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THE EXTENDED FAILURE RATE FAMILY: PROPERTIES AND APPLICATIONS IN THE ENGINEERING AND INSURANCE FIELDS

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ABSTRACT

This paper presents, a new family of distributions, named the generalized linear failure rate family, generated from a generalized linear failure rate random variable is introduced. The new family contains five well-known families as special cases. Four particular distributions stemmed from the new family are explored. Mathematical properties of interest of the new family are derived. The maximum likelihood method is used to estimate the model parameters and its behavior is explored via detailed simulations. Finally, the importance of the new family is illustrated through two applications extracted from practice.

KEYWORDS

Entropy; generalized linear-failure rate distribution; insurance; maximum likelihood estimation; order statistics; stochastic orderings.

1. INTRODUCTION

Analysis of lifetime data is an important subject in many areas for example, reliability, social sciences, biomedical, and engineering, among others, proving the clear need for flexible forms of classical distributions to provide better fits for several real-life data. As a result, many new distributions have been introduced and studied in literature. As well as, there is a good deal of work to generate new families of distributions from existing distributions by adding one or more additional parameter(s) to the baseline distribution, to study the shape behavior of the density and hazard rate functions, and to improve the goodness-of-fit of generated distributions.

Some well-known generators are Marshall-Olkin-G by Marshall and Olkin (1997), the exponentiated-G by Gupta et al. (1998), beta-G by Eugene et al. (2002), Kumaraswamy-G by Cordeiro and de Castro (2011), Weibull-G by Bourguignon et al. (2014), Kumaraswamy-transmuted-G by Afify et al. (2016), Marshall-Olkin alpha-power-G by

Nassar et al. (2019), and odd Dagu m-G by Afify and Alizadeh (2020). Sarhan and Kundu (2009) introduced the three-parameter generalized linear failure rate (GLFR) distribution whose cumulative distribution function (cdf) and probability density function (pdf) are respectively, given by

$$H(t) = \left[1 - e^{-\left(\alpha t + \frac{\beta}{2} t^2\right)}\right]^{\theta}, t \geq 0, \alpha, \beta, \theta > 0 \tag{1}$$

and

$$h(t) = \theta(\alpha + \beta t) e^{-\left(\alpha t + \frac{\beta}{2} t^2\right)} \left[1 - e^{-\left(\alpha t + \frac{\beta}{2} t^2\right)}\right]^{\theta - 1}. \tag{2}$$

It is clear that if θ is an integer, then the GLFR cdf (1) reduces to the cdf of the maximum of a simple random sample of size θ from the linear failure rate (LFR) distribution. That is, when θ is an integer, the GLFR distribution represents the cdf of a parallel system when each component has the LFR distribution. The GLFR distribution is flexible in accommodating all important forms of the hazard rate function (hrf), and it can be used it in a variety of problems to model lifetime data.

The main aim of this article is to propose a new family of distributions based on the GLFR random variable (rv), called generalized linear failure rate-G (GLFR-G) family, to obtain more flexible models compared to the baseline models. The GLFR-G family contains some special sub-families are listed in Table 1. We note this is so since one of the most important features of this new family is the description of the parallel system when θ is an integer, where the GLFR-G family provides the cdf of a parallel system whose components have the linear failure rate-G (LFR-G) family.

Furthermore, the GLFR-G family has some important properties such as, its special models exhibit right skewed, symmetrical, left skewed, concave down, J-shaped or reversed-J shaped densities and increasing, decreasing, unimodal, bathtub reversed-J and J-shaped failure rates. The special models of the GLFR-G class can be used to real data in different applied areas effectively than other competing models.

The T-X family introduced in Alzaatreh et al. (2013) has the cdf

$$F(x) = \int_a^{W[G(x)]} r(t) dt, \tag{3}$$

where $W[G(x)]$ satisfies some conditions (see, Alzaatreh et al., 2013). The pdf corresponding to (3) admits the representation

$$f(x) = \left\{\frac{d}{dx} W[G(x)]\right\} r(W[G(x)]). \tag{4}$$

The new GLFR-G family is proposed by inserting the pdf (2) in the cdf (3) and replacing $W[G(x)]$ by $G(x)/[1 - G(x)]$, where $G(x)$ is a baseline cdf and $\bar{G}(x) = 1 - G(x)$ is the corresponding survival function (sf).

The cdf of the new GLFR-G family is given (for $x \in \mathbb{R}$) by

$$F(x;\xi) = \int_0^{\frac{G(x;\Phi)}{1-G(x;\Phi)}} \theta(\alpha+\beta t)e^{-\alpha t-\frac{\beta}{2}t^2}\left[1-e^{-\left(\alpha t+\frac{\beta}{2}t^2\right)}\right]^{\theta-1} dt = \left\{1-e^{-\left[\alpha\left(\frac{G(x;\Phi)}{1-G(x;\Phi)}\right)+\frac{\beta}{2}\left(\frac{G(x;\Phi)}{1-G(x;\Phi)}\right)^2\right]}\right\}^{\theta}, \quad (5)$$

where $\xi = (\alpha, \beta, \theta, \Phi^T)^T$, $\alpha > 0$ and $\beta > 0$ are scale parameters, and $\theta > 0$ is a shape parameter. The corresponding pdf is

$$f(x;\xi) = \frac{\alpha\theta g(x;\Phi)+\theta(\beta-\alpha)g(x;\Phi)G(x;\Phi)}{[1-G(x;\Phi)]^3} e^{-\left[\alpha\left(\frac{G(x;\Phi)}{1-G(x;\Phi)}\right)+\frac{\beta}{2}\left(\frac{G(x;\Phi)}{1-G(x;\Phi)}\right)^2\right]} \left\{1-e^{-\left[\alpha\left(\frac{G(x;\Phi)}{1-G(x;\Phi)}\right)+\frac{\beta}{2}\left(\frac{G(x;\Phi)}{1-G(x;\Phi)}\right)^2\right]}\right\}^{\theta-1}. \quad (6)$$

The special sub-families of the GLFR-G family are listed in Table 1 can be specified for selected parametric values.

The hrf and reverse hazard rate function (rhrf) of the GLFR-G family take the forms

$$(x;\xi) = \frac{\frac{\alpha\theta g(x;\Phi)+\theta(\beta-\alpha)g(x;\Phi)G(x;\Phi)}{[1-G(x;\Phi)]^3} e^{-\left[\alpha\left(\frac{G(x;\Phi)}{1-G(x;\Phi)}\right)+\frac{\beta}{2}\left(\frac{G(x;\Phi)}{1-G(x;\Phi)}\right)^2\right]}}{\left\{1-e^{-\left[\alpha\left(\frac{G(x;\Phi)}{1-G(x;\Phi)}\right)+\frac{\beta}{2}\left(\frac{G(x;\Phi)}{1-G(x;\Phi)}\right)^2\right]}\right\}^{1-\theta} - \left\{1-e^{-\left[\alpha\left(\frac{G(x;\Phi)}{1-G(x;\Phi)}\right)+\frac{\beta}{2}\left(\frac{G(x;\Phi)}{1-G(x;\Phi)}\right)^2\right]}\right\}} \quad (7)$$

and

$$\tau(x;\xi) = \frac{\alpha\theta g(x;\Phi)+\theta(\beta-\alpha)g(x;\Phi)G(x;\Phi)}{[1-G(x;\Phi)]^3\left\{e^{\alpha\left(\frac{G(x;\Phi)}{1-G(x;\Phi)}\right)+\frac{\beta}{2}\left(\frac{G(x;\Phi)}{1-G(x;\Phi)}\right)^2}-1\right\}}.$$

The cumulative hazard rate function (chrf) of the GLFR-G family reduces to

$$H(x;\xi) = -\log\left(1-\left\{1-e^{-\left[\alpha\left(\frac{G(x;\Phi)}{1-G(x;\Phi)}\right)+\frac{\beta}{2}\left(\frac{G(x;\Phi)}{1-G(x;\Phi)}\right)^2\right]}\right\}^{\theta}\right).$$

It is clear that the GLFR-G family satisfies the proportional reversed hazard rate condition with proportionality constant $\theta > 0$ (see Gupta and Gupta, 2007).

Table 1
Special Sub-Families of the GLFR-G Family

α	β	θ	**Reduced Family**	**Authors**
-	-	1	Linear failure rate-G	New
-	0	1	Exponential-G	Bourguignon et al. (2014)
0	-	1	Rayleigh-G	Bourguignon et al. (2014)
-	0	-	Odd generalized exponential-G	Tahir et al. (2015)
0	-	-	Generalized Rayleigh-G	New

The rest of this paper is structured as follows. Particular models of our family are presented, and the plots of their densities and hazard rate functions are displayed in Section 2. Shapes of the density and hazard functions are provided in Section 3. Section 4 contains fundamental properties of the GLFR-G family. The method of maximum likelihood is used to estimate the parameters in Section 5. Section 6 presents two real-life data sets to demonstrate the usefulness of the proposed family. Concluding remarks are summarized in Section 7.

2. PARTICULAR MODELS OF THE GLFR-G FAMILY

Four particular models of GLFR-G family and display of their density and hazard rate functions plots are shown. We conclude that the special models of the GLFR-G family exhibit right skewed, symmetrical, left skewed, concave down, J-shaped or reversed-J shaped densities and increasing, decreasing, unimodal, bathtub reversed-J and J-shaped failure rates.

2.1 The GLFR-Exponential Distribution

Consider the exponential model with pdf $g(x;a) = ae^{-ax}$, $a > 0$. Hence, the GLFR-Exponential (GLFR-E) distribution has the pdf

$$f(x;\xi) = \theta a e^{ax}[\alpha + \beta(e^{ax} - 1)]e^{-\left[\alpha\,(e^{ax}-1)+\frac{\beta}{2}(e^{ax}-1)^2\right]} \times \left\{1 - e^{-\left[\alpha\,(e^{ax}-1)+\frac{\beta}{2}(e^{ax}-1)^2\right]}\right\}^{\theta-1}, x > 0,$$

where $\xi = (\alpha, \beta, \theta, a)^T$.

Plots of the density, hazard functions, skewness and kurtosis of the GLFR-E distribution for some parametric values are displayed in Figures 1-4, respectively.

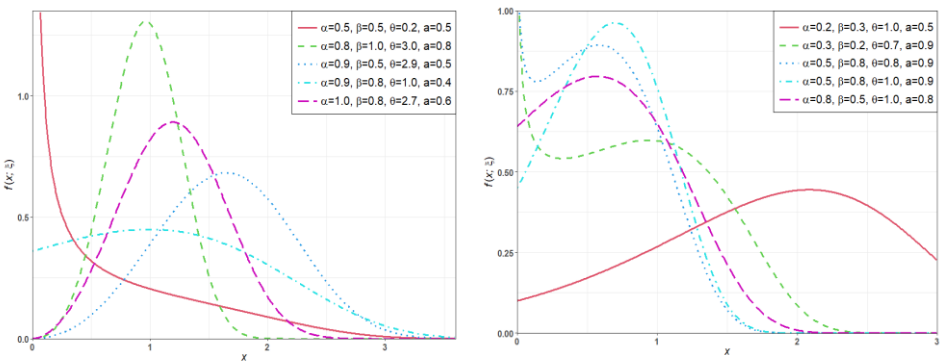

Figure 1: Plots of the GLFR-E pdf for Some Parametric Values

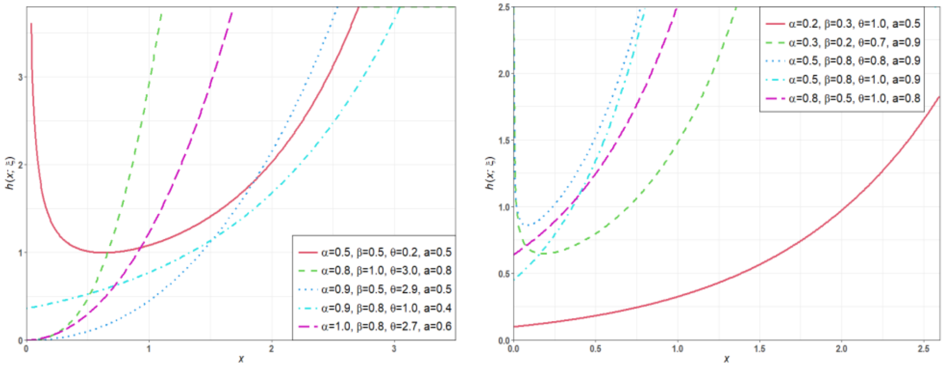

Figure 2: Plots of the GLFR-E hrf for Some Parametric Values

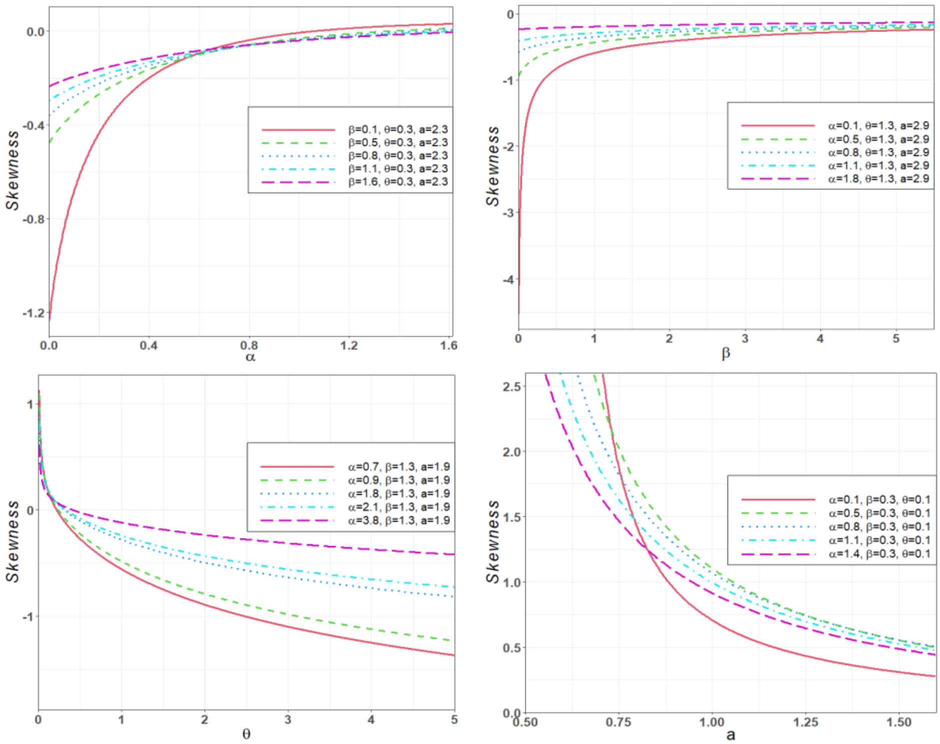

Figure 3: Plots of the GLFR-E skewness for some parametric values

2.2 The GLFR-Lomax Distribution

Using the pdf $g(x;a,b) = \left(\frac{a}{b}\right)\left[1+\left(\frac{x}{b}\right)\right]^{-a-1}, a,b > 0$,of the Lomax distribution, we can write the pdf of GLFR-Lomax (GLFR-Lx) distribution as

$$f(x;\xi) = \theta\left(\frac{a}{b}\right)\left[1+\left(\frac{x}{b}\right)\right]^{a-1}\left(\alpha + \beta\left\{\left[1+\left(\frac{x}{b}\right)\right]^{a}-1\right\}\right)e^{-\left(\alpha\left\{\left[1+\left(\frac{x}{b}\right)\right]^{a}-1\right\}+\frac{\beta}{2}\left\{\left[1+\left(\frac{x}{b}\right)\right]^{a}-1\right\}^{2}\right)} \times\left(1-e^{-\left(\alpha\left\{\left[1+\left(\frac{x}{b}\right)\right]^{a}-1\right\}+\frac{\beta}{2}\left\{\left[1+\left(\frac{x}{b}\right)\right]^{a}-1\right\}^{2}\right)}\right)^{\theta-1}, \qquad x>0,$$

where $\xi = (\alpha, \beta, \theta, a, b)^T$.

