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SOME PROPERTIES OF THE WEIGHTED POWER HAZARD RATE DISTRIBUTION WITH APPLICATION

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ABSTRACT

We introduce the weighted power hazard rate distribution and investigate many of its properties. The weighted power hazard rate distribution derives many lifetime distributions including, weighted Rayleigh, weighted Weibull, and power hazard rate distribution. In addition, maximum likelihood estimates of the model parameters are also obtained. Finally, an application, as well as comparisons of the weighted power hazard rate distribution and its sub-models has been analyzed using the real data set.

KEY WORDS

Weighted, power hazard rate distribution, maximum likelihood estimation.

Mathematics subject classification: 60E05, 62E10, 62F15.

1. INTRODUCTION

In reliability studies, the hazard rate function plays an important role. This function describes the failure rate of the life test. In reliability theory, the lifetime distributions are frequently characterized by selecting a specific hazard rate function. One of these hazard rate functions is the power hazard rate which has the form,

$$h(x) = \alpha x^{\beta}, \quad x > 0, \alpha > 0, \beta > -1. \tag{0.1}$$

The power hazard rate function induces the power hazard rate (PHR) distribution with probability density function (PDF)

$$f(x) = \alpha x^{\beta} \exp\left\{-\frac{\alpha}{\beta+1} x^{\beta+1}\right\}, \quad x > 0, \alpha > 0, \beta > -1, \tag{0.2}$$

and corresponding cumulative density function (CDF)

$$F(x) = 1 - \exp\left\{-\frac{\alpha}{\beta+1} x^{\beta+1}\right\}, \quad x > 0, \alpha > 0, \beta > -1. \tag{0.3}$$

The parameters estimation of the PHR distribution by the least-squares technique was considered by Mugdadi (2005). Bayes estimation of the PHR function was investigated by Mugdadi and Min (2009). The estimation of the stress-strength reliability of the PHR distribution is also studied by Ismail (2014).

In general, PHR distribution can be described as decreasing, increasing, and constant, depending on different choices of the parameter β.

In view of (1.2), we can note that the PHR distribution naturally appears as a special case of the other well-known lifetime distributions, including Weibull, Rayleigh, exponential, and linear exponential are given in Table 1.

Table 1
Sub-Models of PHR Distribution

α	β	**Distribution**
-	$\beta = \alpha - 1$	Weibull
$\alpha = 1/\theta$	$\beta = 1$	Rayleigh
-	$\beta = 0$	Exponential
-	$\beta = 1$	Linear failure rate

Fisher (1934) initiated the concept of the weighted distribution. The weighted distributions can be applied to many experimental problems in the existence of biased samples that appear naturally in numerous situations. The significance of the weighted distributions has been found in many areas such as ecology, medicine, biomedicine, reliability, and branching processes, and water quality. Many researchers investigated several weighted distributions (see, for example,) Rao (1965), Gupta and Kirmani (1990), Gupta and Keating (1985), Oluyede (1999), Patil (2002), Leiva et al. (2009), Gupta and Kundu (2009) and Dey et al. (2014) among others.

Recently, Saghir et al. (2016) proposed the length-biased weighted exponentiated inverted Weibull distribution. Saghir et al. (2017) studied Maxwell -length-biased distribution. Parveen and Ahmad (2018) developed some properties of size-biased weighted Weibull distribution, Al-Omari et al. (2019) suggested power length-biased Suja distribution and, Shakhatreh and Al-Masri (2020) considered the weighted Burr XII distribution, and references cited therein.

The rest of the paper is formulated as follows. In Section 2, the weighted power hazard rate distribution is introduced. Some properties of the introduced distribution are given in Section 3. The estimation of the parameters by the method of maximum likelihood is derived in Section 4. An application, as well as a comparison of the weighted power hazard rate distribution and its sub-models, are given in Section 5. Finally, the concluding remark is given in Section 6.

2. WEIGHTED POWER HAZARD RATE DISTRIBUTION

A weighted class of probability density function on support of X can be expressed as follows

$$f^{w}(x)=\frac{w(x)\,f(x)}{w}, \quad \text{[Patil and Rao (1978)]} \tag{2.1}$$

where $w(x)>0$ and w is normalizing constant that necessitate $f^{w}(x)$ to integrate 1. Let, $w(x)=x^{c}$ the associated pdf and cdf of weighted power hazard rate (WPHR) distribution on the support of X is given by

$$f(x)=\frac{\alpha x^{\beta+c}e^{-\frac{\alpha}{\beta+1}x^{\beta 1}}}{\left(\frac{\beta+1}{\alpha}\right)^{\frac{c}{\beta+1}}\Gamma\left(\frac{\beta+c+1}{\beta+1}\right)}, \quad x>0, \tag{2.2}$$

where $\Gamma(a)=\int_0^{\infty}t^{a-1}e^{-t}dt$ is gamma function.

