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BAYESIAN INFERENCE FOR THE PARAMETERS OF EXPONENTIATED CHEN DISTRIBUTION BASED ON HYBRID CENSORING

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ABSTRACT

This article presents classical and Bayesian inferences based on Type-II hybrid censored data when the lifetime of the items follows the exponentiated Chen distribution. Based on the hybrid censored data, the maximum likelihood estimates and the asymptotic confidence intervals of the involved parameters are derived. Bayes estimates, under squared error loss and linear- exponential loss functions, and highest posterior density intervals are also derived. Monte Carlo simulations are performed to see the effectiveness of the proposed estimation methods. The Application of real data from Vaccination of COVID-19 data for different countries of the region of the Americas is discussed.

KEYWORDS

Hybrid censoring; Exponentiated Chen distribution; Maximum likelihood method; Bayesian inference; Monte Carlo simulation.

ABBREVIATIONS:

EC	Exponentiated Chen
PDF	Probability Density Function
CDF	Cumulative Distribution Function
ML	Maximum Likelihood
MLE	ML Estimate
HCS	Hybrid Censoring Scheme
CI	Confidence Interval
LINEX	Linear- Exponential
MCMC	Markov Chain Monte Carlo
MSE	Mean Squared Error

1. INTRODUCTION

The two-parameter distribution, called the exponentiated Chen distribution EC (α,β) distribution was introduced by Chaubey and Zhang (2015). The EC (α,β) has the probability density function (PDF)

$$f(x)=\alpha\beta x^{\alpha-1}e^{1+x^{\alpha}-e^{x^{\alpha}}}\left(1-e^{1-ex^{\alpha}}\right)^{\beta-1},\quad x>0,\alpha,\beta>0, \tag{1}$$

and cumulative distribution function (CDF)

$$F(x)=\left(1-e^{1-ex^{\alpha}}\right)^{\beta},\quad x>0,\alpha,\beta>0, \tag{2}$$

where α,β are the shape parameters of the EC distribution. Xie et al. (2002) named the extended-Weibull distribution by extending the Chen distribution by adding another parameter. Another branch of the Chen family was introduced by Chaubey and Zhang (2015). Afshari et al. (2021) introduced extended exponentiated Chen distribution with different Applications. They also looked at the challenge of estimating EC distribution parameters, focusing on the maximum likelihood (ML) estimate approach. Khan et al. (2016) proposed the transmuted exponentiated Chen distribution and investigated various structural properties with some applications. Dey et al. (2017) developed statistical properties and a number of estimation strategies for estimating the EC distribution's unknown parameters.

Type-I and Type-II censoring schemes are the most common censoring schemes which are used in practice. A mixture of Type-I and Type-II censoring schemes, known as hybrid censoring scheme (HSC) was introduced by Epstein (1954). Like the standard Type-I and Type-II censoring schemes, HCS is divided into two types: Type-I and Type-II hybrid censoring schemes. The test is stopped at a time $T=min(x_r,T_o)$ in Type-I hybrid censoring, where x_r indicates the failure time of the r^{th} item and T_o is the pre-fixed maximum permissible time of the test. The test is stopped at a time $T=max(x_r,T_o)$ in Type-II hybrid censoring. In the Type-II hybrid censoring scheme, the test contains at least r failure items, whereas, in the Type-I hybrid censoring scheme, the test can never be reached beyond the time T_o.

Type-II hybrid censoring has been discussed extensively in the reliability literature. Banerjee and Kundu (2008) considered Type-II hybrid censoring lifetime data when the lifetime distribution of each unit follows a two-parameter Weibull distribution. Panahi and Asadi (2011) presented the statistical inferences based on Type-II hybrid censored sample from a Burr type XII distribution. Balakrishnan and Kundu (2013) discussed Type-I and Type-II HCSs. Singh et al. (2013) provided the Bayesian procedure for the prediction of the future samples from inverse Weibull distribution under Type-II hybrid censoring scheme. El-Shahat and Ebrahim (2014) considered a simple step-stress model under the Rayleigh distribution when the available data are Type-II hybrid censored. Kohansal et al. (2015) considered the estimation of parameters of weighted exponential distribution based on Type-II hybrid censored data. Koley et al. (2017) considered the analysis of Type-II hybrid censored competing risks data.

