Pak. J. Statist.
2022 Vol. 38(2), 211-218

NEW BOUNDS FOR VARIABLES OF FRACTIONAL ORDER

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ABSTRACT

In this paper, we define some new notion continuous random variables of fractional order. Also, some new generalized inequalities will be computed.

1. INTRODUCTION AND PRELIMINARIES

In day today life, fractional calculus has witnessed a significant role in many domains like applied mathematics, physics, chemistry, applied sciences as can be seen in [2], [12], [14], [16]. Although, it is to be noted that it is more devoted to the topic concerning to Riemann-Liouville and Caputo derivatives. It is the kernel of the integral which is used in the Hadamard derivative is expanded in logarithmic sort. The integral inequalities as in [1], [3]-[11], [15], [22], [23] have been shown to be of significant importance over last few decades. The formation of fractional calculus and its various statements and applications of fractional derivatives have been analyzed. Riemann–Liouville and Grunwald–Letnikov are most famous in this type of study. It was Caputo who reformulated the classical sense of the Riemann–Liouville fractional derivative for getting solutions of fractional differential equations with given conditions. Leibniz notion on fractional calculus was synthesed by Grunwald–Letnikov in a different structure [1], [3], [13], [17], [21], etc.

Quite recently, in [18], [20], the structure of probability theory involving fractional calculus structure are given and hence the classical approach were extended. Further study and observations in this field and other structures have been constructed in the tactile problems which includes applied sciences, mathematics, fluid mechanics, chemistry etc. as in [3], [19], and many others.

Definition 1.1:

For a function $\varphi(s)$ with $s > 1$ and $a \geq 1$, the Hadamard fractional integral of order $\text{פ} \in \mathbb{R}$ is given by

$$\Omega_{\alpha,\lambda}^{\text{פ}}[\varphi(s)] = \frac{1}{\Gamma_\lambda(\text{פ})} \int_a^t \frac{1}{s} \left(log\ \frac{t}{s} \right)^{\frac{\text{פ}}{\lambda} - 1} \varphi(s)\, ds \tag{1}$$

where $\Gamma_\lambda(\text{פ}) = \lambda\, \Gamma(\text{פ})$ and $\Gamma(\text{פ})$ represents classical Gamma function as can be seen in [15], [22], eta cetera.

Definition 1.2:

Let $\varphi: [a,b] \to \mathbb{R}$ be a positive pdf of a random variable Ɣ, then the fractional expectation of order $\text{Ɔ} \in \mathbb{R}$ with $a < v < b$ is

$$E_{\text{Ɣ},\text{Ɔ},\lambda}(s) = \frac{1}{\Gamma_\lambda(\text{Ɔ})}\int_a^v \frac{1}{u}\left(log\ \frac{s}{u}\right)^{\frac{\text{Ɔ}}{\lambda}-1} u\varphi(u)du. \tag{2}$$

Definition 1.3:

Let $\varphi: [a,b] \to \mathbb{R}$ be a positive pdf of a random variable Ɣ, then the fractional expectation of Ɣ – E(Ɣ) of order $\text{Ɔ} \in \mathbb{R}$ with $\alpha < v < b$ is given by

$$E_{\text{Ɣ},\text{Ɔ},\lambda}(s) = \frac{1}{\Gamma_\lambda(\text{Ɔ})}\int_\alpha^v \frac{1}{u}\left(log\ \frac{s}{u}\right)^{\frac{\text{Ɔ}}{\lambda}-1} (u - E(\text{Ɣ})\varphi(u)\ du = \Omega_{a,\lambda}^{\text{Ɔ}}[s\varphi(s)]. \tag{3}$$

Definition 1.4:

Let $\varphi: [a,b] \to \mathbb{R}$ be a positive pdf of a random variable Ɣ, then the fractional variance of order $\text{Ɔ} \in \mathbb{R}$ with $a < v < b$ is given by

$$\begin{aligned}\sigma^2_{\text{Ɣ},\text{Ɔ},\lambda}(s) &= \Omega_{a,\lambda}^{\text{Ɔ}}[(s - \mathrm{E}(\text{Ɣ}))^2\varphi(s)] \\ &= \frac{1}{\Gamma_\lambda(\text{Ɔ})}\int_a^s \frac{1}{u}\left(log\ \frac{s}{u}\right)^{\frac{\text{Ɔ}}{\lambda}-1} (u - E(\text{Ɣ})^2\varphi(u)\ du.\end{aligned} \tag{4}$$

2. MAIN RESULTS FOR NEW FRACTIONAL INEQUALITIES

This portion of the paper belongs to synthesis some the new results of fractional order.

