in the estimation of population variance of study variable y. When the prior information on parameters of auxiliary attribute is available, Singh and Kumar (2011) suggested conventional ratio, regression and exponential estimators of population variance using auxiliary attribute. Adapting the work of Kadilar and Cingi (2004), Singh and Malik (2014) developed a new family of estimators of population variance using information on auxiliary attribute. Adichwal et al. (2015) investigated few improved class of estimators of population variance based on auxiliary attribute whereas Adichwal et al. (2016) adapted Koyuncu (2012) estimator and investigated a generalized class of estimators for population variance using auxiliary attribute. Following Shabbir and Gupta (2006) and Singh and Solanki (2013), Singh and Pal (2018) invoked a new class of estimators for the population variance using known population proportion. This paper considers the problem of estimating population variance S_y^2 using information on auxiliary attribute.

Notations

To estimate the population variance $S_y^2 = (N-1)^{-1}\Sigma_{i=1}^{N}(y_i - \bar{Y})^2$, let a sample of size n be drawn from a finite population $k = (k_1, k_2, \dots, k_N)$ using simple random sampling without replacement (SRSWOR) scheme. Let y_i and ϕ_i be the total number of units on study variable y and auxiliary attribute ϕ for unit i of the population k. It is to be noted that the attribute $\phi_i = 1$ if the unit i possess the attribute ϕ and $\phi_i = 0$, otherwise. Let $A = \Sigma_{i=1}^{N}\phi_i$ and $a = \Sigma_{i=1}^{n}\phi_i$ be the total number of units in the population k and sample respectively possessing attribute ϕ whereas $P = (A/N)$ and $p = (a/n)$ respectively denote the population proportion and sample proportion having attribute ϕ. Let $\bar{y} = n^{-1}\Sigma_{\text{i}=1}^{n}y_i$ and $\bar{Y} = N^{-1}\Sigma_{i=1}^{N}y_i$ respectively be the sample and population means of study variable y; $s_y^2 = (n-1)^{-1}\Sigma_{i=1}^{n}(y_i - \bar{y})^2$ and $S_y^2 = (N-1)^{-1}\Sigma_{i=1}^{N}(y_i - \bar{Y})^2$ be the sample and population variances of the study variable y respectively; $s_\phi^2 = (n-1)^{-1}\Sigma_{i=1}^{n}(\phi_i - p)^2$ and $S_\phi^2 = (N-1)^{-1}\Sigma_{i=1}(\phi_i - P)^2$ be the sample and population variances of auxiliary attribute ϕ respectively.

To find out the bias and mean square error (MSE) of the suggested estimators, let us define $s_y^2 = S_y^2(1+e_0), s_\phi^2 = S_\phi^2(1+e_1)$ provided that $E(e_0) = E(e_1) = 0$ and

$$V_{r,s} = \frac{E\left[\left(s_\phi^2 - S_\phi^2\right)^r\left(s_y^2 - S_y^2\right)^s\right]}{\left(S_\phi^2\right)^r\left(S_y^2\right)^s} \tag{1.1}$$

Using (1.1), we can write

$$E(e_0^2) = \gamma\left(\lambda_{4,0} - 1\right) = V_{0,2} \tag{1.2}$$

$$E(e_1^2) = \gamma\left(\lambda_{0,4} - 1\right) = V_{2,0} \tag{1.3}$$

$$E(e_0, e_1) = \gamma(\lambda_{22} - 1) = V_{1,1} \tag{1.4}$$

where $\gamma = (1-\text{f})/\text{n}, f = n/\text{N}, \lambda_{pq} = \mu_{pq}/\mu_{20}^{p}\mu_{02}^{q}$ and $\mu_{pq} = (N-1)^{-1}\Sigma_{i=1}^{N}(y_i - \bar{Y})^p(\phi_i - P)^q$.

The rest of the article is molded in the following sections. In Section 2, we consider a literature review of the existing estimators with their properties. In Section 3, we develop some modified classes of estimators and studied their properties. The efficiency conditions are derived in Section 4 which are numerically verified using a real data in Section 5. Lastly, the conclusion is given in Section 6.

2. LITERATURE REVIEW

In sequence of getting an estimate of the population variance of study variable y, various authors considered the known population proportion and suggested different class of estimators under simple random sampling (SRS). The variance of traditional mean estimator $s_m^2 = s_y^2$ is given by

$$V(s_m^2) = S_y^4 V_{0,2} \tag{2.1}$$

Singh and Kumar (2011) suggested the following class of estimators as

$$s_r^2 = s_y^2 \frac{S_\phi^2}{s_\phi^2} \tag{2.2}$$

$$s_{lr}^2 = s_y^2 + \beta_\phi \left(S_\phi^2 - s_\phi^2\right) \tag{2.3}$$

$$s_e^2 = s_y^2 \exp\left(\frac{S_\phi^2 - s_\phi^2}{S_\phi^2 + s_\phi^2}\right) \tag{2.4}$$

where β_ϕ is the regression coefficient of y on ϕ.

The MSE of the estimators s_r^2, s_{lr}^2 and s_e^2 are given by

$$MSE(s_r^2) = S_y^4\left(V_{0,2} + V_{2,0} - 2V_{1,1}\right) \tag{2.5}$$

$$MSE(s_{lr}^2) = S_y^4 V_{0,2} + \beta_\phi^2 S_\phi^4 V_{2,0} - 2\beta_\phi^2 S_y^2 S_\phi^2 V_{1,1} \tag{2.6}$$

$$MSE(s_e^2) = S_y^4 (V_{0,2} + \frac{V_{2,0}}{4} - V_{1,1}) \tag{2.7}$$

The MSE of the estimator s_{lr}^2 is minimized for β_ϕ as

$$\beta_{\phi(opt)} = \frac{S_y^2}{S_\phi^2}\frac{V_{1,1}}{V_{2,0}} \tag{2.8}$$

The minimum MSE at optimum value of β_ϕ is given by

$$minMSE(s_{lr}^2) = S_y^4 \left[V_{0,2} - \frac{V_{1,1}^2}{V_{2,0}}\right] \tag{2.9}$$