The PDF plots of the WPHR distribution are shown in Figure 1, for different values of the parameter.

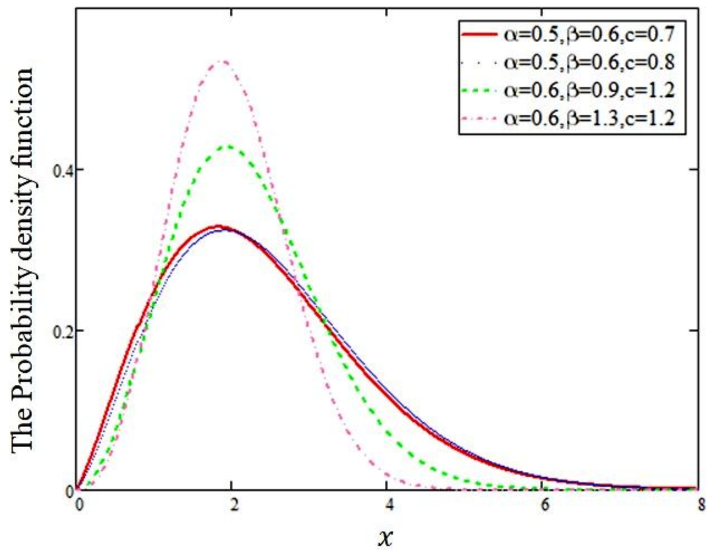

Figure 1: The PDF of the WPHR Distribution for Indicated Values of the Parameters

From Figure 1, the PDF has one mode for given values of α,β and c.

Cumulative distribution function (CDF) is given by

$$F(x)=\frac{\gamma\left(\frac{\beta+c+1}{\beta+1},\frac{\alpha}{\beta+1}x^{\beta+c}\right)}{\Gamma\left(\frac{\beta+c+1}{\beta+1}\right)},\quad x>0, \tag{2.3}$$

where $\gamma(a,x)=\int_0^x t^{a-1}e^{-t}dt$ is an upper incomplete gamma function.

The survival function (reliability) $\bar{F}(x)$ of WPHR distribution is given by

$$\bar{F}(x)=\frac{\Gamma\left(\frac{\beta+c+1}{\beta+1},\frac{\alpha}{\beta+1}x^{\beta+c}\right)}{\Gamma\left(\frac{\beta+c+1}{\beta+1}\right)},\quad x>0, \tag{2.4}$$

where $\Gamma(a,x)=\int_x^\infty t^{a-1}e^{-t}dt$ is an incomplete gamma function.

The hazard rate function $h(x)$ for the WPHR distribution can be presented as follows

$$h(x)=\frac{\alpha x^{\beta+c}e^{-\frac{\alpha}{\beta+1}x^{\beta+1}}}{\left(\frac{\beta+1}{\alpha}\right)^{\frac{c}{\beta+1}}\Gamma\left(\frac{\beta+c+1}{\beta+1},\frac{\alpha}{\beta+1}x^{\beta+c}\right)},\quad x>0, \tag{2.5}$$

Figures 2-3, show the reliability and hazard rate function for WPHR distribution for some different parameter values, respectively.