Theorem 2.1:

Let $\varphi: [a,b] \to \mathbb{R}$ be a positive pdf of a random variable Ɣ, then the inequalities

(i) $\Omega_{a,\lambda}^{\text{Ɔ}}[\varphi(s)]\sigma^2_{\text{Ɣ},\text{Ɔ},\lambda} - \left[E_{\text{Ɣ}-E(\text{Ɣ})}(s)\right]^2 \leq 2||\varphi||^2_\infty \left[\frac{\left(log\ \frac{s}{u}\right)^{\frac{\text{Ɔ}}{\lambda}}}{\Gamma(\text{Ɔ}+1)}\ \Omega_{a,\lambda}^{\text{Ɔ}}[s^2] - \Omega_{a,\lambda}^{\text{Ɔ}}[s^2]\right],$

holds if $\varphi \in L_\infty[a,b]$ for every $a \leq s \leq b, \text{Ɔ} \geq 0$ and $\lambda \geq 0.$

(ii) $\Omega_{a,\lambda}^{\text{Ɔ}}[\varphi(s)]\sigma^2_{\text{Ɣ},\text{Ɔ},\lambda} - \left[E_{\text{Ɣ}-E(\text{Ɣ})}(s)\right]^2 \leq \frac{1}{2}(s-a)^2\left[\ \Omega_{a,\lambda}^{\text{Ɔ}}[\varphi(s)]\right],$
holds.

Proof:

For the proof of the result, we begin by choosing the function

$$\begin{aligned}\text{ɧ}(w,\text{ɳ}) &= \left(\text{ɧ}_1(\mathrm{w}) - \text{ɧ}_1(\text{ɳ})\right)\left(\text{ɧ}_2(\mathrm{w}) - \text{ɧ}_2(\text{ɳ})\right) \\ &= \text{ɧ}_1(\mathrm{w})\text{ɧ}_2(\mathrm{w}) - \text{ɧ}_1(\mathrm{w})\text{ɧ}_2(\text{ɳ}) - \text{ɧ}_1(\text{ɳ})\text{ɧ}_2(\mathrm{w}) - \text{ɧ}_1(\text{ɳ})\text{ɧ}_2(\text{ɳ}).\end{aligned} \tag{5}$$

Now on both sides of (5), we multiply by $\frac{\left(\log \frac{s}{w}\right)^{\frac{\kappa}{\lambda}-1}}{w\Gamma_\lambda(\kappa)}\mathfrak{p}(w)$, where $\mathfrak{p}$ is a function $\mathfrak{p}: [a, b] \to \mathbb{R}^+$, and then integrating the resulting identity from α to v and get

$$\frac{1}{\Gamma_\lambda(\kappa)}\int_a^v \frac{1}{w}\left(\log \frac{s}{w}\right)^{\frac{\kappa}{\lambda}-1} \mathfrak{p}(\mathrm{w})\mathfrak{h}(w, \mathfrak{n})dw$$
$$= \Omega_{a,\lambda}^{\kappa}[\mathfrak{h}_1\mathfrak{h}_2(s)] - \mathfrak{h}_2(\mathfrak{n})\,\Omega_{a,\lambda}^{\kappa}[\mathfrak{p}\mathfrak{h}_1(s)]$$
$$-\mathfrak{h}_1(\mathfrak{n})\Omega_{a,\lambda}^{\kappa}[\mathfrak{p}\mathfrak{h}_2(s)] + \mathfrak{h}_1(\mathfrak{n})\mathfrak{h}_2(\mathfrak{n})\Omega_{a,\lambda}^{\kappa}[\mathfrak{p}(s)]. \quad (6)$$