Plots of the GLFR-Lx pdf and hrf for selected parametric values are displayed in Figures 5 and 6, respectively.

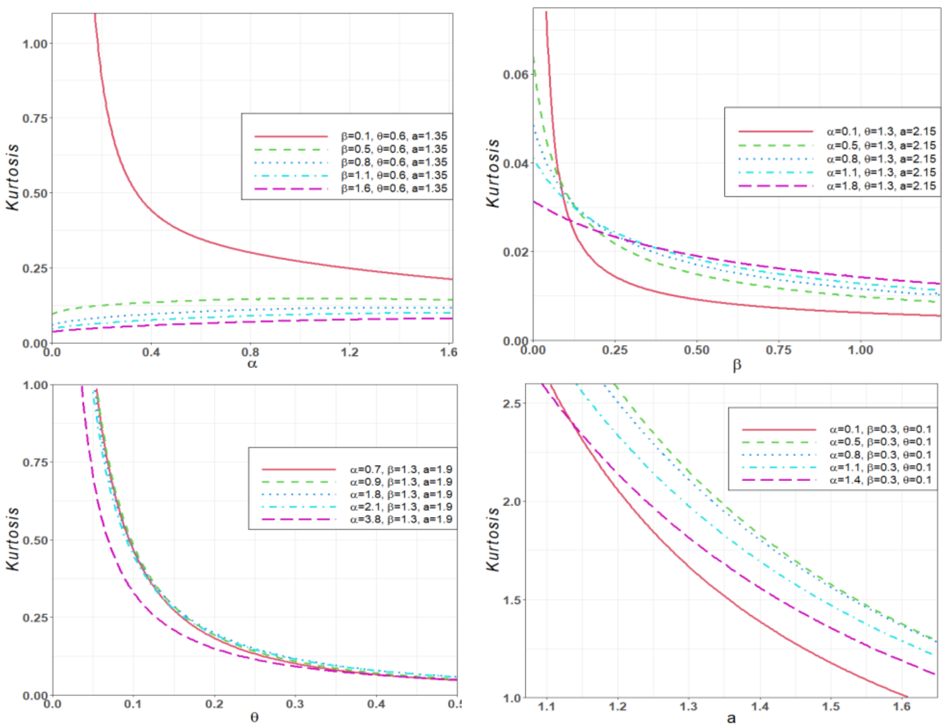

Figure 4: Plots of the GLFR-E Kurtosis for Some Parametric Values

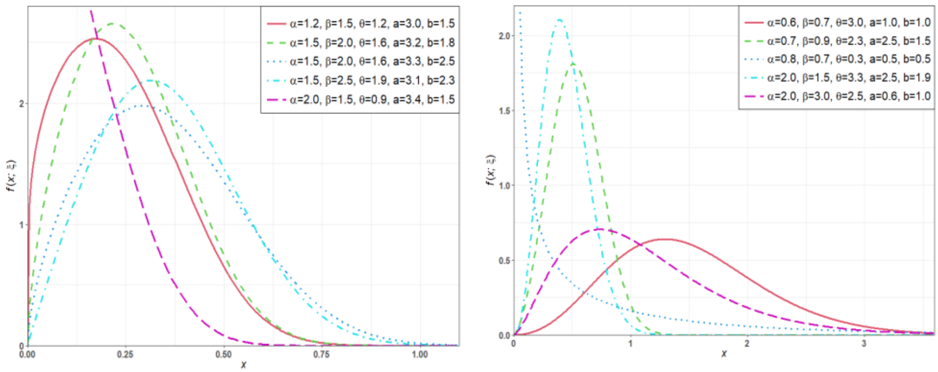

Figure5: Density Plots of the GLFR-Lx Distribution for Some Parametric Values

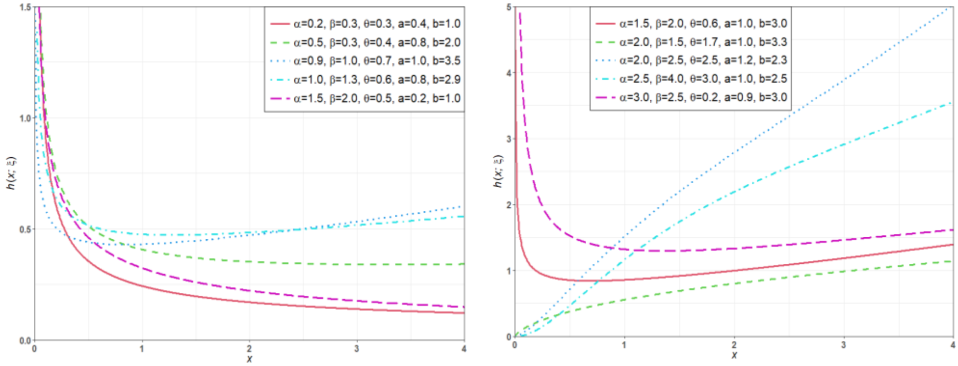

Figure 6: Hazard Rate Plots of the GLFR-Lx Distribution for Some Parametric Values

2.3 The GLFR-Fréchet Distribution

Consider the Fréchet distribution with pdf, $g(x; a, b) = ab^a x^{-(a+1)} e^{-\left(\frac{b}{x}\right)^a}, a, b > 0$, we obtain the pdf of the GLFR-Fréchet (GLFR-Fr) distribution

$$f(x;\xi) = \theta a \left(\frac{b}{x}\right)^a \left(\frac{1}{x}\right) e^{\left(\frac{b}{x}\right)^a} \left(e^{\left(\frac{b}{x}\right)^a} - 1\right)^{-3} \left[\alpha\left(e^{\left(\frac{b}{x}\right)^a} - 1\right) + \beta\right]$$
$$\times e^{-\left[\alpha\left(e^{\left(\frac{b}{x}\right)^a} - 1\right)^{-1} + \frac{\beta}{2}\left(e^{\left(\frac{b}{x}\right)^a} - 1\right)^{-2}\right]}$$
$$\times \left\{1 - e^{-\left[\alpha\left(e^{\left(\frac{b}{x}\right)^a} - 1\right)^{-1} + \frac{\beta}{2}\left(e^{\left(\frac{b}{x}\right)^a} - 1\right)^{-2}\right]}\right\}^{\theta-1}, x > 0,$$

where $\xi = (\alpha, \beta, \theta, a, b)^T$.

If $\beta = 0$, the GLFR-Fr model reduces to the odd generalized exponential Fréchet distribution (Tahir et al., 2015). Plots of the density and hazard functions of the GLFR-Fr distribution are depicted in Figures 7 and 8.

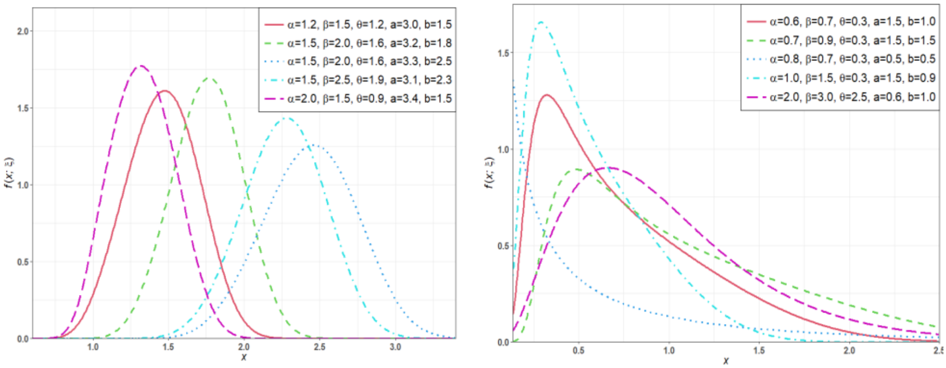

Figure 7: Plots of the GLFR-Fr Density for Some Parametric Values

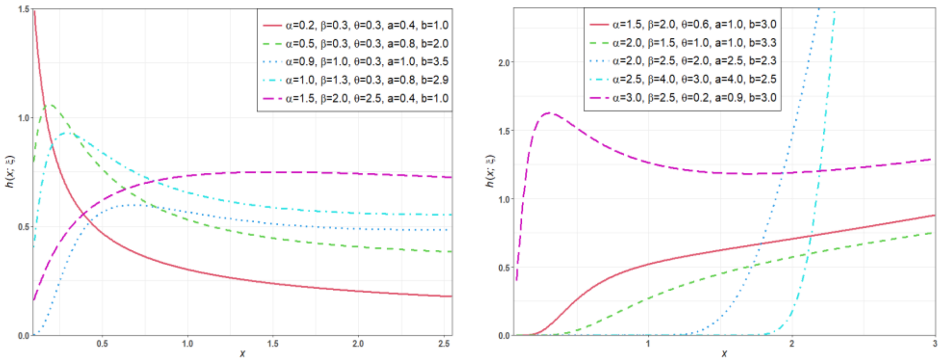

Figure 8: Plots of the GLFR-Fr hrf for Some Parametric Values

2.4 The GLFR-Weibull Distribution

Consider the Weibull distribution with density $g(x; a, b) = \left(\frac{a}{b}\right)\left(\frac{x}{b}\right)^{a-1} e^{-\left(\frac{x}{b}\right)^{a}}, a, b > 0$. Then, the GLFR-Weibull (GLFR-W) distribution is specified by the pdf

$$f(x;\xi) = \theta\left(\frac{a}{b}\right)\left(\frac{x}{b}\right)^{a-1} e^{\left(\frac{x}{b}\right)^{a}}\left\{\alpha+\beta\left[e^{\left(\frac{x}{b}\right)^{a}}-1\right]\right\} \times e^{-\left\{\alpha\left[e^{\left(\frac{x}{b}\right)^{a}}-1\right]+\frac{\beta}{2}\left[e^{\left(\frac{x}{b}\right)^{a}}-1\right]^{2}\right\}}$$
$$\times\left(1-e^{-\left\{\alpha\left[e^{\left(\frac{x}{b}\right)^{a}}-1\right]+\frac{\beta}{2}\left[e^{\left(\frac{x}{b}\right)^{a}}-1\right]^{2}\right\}}\right)^{\theta-1}, x > 0,$$

where $\xi = (\alpha, \beta, \theta, a, b)^{T}$.

The odd generalized exponential Weibull distribution (Tahir et al., 2015) follows as a special case from the GLFR-W model with $\beta = 0$. Figures 9 and 10 display some shapes of the density and hazard rate functions of the GLFR-W distribution.