Motivated by Hansen et al. (1954), Srivastava (1967) and Walsh (1970), one may define a class of difference and ratio type estimators as

$$s_1^2 = s_y^2 + \theta_1\left(s_\phi^{*2} - S_\phi^{*2}\right) \tag{2.10}$$

$$s_2^2 = s_y^2 \left(\frac{S_\phi^{*2}}{s_\phi^{*2}}\right)^{\theta_2} \tag{2.11}$$

$$s_3^2 = s_y^2 \left(\frac{S_\phi^{*2}}{S_\phi^{*2} + \theta_3(s_\phi^{*2} - S_\phi^{*2})}\right) \tag{2.12}$$

where θ_1, θ_2 and θ_3 are suitably chosen scalars. Also, $s_\phi^{*2} = (cs_\phi^2 + d)$ and $S_\phi^{*2} = (cS_\phi^2 + d)$ given that c and d are either real values or function of known parameters of auxiliary attribute ϕ such as standard deviation S_ϕ, coefficient of variation C_ϕ, coefficient of kurtosis $\beta_2(\phi)$ and coefficient of correlation ρ between study variable y and auxiliary attribute ϕ.

The MSE of the estimators $s_i^2, i = 1,2,3$ are given by

$$MSE(s_1^2) = (S_y^4 V_{0,2} + \nu^2\theta_1^2 S_\phi^4 V_{2,0} + 2\nu\theta_1 S_y^2 S_\phi^2 V_{1,1}) \tag{2.13}$$

$$MSE(s_2^2) = S_y^4 (V_{0,2} + \nu^2\theta_2^2 V_{2,0} - 2\nu\theta_2 V_{1,1}) \tag{2.14}$$

$$MSE(s_3^2) = S_y^4 (V_{0,2} + \nu^2\theta_3^2 V_{2,0} - 2\nu\theta_3 V_{1,1}) \tag{2.15}$$

where $\nu = cS_\phi^2/(cS_\phi^2 + d)$

The MSE of the estimators $s_i^2, i = 1, 2, 3$ are minimized for θ_1, θ_2 and θ_3 as

$$\theta_{1(opt)} = -\frac{S_y^2}{\nu S_\phi^2}\frac{V_{1,1}}{V_{2,0}} \tag{2.16}$$

$$\theta_{2(opt)} = \frac{V_{1,1}}{\nu V_{2,0}} = \theta_{3(opt)} \tag{2.17}$$

The minimum MSE at optimum values of θ_1, θ_2 and θ_3 are given by

$$minMSE(s_i^2) = S_y^4 \left[V_{0,2} - \frac{V_{1,1}^2}{V_{2,0}}\right] \tag{2.18}$$

which is the minimum MSE of the conventional regression estimator s_{lr}^2.

On the lines of Kadilar and Cingi (2004), Singh and Malik (2014) adapted the following class of estimators as

$$s_{kc1}^2 = s_y^2 \left(\frac{S_\phi^2 + C_\phi}{s_\phi^2 + C_\phi}\right) \tag{2.19}$$

$$s_{kc2}^2 = s_y^2 \left(\frac{S_\phi^2 + \beta_2(\phi)}{s_\phi^2 + \beta_2(\phi)}\right) \tag{2.20}$$

$$s_{kc3}^2 = s_y^2 \left(\frac{S_\phi^2 \beta_2(\phi) + C_\phi}{s_\phi^2 \beta_2(\phi) + C_\phi}\right) \tag{2.21}$$

$$s_{kc4}^2 = s_y^2 \left(\frac{S_\phi^2 C_\phi + \beta_2(\phi)}{s_\phi^2 C_\phi + \beta_2(\phi)} \right) \quad (2.22)$$

The MSE of the estimators $s_{kc_i}^2, i = 1,2,3,4$ are given by

$$MSE\left(s_{kc_i}^2\right) = s_y^4\left(V_{0,2} + w_i^2 V_{2,0} - 2w_i V_{1,1}\right), i = 1,2,3,4 \quad (2.23)$$

where $w_1 = \frac{S_\phi^2}{\left(S_\phi^2 + C_\phi\right)}, w_2 = \frac{S_\phi^2}{\left(S_\phi^2 + \beta_2(\phi)\right)}, w_3 = \frac{S_\phi^2 \beta_2(\phi)}{\left(S_\phi^2 \beta_2(\phi) + C_\phi\right)}$ and $w_4 = \frac{S_\phi^2 C_\phi}{\left(S_\phi^2 C_\phi + \beta_2(\phi)\right)}$.

Following Singh et al. (2008), Singh and Malik (2014) also adapted the following class of estimators for population variance using information on auxiliary attribute as

$$s_s^2 = \left\{s_y^2 + \beta_\phi\left(S_\phi^2 - s_\phi^2\right)\right\}\left(\frac{aS_\phi^2 + b}{as_\phi^2 + b}\right) \quad (2.24)$$

where a and b are either real numbers or the functions of known parameters of auxiliary attribute ϕ such as, $C_\phi, \rho,\ \beta_2(\phi)$ etc. Few members of the estimators s_s^2 are discussed in Table 1 for ready reference.