The rest of the article is organized as follows. In Section 2, the MLEs of two unknown parameters of EC distribution and the corresponding asymptotic variances and covariance are derived. In Section 3, we discuss the Bayes estimates via Markov Chain Monte Carlo (MCMC) procedure. In Section 4, we present numerical comparisons between the MLEs and the Bayes estimates. The Application of real data from Vaccination of COVID-19 data is discussed in Section 5. Finally, conclusions have been given in Section 6.

2. MAXIMUM LIKELIHOOD ESTIMATION

In this section, we obtain the MLEs of the parameters α, β based on Type-II hybrid censored data and then derive their asymptotic variance-covariance matrix.

2.1 Maximum Likelihood Estimates

Under Type-II HSC mentioned above, we have one of the two following types of observations:

Case I: $X_{1:n} < \cdots < X_{r:n}$ if $X_{r:n} > T_a, r < n$.
Case II:$X_{1:n} < \cdots < X_{k:n}$ if $X_{k:n} < T_a < X_{k+1:n}, r < k < n$.

Thus, the likelihood function of such a Type-II hybrid censored sample is as follows:

Case I: First write the explicit equation

$$l(\alpha,\beta \mid x) = \frac{n!}{(n-r)!}\left(1-\left(1-e^{1-e^{x_r^{\alpha}}}\right)^{\beta}\right)^{n-r} \prod_{i=1}^{r} \alpha\beta x_i^{\alpha-1} e^{1+x_i^{\alpha}-e^{x_i^{\alpha}}}\left(1-e^{1-e^{x_i^{\alpha}}}\right)^{\beta-1} \tag{3}$$

Case II:

$$l(\alpha,\beta \mid x) = \frac{n!}{(n-r)!}\left(1-\left(1-e^{1-e^{T_a^{\alpha}}}\right)^{\beta}\right)^{n-k} \prod_{i=1}^{k} \alpha\beta x_i^{\alpha-1} e^{1+x_i^{\alpha}-e^{x_i^{\alpha}}}\left(1-e^{1-e^{x_i^{\alpha}}}\right)^{\beta-1} \tag{4}$$

where $x_i = x_{i:n}$. Combining Eq. (3) and Eq. (4), the likelihood function may be written as

$$l(\alpha,\beta \mid x) = \frac{n!(\alpha\beta)^{D}}{(n-D)!}\left(1-\left(1-e^{1-e^{W^{\alpha}}}\right)^{\beta}\right)^{n-D} \prod_{i=1}^{D} x_i^{\alpha-1} e^{1+x_i^{\alpha}-e^{x_i^{\alpha}}}\left(1-e^{1-e^{x_i^{\alpha}}}\right)^{\beta-1} \tag{5}$$

where D denotes the number of failures and $W = x_r$ if $D = r$, and $W = T$ if $D > r$. Therefore, the log-likelihood function may be written as

$$\begin{aligned} L(\alpha,\beta \mid x) = {} & \ln\frac{n!}{(n-D)!} + D\ln\alpha\beta + (a-1)\sum_{i=1}^{D}\ln x_i + \sum_{i=1}^{D}\left(1+x^{\alpha}-e^{x^{\alpha}}\right) \\ & +(\beta-1)\sum_{i=1}^{D}\ln \mathrm{A}_1(x_i) + (n-D)\ln A_2(W), \end{aligned} \tag{6}$$

where

$$A_1(x_i) = \left(1 - e^{1-ex^{\alpha}}\right), \tag{7}$$

$$A_2(W) = 1 - \left(A_1(W)\right)^{\beta}. \tag{8}$$

and $A_1(W)$ is as given in Eq. 7.

Taking the derivatives of Eq. (6) with respect to α and β, respectively, we have

$$\frac{\partial L}{\partial \alpha} = \frac{D}{\alpha} + \sum_{i=1}^{D} \left(1 + x_i^{-\alpha} - e^{x_i^{\alpha}} + \frac{(\beta-1)}{A_1(x_i)} e^{1+x_i^{\alpha}-e^{x_i^{\alpha}}}\right) x_i^{\alpha} \ln x_i \\ - \frac{\beta(n-D)}{A_2(W)} \left(A_1(W)\right)^{\beta-1} e^{1+W^{\alpha}-e^{W^{\alpha}}} W^{\alpha} \ln W, \tag{9}$$

$$\frac{\partial L}{\partial \beta} = \frac{D}{\beta} + \sum_{i=1}^{D} \ln A_1(x_i) - \frac{n-D}{A_2(W)} \left(A_1(W)\right)^{\beta} \ln A_1(W). \tag{10}$$

Since Equations Eq. (9) and Eq. (10) cannot be solved analytically for α and β, some numerical methods such as Newton's method must be employed.