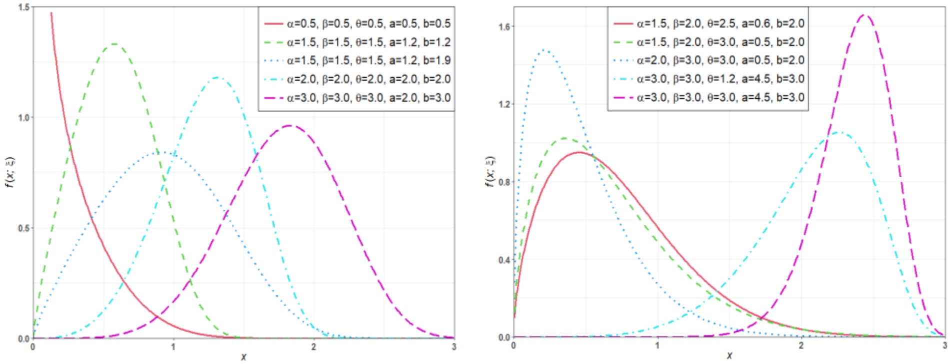

Figure 9: Plots of the GLFR-W pdf for Some Parametric Values

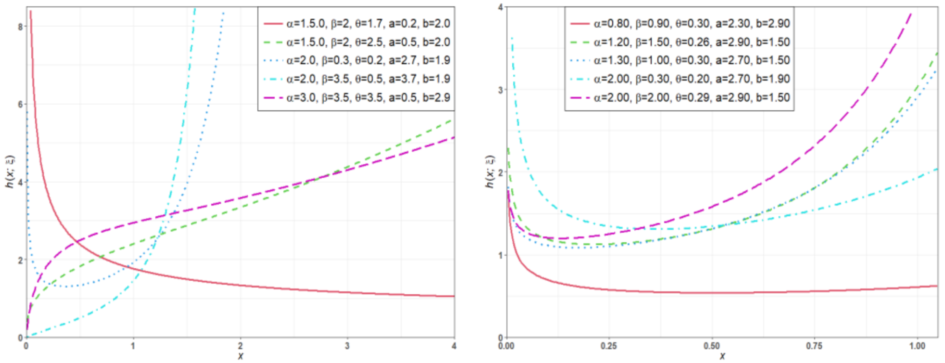

Figure 10: Plots of the GLFR-W hrf for Some Parametric Values

3. SHAPES

The shapes of the pdf and hrf of the GLFR-G family can be described analytically. The critical points of the GLFR-G pdf are the roots of the equation

$$\frac{\theta\alpha g'(x;\Phi)\ +\theta(\beta-\alpha)g'(x;\Phi)G(x;\Phi)\ +\theta(\beta-\alpha)g(x;\Phi)^2}{\theta\alpha g(x;\Phi)+\theta(\beta-\alpha)g(x;\Phi)G(x;\Phi)}$$
$$+\frac{3g(x;\Phi)}{\bar{G}(x;\Phi)}-\frac{\alpha g(x;\Phi)}{\bar{G}(x;\Phi)^2}-\frac{\beta g(x;\Phi)G(x;\Phi)}{\bar{G}(x;\Phi)^3}$$
$$+\frac{(\theta-1)[\alpha g(x;\Phi)\bar{G}(x;\Phi)+\beta g(x;\Phi)G(x;\Phi)]e^{-\left[\alpha\left(\frac{G(x;\Phi)}{1-G(x;\Phi)}\right)+\frac{\beta}{2}\left(\frac{G(x;\Phi)}{1-G(x;\Phi)}\right)^2\right]}}{\bar{G}(x;\Phi)^3\left\{1-e^{-\left[\alpha\left(\frac{G(x;\Phi)}{1-G(x;\Phi)}\right)+\frac{\beta}{2}\left(\frac{G(x;\Phi)}{1-G(x;\Phi)}\right)^2\right]}\right\}}=0.$$

The critical points of $h(x;\xi)$ are obtained from the following equation

$$\frac{\theta\alpha g'(x;\Phi)\ +\theta(\beta-\alpha)g'(x;\Phi)G(x;\Phi)\ +\theta(\beta-\alpha)g(x;\Phi)^2}{\theta\alpha g(x;\Phi)+\theta(\beta-\alpha)g(x;\Phi)G(x;\Phi)}$$
$$-\frac{\alpha g(x;\Phi)}{\bar{G}(x;\Phi)^2}-\frac{\beta g(x;\Phi)G(x;\Phi)}{\bar{G}(x;\Phi)^3}+\frac{3g(x;\Phi)}{\bar{G}(x;\Phi)}$$
$$+\frac{(\theta-1)[\alpha g(x;\Phi)\bar{G}(x;\Phi)+\beta g(x;\Phi)G(x;\Phi)]e^{-\left[\alpha\left(\frac{G(x;\Phi)}{1-G(x;\Phi)}\right)+\frac{\beta}{2}\left(\frac{G(x;\Phi)}{1-G(x;\Phi)}\right)^2\right]}}{\bar{G}(x;\Phi)^3\left\{1-e^{-\left[\alpha\left(\frac{G(x;\Phi)}{1-G(x;\Phi)}\right)+\frac{\beta}{2}\left(\frac{G(x;\Phi)}{1-G(x;\Phi)}\right)^2\right]}\right\}}$$
$$+\frac{\theta[\alpha g(x;\Phi)\bar{G}(x;\Phi)+\beta g(x;\Phi)G(x;\Phi)]e^{-\left[\alpha\left(\frac{G(x;\Phi)}{1-G(x;\Phi)}\right)+\frac{\beta}{2}\left(\frac{G(x;\Phi)}{1-G(x;\Phi)}\right)^2\right]}\left\{1-e^{-\left[\alpha\left(\frac{G(x;\Phi)}{1-G(x;\Phi)}\right)+\frac{\beta}{2}\left(\frac{G(x;\Phi)}{1-G(x;\Phi)}\right)^2\right]}\right\}^{\theta-1}}{\bar{G}(x;\Phi)^3\left(1-\left\{1-e^{-\left[\alpha\left(\frac{G(x;\Phi)}{1-G(x;\Phi)}\right)+\frac{\beta}{2}\left(\frac{G(x;\Phi)}{1-G(x;\Phi)}\right)^2\right]}\right\}^{\theta}\right)}$$
$$=0.$$

Using most symbolic computation software platforms, we can examine the above equations to determine the local maximums and minimums and inflexion points.

4. SOME ANALYTICAL CHARACTERISTICS

This section provides basic analytic characteristics of the GLFR-G family, such as expansions for its pdf and cdf, quantile function (qf), full and incomplete moments, moment generating function (mgf), Shannon entropy, order statistics, and stochastic orderings.

4.1 Useful Expansions

Again, binomial expansions allow us to write

$$(1-z)^n=\sum_{i-0}^{\infty}(-1)^i\binom{n}{i}z^i,$$

where n any real non-integer number and $|z|<1$, we can write the CDF (5) as

$$F(x;\xi)=\sum_{i=0}^{\infty}(-1)^i\binom{\theta}{i}e^{-i\left[\alpha\left(\frac{G(x;\Phi)}{1-G(x;\Phi)}\right)+\frac{\beta}{2}\left(\frac{G(x;\Phi)}{1-G(x;\Phi)}\right)^2\right]},\tag{8}$$

where $\xi=(\alpha,\beta,\theta,\Phi^T)^T$.

Now, by using the exponential series, we obtain

$$e^{-i\alpha\left(\frac{G(x;\Phi)}{1-G(x;\Phi)}\right)-i\frac{\beta}{2}\left(\frac{G(x;\Phi)}{1-G(x;\Phi)}\right)^2}=\sum_{j=0}^{\infty}\frac{\left\{-i\alpha\left(\frac{G(x;\Phi)}{1-G(x;\Phi)}\right)-i\frac{\beta}{2}\left(\frac{G(x;\Phi)}{1-G(x;\Phi)}\right)^2\right\}^j}{j!}$$

$$=\sum_{j=0}^{\infty}\frac{(-i\alpha)^j}{j!}\left(\frac{G(x;\Phi)}{1-G(x;\Phi)}\right)^j\left[1+\frac{\beta}{2\alpha}\left(\frac{G(x;\Phi)}{1-G(x;\Phi)}\right)\right]^j. \qquad (9)$$

Hence, one gets

$$\left[1+\frac{\beta}{2\alpha}\left(\frac{G(x;\Phi)}{1-G(x;\Phi)}\right)\right]^j=\sum_{k=0}^{j}\binom{j}{k}\left(\frac{\beta}{2\alpha}\right)^k\left(\frac{G(x;\Phi)}{1-G(x;\Phi)}\right)^k. \qquad (10)$$

By inserting the expansions (9) and (11) in (8), we have

$$F(x;\xi)=\sum_{i,j=0}^{\infty}\sum_{k=0}^{j}\frac{(-1)^{i+j}(i\alpha)^j\left(\frac{\beta}{2\alpha}\right)^k\binom{\theta}{i}\binom{j}{k}G(x;\Phi)^{j+k}\left(1-G(x;\Phi)\right)^{-j-k}}{j!}. \qquad (11)$$

Furthermore, one can write

$$[1-G(x;\Phi)]^{-j-k}=\sum_{l=0}^{\infty}(-1)^l\binom{-j-k}{l}G(x;\Phi)^l. \qquad (12)$$

By inserting (12) in (11), we can rewrite the cumulative distribution function of the GLFR-G family as an infinite linear combination of exponentiated-G as follows:

$$F(x;\xi)=\sum_{i,j,l=0}^{\infty}\sum_{k=0}^{j}\omega_{i,j,k,l}G(x;\Phi)^{j+k+l}, \qquad (13)$$

where $\omega_{i,j,k,l}=\frac{(-1)^{i+j+l}(i\alpha)^j\beta^k\binom{\theta}{i}\binom{j}{k}\binom{-j-k}{l}}{j!(2\alpha)^k}$.

Now, we can write the GLFR-G family density as a mixture of exp-G densities in the following form

$$f(x;\xi)=\sum_{i,j,l=0}^{\infty}\sum_{k=0}^{j}\omega_{i,j,k,l}h_{J+k+l}(x;\Phi), \qquad (14)$$

where $h_{J+k+l}(x;\Phi)=(j+k+l)g(x;\Phi)G(x;\Phi)^{j+k+l-1}$ is the exp-G pdf where the power parameter is $(j+k+l)$. From (14) some mathematical properties of the GLFR-G family can be derived from those of the exp-G family.

4.2 Quantile and Generating Functions

The GLFR-G family can be directly simulated by inverting $F(x)=u$, where $F(x)$ is the cdf (5) of the GLFR-G family and $u$has a uniform distribution $U(0,1)$, hence the qf of the GLFR-G family has the form

$$x_u = Q(u) = G^{-1}\left[\frac{-\alpha + \sqrt{\alpha^2 - 2\beta \log\left(1 - u^{\frac{1}{\theta}}\right)}}{\beta - \alpha + \sqrt{\alpha^2 - 2\beta \log\left(1 - u^{\frac{1}{\theta}}\right)}}\right].$$

In particular, if $u = 1/2$, we obtain of the median of the GLFR-G family.

Based on Equation (14), we can determine the mgf of X as

$$M_X(t) = E(e^{tX}) = \sum_{i,j,l=0}^{\infty} \sum_{k=0}^{j} \omega_{i,j,k,l} \; M_{(j+k+l)}(t),$$

where $M_{(j+k+l)}(t)$ is the mgf of the $rv\, Z_{(j+k+l)}$ which have the exp-G density function $h_{J+k+l}(x)$.

4.3 Moments

Let $Z_{(j+k+l)}$ be a rv having the exp-G density function $h_{J+k+l}(x;\Phi)$. The r^{th} moment of X~GLFR-G(ξ) family, where $\xi = (\alpha, \beta, \theta, \Phi^T)^T$, is given by

$$\mu_r' = E(X^r) = \sum_{i,j,l=0}^{\infty} \sum_{k=0}^{j} \omega_{i,j,k,l} \, E\left(Z_{(j+k+l)}^r\right).$$

Several statistical characteristics have been discussed. The likelihood ordering is used to do some stochastic comparisons among members of the family. Many of the important features and characteristics of a distribution can be determined using ordinary moments. For example, the variance, skewness, and kurtosis measures which are given by

$$Var(X) = E(X^2) - [E(X)]^2,$$

$$Skewness\ (X) = \frac{E(X^3) - 3E(X)E(X^2) + 2[E(X)]^3}{(E(X^2) - [E(X)]^2)^{\frac{3}{2}}},$$

$$Kurtosis\ (X) = \frac{E(X^4) - 4E(X)E(X^3) + 6E(X^2)[E(X)]^2 - 3[E(X)]^4}{(E(X^2) - [E(X)]^2)^2}.$$

The s^{th} incomplete moment of X is given by

$$\eta_s(y) = \sum_{i,j,l=0}^{\infty} \sum_{k=0}^{j} \omega_{i,j,k,l} I_{(j+k+l)}(y),$$

where $I_{(j+k+l)}(y) = \int_{-\infty}^{y} x^s h_{J+k+l}(x)$.

4.4 Shannon Entropy

The entropy of a rv X a measure of variation of theuncertainty and it has been used in various situations in physics, engineering, and economics. The larger the value of entropy the greater the uncertainty in the data. The Shannon entropy (SE) is one of the most important measures of entropy and stands as an alternative to the kurtosis measure in comparing the shapes of various densities and measuring heaviness of tails. The next Theorem give the general form of the SE of the GLFR-G family.

Theorem 1:

If the rv T has a pdf $r(t)$ and the rv X has GLFR-G (ξ) family, then the rv X has a SE given by

$$I_S = -E\left(\log\left\{g\left[G^{-1}\left(\frac{T}{T+1}\right)\right]\right\}\right) - 2E\left[\log\left(\frac{1}{T+1}\right)\right] + \eta_T,$$

where $\eta_T = -E(\log[r(T)])$ is the SE of the rv T with density $r(t)$.

Proof:

To prove this Theorem, we use a similar argument to that in Alzaatreh et al. (2013). Note that the rv X has a SE defined by

$$I_S = E\{-\log[f(X)]\}. \quad (15)$$

From (4), we can write

$$f(x;\xi) = \left\{\frac{d}{dx}W[G(x;\Phi)]\right\}r\{W[G(x;\Phi)]\} = \frac{g(x;\Phi)}{[1-G(x;\Phi)]^2}\,r\left(\frac{G(x;\Phi)}{1-G(x;\Phi)}\right),$$

where $\xi = (\alpha,\beta,\theta,\Phi^T)^T$.

Therefore,

$$\log[f(x;\xi)] = \log[g(x;\Phi)] - 2\log[1-G(x;\Phi)] + \log\left[r\left(\frac{G(x;\Phi)}{1-G(x;\Phi)}\right)\right]. \quad (16)$$

From (15) and (16), the SE of X with density $f(x)$ takes the form

$$I_S = -E\{\log[g(X)]\} + 2E\{\log[1-G(X)]\} - E\left\{\log\left[r\left(\frac{G(X)}{1-G(X)}\right)\right]\right\}. \quad (17)$$

Using $T = \frac{G(X)}{1-G(X)}$, $X = G^{-1}\left(\frac{T}{T+1}\right)$, thus we can rewrite (17) as

$$I_S = -E\left(log\left\{g\left[G^{-1}\left(\frac{T}{T+1}\right)\right]\right\}\right) - 2E\left[log\left(\frac{1}{T+1}\right)\right] - E\{log[r(T)]\}.$$

Thus, the SE of the GLFR-G family takes the form

$$I_S = -E\left(\log\left\{g\left[G^{-1}\left(\frac{T}{T+1}\right)\right]\right\}\right) - 2E\left[\log\left(\frac{1}{T+1}\right)\right] + \eta_T,$$

where $\eta_T = -E\{\log[r(T)]\}$ is the SE of T with density $r(t)$.