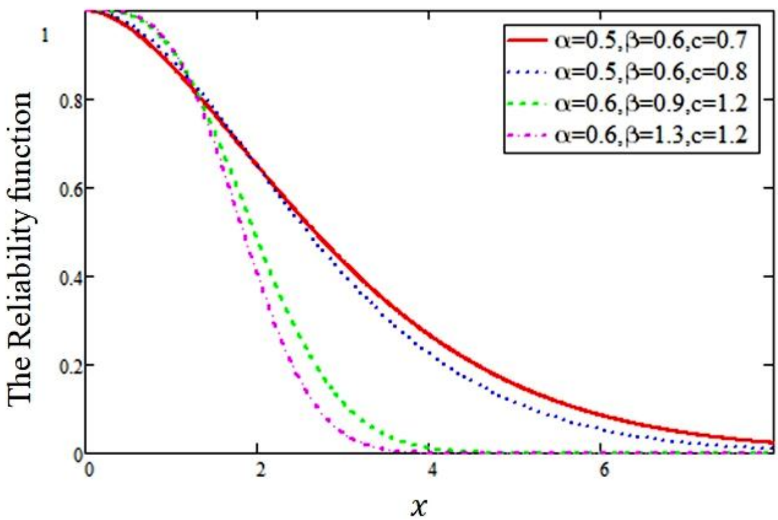

Figure 2: The Reliability Function for Indicated Values of the Parameters

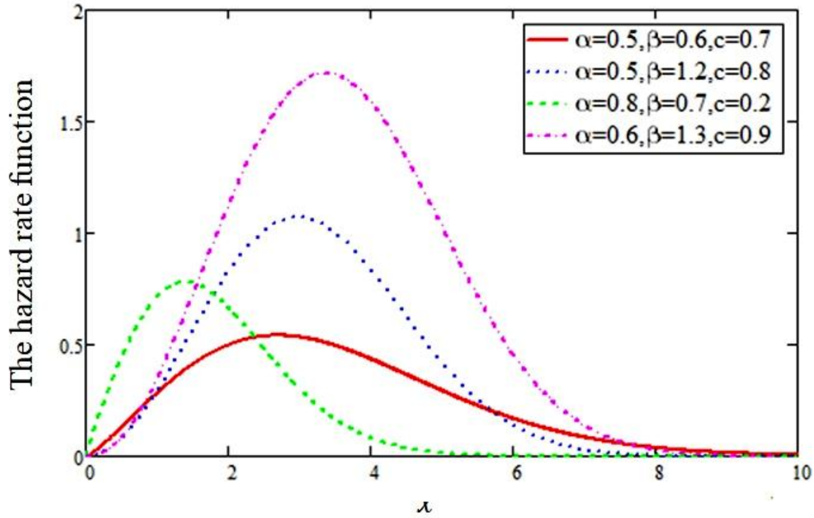

Figure 3: The Hazard Rate Function for Indicated Values of the Parameters

From Figure 3, the hazard rate function increases and then decreasing for the given values of α,β and c.

Sub-Models of WPHR Distribution:

The WPHR distribution is very flexible. It includes various well-known distributions as sub-models based on different values of the parameters. These sub-models are

i) When $\beta=\alpha-1$, the WPHR distribution reduces to weighted Weibull (WW) distribution as obtained by Dey et al. (2014).

ii) When $\alpha=\frac{1}{\theta},\beta=1$, the WPHR distribution reduces to weighted Rayleigh (WR) distribution as obtained by Ajami and Jahanshahi (2017).

iii) When $\beta=0$, the WPHR distribution reduces to weighted exponential (WE) distribution as reported by Gupta and Kundu (2009).

iv) When $c=1$, the WPHR distribution reduces to length-biased power hazard rate (LBPHR) distribution as obtained by Mustafa and Khan (2021).

3. THE STATISTICAL PROPERTIES OF WPHR DISTRIBUTION

In this section, the statistical properties of the WPHR distribution are addressed.

Theorem 3.1

If $X\sim WPHRD(\alpha,\beta,c)$, then the k-th moment of X is related as

$$E_f(X^k)=\frac{\left(\frac{\beta+1}{\alpha}\right)^{\frac{k}{\beta+1}}\Gamma\left(\frac{k+c+\beta+1}{\beta+1}\right)}{\Gamma\left(\frac{\beta+c+1}{\beta+1}\right)}. \tag{3.1}$$

Proof:

Let X be a continuous non-negative random variable with PDF $f(x)$, then the k-th moment of X can be obtained by

$$E_f(X^k)=\int_0^\infty x^k f(x)dx,$$

from the PDF of WPHR distribution in (2.2), then $E_f(X^k)$ can be described as

$$E_f(X^k)=\int_0^\infty \frac{\alpha x^{k+\beta+c} e^{-\frac{\alpha}{\beta+1}x^{\beta+1}}}{\left(\frac{\beta+1}{\alpha}\right)^{\frac{c}{\beta+1}}\Gamma\left(\frac{\beta+c+1}{\beta+1}\right)}dx. \tag{3.2}$$

Let, $u=\frac{\alpha}{\beta+1}x^{\beta+1}, du=\alpha x^{\beta}dx$.