2.2 Asymptotic Variances and Covariance

In this subsection, we compute the observed Fisher information for the MLEs to construct CIs for the parameters.

The observed Fisher information matrix is given by

$$F = -\left(\frac{\partial^2 \hat{L}}{\partial \theta \partial \theta^T}\right), \theta = (\alpha^T, \beta^T)^T,$$

where the caret $\widehat{\ }$ indicates that the derivative is calculated at the MLEs $(\hat{\alpha}, \hat{\beta})$.

As a result, inverting the observed Fisher's information matrix defines the asymptotic variance-covariance matrix for the MLEs of the parameters α and β.

We construct the approximate CIs for α and β based on the asymptotic normality of the MLEs. The $100(1-\gamma)\%$ CIs for α and β are calculated as follows:

$$(\hat{\alpha} \mp z_{1-\gamma/2}\sqrt{var(\hat{\alpha})}) \text{ and } (\hat{\beta} \pm z_{1-\gamma/2}\sqrt{var(\hat{\beta})}),$$

where $z_{1-\gamma/2}$ is the upper $(\gamma/2)$ percentile of the standard normal distribution.

3. BAYES ESTIMATION

We use the Bayesian technique in this section to construct estimates for α and β, as well as the accompanying credible intervals, using Type-II HCS. There are two sorts of loss functions that we have assumed. The first is an asymmetric function called the squared error loss function. The second is the asymmetric LINEX loss function, which was introduced by Varian (1975).

The following are the squared-error and LINEX loss functions, respectively:

$$L_{BS}\left(\theta^*,\theta\right)\propto\left(\theta^*-\theta\right)^2,$$

$$L_{BL}\left(\Delta\right)\propto e^{c\Delta}-c\Delta-1, c\neq 1,$$

where $\Delta=\theta^*-\theta$and θ^* is an estimate of θ.

3.1 Prior and Posterior Distributions

Under the premise that the joint prior density of α and β is as follows, we analyze Bayesian estimation:

$$g\left(\alpha,\beta\right)=g_1\left(\alpha\right)g_2\left(\beta\,|\,\alpha\right),$$

where α and β are dependent on joint prior density, as proposed by Nassar and Eissa (2004), and $\alpha\sim$ Exp(ν) and $\beta|\alpha\sim$ Gamma (μ,α). As a result, the α and β joint prior is stated as:

$$g\left(\alpha,\beta\right)\propto\alpha^{-\mu}\,\beta^{\mu-1}\,e^{-\left(\frac{\alpha}{\nu}+\frac{\beta}{\alpha}\right)}. \tag{11}$$

The joint posterior density function of α and β can be written from Eq. (5) and Eq. (11) as

$$\pi\left(\alpha,\beta\right)\propto l(\alpha,\beta\,|\,x)\;g\left(\alpha,\beta\right)\propto\alpha^{D-\mu}\,\beta^{D+\mu-1}\,e^{-\left(\frac{\alpha}{\nu}+\frac{\beta}{\alpha}\right)}$$
$$\prod_{i=1}^{D}x_i^{\alpha-1}\,e^{1+x_i^{\alpha}-e^{x_i^{\alpha}}}\left(1-e^{1-e^{x_i^{\alpha}}}\right)^{\beta-1}\times\left(1-\left(1-e^{1-e^{W^{\alpha}}}\right)^{\beta}\right)^{n-D}. \tag{12}$$

Under squared-error loss function, the Bayes estimate of any function of α and β say $G(\theta)\equiv G(\alpha,\beta)$ is

$$\hat{G}_{BS}=E(G\left(\theta\right)|\,x)=\int_0^\infty\int_0^\infty G\left(\theta\right)l(\alpha,\beta\,|\,x)\;g\left(\alpha,\beta\right)d\alpha\,d\beta. \tag{13}$$

Based on LINEX loss function, the Bayes estimate is

$$\hat{G}_{BL}=\frac{-1}{c}E(e^{-cG(\theta)}\,|\,x)=\frac{-1}{c}\int_0^\infty\int_0^\infty e^{-cG(\theta)}l(\alpha,\beta\,|\,y)\;g\left(\alpha,\beta\right)d\alpha\,d\beta \tag{14}$$

Eq. (13) and Eq. (14) can't be computed analytically. Therefore, we adopt MCMC procedure to approximate these integrals.