4.5 Order Statistics

Let $X_1, X_2, \ldots, X_n$ be a random sample from the GLFR-G (ξ) family with cdf and pdf given by (5) and (6), respectively. Suppose $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ denote the corresponding order statistics and $X_{s:n}$ is the s^{th} order statistic, then the pdf of the s^{th} order statistic is

$$f_{s:n}(x;\xi) = \sum_{q=0}^{n-s} \frac{(-1)^q \binom{n-s}{q}}{B(s,n-s+1)} f(x;\xi)F(x;\xi)^{s+q-1}, \tag{18}$$

where $\xi = (\alpha,\beta,\theta,\Phi^T)^T$.

Substituting (5) and (6) in (18), we get

$$f_{s:n}(x;\xi) = \sum_{q=0}^{n-s} \frac{(-1)^q \binom{n-s}{q}}{B(s,n-s+1)} \frac{\alpha\theta g(x;\Phi) + \theta(\beta-\alpha)g(x;\Phi)G(x;\Phi)}{[1-G(x;\Phi)]^3} e^{-\alpha\left(\frac{G(x;\Phi)}{1-G(x;\Phi)}\right)-\frac{\beta}{2}\left(\frac{G(x;\Phi)}{1-G(x;\Phi)}\right)^2} \times \left\{1-e^{-\alpha\left(\frac{G(x;\Phi)}{1-G(x;\Phi)}\right)-\frac{\beta}{2}\left(\frac{G(x;\Phi)}{1-G(x;\Phi)}\right)^2}\right\}^{\theta(s+q)-1}. \tag{19}$$

4.6 Stochastic Orderings

Stochastic orders and inequalities have been used in several areas of probability and statistics, such as reliability theory, survival analysis, economics, insurance, actuarial science, queuing theory, biology, operations research, and management science (Shaked and Shanthikumar, 2007). Given two rvs X and, we say that X is smaller than Y in the

- usual stochastic order, denoted by $X \leq_{st} Y$, if $F_X(x) \geq F_Y(x)$, for all x.
- hazard rate order, denoted by $X \leq_{hr} Y$, if $h_X(x) \geq h_Y(x)$, for all x.
- reversed hazard rate order, denoted by $X \leq_{rh} Y$, if $F_X(x)/F_Y(x)$ is decreases in x.
- mean residual life order, denoted by $X \leq_{mrl} Y$, if $m_X(x) \leq m_Y(x)$, for all x.
- likelihood ratio order, denoted by $X \leq_{lr} Y$, if $f_X(x)/f_Y(x)$ is decreases in x.

For all the previous orders, the following chains of implications hold:

$$X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{st} Y,$$
$$X \leq_{lr} Y \Rightarrow X \leq_{rh} Y \Rightarrow X \leq_{st} Y$$

and

$$X \leq_{hr} Y \Rightarrow X \leq_{mrl} Y.$$

For the GLFR-G family, the following theorems provide the stochastic comparisons with respect to the above orderings.

Theorem 2:

Suppose $X\sim$GLFR-G(ξ_1) and $Y\sim$GLFR-G (ξ_2) where $\xi_1 = (\alpha_1,\beta_1,\theta_1,\Phi_1^T)^T$ and $\xi_2 = (\alpha_2,\beta_2,\theta_2,\Phi_2^T)^T$. If $\alpha_1 = \alpha_1$, $\beta_1 = \beta_2$, $\theta_1 \leq \theta_2$ and $\Phi_1^T = \Phi_2^T$, then $X \leq_{lr} Y$.

Proof:

If $\alpha_1 = \alpha_1$, $\beta_1 = \beta_2$, we have

$$\frac{f_X(x;\xi_1)}{f_Y(x;\xi_2)} = \frac{\theta_1\left\{1 - e^{-\alpha_1\left(\frac{G(x;\Phi_1)}{1-G(x;\Phi_1)}\right)-\frac{\beta_1}{2}\left(\frac{G(x;\Phi_1)}{1-G(x;\Phi_1)}\right)^2}\right\}^{\theta_1-1}}{\theta_2\left\{1 - e^{-\alpha_2\left(\frac{G(x;\Phi_2)}{1-G(x;\Phi_2)}\right)-\frac{\beta_2}{2}\left(\frac{G(x;\Phi_2)}{1-G(x;\Phi_2)}\right)^2}\right\}^{\theta_2-1}}$$

$$= \frac{\theta_1\left\{1 - e^{-\alpha_1\left(\frac{G(x;\Phi_1)}{1-G(x;\Phi_1)}\right)-\frac{\beta_1}{2}\left(\frac{G(x;\Phi_1)}{1-G(x;\Phi_1)}\right)^2}\right\}^{\theta_1-1}}{\theta_2\left\{1 - e^{-\alpha_1\left(\frac{G(x;\Phi_1)}{1-G(x;\Phi_1)}\right)-\frac{\beta_1}{2}\left(\frac{G(x;\Phi_1)}{1-G(x;\Phi_1)}\right)^2}\right\}^{\theta_2-1}},$$

therefore,

$$\log\left(\frac{f_X(x;\xi_1)}{f_Y(x;\xi_2)}\right) = \log\theta_1 - \log\theta_2 + (\theta_1-\theta_2)\log\left\{1 - e^{-\alpha_1\left(\frac{G(x;\Phi_1)}{1-G(x;\Phi_1)}\right)-\frac{\beta_1}{2}\left(\frac{G(x;\Phi_1)}{1-G(x;\Phi_1)}\right)^2}\right\},$$

and hence,

$$\frac{d}{dx}\left[\log\left(\frac{f_X(x;\xi_1)}{f_Y(x;\xi_2)}\right)\right] = (\theta_1-\theta_2)\frac{\alpha_1\left(\frac{g(x;\Phi_1)\bar{G}(x;\Phi_1)}{\bar{G}(x;\Phi_1)^3}\right) + \beta_1\left(\frac{g(x;\Phi_1)G(x;\Phi_1)}{\bar{G}(x;\Phi_1)^3}\right)}{1 - e^{-\alpha_1\left(\frac{G(x;\Phi_1)}{\bar{G}(x;\Phi_1)}\right)-\frac{\beta_1}{2}\left(\frac{G(x;\Phi_1)}{\bar{G}(x;\Phi_1)}\right)^2}} e^{-\alpha_1\left(\frac{G(x;\Phi_1)}{\bar{G}(x;\Phi_1)}\right)-\frac{\beta_1}{2}\left(\frac{G(x;\Phi_1)}{\bar{G}(x;\Phi_1)}\right)^2},$$

where

$$\frac{\alpha_1\left(\frac{g(x;\Phi_1)\bar{G}(x;\Phi_1)}{\bar{G}(x;\Phi_1)^3}\right) + \beta_1\left(\frac{g(x;\Phi_1)G(x;\Phi_1)}{\bar{G}(x;\Phi_1)^3}\right)}{1 - e^{-\alpha_1\left(\frac{G(x;\Phi_1)}{\bar{G}(x;\Phi_1)}\right)-\frac{\beta_1}{2}\left(\frac{G(x;\Phi_1)}{\bar{G}(x;\Phi_1)}\right)^2}} e^{-\alpha_1\left(\frac{G(x;\Phi_1)}{\bar{G}(x;\Phi_1)}\right)-\frac{\beta_1}{2}\left(\frac{G(x;\Phi_1)}{\bar{G}(x;\Phi_1)}\right)^2}$$

$$= \frac{\alpha_2\left(\frac{g(x;\Phi_2)\bar{G}(x;\Phi_2)}{\bar{G}(x;\Phi_2)^3}\right) + \beta_2\left(\frac{g(x;\Phi_2)G(x;\Phi_2)}{\bar{G}(x;\Phi_2)^3}\right)}{1 - e^{-\alpha_2\left(\frac{G(x;\Phi_2)}{\bar{G}(x;\Phi_2)}\right)-\frac{\beta_2}{2}\left(\frac{G(x;\Phi_2)}{\bar{G}(x;\Phi_2)}\right)^2}} e^{-\alpha_2\left(\frac{G(x;\Phi_2)}{\bar{G}(x;\Phi_2)}\right)-\frac{\beta_2}{2}\left(\frac{G(x;\Phi_2)}{\bar{G}(x;\Phi_2)}\right)^2}.$$

If $\theta_1 \leq \theta_2$, then $\frac{d}{dx}\left[\log\left(\frac{f_X(x;\xi_1)}{f_Y(x;\xi_2)}\right)\right] < 0$,implies that$X \leq_{lr} Y$.

Theorem 3:

Suppose $X\sim$GLFR-G(ξ_1) and $Y\sim$GLFR-G(ξ_2) where $\xi_1 = (\alpha_1, \beta_1, \theta_1, \Phi_1^T)^T$ and $\xi_2 = (\alpha_2, \beta_2, \theta_2, \Phi_2^T)^T$. If $\alpha_1 > \alpha_1$, $\beta_1 > \beta_2$ and $\theta_1 \leq \theta_2$, then $X \leq_{st} Y$.

Proof:

If $\alpha_1 > \alpha_1$ and $\beta_1 > \beta_2$, we have

$$\alpha_1\left(\frac{G(x;\Phi_1)}{\bar{G}(x;\Phi_1)}\right) > \alpha_2\left(\frac{G(x;\Phi_2)}{\bar{G}(x;\Phi_2)}\right),$$

and

$$\frac{\beta_1}{2}\left(\frac{G(x;\Phi_1)}{\overline{G}(x;\Phi_1)}\right)^2 > \frac{\beta_2}{2}\left(\frac{G(x;\Phi_2)}{\overline{G}(x;\Phi_2)}\right)^2.$$

Thus

$$\alpha_1\left(\frac{G(x;\Phi_1)}{\overline{G}(x;\Phi_1)}\right) + \frac{\beta_1}{2}\left(\frac{G(x;\Phi_1)}{\overline{G}(x;\Phi_1)}\right)^2 > \alpha_2\left(\frac{G(x;\Phi_2)}{\overline{G}(x;\Phi_2)}\right) + \frac{\beta_2}{2}\left(\frac{G(x;\Phi_2)}{\overline{G}(x;\Phi_2)}\right)^2,$$

which implies that

$$-\alpha_1\left(\frac{G(x;\Phi_1)}{\overline{G}(x;\Phi_1)}\right) - \frac{\beta_1}{2}\left(\frac{G(x;\Phi_1)}{\overline{G}(x;\Phi_1)}\right)^2 < -\alpha_2\left(\frac{G(x;\Phi_2)}{\overline{G}(x;\Phi_2)}\right) - \frac{\beta_2}{2}\left(\frac{G(x;\Phi_2)}{\overline{G}(x;\Phi_2)}\right)^2.$$

Therefore,

$$1 - e^{-\alpha_1\left(\frac{G(x;\Phi_1)}{\overline{G}(x;\Phi_1)}\right) - \frac{\beta_1}{2}\left(\frac{G(x;\Phi_1)}{\overline{G}(x;\Phi_1)}\right)^2} > 1 - e^{-\alpha_2\left(\frac{G(x;\Phi_2)}{\overline{G}(x;\Phi_2)}\right) - \frac{\beta_2}{2}\left(\frac{G(x;\Phi_2)}{\overline{G}(x;\Phi_2)}\right)^2}.$$

If $\theta_1 \leq \theta_2$, then

$$\left\{1 - e^{-\alpha_1\left(\frac{G(x;\Phi_1)}{\overline{G}(x;\Phi_1)}\right) - \frac{\beta_1}{2}\left(\frac{G(x;\Phi_1)}{\overline{G}(x;\Phi_1)}\right)^2}\right\}^{\theta_1} > \left\{1 - e^{-\alpha_2\left(\frac{G(x;\Phi_2)}{\overline{G}(x;\Phi_2)}\right) - \frac{\beta_2}{2}\left(\frac{G(x;\Phi_2)}{\overline{G}(x;\Phi_2)}\right)^2}\right\}^{\theta_2},$$

is true for all x, this means that $F_X(x;\xi_1) \geq F_Y(x;\xi_2)$ and hence $X \leq_{st} Y$.

5. ESTIMATION

Here, we determine the maximum likelihood estimates (MLEs) of the GLFR-G parameters. Let $x_1, x_2, \ldots, x_n$ be observed values from the GLFR-G family with parameters α, β, θ and Φ. Let $\xi = (\alpha, \beta, \theta, \Phi^T)^T$ be the parameters vector. The total log-likelihood function for ξ is given by

$$\begin{aligned} l(\xi) = &\sum_{i=1}^{n} \log[\alpha\theta g(x_i,\Phi) + \theta(\beta-\alpha)g(x_i,\Phi)G(x_i,\Phi)] \\ &- 3\sum_{i=1}^{n}\log[1-G(x_i,\Phi)] - \alpha\sum_{i=1}^{n}H(x_i,\Phi) - \frac{\beta}{2}\sum_{i=1}^{n}H(x;\Phi)^2 \\ &+ (\theta-1)\sum_{i=1}^{n}\log\left[1-e^{-\alpha H(x_i,\Phi)-\frac{\beta}{2}H(x;\Phi)^2}\right], \end{aligned}$$

where $(x_i,\Phi) = \frac{G(x_i,\Phi)}{1-G(x_i,\Phi)}$.