Upon simplification, (3.2) leads to

$$E_f(X^k)=\frac{\left(\frac{\beta+1}{\alpha}\right)^{\frac{k}{\beta+1}}\Gamma\left(\frac{k+c+\beta+1}{\beta+1}\right)}{\Gamma\left(\frac{\beta+c+1}{\beta+1}\right)}. \tag{3.3}$$

From (3.3), we can calculate the mean and variance for WPHR distribution as following

- At $k=1$, in (3.3), the expected value of X can be attained as

$$E_f(X)=\frac{\left(\frac{\beta+1}{\alpha}\right)^{\frac{1}{\beta+1}}\Gamma\left(\frac{c+\beta+2}{\beta+1}\right)}{\Gamma\left(\frac{\beta+c+1}{\beta+1}\right)}. \tag{3.4}$$

- If $k=2$, in (3.3), the second moment can be written as

$$E_f(X^2)=\frac{\left(\frac{\beta+1}{\alpha}\right)^{\frac{2}{\beta+1}}\Gamma\left(\frac{c+\beta+3}{\beta+1}\right)}{\Gamma\left(\frac{\beta+c+1}{\beta+1}\right)}. \tag{3.5}$$

- The variance of X may be obtained as

$$Var(X)=\frac{\left(\frac{\beta+1}{\alpha}\right)^{\frac{2}{\beta+1}}\Gamma\left(\frac{c+\beta+3}{\beta+1}\right)}{\Gamma\left(\frac{\beta+c+1}{\beta+1}\right)}-\left[\frac{\left(\frac{\beta+1}{\alpha}\right)^{\frac{1}{\beta+1}}\Gamma\left(\frac{c+\beta+2}{\beta+1}\right)}{\Gamma\left(\frac{\beta+c+1}{\beta+1}\right)}\right]^2. \tag{3.6}$$

Harmonic Mean:

The harmonic mean (H.M.) of WPHR distribution is given as

$$H.M.=\frac{\left(\frac{\beta+1}{\alpha}\right)^{\frac{1}{\beta+1}}\Gamma\left(\frac{\beta+c+1}{\beta+1}\right)}{\Gamma\left(\frac{\beta+c}{\beta+1}\right)}.$$

Skewness and Kurtosis:

The statistical measures of skewness and kurtosis play important role in describing shape characteristics of the probability distributions. From Theorem 3.1, skewness and kurtosis can be calculated by the following relations

$$Su=E\left(\frac{X-\mu_1^{'}}{\sigma_X}\right)^3=\frac{\mu_3^{'}-3\mu_1^{'}\mu_2^{'}+2\mu_1^{'2}}{(\mu_2^{'}-\mu_1^{'2})^{3/2}},$$

$$Ku=E\left(\frac{X-\mu_1^{'}}{\sigma_X}\right)^4=\frac{\mu_4^{'}-4\mu_1^{'}\mu_3^{'}+6\mu_1^{'2}\mu_2^{'}-3\mu_1^{'4}}{(\mu_2^{'}-\mu_1^{'2})^{2}},$$

where $\mu_k^{'}=E_f(X^k)$ and $\sigma_X=\sqrt{Var(X)}$.

Mode:

The mode of the WPHR distribution can be derived as follows.

Taking the logarithm of (2.2), we have

$$\ln f(x)=\ln(\alpha)+(\beta+c)\ln(x)-\frac{\alpha}{\beta+1}x^{\beta+1}-\ln\left[\left(\frac{\beta+1}{\alpha}\right)^{\frac{c}{\beta+1}}\Gamma\left(\frac{\beta+c+1}{\beta+1}\right)\right] \tag{3.7}$$

Differentiate (3.7) with respect to x, and equating to zero, we arrive at

$$\frac{d}{dx}\ln f(x)=\frac{\beta+c}{x}-\alpha x^{\beta}=0, \tag{3.8}$$

then $x = \left(\frac{\beta + c}{\alpha}\right)^{\frac{1}{\beta+1}}$.

Differentiate (3.8) with respect to x, we obtain

$$\frac{d^2}{dx^2}\ln f(x) = -\frac{\beta + c}{x^2} - \alpha\beta x^{\beta-1} = -\frac{(\beta + c) + \alpha\beta x^{\beta+1}}{x^2},$$

at $x = \left(\frac{\beta + c}{\alpha}\right)^{\frac{1}{\beta+1}}$, then $\frac{d^2}{dx^2}\ln f(x) = -\frac{(\beta + c) + (\beta + 1)}{x^2} < 0.$

Therefore, $f(x)$ has a maximum at $x = \left(\frac{\beta + c}{\alpha}\right)^{\frac{1}{\beta+1}}$.

Hence,

$$\text{Mode}(X) = \left(\frac{\beta + c}{\alpha}\right)^{\frac{1}{\beta+1}}. \tag{3.9}$$

The following table contains some statistical measures for selected values of α, β and c.