The MSE of the estimator s_s^2 is given by

$$\begin{aligned} MSE(s_s^2) = \big[&S_y^4 V_{0,2} + V_{2,0}\left\{\beta_\phi^2 S_y^4 + A_1^2 S_y^4 + 2A_1\beta_\phi S_y^2 S_\phi^2\right\} \\ &- 2S_y^2 V_{1,1}\left\{\beta_\phi S_\phi^2 + A_1 S_y^2\right\}\big] \end{aligned} \quad (2.25)$$

where $A_1 = aS_\phi^2/(aS_\phi^2 + b)$

Singh and Malik (2014) suggested the following class of estimators given as

Table 1
Some Members of Singh et al. (2008) Estimators s_s^2

Members of estimators $s^2_{s(j)}\ j = 1, 2, \ldots, 10$	Values of	
	a	b
$s^2_{s(1)} = \left[s_y^2 + \beta_\phi\left(S_\phi^2 - s_\phi^2\right)\right]\left[\frac{S_\phi^2 + \beta_2(\phi)}{s_\phi^2 + \beta_2(\phi)}\right]$	1	$\beta_2(\phi)$
$s^2_{s(2)} = \left[s_y^2 + \beta_\phi(S_\phi^2 - s_\phi^2\right]\left[\frac{S_\phi^2 + C_\phi}{s_\phi^2 + C_\phi}\right]$	1	C_ϕ
$s^2_{s(3)} = \left[s_y^2 + \beta_\phi\left(S_\phi^2 - s_\phi^2\right)\right]\left[\frac{S_\phi^2\beta_2(\phi) + C_\phi}{s_\phi^2\beta_2(\phi) + C_\phi}\right]$	$\beta_2(\phi)$	C_ϕ
$s^2_{s(4)} = \left[s_y^2 + \beta_\phi\left(S_\phi^2 - s_\phi^2\right)\right]\left[\frac{S_\phi^2 C_\phi + \beta_2(\phi)}{s_\phi^2 C_\phi + \beta_2(\phi)}\right]$	C_ϕ	$\beta_2(\phi)$
$s^2_{s(5)} = \left[s_y^2 + \beta_\phi\left(S_\phi^2 - s_\phi^2\right)\right]\left[\frac{S_\phi^2 + \rho}{s_\phi^2 + \rho}\right]$	1	ρ
$s^2_{s(6)} = \left[s_y^2 + \beta_\phi\left(S_\phi^2 - s_\phi^2\right)\right]\left[\frac{S_\phi^2 C_\phi + \rho}{s_\phi^2 C_\phi + \rho}\right]$	C_ϕ	ρ
$s^2_{s(7)} = \left[s_y^2 + \beta_\phi\left(S_\phi^2 - s_\phi^2\right)\right]\left[\frac{S_\phi^2 \rho + C_\phi}{s_\phi^2 \rho + C_\phi}\right]$	ρ	C_ϕ
$s^2_{s(8)} = \left[s_y^2 + \beta_\phi\left(S_\phi^2 - s_\phi^2\right)\right]\left[\frac{S_\phi^2\beta_2(\phi) + \rho}{s_\phi^2\beta_2(\phi) + \rho}\right]$	$\beta_2(\phi)$	ρ
$s^2_{s(9)} = \left[s_y^2 + \beta_\phi\left(S_\phi^2 - s_\phi^2\right)\right]\left[\frac{S_\phi^2\rho + \beta_2(\phi)}{s_\phi^2\rho + \beta_2(\phi)}\right]$	ρ	$\beta_2(\phi)$
$s^2_{s(10)} = \left[s_y^2 + \beta_\phi\left(S_\phi^2 - s_\phi^2\right)\right]\left[\frac{S_\phi^2 S_\phi + \beta_2(\phi)}{s_\phi^2 S_\phi + \beta_2(\phi)}\right]$	S_ϕ	$\beta_2(\phi)$

$$s_{sm}^2 = s_y^2\left[m_1 + m_2(S_\phi^2 - s_\phi^2)\right]exp\left\{\frac{\delta\left(\left(cS_\phi^2 + d\right) - \left(cs_\phi^2 + d\right)\right)}{\left(cS_\phi^2 + d\right) + \left(cs_\phi^2 + d\right)}\right\} \tag{2.26}$$

where m_1 and m_2 are suitably chosen scalars and δ assumes values +1 and -1 to design different estimators. Few unknown members of the estimators s_{sm}^2 are discussed in Table 2 for ready reference.

The MSE of the estimator s_{sm}^2 up to the first order approximation is given by

$$MSE(s_{sm}^2) = S_y^4\left[1 + m_1^2R_1 + m_2^2R_2 + 2m_1m_2R_3 - 2m_1R_4 - 2m_2R_5\right] \tag{2.27}$$

where

$$R_1 = 1 + V_{0,2} + \delta^2\nu^2V_{2,0} + 2\delta\left(1 + \frac{\delta}{2}\right)\nu^2V_{2,0} - 4\delta\nu V_{1,1}$$
$$R_2 = S_\phi^4V_{2,0}$$
$$R_3 = S_\phi^2\left[2V_{1,1} + 2\delta\nu V_{2,0}\right]$$

$$R_4 = 1 + \delta\left(1 + \frac{\delta}{2}\right)\nu^2 V_{2,0} - \delta\nu V_{1,1}$$
$$R_5 = S_\phi^2\left[\delta\nu V_{2,0} - V_{1,1}\right]$$
$$\nu = cS_\phi^2/2\left(cS_\phi^2 + d\right)$$

The MSE of the estimator s_{sm}^2 is minimized for optimum values of m_1 and m_2 as

$$m_{1(opt)} = \frac{(R_2R_4 - R_3R_5)}{(R_1R_2 - R_3^2)} \tag{2.28}$$

$$m_{2(opt)} = \frac{(R_1R_5 - R_3R_4)}{(R_1R_2 - R_3^2)} \tag{2.29}$$

The minimum MSE at the optimum values of m_1 and m_2 is given by

$$minMSE(s_{sm}^2) = S_y^4\left[1 - \frac{(R_1R_5^2 + R_2R_4^2 - 2R_3R_4R_5)}{(R_1R_2 - R_3^2)}\right] \tag{2.30}$$

On the lines of Bhushan and Gupta (2016, 2020), we define log type estimator for population variance using attribute as

$$s_{bg}^2 = s_y^2\left[1 - \log\left(\frac{s_\phi^{*2}}{S_\phi^{*2}}\right)\right]^{\theta_4} \tag{2.31}$$

The MSE of the estimator s_{bg}^2 is given by

$$MSE\left(s_{bg}^2\right) = S_y^2\left(V_{0,2} + \nu^2\theta_4^2 V_{2,0} + 2\nu\theta_4 V_{1,1}\right) \tag{2.32}$$

On differentiating the above MSE expression w.r.t. to scalar θ_4 and equating to zero, we get

$$\theta_{4(opt)} = -\frac{V_{1,1}}{\nu V_{2,0}} \tag{2.33}$$

The minimum MSE at optimum value of θ_4 is given by

$$minMSE\left(s_{bg}^2\right) = S_y^4\left[V_{0,2} - \frac{V_{1,1}^2}{V_{2,0}}\right] \tag{2.34}$$

which is similar to the minimum MSE of the conventional regression estimator s_{lr}^2.