The components of the score vector $U(\xi)$ are given by

$$\begin{aligned} U_\alpha = &\sum_{i=1}^{n}\frac{\theta g(x_i,\Phi) - \theta g(x_i,\Phi)G(x_i,\Phi)}{\alpha\theta g(x_i,\Phi) + \theta(\beta-\alpha)g(x_i,\Phi)G(x_i,\Phi)} - \sum_{i=1}^{n}H(x_i,\Phi) \\ &+ (\theta-1)\sum_{i=1}^{n}\frac{H(x_i,\Phi)e^{-\alpha H(x_i,\Phi)-\frac{\beta}{2}H(x_i,\Phi)^2}}{1-e^{-\alpha H(x_i,\Phi)-\frac{\beta}{2}H(x_i,\Phi)^2}}, \end{aligned}$$

$$U_\beta = \sum_{i=1}^{n} \frac{\theta g(x_i,\Phi)G(x_i,\Phi)}{\alpha\theta g(x_i,\Phi) + \theta(\beta-\alpha)g(x_i,\Phi)G(x_i,\Phi)} - \frac{1}{2}\sum_{i=1}^{n} H(x_i,\Phi)^2 + \frac{(\theta-1)}{2}\sum_{i=1}^{n} \frac{H(x_i,\Phi)^2 e^{-\alpha H(x_i,\Phi)-\frac{\beta}{2}H(x_i,\Phi)^2}}{1-e^{-\alpha H(x_i,\Phi)-\frac{\beta}{2}H(x_i,\Phi)^2}},$$

$$U_\theta = \sum_{i=1}^{n} \frac{\alpha g(x_i,\Phi) + (\beta-\alpha)g(x_i,\Phi)G(x_i,\Phi)}{\alpha\theta g(x_i,\Phi) + \theta(\beta-\alpha)g(x_i,\Phi)G(x_i,\Phi)} + \sum_{i=1}^{n} \log\left[1-e^{-\alpha H(x_i,\Phi)-\frac{\beta}{2}H(x_i,\Phi)^2}\right]$$

and

$$U_{\Phi_k} = \sum_{i=1}^{n} \frac{\alpha\theta \frac{\partial g(x_i,\Phi)}{\partial \Phi_k} + \theta(\beta-\alpha)\frac{\partial}{\partial \Phi_k}[g(x_i,\Phi)G(x_i,\Phi)]}{\alpha\theta g(x_i,\Phi) + \theta(\beta-\alpha)g(x_i,\Phi)G(x_i,\Phi)} + 3\sum_{i=1}^{n} \frac{\frac{\partial G(x_i,\Phi)}{\partial \Phi_k}}{1-G(x_i,\Phi)} - \alpha\sum_{i=1}^{n}\frac{\partial H(x_i,\Phi)}{\partial \Phi_k} - \frac{\beta}{2}\sum_{i=0}^{n}\frac{\partial H(x_i,\Phi)^2}{\partial \Phi_k} + \sum_{i=1}^{n}\left[\frac{(\theta-1)e^{-\alpha H(x_i,\Phi)-\frac{\beta}{2}H(x_i,\Phi)^2}\frac{\partial}{\partial \Phi_k}\left[\alpha H(x_i,\Phi)+\frac{\beta}{2}H(x_i,\Phi)^2\right]}{1-e^{-\alpha H(x_i,\Phi)-\frac{\beta}{2}H(x_i,\Phi)^2}}\right].$$

Setting $U_\alpha, U_\beta, U_\theta$ and U_Φ equal to zero and solving the equations simultaneously yields the MLE $\hat{\xi} = \left(\hat{\alpha},\hat{\beta},\hat{\theta},\hat{\Phi}^T\right)^T$ of $\xi = (\alpha,\beta,\theta,\Phi^T)^T$. These equations cannot be solved analytically, and statistical software such as R, Maple, Mathematica, or SAS can be used to solve them numerically using iterative methods such as the Newton-Raphson type algorithms.

5.1 Maximum Likelihood Method for GLFR-E Model

This subsection deals with the maximum likelihood estimates of the unknown parameters for GLFR-E distribution based on complete samples. Let $X_1, X_2, \ldots, X_n$ be a random sample from the GLFR-E with set of parameters $\xi = (\alpha,\beta,\theta,a)^T$. The log-likelihood function for parameter vector $\xi = (\alpha,\beta,\theta,a)^T$ is obtained as follows

$$\ell(\xi) = n\log\theta + n\log a + a\sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\log[\alpha+\beta(e^{ax_i}-1)] - \sum_{i=1}^{n}\left(\alpha(e^{ax_i}-1)+\frac{\beta}{2}(e^{ax_i}-1)^2\right) + (\theta-1)\sum_{i=1}^{n}\log\left[1-e^{-\left(\alpha(e^{ax_i}-1)+\frac{\beta}{2}(e^{ax_i}-1)^2\right)}\right].$$

The elements of the score function $U(\xi) = \left(U_\alpha, U_\beta, U_\theta, U_a\right)$ are given by

$$U_\alpha = (\theta-1)\sum_{i=1}^{n}(e^{ax_i}-1)\frac{e^{-\left(\alpha(e^{ax_i}-1)+\frac{\beta}{2}(e^{ax_i}-1)^2\right)}}{1-e^{-\left(\alpha(e^{ax_i}-1)+\frac{\beta}{2}(e^{ax_i}-1)^2\right)}}+\sum_{i=1}^{n}\frac{1}{\alpha+\beta(e^{ax_i}-1)} - \sum_{i=1}^{n}(e^{ax_i}-1),$$

$$U_\beta = \sum_{i=1}^{n}\frac{e^{ax_i}-1}{\alpha+(e^{ax_i}-1)\beta}+(\theta-1)\sum_{i=1}^{n}\left(\frac{1}{2}-e^{ax_i}+\frac{1}{2}e^{2ax_i}\right) \frac{e^{-\left(\alpha(e^{ax_i}-1)+\frac{\beta}{2}(e^{ax_i}-1)^2\right)}}{1-e^{-(e^{ax_i}-1)\alpha-\frac{1}{2}(e^{ax_i}-1)^2\beta}}-\frac{1}{2}\sum_{i=1}^{n}(e^{ax_i}-1)^2,$$

$$U_\theta = \frac{n}{\theta}+\sum_{i=1}^{n}\log\left(1-e^{-\left(\alpha(e^{ax_i}-1)+\frac{\beta}{2}(e^{ax_i}-1)^2\right)}\right),$$

and

$$U_a = \frac{n}{a}+\sum_{i=1}^{n}x_i+\beta\sum_{i=1}^{n}\frac{x_ie^{ax_i}}{\alpha+\beta(e^{ax_i}-1)}-\sum_{i=1}^{n}x_ie^{ax_i}\left(\alpha x_i+\beta(e^{ax_i}-1)\right) +(\theta-1)\sum_{i=1}^{n}x_ie^{ax_i}\left(\alpha+\beta(e^{ax_i}-1)\right)\frac{e^{-\left(\alpha(e^{ax_i}-1)+\frac{\beta}{2}(e^{ax_i}-1)^2\right)}}{1-e^{-\left(\alpha(e^{ax_i}-1)+\frac{\beta}{2}(e^{ax_i}-1)^2\right)}}.$$

Setting U_α, U_β, U_θ and U_a equal to zero and solving these equations simultaneously yield the maximum likelihood estimate (MLE) $\hat{\xi}=\left(\hat{\alpha},\hat{\beta},\hat{\theta},\hat{a}\right)^T$ of $\xi=(\alpha,\beta,\theta,a)^T$. These equations cannot be solved analytically, and statistical software can be used to solve them numerically using iterative methods.

6. SIMULATION STUDY

To investigate the behavior of the MLEs in the previous section, we propose some simulations in terms of the sample size n for the GLFR-E. The GLFR-E rv can be simulated by using $X=Q(U)$ in section 4.2, where U is a uniformly rv on the interval (0,1).

By using R 4.0.3 programming language R (R Core Team (2020)), 3,000 random samples from the distribution GLFR-E have been generated with different sample sizes $n=30,50,80,120$ and $n=200$. We selected the true values of the parameters as follows: $\alpha=(0.4,1.0,3.5)$, $\beta=(0.5,1.0,4.7)$, $\theta=(0.6,1.0,2.9)$ and $a=(0.2,1.0,2.1)$. Tables 2-7 show the average of: mean square errors ($MSEs$), $MSEs=\frac{1}{N}\sum_{i=1}^{N}\left(\hat{\xi}-\xi\right)^2$, and mean relative estimates ($MREs$), $MREs=\frac{1}{N}\sum_{i=1}^{N}\left|\hat{\xi}-\xi\right|/\xi$, were computed for each sample size and each parameter, it can be seen that the estimates are stable, and as the sample size increases the $MSEs$ decreases in all cases.

Table 2
Simulation Results for the GLFR-E Distribution for Different Values of θ and a

ξ				n	$MSEs$				$MREs$			
α	β	θ	a		$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{a}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{a}$
0.4	0.5	0.6	0.2	30	0.09556	0.17813	0.09154	0.02680	0.66666	0.79265	0.36021	0.56620
				50	0.08524	0.17075	0.04103	0.01717	0.63235	0.77829	0.25580	0.46714
				80	0.06760	0.16447	0.02406	0.01168	0.55005	0.76117	0.20047	0.39474
				120	0.05596	0.15797	0.01686	0.00862	0.49690	0.73851	0.16963	0.34275
				200	0.04114	0.14942	0.01063	0.00624	0.41236	0.70700	0.13193	0.29375
			1.0	30	0.08489	0.14982	0.08308	0.13246	0.61194	0.72298	0.35014	0.30896
				50	0.07573	0.15555	0.04221	0.12898	0.57829	0.74632	0.25655	0.30873
				80	0.06256	0.15673	0.02361	0.12294	0.51962	0.75147	0.19974	0.30098
				120	0.04987	0.15832	0.01707	0.11630	0.45372	0.75217	0.16728	0.29072
				200	0.03636	0.15396	0.01047	0.10333	0.38098	0.73131	0.13246	0.26836
			2.1	30	0.08362	0.10779	0.07636	0.17611	0.59683	0.58280	0.33468	0.18217
				50	0.07533	0.11382	0.04024	0.17631	0.57256	0.61256	0.25096	0.18461
				80	0.06019	0.11615	0.02356	0.17355	0.50096	0.62329	0.19626	0.18352
				120	0.04963	0.11630	0.01739	0.16899	0.44419	0.62610	0.17070	0.18078
				200	0.03589	0.11999	0.01065	0.15909	0.36800	0.63684	0.13322	0.17403
		1.0	0.2	30	0.10732	0.18848	0.30827	0.02832	0.72651	0.83591	0.40327	0.59822
				50	0.09068	0.17402	0.17457	0.01687	0.65543	0.79196	0.31889	0.46599
				80	0.07352	0.16117	0.11237	0.01060	0.57384	0.75008	0.25779	0.38129
				120	0.06292	0.15469	0.08006	0.00814	0.52168	0.72716	0.21915	0.33658
				200	0.04697	0.14459	0.05151	0.00593	0.43883	0.69245	0.17782	0.28956
			1.0	30	0.09506	0.16533	0.29107	0.14584	0.65880	0.78832	0.38571	0.33953
				50	0.07995	0.15935	0.16520	0.13026	0.59568	0.76497	0.30389	0.31400
				80	0.06649	0.15532	0.11026	0.12065	0.53632	0.74649	0.25494	0.29861
				120	0.05734	0.15331	0.07905	0.11073	0.48950	0.73481	0.22024	0.28242
				200	0.04338	0.15148	0.05003	0.10258	0.41759	0.72477	0.17567	0.26845
			2.1	30	0.09291	0.11474	0.24653	0.19026	0.65163	0.62196	0.36104	0.19588
				50	0.07614	0.11626	0.15799	0.18419	0.57659	0.63068	0.29514	0.19197
				80	0.06835	0.11616	0.10753	0.17397	0.54071	0.62819	0.25457	0.18459
				120	0.05459	0.11407	0.07316	0.16919	0.47403	0.62272	0.21327	0.18037
				200	0.04281	0.11560	0.05091	0.15486	0.40892	0.62191	0.17731	0.17052
		2.9	0.2	30	0.14345	0.17750	7.71520	0.02440	0.86432	0.79316	0.73734	0.53969
				50	0.12519	0.16070	4.97845	0.01282	0.78418	0.74259	0.60504	0.41104
				80	0.11084	0.14476	3.71274	0.00755	0.72474	0.69088	0.52141	0.32974
				120	0.09293	0.13461	2.61144	0.00512	0.64300	0.65986	0.43556	0.28179
				200	0.07314	0.11671	1.60846	0.00361	0.55728	0.59569	0.35083	0.23539
			1.0	30	0.13778	0.17025	8.05937	0.14103	0.83954	0.80027	0.71704	0.34216
				50	0.12134	0.16287	4.94312	0.13125	0.76759	0.77699	0.59644	0.32742
				80	0.10164	0.15583	3.37424	0.11525	0.67864	0.74740	0.49934	0.30039
				120	0.08614	0.14779	2.58770	0.10171	0.60915	0.71782	0.43341	0.27772
				200	0.06742	0.14234	1.63237	0.09442	0.52159	0.69547	0.35098	0.26263
			2.1	30	0.14052	0.11543	6.64209	0.19831	0.86309	0.62137	0.66989	0.20279
				50	0.12492	0.11432	4.29066	0.18999	0.79395	0.61944	0.56190	0.19717
				80	0.11190	0.11178	3.24643	0.18193	0.73862	0.60818	0.49586	0.19203
				120	0.09417	0.10772	2.43978	0.17345	0.66252	0.59510	0.42783	0.18483
				200	0.07529	0.10831	1.71443	0.15946	0.57359	0.59370	0.35895	0.17464

Table 3
Simulation results for the GLFR-E distribution for different Values of θ and a