Table 2
Some Statistical Measures for $\alpha = 0.8$ and different values of β and c

β	C	Mean	Mode	H. M.	Variance	C.V.	Sk	Ku
0.6	0.7	1.790	1.355	1.187	0.903	0.531	0.786	3.675
0.6	1.0	1.949	1.542	1.383	0.944	0.499	0.734	3.584
0.6	1.5	2.199	1.828	1.679	1.004	0.456	0.666	3.475
1.0	0.7	1.678	1.458	1.267	0.560	0.446	0.519	3.134
1.0	1.0	1.784	1.581	1.401	0.567	0.422	0.486	3.108
1.0	1.5	1.949	1.768	1.603	0.576	0.389	0.441	3.078
1.5	0.7	1.585	1.499	1.309	0.351	0.374	0.290	2.859
1.5	1.0	1.656	1.577	1.400	0.348	0.356	0.269	2.864
1.5	1.5	1.765	1.697	1.535	0.343	0.332	0.241	2.873

From Table 2, we can conclude that:

1. The central tendency measures are increasing for c is increasing and β fixed.
2. The mean is decreasing when β increasing and c is fixed.
3. The mode and harmonic mean are increasing when c increasing and c is fixed.

4. The variance is increasing when c increasing and decreasing when β increasing.
5. The measures coefficient of variation, skewness and kurtosis are decreasing when c or β is increasing.
6. The skewness is positive for all values of β and c and, also less than 1, so it can be concluded that the distribution is not significantly skewed.
7. The kurtosis is decreasing when β or c increasing and for $\beta \le 1$, $Ku \ge 3$, but when $\beta > 1$, $Ku < 3$.

pth Percentile (x_p):

The pth percentile can be obtained by using the following relation

$$F(x_p, \alpha, \beta, c) = p. \tag{3.10}$$

By using equation (2.3) and (3.10), x_p satisfies the equation

$$\Gamma\left(\frac{\beta+c+1}{\beta+1}, \frac{\alpha}{\beta+1} x_p^{\beta+1}\right) - p\Gamma\left(\frac{\beta+c+1}{\beta+1}\right) = 0. \tag{3.11}$$

The pth percentile can be calculated by solving Equation (3.11) with respect to, x_p for α, β, c and p.

From equation (3.11), when $p = 0.5$, the median can be calculated.

Mean Residual Life Function:

Let X be a WPHR random variable, then the mean residual life (MRL) function denoted by $m(t)$ is given by

$$m(t) = E(X - t \mid X > t) = \frac{I(t)}{S(t)} - t, \tag{3.12}$$

where $I(t) = \int_t^\infty x f(x) dx$ is simplified as,

$$I(t) = \int_t^\infty x f(x) dx = \frac{\left(\frac{\beta+1}{\alpha}\right)^{\frac{1}{\beta+1}}}{\Gamma\left(\frac{\beta+c+1}{\beta+1}\right)} \Gamma\left(\frac{\beta+c+2}{\beta+1}, \frac{\alpha}{\beta+1} t^{\beta+1}\right).$$

Putting the above integral value into (3.12), the MRL function is obtained

$$m(t) = E(X - t \mid X > t) = \left(\frac{\beta+1}{\alpha}\right)^{\frac{1}{\beta+1}} \frac{\Gamma\left(\frac{\beta+c+2}{\beta+1}, \frac{\alpha}{\beta+1} t^{\beta+1}\right)}{\Gamma\left(\frac{\beta+c+1}{\beta+1}, \frac{\alpha}{\beta+1} t^{\beta+c}\right)} - t.$$

4. ESTIMATION

In this section, the method of maximum likelihood is adopted to estimate the parameters. The corresponding likelihood function of (2.2) is given by

$$L(x_1,\cdots,x_n \mid \alpha,\beta,c)=\alpha^n\left(\prod_{i=1}^{n}x_i^{\beta+c}\right)e^{-\frac{\alpha}{\beta+1}\sum_{i=1}^{n}x_i^{\beta+1}}\left(\frac{\beta+1}{\alpha}\right)^{-\frac{nc}{\beta+1}}\left[\Gamma\left(\frac{\beta+c+1}{\beta+1}\right)\right]^{-n}. \quad (4.1)$$

The log-likelihood function could be expressed as

$$\mathcal{L}=n\ln(\alpha)+(\beta+c)\sum_{i=1}^{n}\ln(x_i)-\frac{\alpha}{\beta+1}\sum_{i=1}^{n}x_i^{\beta+1}-\frac{nc}{\beta+1}\ln\left(\frac{\beta+1}{\alpha}\right)-n\ln\Gamma\left(\frac{\beta+c+1}{\beta+1}\right). \quad (4.2)$$