3.2 Markov Chain Monte Carlo

The MCMC method is a method for calculating parameters that are different from the traditional method. When compared to traditional procedures, it is more adaptable. Probability intervals are also offered. The probability intervals provide us a decent estimation of the unknown parameter's interval. The MCMC approach is proposed for drawing samples from the posterior density function, computing Bayes estimates, and constructing the highest posterior density credible intervals.

The conditional posterior distributions of the parameters α and β are written as $\pi(\alpha|\beta,x)$ and $\pi(\beta|\alpha,x)$, respectively.

$$\pi(\alpha\,|\,\beta,x)\propto\alpha^{D-\mu}\,e^{-\left(\frac{\alpha}{\nu}+\frac{\beta}{\alpha}\right)}\prod_{i=1}^{D}x_i^{\alpha}\,e^{x_i^{\alpha}-e^{x_i^{\alpha}}}\left(1-e^{1-e^{x_i^{\alpha}}}\right)^{\beta-1}\left(1-\left(1-e^{1-e^{W^{\alpha}}}\right)^{\beta}\right)^{n-D},$$

$$(\beta\,|\,\alpha,x)\propto\beta^{D+\mu-1}\,e^{-\frac{\beta}{\alpha}}\prod_{i=1}^{D}\left(1-e^{1-e^{x_i^{\alpha}}}\right)^{\beta}\left(1-\left(1-e^{1-e^{W^{\alpha}}}\right)^{\beta}\right)^{n-D}.$$

It can be seen that, both conditional posterior distributions of α and β cannot be reduced analytically to well-known distributions and therefore it is not possible to sample directly by standard methods, but the plots of them show that they are similar to normal distribution. The Metropolis-Hastings method will be used to generate random numbers from this distribution. The following is the proposed scheme for generating α and β from posterior density functions, and then obtaining Bayes estimates and credible intervals:

Step 1. Choose the MLEs $\hat{\alpha}$ and $\hat{\beta}$, as the starting values $(\alpha^{(0)},\beta^{(0)})$ of α.

Step 2. Set $i=1$

Step 3. Generate $\alpha^{(*)}$ from $\mathrm{N}(\alpha^{(i-1)},V(\hat{\alpha}))$.

Step 4. Calculate the acceptance probabilities

$$U_1^*=min\left(1,\frac{\pi\left(\alpha^{(*)}|\beta^{(i-1)},x\right)}{\pi(\alpha^{(i-1)}|\beta^{(i-1)},x)}\right)$$

Step 5. Generate U_1 follow a Uniform (0, 1) distribution.

Step 6. If $U_1\leq U_1^*$, set $\alpha^{(i)}=\alpha^{(*)}$, else set $\alpha^{(i)}=\alpha^{(i-1)}$.

Step 7. Generate $\beta^{(*)}$ from $\mathrm{N}(\beta^{(i-1)},V(\hat{\beta}))$.

Step 8. Calculate the acceptance probabilities

$$U_2^*=min\left(1,\frac{\pi\left(\beta^{(*)}|\alpha^{(i-1)},x\right)}{\pi(\beta^{(i-1)}|\alpha^{(i-1)},x)}\right)$$

Step 9. Generate U_2 follow a Uniform (0, 1) distribution.

Step 10. If $U_2\leq U_2^*$, set $\beta^{(i)}=\beta^{(*)}$, else set $\beta^{(i)}=\beta^{(i-1)}$.

Step 11. Set $i=i+1$.

Step 12. Repeat steps 3-11 N times and obtain $\alpha^{(i)}$ and $\beta^{(i)}, i=1,\dots,N$.

Step 13. Remove the first M values for α and β, which is the burn-in period, to construct the credible intervals for α and β based on the generated values, then sort the $(N-M)$ remaining values for ascending to be $\alpha_1,\alpha_2,\dots,\alpha_{N-M}$, and also for β to be $\beta_1,\beta_2,\dots,\beta_{N-M}$, where M is the burn-in period.