ξ				n	*MSEs*				*MREs*			
α	β	θ	a		$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{a}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{a}$
0.4	1.0	0.6	0.2	30	0.10650	0.69570	0.07830	0.04184	0.72326	0.78440	0.34613	0.76223
				50	0.09186	0.63180	0.04615	0.02846	0.66130	0.74137	0.27244	0.62379
				80	0.08007	0.57340	0.02952	0.02029	0.60587	0.69901	0.22212	0.52812
				120	0.06801	0.49470	0.02013	0.01416	0.54955	0.63625	0.18776	0.42812
				200	0.05423	0.42622	0.01419	0.00950	0.47686	0.58114	0.15760	0.34723
			1.0	30	0.09017	0.53935	0.07402	0.16790	0.63446	0.69310	0.33371	0.37193
				50	0.08251	0.52054	0.04292	0.15930	0.60867	0.68179	0.26174	0.35815
				80	0.07286	0.48177	0.02927	0.14184	0.56103	0.65153	0.22491	0.32829
				120	0.06084	0.45919	0.02078	0.13319	0.50105	0.63001	0.19104	0.31253
				200	0.04610	0.41097	0.01348	0.11415	0.42273	0.58969	0.15294	0.28138
			2.1	30	0.09712	0.32745	0.07102	0.20240	0.66897	0.51750	0.32708	0.20191
				50	0.08915	0.31627	0.03822	0.19089	0.64191	0.51759	0.25177	0.19477
				80	0.07651	0.30026	0.02566	0.18103	0.58261	0.50787	0.21138	0.18799
				120	0.06733	0.28574	0.01969	0.16921	0.54134	0.49713	0.18695	0.17972
				200	0.05184	0.26643	0.01394	0.15362	0.46146	0.48085	0.15495	0.16840
		1.0	0.2	30	0.11505	0.71613	0.30246	0.04243	0.75336	0.80411	0.41190	0.77276
				50	0.10023	0.64449	0.19150	0.02725	0.69090	0.74920	0.33475	0.61540
				80	0.08826	0.57551	0.13249	0.01696	0.63803	0.69819	0.28453	0.48898
				120	0.07498	0.51558	0.09766	0.01238	0.57405	0.65393	0.24826	0.41608
				200	0.06013	0.44138	0.06949	0.00837	0.50332	0.59072	0.21381	0.33719
			1.0	30	0.10254	0.57774	0.26906	0.17466	0.69382	0.72802	0.38018	0.38639
				50	0.09328	0.54635	0.19196	0.15694	0.64966	0.70328	0.32925	0.35780
				80	0.08027	0.51206	0.12965	0.14402	0.59138	0.67424	0.28197	0.33589
				120	0.06696	0.48570	0.09630	0.13382	0.53200	0.64970	0.24439	0.31724
				200	0.05765	0.43304	0.07068	0.11638	0.48507	0.60429	0.21324	0.28714
			2.1	30	0.11042	0.35698	0.22168	0.20730	0.72845	0.55005	0.35213	0.20717
				50	0.09767	0.33454	0.15806	0.19854	0.67627	0.53814	0.30366	0.20093
				80	0.08884	0.32352	0.11267	0.18724	0.64191	0.53229	0.26316	0.19350
				120	0.07674	0.30730	0.08953	0.17506	0.58862	0.51633	0.23956	0.18434
				200	0.06502	0.28080	0.06654	0.15736	0.53376	0.49157	0.20703	0.17150
		2.9	0.2	30	0.16072	0.53035	4.43651	0.02715	0.94215	0.66114	0.59062	0.56258
				50	0.14748	0.47734	2.97028	0.01506	0.88648	0.61779	0.50205	0.43247
				80	0.13326	0.43759	2.38542	0.00899	0.82873	0.58155	0.45721	0.35120
				120	0.12625	0.38886	2.01995	0.00620	0.79845	0.53876	0.42079	0.30018
				200	0.10799	0.34536	1.60392	0.00431	0.72212	0.49336	0.37545	0.25224
			1.0	30	0.15799	0.47673	4.44744	0.14207	0.94203	0.65615	0.56511	0.33434
				50	0.14760	0.46327	3.09997	0.13268	0.90104	0.64206	0.50613	0.32212
				80	0.13452	0.44336	2.51722	0.12224	0.84578	0.61949	0.46581	0.30520
				120	0.12102	0.42257	1.99845	0.10940	0.78661	0.59630	0.42208	0.28403
				200	0.10443	0.39716	1.64892	0.09885	0.71707	0.56976	0.38129	0.26603
			2.1	30	0.16596	0.26967	3.60303	0.18052	0.97572	0.47857	0.51127	0.18999
				50	0.15740	0.27193	2.62997	0.17686	0.94187	0.47868	0.46196	0.18792
				80	0.14342	0.26526	2.03924	0.17025	0.88450	0.47057	0.41801	0.18350
				120	0.13250	0.26002	1.75729	0.16149	0.84048	0.46113	0.39289	0.17723
				200	0.11886	0.25606	1.44658	0.15060	0.78298	0.44892	0.36047	0.16841

Table 4
Simulation Results for the GLFR-E Distribution for Different Values of θ and a

ξ				n	$MSEs$				$MREs$			
α	β	θ	a		$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{a}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{a}$
0.4	4.7	0.6	0.2	30	0.11899	13.82084	0.06055	0.08834	0.76949	0.68271	0.31763	1.16755
				50	0.11510	12.55792	0.04443	0.06520	0.75612	0.64060	0.27455	0.96316
				80	0.10829	11.18735	0.03407	0.04648	0.72329	0.59461	0.24541	0.78005
				120	0.10425	10.12931	0.02664	0.03544	0.70582	0.55651	0.22220	0.66457
				200	0.09466	8.14225	0.02039	0.02144	0.66514	0.48704	0.19609	0.48620
			1.0	30	0.12403	7.77036	0.03888	0.17085	0.79019	0.51072	0.25345	0.36752
				50	0.11988	7.67037	0.02933	0.16350	0.77449	0.51053	0.22562	0.35430
				80	0.12125	7.45049	0.02562	0.15183	0.77727	0.49778	0.21593	0.33386
				120	0.11298	7.16000	0.02038	0.14130	0.74516	0.48531	0.19581	0.31554
				200	0.10765	6.51254	0.01799	0.12480	0.72307	0.45687	0.18538	0.28936
			2.1	30	0.13113	3.81391	0.03691	0.18902	0.81212	0.34096	0.24614	0.19287
				50	0.12499	3.79969	0.02563	0.17977	0.78908	0.34891	0.21385	0.18468
				80	0.11835	3.63152	0.01949	0.17123	0.76369	0.34271	0.18972	0.17786
				120	0.11674	3.40925	0.01709	0.16107	0.75585	0.33247	0.18062	0.16963
				200	0.10422	3.48759	0.01351	0.15866	0.70508	0.34163	0.16103	0.16726
		1.0	0.2	30	0.14081	10.77631	0.13523	0.06406	0.86075	0.56084	0.28527	0.89396
				50	0.13436	9.92813	0.11364	0.04636	0.82613	0.53205	0.26826	0.74024
				80	0.12958	9.31047	0.09227	0.03402	0.80027	0.51423	0.25072	0.62447
				120	0.12412	8.57668	0.08615	0.02599	0.77767	0.48996	0.24358	0.53804
				200	0.11735	7.82427	0.07584	0.01808	0.75273	0.46890	0.22979	0.44610
			1.0	30	0.13478	6.30475	0.10031	0.14121	0.83503	0.43935	0.23553	0.31550
				50	0.13596	6.45333	0.07798	0.13592	0.83364	0.44446	0.22315	0.30416
				80	0.13286	6.45973	0.07053	0.13021	0.82032	0.44585	0.21907	0.29453
				120	0.13041	6.35117	0.06465	0.12511	0.81164	0.44227	0.21248	0.28615
				200	0.12877	6.01743	0.06047	0.11301	0.79971	0.42575	0.20764	0.26645
			2.1	30	0.13943	3.00535	0.09129	0.16094	0.83121	0.29335	0.22177	0.17080
				50	0.13487	3.15627	0.06658	0.15843	0.81420	0.31096	0.20301	0.16826
				80	0.13226	3.17652	0.05682	0.15189	0.80471	0.31424	0.19417	0.16240
				120	0.12884	3.17848	0.05081	0.14954	0.79209	0.31756	0.18716	0.16047
				200	0.12437	3.04845	0.04663	0.13947	0.77656	0.31011	0.18324	0.15217
		2.9	0.2	30	0.17951	7.95153	2.45940	0.04290	1.02344	0.44656	0.42769	0.64137
				50	0.17381	7.19285	1.51339	0.02621	0.99755	0.42700	0.35268	0.49576
				80	0.16933	6.05099	1.09783	0.01473	0.97383	0.38700	0.30711	0.37004
				120	0.16697	5.10745	0.94117	0.00894	0.96031	0.35410	0.28831	0.29085
				200	0.15887	4.13680	0.69138	0.00516	0.92471	0.31616	0.25194	0.22601
			1.0	30	0.17370	4.83231	1.84641	0.11038	1.00427	0.36459	0.35035	0.25648
				50	0.17101	4.70194	1.24627	0.10210	0.98937	0.35897	0.30872	0.24227
				80	0.16848	4.43052	0.93814	0.09196	0.97678	0.34597	0.28206	0.22599
				120	0.16452	4.14542	0.79372	0.08225	0.95865	0.33149	0.26640	0.21084
				200	0.15784	3.66204	0.65126	0.06828	0.92787	0.31061	0.24696	0.18942
			2.1	30	0.17149	2.09822	1.81751	0.13075	0.99117	0.24260	0.33933	0.14480
				50	0.16755	2.12769	1.16057	0.12337	0.96736	0.24763	0.29057	0.13822
				80	0.16239	2.10199	0.86927	0.11692	0.94403	0.24585	0.26242	0.13290
				120	0.15876	2.06792	0.71368	0.11345	0.92780	0.24406	0.24584	0.13072
				200	0.15433	1.94168	0.59878	0.10295	0.91211	0.23395	0.23264	0.12299

Table 5
Simulation Results for the GLFR-E Distribution for different Values of α, β, θ and a

ξ				n	$MSEs$				$MREs$			
α	β	θ	a		$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{a}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{a}$
1.0	0.5	0.6	0.2	30	0.09556	0.17813	0.09154	0.02680	0.66666	0.79265	0.36021	0.56620
				50	0.08524	0.17075	0.04103	0.01717	0.63235	0.77829	0.25580	0.46714
				80	0.06760	0.16447	0.02406	0.01168	0.55005	0.76117	0.20047	0.39474
				120	0.05596	0.15797	0.01686	0.00862	0.49690	0.73851	0.16963	0.34275
				200	0.04114	0.14942	0.01063	0.00624	0.41236	0.70700	0.13193	0.29375
			1.0	30	0.08489	0.14982	0.08308	0.13246	0.61194	0.72298	0.35014	0.30896
				50	0.07573	0.15555	0.04221	0.12898	0.57829	0.74632	0.25655	0.30873
				80	0.06256	0.15673	0.02361	0.12294	0.51962	0.75147	0.19974	0.30098
				120	0.04987	0.15832	0.01707	0.11630	0.45372	0.75217	0.16728	0.29072
				200	0.03636	0.15396	0.01047	0.10333	0.38098	0.73131	0.13246	0.26836
			2.1	30	0.08362	0.10779	0.07636	0.17611	0.59683	0.58280	0.33468	0.18217
				50	0.07533	0.11382	0.04024	0.17631	0.57256	0.61256	0.25096	0.18461
				80	0.06019	0.11615	0.02356	0.17355	0.50096	0.62329	0.19626	0.18352
				120	0.04963	0.11630	0.01739	0.16899	0.44419	0.62610	0.17070	0.18078
				200	0.03589	0.11999	0.01065	0.15909	0.36800	0.63684	0.13322	0.17403
		1.0	0.2	30	0.10732	0.18848	0.30827	0.02832	0.72651	0.83591	0.40327	0.59822
				50	0.09068	0.17402	0.17457	0.01687	0.65543	0.79196	0.31889	0.46599
				80	0.07352	0.16117	0.11237	0.01060	0.57384	0.75008	0.25779	0.38129
				120	0.06292	0.15469	0.08006	0.00814	0.52168	0.72716	0.21915	0.33658
				200	0.04697	0.14459	0.05151	0.00593	0.43883	0.69245	0.17782	0.28956
			1.0	30	0.09506	0.16533	0.29107	0.14584	0.65880	0.78832	0.38571	0.33953
				50	0.07995	0.15935	0.16520	0.13026	0.59568	0.76497	0.30389	0.31400
				80	0.06649	0.15532	0.11026	0.12065	0.53632	0.74649	0.25494	0.29861
				120	0.05734	0.15331	0.07905	0.11073	0.48950	0.73481	0.22024	0.28242
				200	0.04338	0.15148	0.05003	0.10258	0.41759	0.72477	0.17567	0.26845
			2.1	30	0.09291	0.11474	0.24653	0.19026	0.65163	0.62196	0.36104	0.19588
				50	0.07614	0.11626	0.15799	0.18419	0.57659	0.63068	0.29514	0.19197
				80	0.06835	0.11616	0.10753	0.17397	0.54071	0.62819	0.25457	0.18459
				120	0.05459	0.11407	0.07316	0.16919	0.47403	0.62272	0.21327	0.18037
				200	0.04281	0.11560	0.05091	0.15486	0.40892	0.62191	0.17731	0.17052
		2.9	0.2	30	0.14345	0.17750	7.71520	0.02440	0.86432	0.79316	0.73734	0.53969
				50	0.12519	0.16070	4.97845	0.01282	0.78418	0.74259	0.60504	0.41104
				80	0.11084	0.14476	3.71274	0.00755	0.72474	0.69088	0.52141	0.32974
				120	0.09293	0.13461	2.61144	0.00512	0.64300	0.65986	0.43556	0.28179
				200	0.07314	0.11671	1.60846	0.00361	0.55728	0.59569	0.35083	0.23539
			1.0	30	0.13778	0.17025	8.05937	0.14103	0.83954	0.80027	0.71704	0.34216
				50	0.12134	0.16287	4.94312	0.13125	0.76759	0.77699	0.59644	0.32742
				80	0.10164	0.15583	3.37424	0.11525	0.67864	0.74740	0.49934	0.30039
				120	0.08614	0.14779	2.58770	0.10171	0.60915	0.71782	0.43341	0.27772
				200	0.06742	0.14234	1.63237	0.09442	0.52159	0.69547	0.35098	0.26263
			2.1	30	0.14052	0.11543	6.64209	0.19831	0.86309	0.62137	0.66989	0.20279
				50	0.12492	0.11432	4.29066	0.18999	0.79395	0.61944	0.56190	0.19717
				80	0.11190	0.11178	3.24643	0.18193	0.73862	0.60818	0.49586	0.19203
				120	0.09417	0.10772	2.43978	0.17345	0.66252	0.59510	0.42783	0.18483
				200	0.07529	0.10831	1.71443	0.15946	0.57359	0.59370	0.35895	0.17464