Differentiate Equation (4.2) with respect to (α,β,c) and equating the first-order partial derivatives to zero. To obtain the MLE of the parameters,

$$\frac{\partial\mathcal{L}}{\partial\alpha}=\frac{n}{\alpha}-\frac{1}{\beta+1}\sum_{i=1}^{n}x_i^{\beta+1}+\frac{nc}{(\beta+1)}, \quad (4.3)$$

$$\frac{\partial\mathcal{L}}{\partial\beta}=\sum_{i=1}^{n}\ln(x_i)+\frac{\alpha}{(\beta+1)^2}\sum_{i=1}^{n}x_i^{\beta+1}-\frac{\alpha}{\beta+1}\sum_{i=1}^{n}x_i^{\beta+1}\ln(x_i)$$
$$+\frac{nc}{(\beta+1)^2}\ln\left(\frac{\beta+1}{\alpha}\right)-\frac{nc}{(\beta+1)^2}-n\Psi\left(\frac{\beta+c+1}{\beta+1}\right)\left(\frac{1}{\beta+1}-\frac{\beta+c+1}{(\beta+1)^2}\right), \quad (4.4)$$

$$\frac{\partial\mathcal{L}}{\partial c}=\sum_{i=1}^{n}\ln(x_i)-\frac{n}{\beta+1}\ln\left(\frac{\beta+1}{\alpha}\right)-\frac{n}{\beta+1}\Psi\left(\frac{\beta+c+1}{\beta+1}\right) \quad (4.5)$$

where $\Psi(x)=\frac{d}{dx}\ln\Gamma(x)$.

When the estimates of α,β and c are obtained, one can relate the asymptotic normality of the MLEs to evaluate the approximate confidence interval (CIs) for the parameters. The observed variance-covariance matrix for MLEs of the unknown parameters $\Theta=(\alpha,\beta,c)$ is the inverse of the sample information matrix, that is,

$$V=I^{-1}=\begin{pmatrix}-I_{11} & -I_{12} & -I_{13}\\ -I_{21} & -I_{22} & -I_{23}\\ -I_{31} & -I_{32} & -I_{33}\end{pmatrix}^{-1}, \quad (4.6)$$

where $I_{i,j}=\frac{\partial^2\mathcal{L}}{\partial\theta_i\partial\theta_j}$, the second partial derivatives are as follows

$$\frac{\partial^2\mathcal{L}}{\partial\alpha^2}=-\frac{n}{\alpha^2}-\frac{nc}{(\beta+1)\alpha^2} \tag{4.7}$$

$$\frac{\partial^2\mathcal{L}}{\partial\alpha\partial\beta}=\frac{1}{(\beta+1)^2}\sum_{i=1}^{n}x_i^{\beta+1}-\frac{1}{\beta+1}\sum_{i=1}^{n}x_i^{\beta+1}\ln(x_i)-\frac{nc}{(\beta+1)^2\alpha}, \tag{4.8}$$

$$\frac{\partial^2\mathcal{L}}{\partial\alpha\partial c}=\frac{n}{(\beta+1)\alpha}, \tag{4.9}$$

$$\begin{aligned}\frac{\partial^2\mathcal{L}}{\partial\beta^2}=&-\frac{2\alpha}{(\beta+1)^3}\sum_{i=1}^{n}x_i^{\beta+1}+\frac{2\alpha}{(\beta+1)^2}\sum_{i=1}^{n}x_i^{\beta+1}\ln(x_i)-\frac{\alpha}{\beta+1}\sum_{i=1}^{n}x_i^{\beta+1}(\ln(x_i))^2\\&-\frac{2nc}{(\beta+1)^3}\ln\left(\frac{\beta+1}{\alpha}\right)+\frac{3nc}{(\beta+1)^3}-n\Psi\left(1,\frac{\beta+c+1}{\beta+1}\right)\left(\frac{1}{\beta+1}-\frac{\beta+c+1}{(\beta+1)^2}\right)^2\\&-n\Psi\left(\frac{\beta+c+1}{\beta+1}\right)\left(\frac{2(\beta+c+1)}{(\beta+1)^3}-\frac{2}{(\beta+1)^2}\right)\end{aligned} \tag{4.10}$$

$$\begin{aligned}\frac{\partial^2\mathcal{L}}{\partial\beta\partial c}=&\frac{n}{(\beta+1)^2}\ln\left(\frac{\beta+1}{\alpha}\right)-\frac{n}{(\beta+1)^2}-\frac{n}{\beta+1}\Psi\left(1,\frac{\beta+c+1}{\beta+1}\right)\left(\frac{1}{\beta+1}-\frac{\beta+c+1}{(\beta+1)^2}\right)\\&+\frac{n}{(\beta+1)^2}\Psi\left(\frac{\beta+c+1}{\beta+1}\right),\end{aligned} \tag{4.11}$$