Now multiplying $\frac{\left(\log \frac{s}{y}\right)^{\frac{\kappa}{\lambda}-1}}{\mathfrak{n}\Gamma_\lambda(\kappa)}\mathfrak{p}(y)$ on both sides of (6) for $\mathfrak{n} \in (\alpha, v)$, and then integrating the resulting identity from α to s w. r. t. '$\mathfrak{n}$' and note that ${\Gamma_\lambda}^2 = \lambda^2\Gamma^2$ yields

$$\frac{1}{{\Gamma_\lambda}^2(\kappa)}\int_a^s\int_a^s \left(\log \frac{s}{w}\right)^{\frac{\kappa}{\lambda}-1}\left(\log \frac{s}{\mathfrak{n}}\right)^{\frac{\kappa}{\lambda}-1} \mathfrak{p}(\mathrm{y})\mathfrak{p}(\mathrm{w})\mathfrak{h}(w, y)\frac{dw}{w}\frac{dy}{y}$$
$$= 2\Omega_{a,\lambda}^{\kappa}[\mathfrak{p}(s)]\Omega_{a,\lambda}^{\kappa}[\mathfrak{p}\mathfrak{h}_1\mathfrak{h}_2(s)] - 2\,\Omega_{a,\lambda}^{\kappa}[\mathfrak{p}\mathfrak{h}_2(s)]\,\mathfrak{h}_2(\mathfrak{n})\Omega_{a,\lambda}^{\kappa}[\mathfrak{p}\mathfrak{h}_1(s)]. \quad (7)$$

Choosing $\mathfrak{p}(\mathrm{s}) = \varphi(\mathrm{s})$ and $\mathfrak{h}_1(\mathrm{s}) = \mathfrak{h}_2(s) = s - E(\gamma)$, $s \in (a, b)$ in (7), we see

$$\frac{1}{{\Gamma_\lambda}^2(\kappa)}\int_a^s\int_a^s \frac{1}{w}\left(\log \frac{s}{w}\right)^{\frac{\kappa}{\lambda}-1}\frac{1}{\mathfrak{n}}\left(\log \frac{s}{\mathfrak{n}}\right)^{\frac{\kappa}{\lambda}-1} \varphi(\mathfrak{n})\varphi(\mathfrak{n})(w - \mathfrak{n})^2 dw\, dy$$
$$= 2\Omega_{a,\lambda}^{\kappa}[\varphi(s)]\Omega_{a,\lambda}^{\kappa}[\varphi(s)(s - E(\gamma)^2)] - 2\Omega_{a,\lambda}^{\kappa}[\varphi(s)[\varphi(s)(s - E(\gamma))]^2. \quad (8)$$

Now we can write

$$\frac{1}{{\Gamma_\lambda}^2(\kappa)}\int_a^v\int_a^v \frac{1}{w}\frac{1}{\mathfrak{n}}\left(\log \frac{s}{w}\right)^{\frac{\kappa}{\lambda}-1}\left(\log \frac{s}{\mathfrak{n}}\right)^{\frac{\kappa}{\lambda}-1} \varphi(\mathfrak{n})\varphi(\mathrm{w})(w - \mathfrak{n})^2 dw\, d\mathfrak{n}$$
$$\leq 2||\varphi||_\infty^2\left[\frac{\left(\log \frac{s}{a}\right)^{\frac{\kappa}{\lambda}-1}}{\Gamma(\kappa + 1)}\,\Omega_{a,\lambda}^{\kappa}[s^2] - \left(\Omega_{a,\lambda}^{\kappa}[s]\right)^2\right]. \quad (9)$$

Thus, we conclude part (i) of the theorem from (8) and (9).