Table 6
Simulation Results for the GLFR-E Distribution for different Values of α, β, θ and a

ξ				n	MSEs				MREs			
α	β	θ	a		$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{a}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{a}$
1.0	1.0	0.6	0.2	30	0.10650	0.69570	0.07830	0.04184	0.72326	0.78440	0.34613	0.76223
				50	0.09186	0.63180	0.04615	0.02846	0.66130	0.74137	0.27244	0.62379
				80	0.08007	0.57340	0.02952	0.02029	0.60587	0.69901	0.22212	0.52812
				120	0.06801	0.49470	0.02013	0.01416	0.54955	0.63625	0.18776	0.42812
				200	0.05423	0.42622	0.01419	0.00950	0.47686	0.58114	0.15760	0.34723
			1.0	30	0.09017	0.53935	0.07402	0.16790	0.63446	0.69310	0.33371	0.37193
				50	0.08251	0.52054	0.04292	0.15930	0.60867	0.68179	0.26174	0.35815
				80	0.07286	0.48177	0.02927	0.14184	0.56103	0.65153	0.22491	0.32829
				120	0.06084	0.45919	0.02078	0.13319	0.50105	0.63001	0.19104	0.31253
				200	0.04610	0.41097	0.01348	0.11415	0.42273	0.58969	0.15294	0.28138
			2.1	30	0.09712	0.32745	0.07102	0.20240	0.66897	0.51750	0.32708	0.20191
				50	0.08915	0.31627	0.03822	0.19089	0.64191	0.51759	0.25177	0.19477
				80	0.07651	0.30026	0.02566	0.18103	0.58261	0.50787	0.21138	0.18799
				120	0.06733	0.28574	0.01969	0.16921	0.54134	0.49713	0.18695	0.17972
				200	0.05184	0.26643	0.01394	0.15362	0.46146	0.48085	0.15495	0.16840
		1.0	0.2	30	0.11505	0.71613	0.30246	0.04243	0.75336	0.80411	0.41190	0.77276
				50	0.10023	0.64449	0.19150	0.02725	0.69090	0.74920	0.33475	0.61540
				80	0.08826	0.57551	0.13249	0.01696	0.63803	0.69819	0.28453	0.48898
				120	0.07498	0.51558	0.09766	0.01238	0.57405	0.65393	0.24826	0.41608
				200	0.06013	0.44138	0.06949	0.00837	0.50332	0.59072	0.21381	0.33719
			1.0	30	0.10254	0.57774	0.26906	0.17466	0.69382	0.72802	0.38018	0.38639
				50	0.09328	0.54635	0.19196	0.15694	0.64966	0.70328	0.32925	0.35780
				80	0.08027	0.51206	0.12965	0.14402	0.59138	0.67424	0.28197	0.33589
				120	0.06696	0.48570	0.09630	0.13382	0.53200	0.64970	0.24439	0.31724
				200	0.05765	0.43304	0.07068	0.11638	0.48507	0.60429	0.21324	0.28714
			2.1	30	0.11042	0.35698	0.22168	0.20730	0.72845	0.55005	0.35213	0.20717
				50	0.09767	0.33454	0.15806	0.19854	0.67627	0.53814	0.30366	0.20093
				80	0.08884	0.32352	0.11267	0.18724	0.64191	0.53229	0.26316	0.19350
				120	0.07674	0.30730	0.08953	0.17506	0.58862	0.51633	0.23956	0.18434
				200	0.06502	0.28080	0.06654	0.15736	0.53376	0.49157	0.20703	0.17150
		2.9	0.2	30	0.16072	0.53035	4.43651	0.02715	0.94215	0.66114	0.59062	0.56258
				50	0.14748	0.47734	2.97028	0.01506	0.88648	0.61779	0.50205	0.43247
				80	0.13326	0.43759	2.38542	0.00899	0.82873	0.58155	0.45721	0.35120
				120	0.12625	0.38886	2.01995	0.00620	0.79845	0.53876	0.42079	0.30018
				200	0.10799	0.34536	1.60392	0.00431	0.72212	0.49336	0.37545	0.25224
			1.0	30	0.15799	0.47673	4.44744	0.14207	0.94203	0.65615	0.56511	0.33434
				50	0.14760	0.46327	3.09997	0.13268	0.90104	0.64206	0.50613	0.32212
				80	0.13452	0.44336	2.51722	0.12224	0.84578	0.61949	0.46581	0.30520
				120	0.12102	0.42257	1.99845	0.10940	0.78661	0.59630	0.42208	0.28403
				200	0.10443	0.39716	1.64892	0.09885	0.71707	0.56976	0.38129	0.26603
			2.1	30	0.16596	0.26967	3.60303	0.18052	0.97572	0.47857	0.51127	0.18999
				50	0.15740	0.27193	2.62997	0.17686	0.94187	0.47868	0.46196	0.18792
				80	0.14342	0.26526	2.03924	0.17025	0.88450	0.47057	0.41801	0.18350
				120	0.13250	0.26002	1.75729	0.16149	0.84048	0.46113	0.39289	0.17723
				200	0.11886	0.25606	1.44658	0.15060	0.78298	0.44892	0.36047	0.16841

Table 7
Simulation Results for the GLFR-E Distribution for different Values of α, β, θ and a

ξ				n	$MSEs$				$MREs$			
α	β	θ	a		$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{a}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{a}$
1.0	4.7	0.6	0.2	30	0.11899	13.82084	0.06055	0.08834	0.76949	0.68271	0.31763	1.16755
				50	0.11510	12.55792	0.04443	0.06520	0.75612	0.64060	0.27455	0.96316
				80	0.10829	11.18735	0.03407	0.04648	0.72329	0.59461	0.24541	0.78005
				120	0.10425	10.12931	0.02664	0.03544	0.70582	0.55651	0.22220	0.66457
				200	0.09466	8.14225	0.02039	0.02144	0.66514	0.48704	0.19609	0.48620
			1.0	30	0.12403	7.77036	0.03888	0.17085	0.79019	0.51072	0.25345	0.36752
				50	0.11988	7.67037	0.02933	0.16350	0.77449	0.51053	0.22562	0.35430
				80	0.12125	7.45049	0.02562	0.15183	0.77727	0.49778	0.21593	0.33386
				120	0.11298	7.16000	0.02038	0.14130	0.74516	0.48531	0.19581	0.31554
				200	0.10765	6.51254	0.01799	0.12480	0.72307	0.45687	0.18538	0.28936
			2.1	30	0.13113	3.81391	0.03691	0.18902	0.81212	0.34096	0.24614	0.19287
				50	0.12499	3.79969	0.02563	0.17977	0.78908	0.34891	0.21385	0.18468
				80	0.11835	3.63152	0.01949	0.17123	0.76369	0.34271	0.18972	0.17786
				120	0.11674	3.40925	0.01709	0.16107	0.75585	0.33247	0.18062	0.16963
				200	0.10422	3.48759	0.01351	0.15866	0.70508	0.34163	0.16103	0.16726
		1.0	0.2	30	0.14081	10.77631	0.13523	0.06406	0.86075	0.56084	0.28527	0.89396
				50	0.13436	9.92813	0.11364	0.04636	0.82613	0.53205	0.26826	0.74024
				80	0.12958	9.31047	0.09227	0.03402	0.80027	0.51423	0.25072	0.62447
				120	0.12412	8.57668	0.08615	0.02599	0.77767	0.48996	0.24358	0.53804
				200	0.11735	7.82427	0.07584	0.01808	0.75273	0.46890	0.22979	0.44610
			1.0	30	0.13478	6.30475	0.10031	0.14121	0.83503	0.43935	0.23553	0.31550
				50	0.13596	6.45333	0.07798	0.13592	0.83364	0.44446	0.22315	0.30416
				80	0.13286	6.45973	0.07053	0.13021	0.82032	0.44585	0.21907	0.29453
				120	0.13041	6.35117	0.06465	0.12511	0.81164	0.44227	0.21248	0.28615
				200	0.12877	6.01743	0.06047	0.11301	0.79971	0.42575	0.20764	0.26645
			2.1	30	0.13943	3.00535	0.09129	0.16094	0.83121	0.29335	0.22177	0.17080
				50	0.13487	3.15627	0.06658	0.15843	0.81420	0.31096	0.20301	0.16826
				80	0.13226	3.17652	0.05682	0.15189	0.80471	0.31424	0.19417	0.16240
				120	0.12884	3.17848	0.05081	0.14954	0.79209	0.31756	0.18716	0.16047
				200	0.12437	3.04845	0.04663	0.13947	0.77656	0.31011	0.18324	0.15217
		2.9	0.2	30	0.17951	7.95153	2.45940	0.04290	1.02344	0.44656	0.42769	0.64137
				50	0.17381	7.19285	1.51339	0.02621	0.99755	0.42700	0.35268	0.49576
				80	0.16933	6.05099	1.09783	0.01473	0.97383	0.38700	0.30711	0.37004
				120	0.16697	5.10745	0.94117	0.00894	0.96031	0.35410	0.28831	0.29085
				200	0.15887	4.13680	0.69138	0.00516	0.92471	0.31616	0.25194	0.22601
			1.0	30	0.17370	4.83231	1.84641	0.11038	1.00427	0.36459	0.35035	0.25648
				50	0.17101	4.70194	1.24627	0.10210	0.98937	0.35897	0.30872	0.24227
				80	0.16848	4.43052	0.93814	0.09196	0.97678	0.34597	0.28206	0.22599
				120	0.16452	4.14542	0.79372	0.08225	0.95865	0.33149	0.26640	0.21084
				200	0.15784	3.66204	0.65126	0.06828	0.92787	0.31061	0.24696	0.18942
			2.1	30	0.17149	2.09822	1.81751	0.13075	0.99117	0.24260	0.33933	0.14480
				50	0.16755	2.12769	1.16057	0.12337	0.96736	0.24763	0.29057	0.13822
				80	0.16239	2.10199	0.86927	0.11692	0.94403	0.24585	0.26242	0.13290
				120	0.15876	2.06792	0.71368	0.11345	0.92780	0.24406	0.24584	0.13072
				200	0.15433	1.94168	0.59878	0.10295	0.91211	0.23395	0.23264	0.12299

7. MODELING REAL-LIFE DATA

In this section, we explore the importance and flexibility of the GLFR-E distribution by modeling two real-life data sets from engineering and insurance fields. The MLEs of the model parameters are computed and analytical measures for the GLFR-E model are compared with some competing models. The first data consist of 63-observation and represent strengths of 1.5 cm glass fibers, originally obtained by workers at the UK National Physical Laboratory. This data set was studied by Mead et al. (2019, 2020), and the data are:

0.55, 0.93,1.25, 1.36, 1.49, 1.52, 1.58, 1.66,1.69,1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.5, 1.55, 1.61, 1.62,1.81, 2, 0.74, 1.04, 1.62, 1.66,1.7, 1.77, 1.84,1.27, 1.39, 1.49, 1.53,1.59, 1.61,1.66,1.68, 1.76, 1.82, 2.01, 0.77,1.61, 1.64,1.68, 1.73, 1.11, 1.28, 1.42, 1.5, 1.54, 1.6, 0.84, 1.24, 1.3, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67,1.7,1.78, 1.89.

The second data consist of 58-observation and represent monthly metrics on unemployment insurance from July 2008 to April 2013 as reported by the department of Labor, Licensing and Regulation. The data contain 21-variable and here we consider the variable number 6 in the data file which is available at: https://catalog.data.gov/dataset/unemployment-insurance-data-july-2008-to-april-2013. The data are:

8.0, 18.2, 22.7, 16.3, 11.1, 26.9, 26.6, 18.9, 24.3, 20.6, 40.7, 56.9, 49.3, 55.0, 49.4, 50.7, 46.6, 87.9, 79.2, 78.9, 82.1, 63.7, 61.7, 68.8, 60.8, 69.2, 55.9, 54.8, 65.4, 53.7, 65.5, 52.7, 52.9, 50.7, 61.3, 49.6, 47.5, 58.9, 47.4, 56.0, 46.9, 46.5, 57.9, 45.7, 44.5, 53.1, 44.1, 41.8, 48.2, 37.1, 32.7, 37.6, 42.8, 47.4, 35.6, 32.2, 30.1, 31.2.

The GLFR-E model is compared with some comparative models including the beta generalized-E (BG-E) (Barreto-Souzaet al., 2010), Marshall-Olkin alpha-power-E (MOAP-E) (Nassar et al., 2019), beta-E (B-E) (Jones, 2004), transmuted generalized-E (TG-E) (Khan et al. 2017), alpha-power-E (AP-E) (Mahdavi and Kundu, 2017) and E distributions.

The analytical measures are computed to compare the fitted models, include the Akaike information (AIC), consistent Akaike information (CAIC), Bayesian information (BIC), and Hannan-Quinn information (HQIC) criterions, along with Anderson–Darling (A^*) and Cramér–von Mises (W^*) statistics.

The numerical values of these measures are reported in Tables 8 and 9, for both data sets, respectively. The MLEs of the parameters of all studied models and associated standard errors (SEs) are reported in Tables 10 and 11, for both data sets, respectively.

Figures 11 and 12 depicted the estimated pdf, cdf, sf, and PP plots of the GLFR-E model for both data sets. The PP plots of the GLFR-E model and other competing models are shown in Figure 13 (for first data) and Figure 14 (for second data). The visual comparisons in these plots support the numerical results in Tables 2 and 3.

Figures 15 and 16show the plots the profile likelihood functions of parameters of the GLFR-E model for both data, respectively. One can see the plots the profile likelihood functions are unimodal of all estimated parameters.