Step 14. The approximate means of $G(\alpha;\beta)$ and $e^{-cG(\theta)}$ are given, respectively, by

$$\hat{G}_{BS} = E(G(\alpha,\beta)|data) = \frac{1}{N-M}\sum_{i=M+1}^{N} G(\alpha,\beta)^{(i)},$$

$$\hat{G}_{BL} = \frac{-1}{c}E\left(e^{-cG(\alpha;\,\beta)}|data\right) = \frac{-1}{c}\,ln\left[\frac{1}{N-M}\sum_{i=M+1}^{N} e^{-cG(\alpha;\,\beta)^{(i)}}\right].$$

To obtain Bayes credible intervals of the parameters as follows:

1. Arrange $\alpha^{[1]},\alpha^{[2]},\ldots,\alpha^{[L]}$, and $\beta^{[1]},\beta^{[2]},\ldots,\theta^{[L]}$ where L is length of simulation generated.
2. The $100(1-\gamma)\%$ symmetric credible intervals ofα and βbecome
   $\left\{\alpha^{\left[M\frac{\gamma}{2}\right]},\alpha^{\left[M\left(1-\frac{\gamma}{2}\right)\right]}\right\}$ and $\left\{\beta^{\left[M\frac{\gamma}{2}\right]},\beta^{\left[M\left(1-\frac{\gamma}{2}\right)\right]}\right\}$.

4. SIMULATION STUDY

We use Monte Carlo simulations to compare the results of various estimates of an EC distribution's unknown parameters. We primarily compare the average point estimation (mean) and mean squared errors (MSEs) of MLEs with Bayes estimates derived using the MCMC technique based on squared error and LINEX loss functions. Using the asymptotic distribution of the MLE, we estimated approximately 95% CIs for the unknown parameters of EC distribution, as well as the highest probability density (HPD) credible intervals by their coverage percentages. We repeat the process 1000 times and return the average estimates, MSEs, confidence/credible lengths, and percentage coverage for MLE. Tables 1-6 summarize all of the findings. The point estimate for MLE and Bayesian estimation based on squared-error and LINEX loss function where BL1 is LINEX with $c = 0.5$ and BL2 is LINEX with $c = 1.5$ are shown in Tables 1, 2 and 3, as well as average estimation (Av.) for 1000 loop and MSE. Under squared-error and LINEX loss functions, intervals estimation for MLE based on asymptotic CIs (ACI) and Bayesian estimate based on credible intervals are shown in Tables 4, 5, and 6. The length of CIs is determined (L.CI).

Monte-Carlo experiments are performed on the basis of data-generated 10000 random EC distribution samples, where x has EC distribution for various parameter actual values and different times of each case of model:

Case 1: $\alpha = 0.5, \beta = 0.5$ and T are chosen as 0.25 and 0.7.
Case 2: $\alpha = 0.5, \beta = 2$ and T are chosen as 0.7 and 1.5.
Case 3: $\alpha = 2, \beta = 2$ and T are chosen as 0.7 and 1.5.

We looked at different n, r, and T values. The sample sizes of 40, 60, and 80 were chosen. The value of r has been taken to be equal to 60% and 80% of the sample size n.

The same approach as described in Dey et al. (2016) is used to elicit the hyper-parameters of the informative priors. As a result, these informative prior hyper-parameters are generated from the MLE of (α,β) by equating their values with the mean and variance using the inverse Fisher information matrix of (α,β). For more information

see Bantan et al. (2021), El-Sherpieny et al. (2021), Haj Ahmad et al. (2021), Ahmad and Almetwally (2020) and Almetwally et al. (2020).
Tables 1-6 are also summarized as follows in the subsequent observations.

1. For fixed values of T and r, the MSE and L.CI decrease as the sample size n increases.
2. For fixed values of n and T, the MSE and L.CI decrease as the number of failure items r increases.
3. For fixed values of n and r, the MSE and L.CI decrease as the item T increases.
4. For fixed values of n, r, T and α, the MSE and L.CI are increase as the item β increases.
5. For fixed values of n, r, T and β, the MSE and L.CI are increase as the item α increases.
6. In all the considered cases, the Bayes estimates have the smallest MSE as compared with their corresponding MLEs.
7. The Bayesian estimates based on LINEX loss function are efficient than anther estimation methods for most cases of the EC distribution under Type-II hybrid censored samples.
8. Estimates obtained from the LINEX loss function $c = 1.5$ gives a better choice than $c = 0.5$ choice.
9. The MSE and L.CI decrease as the value constant of LINEX loss function c increases.
10. Moreover, the width of the HPD credible intervals is smaller than the width of the asymptotic CIs in all the cases.