In order to establish part (ii), we have

$$\frac{1}{{\Gamma_\lambda}^2(\kappa)}\int_a^v\int_a^v \frac{1}{w}\frac{1}{\mathfrak{n}}\left(\log \frac{s}{w}\right)^{\frac{\kappa}{\lambda}-1}\left(\log \frac{s}{y}\right)^{\frac{\kappa}{\lambda}-1} \mathrm{g}(\mathfrak{n})\mathrm{g}(\mathrm{w})(w - \mathfrak{n})^2 dw\, d\mathfrak{n}$$
$$\leq \max_{w,\mathfrak{n}\in[a,s]}|w - \mathfrak{n}|^2\left(\Omega_{a,\lambda}^{\kappa}[s]\right)^2, \quad (10)$$

yielding part(ii) of the result from (8) and (10).

Theorem 2.2

Let $\varphi: [a,b] \to \mathbb{R}$ be a positive pdf of a random variable Ɣ, then

(i) The inequalities

$$\Omega^{Ꝺ}_{a,\lambda}[\varphi(s)]\sigma^2_{Ɣ,\mu,\lambda} + \Omega^{Ꝺ}_{a,\lambda}[\varphi(s)]\sigma^2_{Ɣ,Ꝺ,\lambda} - \left[E_{Ɣ-E(Ɣ),Ꝺ,\lambda}(s)\right]\left[E_{Ɣ-E(Ɣ),\mu,\lambda}(s)\right]$$
$$\leq ||g||^2_\infty \left[\frac{\left(log\ \frac{s}{u}\right)^{\frac{Ꝺ}{\lambda}}}{\Gamma(Ꝺ+1)}\Omega^{Ꝺ}_{a,\lambda}[s^2] + \frac{\left(log\ \frac{s}{u}\right)^{\frac{\mu}{\lambda}}}{\Gamma(Ꝺ+1)}\Omega^{Ꝺ}_{a,\lambda}[s^2] - \left(\Omega^{Ꝺ}_{a,\lambda}[s]\right)\left(\Omega^{Ꝺ}_{a,\lambda}[s]\right)\right],$$

holds if $\varphi \in L_\infty[a,b]$ for every $a < v \leq b, Ꝺ, \mu \geq 0$ and

(ii) the inequality

$$\Omega^{Ꝺ}_{a,\lambda}[\varphi(s)]\sigma^2_{Ɣ,\mu,\lambda} + \Omega^{Ꝺ}_{a,\lambda}[\varphi(s)]\sigma^2_{Ɣ,Ꝺ,\lambda} - \left[E_{Ɣ-E(Ɣ),Ꝺ,\lambda}(s)\right]\left[E_{Ɣ-E(Ɣ),\mu,\lambda}(s)\right]$$
$$\leq (s-a)^2\left[\ \Omega^{Ꝺ}_{a,\lambda}[\varphi(s)]\right]\left[\ \Omega^{Ꝺ}_{a,\lambda}[\varphi(s)]\right],$$

holds for $a < s \leq b$ and $\mu \geq 0$.

Proof:

In order to establish part (i), multiply (6) by $\frac{\left(log\,\frac{s}{ɳ}\right)^{\frac{\mu}{\lambda}-1}}{ɳ\lambda\Gamma(Ꝺ)}$ ꝓ(y) with $ɳ \in (\alpha, s)$, and then integrating the resulting identity from a to s w. r. t. 'ɳ' giving

$$\frac{1}{\Gamma_\lambda(Ꝺ)\Gamma_\lambda(Ꝺ)}\int_a^s\int_a^s \frac{1}{w}\frac{1}{ɳ}\left(log\ \frac{s}{w}\right)^{\frac{\kappa}{\lambda}-1}\left(log\ \frac{s}{ɳ}\right)^{\frac{\mu}{\lambda}-1} ꝓ(ɳ)ꝓ(\mathrm{w})ɧ(w,ɳ)dw\ dɳ$$
$$= \Omega^{Ꝺ}_{a,\lambda}[ꝓ(s)]\Omega^{Ꝺ}_{a,\lambda}[ꝓ(\mathrm{s})ɧ_1(s)ɧ_2(s)] + \Omega^{Ꝺ}_{a,\lambda}[ꝓ(s)]\Omega^{\mu}_{a,\lambda}[ꝓ(\mathrm{s})ɧ_1(s)ɧ_2(s)]$$
$$-\Omega^{\mu}_{a,\lambda}[ꝓ(\mathrm{s})ɧ_2(s)]\Omega^{Ꝺ}_{a,\lambda}[ꝓ(\mathrm{s})ɧ_1(s)] - \Omega^{Ꝺ}_{a,\lambda}[ꝓ(s)ɧ_2(s)]\Omega^{Ꝺ}_{a,\lambda}[ꝓ(\mathrm{s})ɧ_1(s)]. \quad (11)$$