$$\frac{\partial^2\mathcal{L}}{\partial c^2}=-\frac{n}{(\beta+1)^2}\Psi\left(1,\frac{\beta+c+1}{\beta+1}\right). \tag{4.12}$$

where $\Psi(n,x)=\frac{d^n}{dx^n}\Psi(x)$ be a digamma function, which is the logarithmic derivative of the gamma function. Therefore, the above technique is used to obtain the approximate $100(1-\delta)\%$ confidence intervals of the parameters $\Theta=(\alpha,\beta,c)$ as in the following form

$$\hat{\alpha}\pm z_{\delta/2}\sqrt{Var(\hat{\alpha})},\qquad \hat{\beta}\pm z_{\delta/2}\sqrt{Var(\hat{\beta})},\qquad \hat{c}\pm z_{\delta/2}\sqrt{Var(\hat{c})}.$$

Here $z_{\delta/2}$ is the upper $(\delta/2)$ th percentile of the standard normal distribution.

5. APPLICATION

We consider the data set from a study carried out by Bjerkedal (1960). The data are given below:

The data set comprise survival times in days of 72 Guinea pigs injected with various amount of tubercle.	12, 15, 22, 24, 24, 32, 32, 33, 34, 38, 38, 43, 44, 48, 52, 53, 54, 54, 55, 56, 57, 58, 58, 59, 60, 60, 60, 60, 61, 62, 63, 65, 65, 67, 68, 70, 70, 72, 73, 75, 76, 76, 81, 83, 84, 85, 87, 91, 95, 95, 96, 98, 99, 109, 110, 121, 127, 129, 131, 143, 146, 146, 175, 175, 211, 233, 258, 258, 263, 297, 341, 376

The MLEs of the unknown parameters of PHR distribution, WW distribution, WR distribution and WPHR distribution are given along with criterion such as [log-likelihood, Akaike information criterion (AIC), Akaike information criterion corrected (AICC), Bayesian information criterion (BIC) and Hannan-Quinn information criterion (HQIC)] in Table 3, where

$$\text{AIC} = -2\mathcal{L} + 2k, \qquad \text{AICC} = \text{AIC} + \frac{2k(k+1)}{n-k-1},$$

$$\text{BIC} = -2\mathcal{L} + k\ln(n), \quad \text{HQIC} = -2\mathcal{L} + 2k\ln(\ln(n)),$$

where $\mathcal{L}$ is the log-likelihood, n is the number of observations and k is the number of parameters.

Table 3
MLEs of the Model Parameters and the Log Likelihood, AIC, AICC, BIC and HQIC

	Model			
	PHRD	**WWD**	**WRD**	**WPHRD**
$\hat{\alpha}$	1.981×10^{-3}	–	0.113	0.013
$\hat{\beta}$	0.393	1.368	5.033×10^{3}	0.079
$\hat{\theta}$	–	1.661×10^{-3}	–	–
$\hat{c}$	–	0.028	–	0.834
$\mathcal{L}$	-397.148	-397.004	-418.587	-394.748
AIC	798.295	800.008	841.174	795.495
AICC	798.382	800.184	841.260	795.672
BIC	802.849	806.838	845.727	802.325
HQIC	800.108	802.727	842.986	798.214

As the value of $\mathcal{L}$, AIC, AICC, BIC and HQIC are smaller for WPHR distribution as compare to PHR distribution, WW distribution and WR distribution. Therefore, WPHR distribution fits better for given data set.

Variance and covariance matrix are given as

$$V = \begin{pmatrix} 2.668\times10^{-3} & -0.036 & 0.091 \\ -0.036 & 0.484 & -1.212 \\ 0.091 & -1.212 & 4.124 \end{pmatrix}.$$

Then the 95% confidence interval for α, β and c for WPHR distribution are (0, 0.11427), (0, 1.4426) and (0, 4.29782), respectively.

Table 4 gives some descriptive summary statistics for distribution by using the values of $\hat{\alpha}$, $\hat{\beta}$ and $\hat{c}$.

Table 4

Some Statistical Measures for WPHRD, for $\hat{\alpha}$, $\hat{\beta}$ and $\hat{c}$.

Mean	Mode	H. M.	Standard Deviation	Sk	Ku
100.97	51.325	49.627	69.70	1.339	5.618

From Table 4, we notice that positive values of skewness and kurtosis but having a large value of standard deviation. This confirms that data are positively skewed, and hence proposed distribution could be used to model this data.