3. PROPOSED ESTIMATORS

Adapting the procedure of Kadilar and Cingi (2006), we develop some modified classes of estimators by combining respectively the class of difference, Srivastava and Walsh type estimators given in (2.10), (2.11), (2.12) with the log type estimator defined on the lines of Bhushan and Gupta (2016, 2020) given in (2.31).

$$s_{b1}^2 = \zeta_1\left\{s_y^2 + \theta_1\left(s_\phi^{*2} - S_\phi^{*2}\right)\right\} + \psi_1 s_y^2\left[1 + \log\left(\frac{s_\phi^{*2}}{S_\phi^{*2}}\right)\right]^{\theta_4} \tag{3.1}$$

$$s_{b2}^2 = \zeta_2 s_y^2 \left(\frac{s_\phi^{*2}}{S_\phi^{*2}}\right)^{\theta_2} + \psi_2 s_y^2 \left[1 + \log\left(\frac{s_\phi^{*2}}{S_\phi^{*2}}\right)\right]^{\theta_4} \tag{3.2}$$

$$s_{b3}^2 = \zeta_3 s_y^2 \left(\frac{S_\phi^{*2}}{S_\phi^{*2} + \theta_3\left(s_\phi^{*2} - S_\phi^{*2}\right)}\right) + \psi_3 s_y^2 \left[1 + \log\left(\frac{s_\phi^{*2}}{S_\phi^{*2}}\right)\right]^{\theta_4} \tag{3.3}$$

where ζ_i and $\psi_i, i = 1,2,3$ are suitably chosen characterizing scalars, For different values of characterizing scalars, the classes of estimators $s_{bi}^2, i = 1,2,3$ will reduce into the following known estimators.

i. For $(\zeta_1, \psi_1, \theta_1, \theta_4) = (1, 0, 0,0), s_{b1}^2 \to s_m^2$
ii. For $(\zeta_1, \psi_1, \theta_1) = \left(1, 0, -\beta_\phi\right), s_{b1}^2 \to s_{lr}^2$
iii. For $(\zeta_2, \psi_2, \theta_2) = (1, 0, 1), s_{b2}^2 \to s_r^2$
iv. For $(\zeta_3, \psi_3, \theta_3, c, d) = \left(1, 0, 1,1, C_\phi\right), s_{b3}^2 \to s_{kc1}^2$
v. For $(\zeta_3, \psi_3, \theta_3, c, d) = (1, 0, 1, 1, \beta_2(\phi)), s_{b3}^2 \to s_{kc2}^2$
vi. For $(\zeta_3, \psi_3, \theta_3, c, d) = (1, 0, 1, \beta_2(\phi), C_\phi), s_{b3}^2 \to s_{kc3}^2$
vii. For $(\zeta_3, \psi_3, \theta_3, c, d) = \left(1, 0, 1, 1, C_\phi, \beta_2(\phi)\right), s_{b3}^2 \to s_{kc4}^2$

Several other estimators can be designed for various values of scalars. Further, for different values of characterizing scalars, some unknown members of the proposed classes of estimators $s_{b_i}^2, i = 1, 2, 3$ are given in table 2 for ready reference.

Theorem 3.1

The bias and MSE of the suggested classes of estimators s_{bi}^2 , $i = 1, 2, 3$ are given by

$$Bias(s_{bi}^2) = S_y^2[\zeta_i D_i + \psi_i E_i - 1] \tag{3.4}$$

$$MSE(s_{bi}^2) = S_y^4[1 + \zeta_i^2 A_i + \psi_i^2 B_i + 2\zeta_i\psi_i C_i - 2\zeta_i D_i - 2\psi_i E_i] \tag{3.5}$$

$$minMSE(s_{bi}^2) = S_y^4 \left[1 - \frac{(A_i E_i^2 + B_i D_i^2 - 2C_i D_i E_i)}{(A_i B_i - C_i^2)}\right] \tag{3.6}$$

Proof:

Consider the first suggested estimator as

$$s_{b1}^2 = \zeta_1\left\{s_y^2 + \theta_1\left(s_\phi^{*2} - S_\phi^{*2}\right)\right\} + \psi_1 s_y^2 \left[1 + \log\left(\frac{s_\phi^{*2}}{S_\phi^{*2}}\right)\right]^{\theta_4} \tag{3.7}$$

Express the above estimator in terms of $e's$ by using the notations defined in earlier section as

$$\begin{aligned} s_{b1}^2 - S_y^2 = S_y^2 \Bigg[\zeta_1 &\left\{1 + e_0 + \left(\frac{\theta_1}{R}\right) c e_1\right\} \\ &+ \psi_1 \Bigg\{1 + e_0 + \theta_4 \nu e_1 - \theta_4 \nu^2 e_1^2 + \left(\frac{\theta_4^2}{2}\right)\nu^2 e_1^2 \\ &+ \theta_4 \nu e_0 e_1\Bigg\} - 1\Bigg] \end{aligned} \tag{3.8}$$

where $R = S_y^2/S_\phi^2$.

Taking expectation both the sides of (3.8), we get the bias of the estimator s_{b1}^2 up to first order of approximation as

$$Bias(s_{b1}^2) = S_y^2[\zeta_1 D_1 + \psi_1 E_1 - 1] \tag{3.9}$$

Similarly, we can obtain the bias of the other proposed estimators.