Now choosing $ꝓ(\mathrm{s}) = \varphi(\mathrm{s}), ꝓ(\mathrm{w}) = \varphi(\mathrm{w})$ and $ɧ_1(\mathrm{s}) = ɧ_2(s) = s - E(Ɣ),\ s \in (\alpha, \beta)$ in (11) yields

$$\frac{1}{\Gamma_\lambda(Ꝺ)\Gamma_\lambda(\mu)}\int_a^s\int_a^s \frac{1}{w}\frac{1}{ɳ}\left(log\ \frac{s}{w}\right)^{\frac{Ꝺ}{\lambda}-1}\left(log\ \frac{s}{ɳ}\right)^{\frac{\mu}{\lambda}-1} \varphi(ɳ)\mathrm{g}(\mathrm{w})(w-ɳ)^2 dw\ dɳ$$
$$= \Omega^{Ꝺ}_{a,\lambda}[\varphi(s)]\Omega^{Ꝺ}_{a,\lambda}[\varphi(\mathrm{s})(s-E(Ɣ)^2)] + \Omega^{Ꝺ}_{a,\lambda}[\varphi(s)]\Omega^{Ꝺ}_{a,\lambda}[\varphi(\mathrm{s})(s-E(Ɣ)^2)]$$
$$-2\Omega^{Ꝺ}_{a,\lambda}[\varphi(\mathrm{s})(s-E(Ɣ))]\ \Omega^{Ꝺ}_{a,\lambda}[\varphi(\mathrm{s})(s-E(Ɣ))] \quad (12)$$

Now we can write

$$\frac{1}{\Gamma_\lambda(Ꝺ)\Gamma_\lambda(\mu)}\int_a^s\int_a^s \frac{1}{w}\frac{1}{ɳ}\left(log\ \frac{s}{w}\right)^{\frac{Ꝺ}{\lambda}-1}\left(log\ \frac{s}{ɳ}\right)^{\frac{\mu}{\lambda}-1} \varphi(ɳ)\varphi(\mathrm{w})(w-ɳ)^2 dw\ dɳ$$
$$\leq ||\varphi||^2_\infty \frac{1}{\Gamma_\lambda(Ꝺ)\Gamma_\lambda(\mu)}\int_a^s\int_a^s \frac{1}{w}\frac{1}{ɳ}\left(log\ \frac{s}{w}\right)^{\frac{Ꝺ}{\lambda}-1}\left(log\ \frac{s}{ɳ}\right)^{\frac{\mu}{\lambda}-1}(w-ɳ)^2 dw\ dɳ$$

$$\leq ||\varphi||_{\infty}^{2}\left[\frac{\left(log\ \frac{s}{a}\right)^{\frac{פ}{\lambda}-1}}{\Gamma(פ+1)}\Omega_{a,\lambda}^{פ}[s^2]+\frac{\left(log\ \frac{s}{\alpha}\right)^{\frac{\mu}{\lambda}}}{\Gamma(\mu+1)}\Omega_{a,\lambda}^{פ}[s^2]-2\left(\Omega_{a,\lambda}^{פ}[s]\right)\left(\Omega_{a,\lambda}^{פ}[s]\right)\right], \tag{13}$$

establishing part (i) of the result from (12) and (13).