Figure 4 – 6, show that the likelihood function has unique solution for α, β and c.

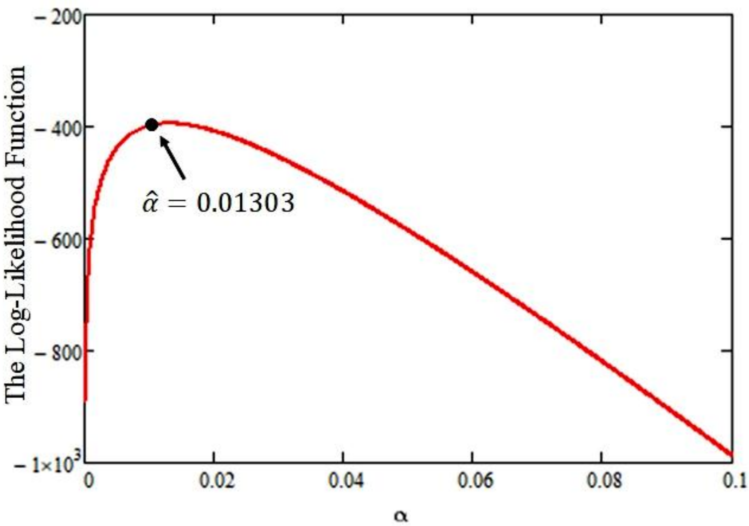

Figure 4: The Profile of the Log-Likelihood Function of α

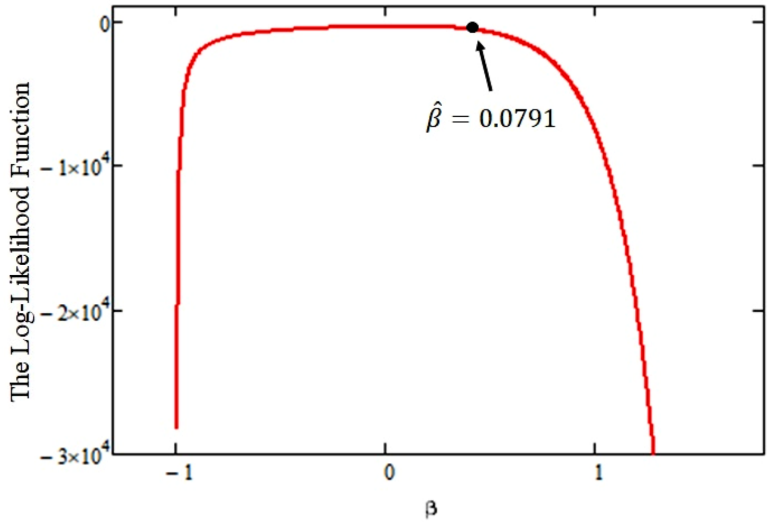

Figure 5: The Profile of the Log-Likelihood Function of β

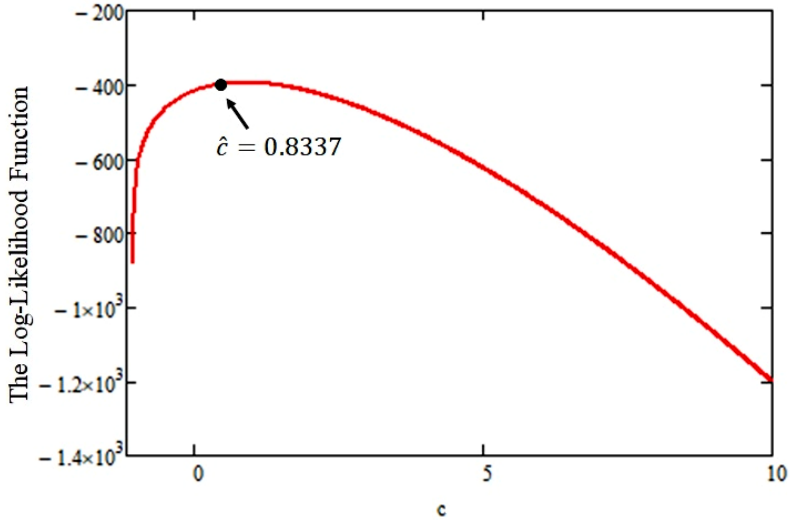

Figure 6: The Profile of the Log-Likelihood Function of c

6. CONCLUSION

We propose a new model called the weighted power hazard rate distribution. Some statistical properties are obtained. The maximum likelihood estimates for model parameters are presented. Finally, the practicality of the WPHR distribution is exhibited using the established data given by Bjerkedal (1960).

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