Now, squaring and taking expectation both the sides of (3.8), we get the MSE of the estimator s_{b1}^2 up to first order of approximation as

$$MSE(s_{b1}^2) = S_y^4[1 + \zeta_1^2 A_1 + \psi_1^2 B_1 + 2\zeta_1\psi_1 C_1 - 2\zeta_1 D_1 - 2\psi_1 E_1] \tag{3.10}$$

On differentiating the $MSE(s_{b1}^2)$ w.r.t. ζ_1 and ψ_1 and equating to zero, we get

$$\zeta_{1(opt)} = \frac{(B_1 D_1 - C_1 E_1)}{(A_1 B_1 - C_1^2)} \tag{3.11}$$

$$\psi_{1(opt)} = \frac{(A_1 E_1 - C_1 D_1)}{(A_1 B_1 - C_1^2)} \tag{3.12}$$

The minimum MSE at $\zeta_{1(opt)}$ and $\psi_{1(opt)}$ is given by

$$minMSE(s_{b1}^2) = S_y^4\left[1 - \frac{A_1 E_1^2 + B_1 D_1^2 - 2C_1 D_1 E_1}{A_1 B_1 - C_1^2}\right] \tag{3.13}$$

Similarly, we can obtain the MSE of other proposed estimators. In general, we can write

$$MSE\left(s_{b_i}^2\right) = S_y^4[1 + \zeta_i^2 A_i + \psi_i^2 B_i + 2\zeta_i\psi_i C_i - 2\zeta_i D_i - 2\psi_i E_i], i = 1,2,3 \tag{3.14}$$

The MSE of above estimator is minimized for

$$\zeta_{i(opt)} = \frac{(B_i D_i - C_i E_i)}{(A_i B_i - C_i^2)} \tag{3.15}$$

$$\psi_{i(opt)} = \frac{(A_i E_i - C_i D_i)}{(A_i B_i - C_i^2)} \tag{3.16}$$

The minimum MSE at $\zeta_{i(opt)}$ and $\psi_{i(opt)}$ is expressed as

$$minMSE(s_{bi}^2) = S_y^4\left[1 - \frac{(A_i E_i^2 + B_i D_i^2 - 2C_i D_i E_i)}{(A_i B_i - C_i^2)}\right] \tag{3.17}$$

where

$$A_1 = 1 + V_{0,2} + \left(\frac{\theta_1}{R}\right)^2 c^2 V_{2,0} + 2\left(\frac{\theta_1}{R}\right) c V_{1,1}$$

$$B_1 = 1 + V_{0,2} + (2\theta_4^2 \nu^2 - 2\theta_4 \nu^2) V_{2,0} + 4\theta_4 \nu V_{1,1}$$

$$C_1 = 1 + V_{0,2} + \left\{\frac{\theta_4^2}{2}\nu^2 - \theta_4 \nu^2 + \left(\frac{\theta_1\theta_4}{R}\right) c\nu\right\} V_{2,0} + \left(\frac{\theta_1}{R} c + 2\theta_4 \nu\right) V_{1,1}$$

$$D_1 = 1$$

$$E_1 = 1 + \theta_4 \nu V_{1,1} + \left(\frac{\theta_4^2}{2}\nu^2 - \theta_4 \nu^2\right) V_{2,0}$$

$$A_2 = 1 + V_{0,2} + (2\theta_2^2 + \theta_2)\nu^2 V_{2,0} - 4\theta_2 \nu V_{1,1}$$

$$B_2 = 1 + V_{0,2} + (2\theta_2^2 + \theta_2)\nu^2 V_{2,0} + 4\theta_4 \nu V_{1,1}$$

$$C_2 = 1 + V_{0,2} + \left(\frac{\theta_4^2}{2}\nu^2 - \theta_4 \nu^2 + \frac{\theta_2(\theta_2 + 1)}{2}\nu^2 - \theta_2\theta_4\nu^2\right) V_{2,0} + (2\theta_4 \nu - 2\theta_2 \nu) V_{1,1}$$

$$D_2 = 1 + \frac{\theta_2(\theta_2 + 1)}{2}\nu^2 V_{2,0} - \theta_2 \nu V_{1,1}$$

$$E_2 = 1 + \theta_4 \nu V_{1,1} + \left(\frac{\theta_4^2}{2} - \theta_4\right)\nu^2 V_{2,0}$$

$$A_3 = 1 + V_{0,2} + 3\theta_3^2 \nu^2 V_{2,0} - 4\theta_3 \nu V_{1,1}$$

$$B_3 = 1 + V_{0,2} + (2\theta_4^2 - 2\theta_4)\nu^2 V_{2,0} + 4\theta_4 \nu V_{1,1}$$

$$C_3 = 1 + V_{0,2} + \left(\theta_3^2 + \frac{\theta_4^2}{2} - \theta_4 - \theta_3\theta_4\right)\nu^2 V_{2,0} + (2\theta_4 \nu - 2\theta_3 \nu) V_{1,1}$$

$$D_3 = 1 + \theta_3^2 \nu^2 V_{2,0} - \theta_3 \nu V_{1,1}$$

$$E_3 = 1 + \theta_4 \nu V_{1,1} + \left(\frac{\theta_4^2}{2} - \theta_4\right)\nu^2 V_{2,0}.$$

Table 2
Few Members of Singh and Malik (2014) Estimator s_{sm}^2 and Proposed Estimators s_{bi}^2