Now in order to establish part (ii), we note $\max_{w,y\in[\alpha,v]}|w-ռ|^2=(w-a)^2$, and

$$\frac{1}{\Gamma_{\lambda}{}^{2}(פ)}\int_{a}^{s}\int_{a}^{s}\frac{1}{w}\frac{1}{s}\left(log\ \frac{s}{w}\right)^{\frac{פ}{\lambda}-1}\left(log\ \frac{s}{ռ}\right)^{\frac{\mu}{\lambda}-1}\varphi(ռ)\varphi(\mathrm{w})(w-ռ)^2dw\ dռ \tag{14}$$
$$\leq |w-a|^2\left(\Omega_{a,\lambda}^{פ}[\varphi(s)]\right)\left(\Omega_{a,\lambda}^{\mu}[\varphi(s)]\right).$$

So that part (ii) of the result is obtained from (12) and (14).

Theorem 2.1:

Let $\varphi:[a,b]\to\mathbb{R}$ be a positive pdf of a random variable $Ɣ$, then for every $פ\geq 0$, we have

$$\sigma_{Ɣ,פ,\lambda}^{2}=E_{Ɣ,פ,\lambda}-2E(Ɣ)E_{Ɣ,פ,\lambda}+E(Ɣ)^2\Omega_{a,\lambda}^{פ}[g(\beta)],$$

Proof:

We have

$$\sigma_{Ɣ,פ,\lambda}^{2}=\frac{1}{\Gamma_{\lambda}(פ)}\int_{a}^{b}\frac{1}{u}\left(log\ \frac{s}{u}\right)^{\frac{פ}{\lambda}-1}(u-E(Ɣ))^2\varphi(u)\ du$$
$$=\frac{1}{\Gamma_{\lambda}(פ)}\int_{a}^{b}\frac{1}{u}\left(log\ \frac{s}{u}\right)^{\frac{פ}{\lambda}-1}(u-2E(Ɣ)+E(Ɣ)^2)\varphi(u)\ du$$
$$=\frac{1}{\Gamma_{\lambda}(פ)}\int_{a}^{b}\frac{1}{u}\left(log\ \frac{s}{u}\right)^{\frac{פ}{\lambda}-1}u^2\ \varphi(u)du-2E(Ɣ).\frac{1}{\Gamma_{\lambda}(פ)}\int_{\alpha}^{\beta}\frac{1}{u}\left(log\ \frac{s}{u}\right)^{\frac{פ}{\lambda}-1}u\ \varphi(u)du$$
$$+E(Ɣ)^2.\frac{1}{\Gamma_{\lambda}(פ)}\int_{a}^{b}\frac{1}{u}\left(log\ \frac{v}{u}\right)^{\frac{פ}{\lambda}-1}\varphi(u)\ du$$
$$=E_{Ɣ,פ,\lambda}-2E(Ɣ)E_{Ɣ,פ,\lambda}+E(Ɣ)^2\Omega_{a,\lambda}^{פ}[\varphi(b)].$$

Definition 2.4:

To extend the factorial to any <u>real number</u> **δ** > 0 (whether **δ** is a whole number), the gamma function is defined as

$$\Gamma(\delta)=\int_{0}^{\infty}e^{-\mu}\ \mu^{\delta-1}\ d\mu$$

$$= \int_0^{\sigma} e^{-\mu}\, \mu^{\delta-1}\, d\mu + \int_{\sigma}^{\infty} e^{-\mu}\, \mu^{\delta-1}\, d\mu$$
$$= \gamma(\delta,\sigma) + \Gamma(\delta,\sigma),$$

where

$$\gamma(\delta,\sigma) = \int_0^{\sigma} e^{-\mu}\, \mu^{\delta-1}\, d\mu \text{ and } \Gamma(\delta,\sigma) = \int_{\sigma}^{\infty} e^{-\mu}\, \mu^{\delta-1}\, d\mu.$$

Now we give some examples of the observed results and we have following definitions in this case:

Definition 2.5:

The beta function, symbolized by $\beta(k,m)$ [15], is defined as

$$\beta(k,m) = \int_0^1 \tau^{k-m}(1-\tau)^{m-1}\, d\tau = \frac{\Gamma(k)\Gamma(m)}{\Gamma(k+m)}$$

and we define $\beta\left(\frac{k}{n},\frac{m}{n}\right) = \frac{1}{n}\beta(k,m)$.