Table 8
Analytical Measures for First Data Set

Model	$-l$	AIC	CAIC	BIC	HQIC	A*	W*
GLFR-E	14.0675	36.1351	36.8247	44.7076	39.5067	0.89203	0.15809
BG-E	14.6583	37.3166	38.0062	45.8891	40.6882	1.12115	0.20211
MOAP-E	16.0037	38.0074	38.4142	44.4368	40.5361	1.44137	0.26181
B-E	23.9592	53.9183	54.3251	60.3477	56.4471	3.11881	0.56866
TG-E	28.4755	62.9510	63.3578	69.3804	65.4797	3.82655	0.69923
AP-E	35.0383	74.0767	74.2767	78.3629	75.7625	3.52412	0.64175
E	88.8303	179.661	179.726	181.804	180.504	3.12704	0.57020

Table 9
Analytical Measures for Second Data Set

Model	$-l$	AIC	CAIC	BIC	HQIC	A*	W*
GLFR-E	247.721	503.444	504.198	511.685	506.654	0.41360	0.07618
BG-E	247.924	503.849	504.603	512.090	507.059	0.49244	0.09536
MOAP-E	250.144	506.289	506.733	512.470	508.694	0.69612	0.13547
AP-E	252.852	509.704	509.922	513.825	511.309	1.34821	0.25789
B-E	252.345	510.690	511.135	516.872	513.098	1.36049	0.26134
TG-E	252.965	511.931	512.375	518.112	514.338	1.45801	0.27882
E	281.228	564.455	564.527	566.516	565.258	1.32868	0.25559

Table 10
Estimated Parameters and their SEs for the First Data Set

Model	Estimates(SEs)			
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{a}$
GLFR-E	0.051331 (0.072384)	0.014579 (0.021386)	1.910410 (1.015134)	1.534822 (0.302516)
BG-E	0.558389 (0.251316)	224589.2 (72373.33)	0.181090 (0.095530)	9.598947 (3.656231)
MOAP-E	528.9524 (462.9113)	5.843255 (0.256699)	1297.571 (490.0657)	
B-E	17.44929 (3.081843)	134.5357 (243.8003)	0.081217 (0.138802)	
TG-E	31.15388 (11.14173)	2.909722 (0.257874)	-0.695736 (0.178694)	
AP-E	1920499 (23726.92)	2.071124 (0.098884)		
E				0.663647 (0.083611)

Table 11
Estimated Parameters and their SEs for the Second Data Set

Model	Estimates(SEs)			
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{a}$
GLFR-E	0.551220 (1.074932)	0.741379 (7.266695)	1.868851 (2.521885)	0.016517 (0.034208)
BG-E	0.344773 (0.365885)	42.01581 (164.7110)	0.015541 (0.025777)	9.035319 (13.20707)
MOAP-E	54515.40 (5938.390)	0.097253 (0.013415)	7.939736 (5.859727)	
B-E	5.691805 (1.057629)	5.855034 (3.669163)	0.015382 (0.007579)	
TG-E	5.563676 (1.727998)	0.058321 (0.006392)	-0.641725 (0.251006)	
AP-E	5120.593 (7544.148)	0.056671 (0.004904)		
E				0.021306 (0.002791)

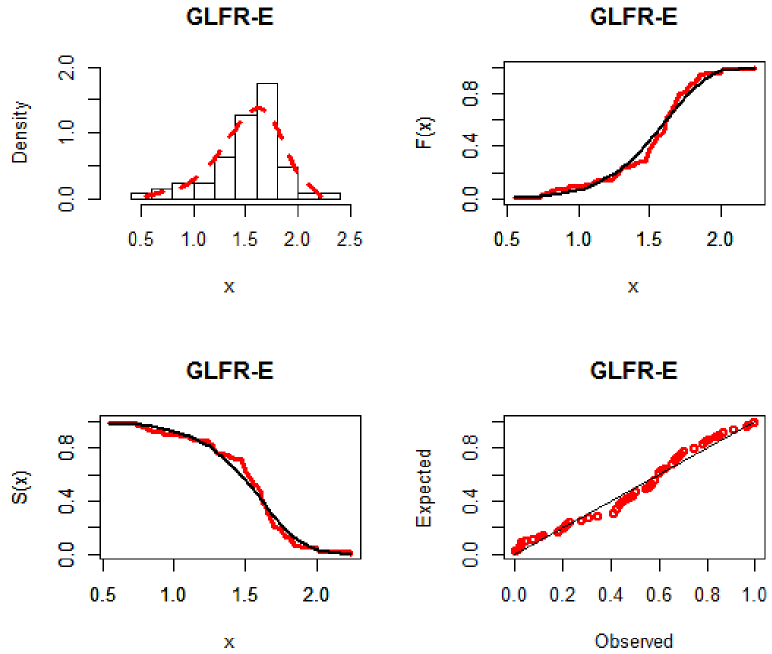

Figure 11: The Estimated pdf, cdf, sf and PP Plots of the GLFR-E Model for the First Data

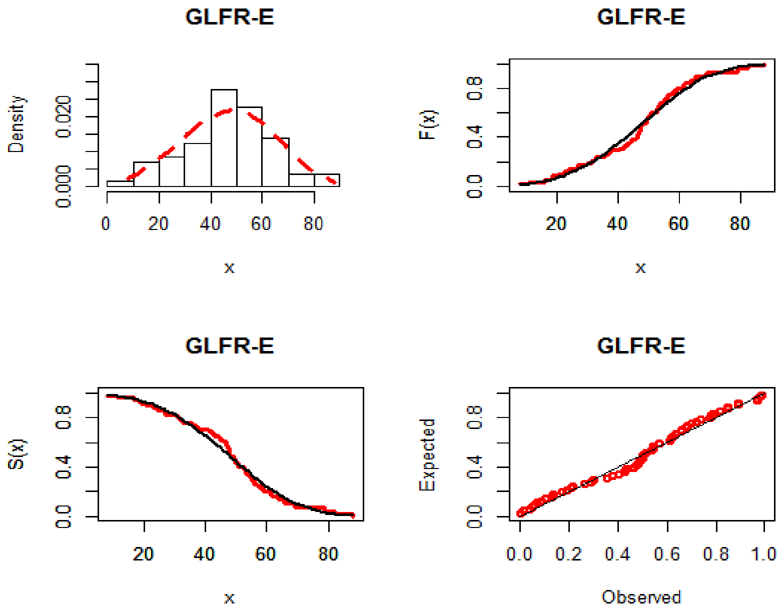

Figure 12: The Estimated pdf, cdf, sf and PP plots of the GLFR-E Model for the Second Data

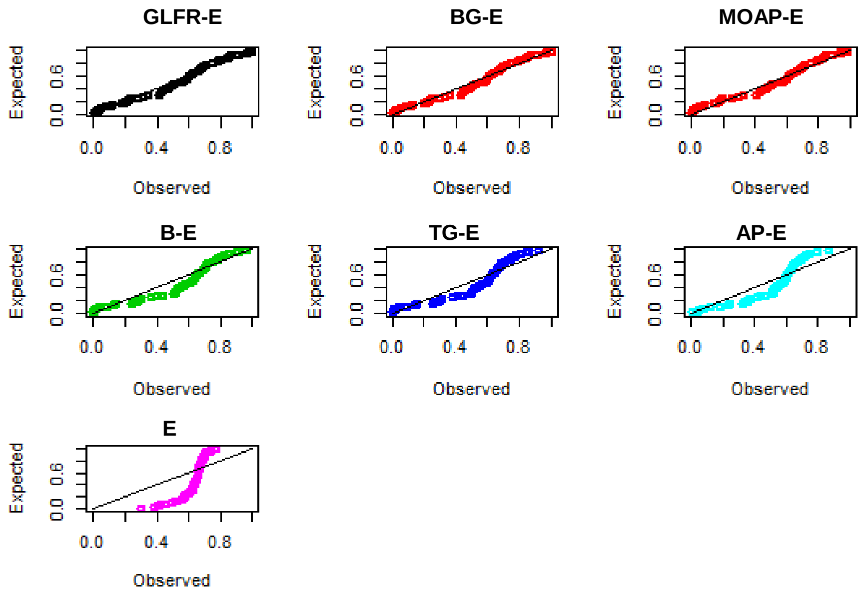

Figure 13: PP Plots of the GLFR-E Model and other Competing Models for the First Data

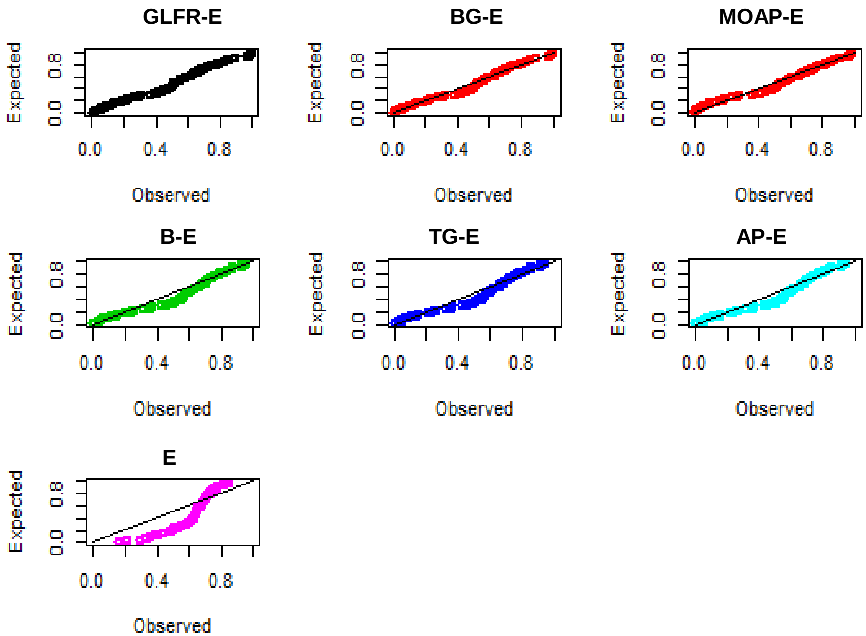

Figure 14: PP Plots of the GLFR-E Model and other Competing Models for the Second Data

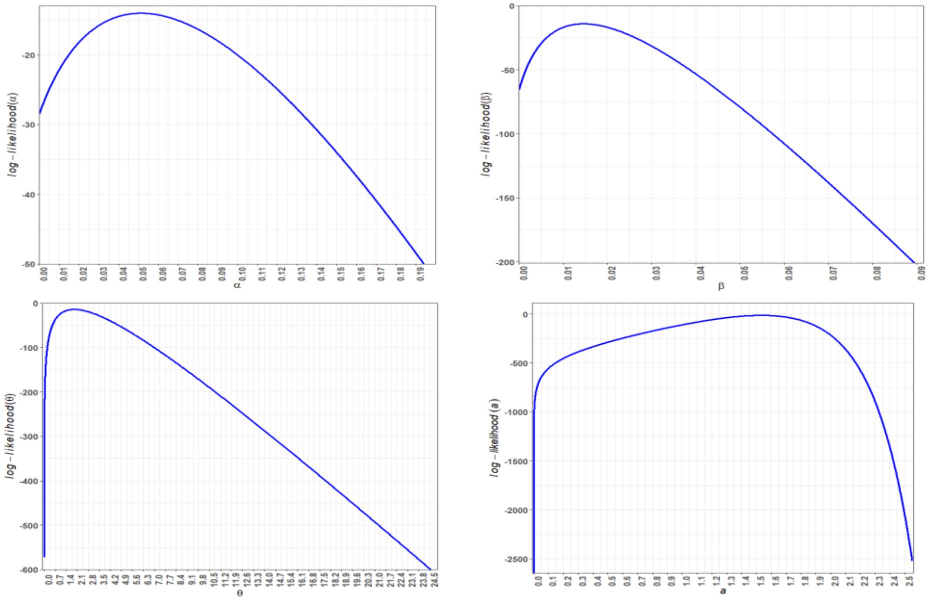

Figure 15: Plots of the Profile Likelihood Functions of Parameters for the First Data (GLFR-E Model)

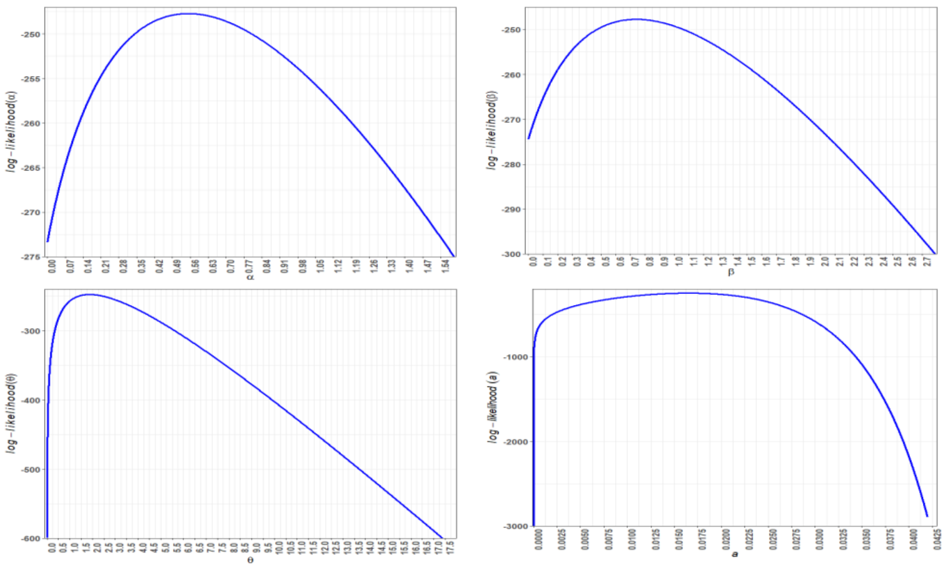

Figure 16: Plots of the Profile Likelihood Functions of Parameters for the Second Data (GLFR-E Model)

8. CONCLUDING REMARKS

This paper introduces a new family to generate more flexible distributions than the classical distributions by adding three additional parameters to the baseline distribution. The proposed family called generalized linear failure rate (GLFR-G) family. Some mathematical properties of the introduced family are provided. Comparisons of ordered families were discussed by the likelihood ratio ordering, which is the strongest well-known ordering in the reliability theory. We provide four special sub-models correspond to the baseline exponential, Lomax, Fréchet, and Weibull distributions. The maximum likelihood method is used for estimating the model parameters. The GLFR-exponential distribution is adopted to model two real data sets from engineering and insurance sciences. The GLFR-exponential distribution provides consistently better fit than other comparative distributions for the analyzed data.

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