Member of $s_{sm(j)}^2$ ($\delta = 1$) $j = 1, 2, \dots, 9$	Member of $s_{b1(j)}^2$ $j = 1, 2, \dots, 9$	Member of $s_{b2(j)}^2$ $j = 1, 2, \dots, 9$	Member of $s_{b3(j)}^2$ $j = 1, 2, \dots, 9$	**Values of** c	d
$s_{sm(1)}^2$	$s_{b1(1)}^2$	$s_{b2(1)}^2$	$s_{b3(1)}^2$	N	1
$s_{sm(2)}^2$	$s_{b1(2)}^2$	$s_{b2(2)}^2$	$s_{b3(2)}^2$	N	f
$s_{sm(3)}^2$	$s_{b1(3)}^2$	$s_{b2(3)}^2$	$s_{b3(3)}^2$	N	$1 - f$
$s_{sm(4)}^2$	$s_{b1(4)}^2$	$s_{b2(4)}^2$	$s_{b3(4)}^2$	$\beta_2(\phi)$	$\rho(C_y/C_\phi)$
$s_{sm(5)}^2$	$s_{b1(5)}^2$	$s_{b2(5)}^2$	$s_{b3(5)}^2$	N	ρ
$s_{sm(6)}^2$	$s_{b1(6)}^2$	$s_{b2(6)}^2$	$s_{b3(6)}^2$	$\beta_2(\phi)$	C_ϕ
$s_{sm(7)}^2$	$s_{b1(7)}^2$	$s_{b2(7)}^2$	$s_{b3(7)}^2$	C_ϕ	$\beta_2(\phi)$
$s_{sm(8)}^2$	$s_{b1(8)}^2$	$s_{b2(8)}^2$	$s_{b3(8)}^2$	N	$\rho(C_y/C_\phi)$
$s_{sm(9)}^2$	$s_{b1(9)}^2$	$s_{b2(9)}^2$	$s_{b3(9)}^2$	n	f

4. ANALYTICAL COMPARISON

On comparing the minimum MSE of the proposed classes of estimators $s_{bi}^2, i = 1, 2, 3$ from (3.6) with the minimum MSE of existing estimators from (2.1), (2.5), (2.9), (2.18), (2.34), (2.7), (2.23), (2.25) and (2.30), we get the following efficiency conditions.

$$MSE(s_m^2) > MSE(s_{bi}^2)$$

$$\frac{(A_iE_i^2 + B_iD_i^2 - 2C_iD_iE_i)}{(A_iB_i - C_i^2)} > 1 - V_{0,2} \tag{4.1}$$

$$MSE(s_r^2) > MSE(s_{bi}^2)$$

$$\frac{(A_iE_i^2 + B_iD_i^2 - 2C_iD_iE_i)}{(A_iB_i - C_i^2)} > 1 - V_{0,2} - V_{2,0} + 2V_{1,1} \tag{4.2}$$

$MSE(s_*^2) > MSE(s_{bi}^2)$ where $s_*^2 = s_{lr}^2, s_i^2, i = 1,2,3$ and s_{bg}^2

$$\frac{(A_iE_i^2 + B_iD_i^2 - 2C_iD_iE_i)}{(A_iB_i - C_i^2)} > 1 - V_{0,2} + \frac{V_{1,1}^2}{V_{2,0}} \tag{4.3}$$

$$MSE(s_e^2) > MSE(s_{bi}^2)$$

$$\frac{(A_iE_i^2 + B_iD_i^2 - 2C_iD_iE_i)}{(A_iB_i - C_i^2)} > 1 - V_{0,2} - \frac{V_{2,0}}{4} + V_{1,1} \tag{4.4}$$

$$MSE(s_{kci}^2) > MSE(s_{bi}^2)$$

$$\frac{(A_iE_i^2 + B_iD_i^2 - 2C_iD_iE_i)}{(A_iB_i - C_i^2)} > 1 - V_{0,2} - w_i^2V_{2,0} + 2w_iV_{1,1} \tag{4.5}$$

$$MSE(s_s^2) > MSE(s_{bi}^2)$$

$$\frac{(A_iE_i^2 + B_iD_i^2 - 2C_iD_iE_i)}{(A_iB_i - C_i^2)} > 1 - \frac{1}{S_y^4}\left[S_y^4V_{0,2} + V_{2,0}\left\{\beta_\phi^2S_\phi^4 + A_1^2S_y^4 + 2A_1\beta_\phi S_y^2S_\phi^2\right\} - 2S_y^2V_{1,1}\left\{\beta_\phi S_\phi^2 + A_1S_y^2\right\}\right] \tag{4.6}$$

$$MSE(s_{sm}^2) > MSE(s_{bi}^2)$$

$$\frac{(A_iE_i^2 + B_iD_i^2 - 2C_iD_iE_i)}{(A_iB_i - C_i^2)} > \frac{(R_1R_5^2 + R_2R_4^2 - 2R_3R_4R_5)}{(R_1R_2 - R_3^2)} \tag{4.7}$$

Under the above conditions, the proposed classes of estimators $s_{bi}^2, i = 1, 2, 3$ perform better than the traditional mean estimator, classical ratio, regression and exponential estimators suggested by Singh and Kumar (2011), Singh and Malik (2014) estimator and log type estimators defined on the lines of Bhushan and Gupta (2016, 2020). Further, these conditions are verified by a numerical study using a real population.

5. NUMERICAL STUDY

In the present section, we have performed a numerical study over a real population taken from Sukhatme and Sukhatme (1970, pp. 256). The description about the population is given below.

y: Number of villages in the circles,
ϕ: A circle consisting of more than five villages.

$$N = 89, n = 23, S_y^2 = 4.074, S_\phi^2 = 0.11, C_y = 0.601, C_\phi = 2.678, \rho = 0.766,\ \beta_2(\phi) = 6.162,\ \lambda_{22} = 3.996,\ \lambda_{4,0} = 3.811, \lambda_{0,4} = 6.162.$$

The percent relative efficiency (PRE) of several estimators s_t^2 regarding the traditional mean estimator s_m^2 for the above population is calculated using the following expression.

$$PRE = \frac{MSE(s_m^2)}{MSE(s_t^2)} * 100$$

The numerical results are disclosed by means of PRE in Table 3. It has been observed from Table 3 that the members $s_{bi(j)}^2, i = 1, 2, 3;\ j = 1, 2, \ldots, 9$ of the proposed classes of estimators $s_{b_i}^2$ dominate:

i. the traditional mean estimator s_m^2, ratio estimator s_r^2, regression estimator s_{lr}^2 and exponential estimator s_e^2 envisaged by Singh and Kumar (2011), class of difference, Srivastava (1967) and Walsh (1970) type estimators $s_i^2, i = 1,2,3,$ the estimators $s_{kc_i}^2, i = 1, 2, 3, 4$ adapted by Singh and Malik (2014) and log type estimators s_{bg}^2 defined on the lines of Bhushan and Gupta (2016, 2020).
ii. the members $s_{s(j)}^2, j = 1$ to 10; of the class of estimators s_s^2 adapted by Singh and Malik (2014).
iii. the corresponding members $s_{sm(j)}^2, j = 1$ to 9; of the class of estimators s_{sm}^2 suggested by Singh and Malik (2014).