Example 2.6:

Consider the function $\varphi(s) = \left(log\ \frac{s}{\alpha}\right)^{\frac{\mu}{\lambda}-1}$, we see

$$\Omega_{\alpha,\lambda}^{\text{פ}}\left(log\ \frac{s}{\alpha}\right)^{\frac{\eta}{\lambda}-1} = \frac{1}{\Gamma_\lambda(\text{פ})}\int_\alpha^v \frac{1}{w}\left(log\,\frac{w}{s}\right)^{\frac{\kappa}{\lambda}-1}\left(log\ \frac{s}{\alpha}\right)^{\frac{\eta}{\lambda}-1}\ ds.$$

Choosing $\tau = \frac{log\left(\frac{w}{s}\right)}{log\left(\frac{s}{\alpha}\right)}$ for $\tau \in (\alpha,\beta]$ with $\eta,\kappa > 0$, we see

$$\Omega_1^{\kappa,\lambda}\left(log\ \frac{s}{\alpha}\right)^{\frac{\eta}{\lambda}-1} = \left(log\ \frac{v}{\alpha}\right)^{\frac{\kappa+\eta}{\lambda}-1}\frac{1}{\Gamma_\lambda(\text{פ})}\int_0^1 (1-\tau)^{\frac{\eta}{\lambda}-1}(1-\tau)^{\frac{\text{פ}}{\lambda}-1}\, d\tau$$
$$= \left(log\ \frac{v}{\alpha}\right)^{\frac{\text{פ}+\eta}{\lambda}-1}\frac{\beta\left(\frac{\text{פ}}{\lambda},\frac{\eta}{\lambda}\right)}{\Gamma_\lambda(\text{פ})}$$
$$= \left(log\ \frac{v}{\alpha}\right)^{\frac{\text{פ}+\eta}{\lambda}-1}\frac{\beta(\text{פ},\eta)}{\Gamma(\text{פ})}.$$

Example 2.7:

Consider the function $\varphi(s) = s^c$ for $c \in R - \{0\}$, then, we see

$$\Omega_{\alpha,\lambda}^{\text{פ}}[s^c] = \frac{1}{\Gamma_\lambda(\text{פ})}\int_\alpha^t \frac{1}{s}\left(log\ \frac{t}{s}\right)^{\frac{\text{פ}}{\lambda}-1}.\, s^c\ ds = \frac{1}{\Gamma_\lambda(\text{פ})}\int_\alpha^t \left(log\ \frac{t}{s}\right)^{\frac{\text{פ}}{\lambda}-1} s^{c-1} ds.$$

Choosing $z = log\ \left(\frac{t}{s}\right)$ yielding $t = se^{-z}$ and hence

$$\Omega_{\alpha,\lambda}^{פ}[s^{c}] = \frac{1}{\Gamma_{\lambda}(פ)} \int_{0}^{log\left(\frac{t}{\alpha}\right)} e^{-cz}\, y^{\frac{פ}{\lambda}-1} dz.$$

Again, choose $p = cz$ yielding $dz = \frac{1}{c} dp$ and hence by Definition 2.4, we see

$$\Omega_{\alpha,\lambda}^{פ}[s^{c}] = \frac{t^{c} c^{\frac{פ}{\lambda}}}{\Gamma_{\lambda}(פ)} \int_{0}^{c\, log\left(\frac{t}{\alpha}\right)} e^{-p}\, p^{\frac{פ}{\lambda}-1} dp$$

$$= \frac{t^{c} c^{\frac{פ}{\lambda}}}{\Gamma_{\lambda}(פ)} \gamma\left(\frac{פ}{\lambda}, c\, log\left(\frac{t}{\alpha}\right)\right).$$

Example 2.8:

Consider the function $\varphi(s) = s$, then, we see using Example 2.7 by choosing $c = 1$, that

$$\Omega_{\alpha,\lambda}^{פ}[s] = \frac{1}{\Gamma_{\lambda}(פ)} \gamma\left(\frac{פ}{\lambda}, log\left(\frac{t}{\alpha}\right)\right).$$

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