It is observed from Table 3 that the members $s_{bi(j)}^2, i = 1, 2, 3;\ j = 1, 2, \ldots, 9$ of the proposed classes of estimators s_{bi}^2 show their superiority over the existing estimators. Moreover, it is also observed that the member $s_{b_2(8)}^2$ of the suggested estimator $s_{b_2}^2$ based on the information $\left(N, \rho\left(C_y/C_\phi\right)\right)$ is found to be the most efficient among the proposed classes of estimators.

These results are expected because the conditions (4.1) to (4.7) are satisfied for the above data set.

We have numerically obtained the conditions derived in previous section using the above data set under which the proposed classes of estimators $s_{bi}^2, i = 1, 2, 3$ dominate the existing estimators.

Condition (4.1)

$$\frac{A_1E_1^2+B_1D_1^2-2C_1D_1E_1}{A_1B_1-C_1^2}>1-V_{0,2}\Rightarrow 0.9719>0.9093 \tag{5.1}$$

$$\frac{A_2E_2^2+B_2D_2^2-2C_2D_2E_2}{A_2B_2-C_2^2}>1-V_{0,2}\Rightarrow 0.9721>0.9093 \tag{5.2}$$

$$\frac{A_3E_3^2+B_3D_3^2-2C_3D_3E_3}{A_3B_3-C_3^2}>1-V_{0,2}\Rightarrow 0.9717>0.9093 \tag{5.3}$$

Condition (4.2)

$$\frac{A_1E_1^2+B_1D_1^2-2C_1D_1E_1}{A_1B_1-C_1^2}>1-V_{0,2}-V_{2,0}+2V_{1,1}\Rightarrow 0.9719>0.9361 \tag{5.4}$$

$$\frac{A_2E_2^2+B_2D_2^2-2C_2D_2E_2}{A_2B_2-C_2^2}>1-V_{0,2}-V_{2,0}+2V_{1,1}\Rightarrow 0.9721>0.9361 \tag{5.5}$$

$$\frac{A_3E_3^2+B_3D_3^2-2C_3D_3E_3}{A_3B_3-C_3^2}>1-V_{0,2}-V_{2,0}+2V_{1,1}\Rightarrow 0.9717>0.9361 \tag{5.6}$$

Condition (4.3)

$$\frac{A_1E_1^2+B_1D_1^2-2C_1D_1E_1}{A_1B_1-C_1^2}>1-V_{0,2}+\frac{V_{1,1}^2}{V_{2,0}}\Rightarrow 0.9719>0.9654 \tag{5.7}$$

$$\frac{A_2E_2^2+B_2D_2^2-2C_2D_2E_2}{A_2B_2-C_2^2}>1-V_{0,2}+\frac{V_{1,1}^2}{V_{2,0}}\Rightarrow 0.9721>0.9654 \tag{5.8}$$

$$\frac{A_3E_3^2+B_3D_3^2-2C_3D_3E_3}{A_3B_3-C_3^2}>1-V_{0,2}+\frac{V_{1,1}^2}{V_{2,0}}\Rightarrow 0.9717>0.9654 \tag{5.9}$$

Condition (4.4)

$$\frac{A_1E_1^2+B_1D_1^2-2C_1D_1E_1}{A_1B_1-C_1^2}>1-V_{0,2}-\frac{V_{2,0}}{4}+V_{1,1}\Rightarrow 0.9719>0.9643 \tag{5.10}$$

$$\frac{A_2E_2^2+B_2D_2^2-2C_2D_2E_2}{A_2B_2-C_2^2}>1-V_{0,2}-\frac{V_{2,0}}{4}+V_{1,1}\Rightarrow 0.9721>0.9643 \tag{5.11}$$

$$\frac{A_3E_3^2+B_3D_3^2-2C_3D_3E_3}{A_3B_3-C_3^2}>1-V_{0,2}-\frac{V_{2,0}}{4}+V_{1,1}\Rightarrow 0.9717>0.643 \tag{5.12}$$

Condition (4.5)

$$\frac{A_1E_1^2+B_1D_1^2-2C_1D_1E_1}{A_1B_1-C_1^2}>1-V_{0,2}-w_1^2V_{2,0}+2w_1V_{1,1}\Rightarrow 0.9719>0.9167 \tag{5.13}$$

$$\frac{A_2E_2^2+B_2D_2^2-2C_2D_2E_2}{A_2B_2-C_2^2}>1-V_{0,2}-w_1^2V_{2,0}+2w_1V_{1,1}\Rightarrow 0.9721>0.9167 \tag{5.14}$$

$$\frac{A_3E_3^2+B_3D_3^2-2C_3D_3E_3}{A_3B_3-C_3^2}>1-V_{0,2}-w_1^2V_{2,0}+2w_1V_{1,1}\Rightarrow 0.9717>0.9167 \tag{5.15}$$

Condition (4.6)

$$\frac{A_1E_1^2+B_1D_1^2-2C_1D_1E_1}{A_1B_1-C_1^2}>1-\frac{1}{S_y^4}\left[S_y^2V_{0,2}+V_{2,0}\left\{\beta_\phi^2S_\phi^4+A_1^2S_y^4+2A_1\beta_\phi S_y^2S_\phi^2\right\}-2S_y^2V_{1,1}\left\{\beta_\phi S_\phi^2+A_1S_y^2\right\}\right]\Rightarrow 0.9719>0.9167 \tag{5.16}$$

$$\frac{A_2E_2^2+B_2D_2^2-2C_2D_2E_2}{A_2B_2-C_2^2}>1-\frac{1}{S_y^4}\left[S_y^2V_{0,2}+V_{2,0}\left\{\beta_\phi^2S_\phi^4+A_1^2S_y^4+2A_1\beta_\phi S_y^2S_\phi^2\right\}-2S_y^2V_{1,1}\left\{\beta_\phi S_\phi^2+A_1S_y^2\right\}\right]\Rightarrow 0.9721>0.9167 \tag{5.17}$$

$$\frac{A_3E_3^2+B_3D_3^2-2C_3D_3E_3}{A_3B_3-C_3^2}>1-\frac{1}{S_y^4}\left[S_y^2V_{0,2}+V_{2,0}\left\{\beta_\phi^2S_\phi^4+A_1^2S_y^4+2A_1\beta_\phi S_y^2S_\phi^2\right\}-2S_y^2V_{1,1}\left\{\beta_\phi S_\phi^2+A_1S_y^2\right\}\right]\Rightarrow 0.9717>0.9167 \tag{5.18}$$

Condition (4.7)

$$\frac{A_1E_1^2+B_1D_1^2-2C_1D_1E_1}{A_1B_1-C_1^2}>\frac{R_1R_5^2+R_2R_4^2-2R_3R_4R_5}{R_1R_2-R_3^2}\Rightarrow 0.9719>0.9669 \tag{5.19}$$

$$\frac{A_2E_2^2+B_2D_2^2-2C_2D_2E_2}{A_2B_2-C_2^2}>\frac{R_1R_5^2+R_2R_4^2-2R_3R_4R_5}{R_1R_2-R_3^2}\Rightarrow 0.9721>0.9669 \tag{5.20}$$

$$\frac{A_3E_3^2+B_3D_3^2-2C_3D_3E_3}{A_3B_3-C_3^2}>\frac{R_1R_5^2+R_2R_4^2-2R_3R_4R_5}{R_1R_2-R_3^2}\Rightarrow 0.9717>0.9669 \tag{5.21}$$

These results are valid for every data set that satisfies the inequalities (4.1) to (4.7).

Table 3
PRE of Different Estimators

Estimators	PRE	Estimators	PRE
s_m^2	100	$s_{b1(1)}^2$	**323.3259**
s_r^2	141.898	$s_{b1(2)}^2$	**345.0904**
s_{lr}^2	262.1869	$s_{b1(3)}^2$	**329.7962**
s_e^2	254.2741	$s_{b1(4)}^2$	**298.2818**
s_{kc1}^2	108.8429	$s_{b1(5)}^2$	**329.1406**
s_{kc2}^2	103.8228	$s_{b1(6)}^2$	**268.6435**
s_{kc3}^2	155.1915	$s_{b1(7)}^2$	**270.4988**
s_{kc4}^2	110.3062	$s_{b1(8)}^2$	**348.386**
$s_{s(1)}^2$	261.7992	$s_{b1(9)}^2$	**323.0696**
$s_{s(2)}^2$	260.2364	$s_{b2(1)}^2$	**324.9893**
$s_{s(3)}^2$	219.1417	$s_{b2(2)}^2$	**346.2487**
$s_{s(4)}^2$	259.5852	$s_{b2(3)}^2$	**331.4395**
$s_{s(5)}^2$	243.6867	$s_{b2(4)}^2$	**300.1637**
$s_{s(6)}^2$	191.1765	$s_{b2(5)}^2$	**330.7914**
$s_{s(7)}^2$	261.0175	$s_{b2(6)}^2$	**263.3708**
$s_{s(8)}^2$	127.2051	$s_{b2(7)}^2$	**265.0370**
$s_{s(9)}^2$	261.9574	$s_{b2(8)}^2$	**349.3680**
$s_{s(10)}^2$	262.1432	$s_{b2(9)}^2$	**324.9893**
$s_{sm(1)}^2$	274.0323	$s_{b3(1)}^2$	**320.4494**
$s_{sm(2)}^2$	277.8316	$s_{b3(2)}^2$	**344.8036**
$s_{sm(3)}^2$	275.2199	$s_{b3(3)}^2$	**327.7019**
$s_{sm(4)}^2$	269.1781	$s_{b3(4)}^2$	**293.8393**
$s_{sm(5)}^2$	275.1019	$s_{b3(5)}^2$	**326.9675**
$s_{sm(6)}^2$	263.1846	$s_{b3(6)}^2$	**263.1888**
$s_{sm(7)}^2$	265.0181	$s_{b3(7)}^2$	**265.6184**
$s_{sm(8)}^2$	270.9300	$s_{b3(8)}^2$	**348.4820**
$s_{sm(9)}^2$	274.0323	$s_{b3(9)}^2$	**320.4494**

6. CONCLUSION

In this study, we have investigated some modified classes of estimators by combining respectively the class of difference, Srivastava (1967) and Walsh (1970) type estimators with the log type estimator for the estimation of population variance using known population proportion under simple random sampling. The usual mean, classical ratio and regression estimators and Kadilar and Cingi (2004) type estimators adapted by Singh and Malik (2014) are identified as the members of the suggested classes of estimators for suitably chosen values of characterizing scalars. The properties like bias and mean square error of the proposed classes of estimators are obtained up to first order of approximation. The efficiency conditions are obtained under which the proposed classes of estimators dominate the existing estimators which are further verified numerically using a real data set. The numerical results reported in Table 3 show the superiority of the members of proposed classes of estimators in terms of greater PRE over the estimators existing till date such as usual mean estimator, classical ratio, regression and exponential estimators, Kadilar and Cingi (2004) and Singh et al. (2008) types of estimators adapted by Singh and Malik (2014), the estimators suggested by Singh and Malik (2014) and log type estimators defined on the lines of Bhushan and Gupta (2016, 2020). Thus, the proposed classes of estimators are highly justified for the estimation of population variance when the information is available in the form of auxiliary attribute.

In forthcoming studies, we hope to develop the proposed classes of estimators for the estimation of population variance using two-phase sampling.

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