5. APPLICATION

In this section, the real data from Vaccination of COVID-19 data for different countries of the region of the Americas (AMRO) are given in 18.7.2021 from World Health Organization- https (WHO) see this link (https://covid19.who.int/who-data/vaccination-data.csv). Many Authors discussed application of COVID-19 data as Hassan et al. (2021), Almetwally (2021), and Almongy et al. (2021). We used data of cumulative total vaccine doses ad- ministered per 100 population for AMRO. The data was presented in Table 7. The goodness of fit test is done by using Kolmogorov-Smirnov. We obtained the Kolmogorov-Smirnov statistics (KS) with P-value for different models to find the best model fit on this data. Also, we used different measures as the Akaike information criterion (AIC), corrected AIC (CAIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC) to be sure this model is the best model fit for this data see Table 8. The EC distribution is compared with other related models such as Weibull (W), inverse Weibull (IW), inverse Kumaraswamy (IK), generalized Rayleigh (GR), Lomax, inverse Lomax (IL), Kumaraswamy inverse Topp–Leone (KITL) [Hassan et al. (2021)]. By Table 8 and Figure 1, we concluded the EC distribution is best model to fit of this data. According this concluding, we can use this distribution to estimate parameter under different censored sample. Table 9 show the MLE and Bayesian estimation method point and interval estimation based on Type-II hybrid censored data. In Figure 2 History plots, MCMC convergence of α and β are represented for EC model based on Type-II hybrid censored data when $m = 40$ and $T = 120$.

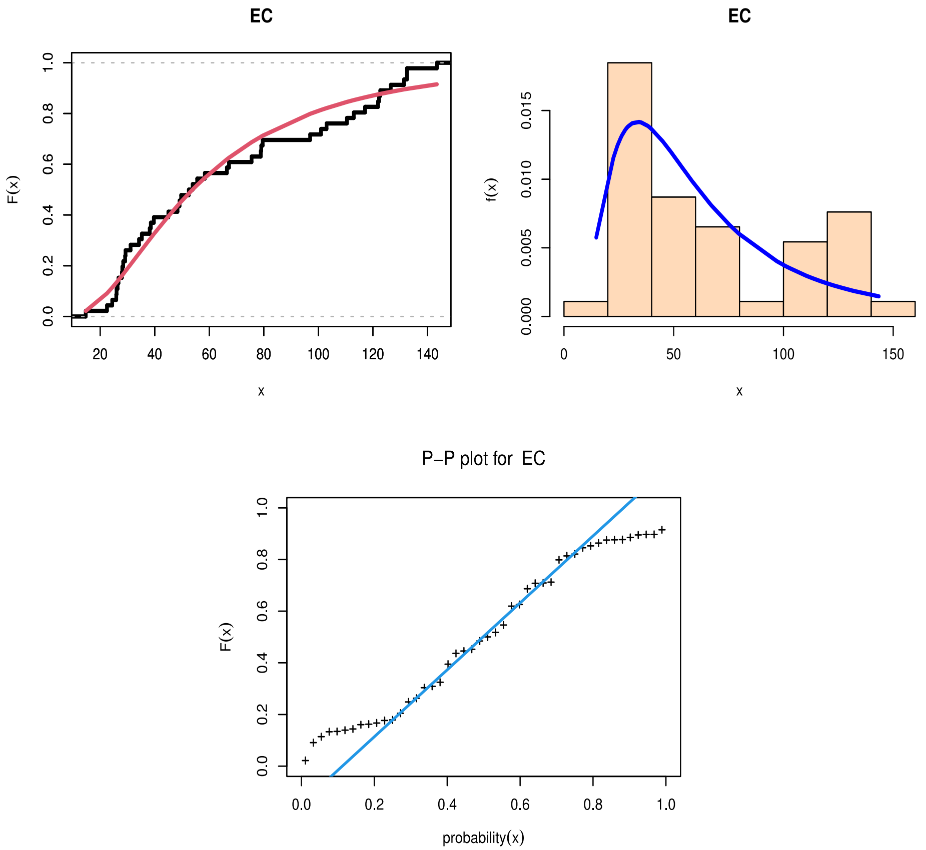

Figure 1: The cdf with Empirical cdf, Histogram with Estimated pdf and PP-Plot of EC Distribution

Table 1
The Average and the MSE of MLE, BS and BL for $\alpha = 0.5, \beta = 0.5$

α = 0.5, β = 0.5				MLE		Bayesian					
						SE		LINEX (c=0.5)		LINEX (c=1.5)	
n	T	R		Av.	MSE	Av.	MSE	Av.	MSE	Av.	MSE
40	0.25	25	α	0.5967	0.0744	0.5150	0.0138	0.5128	0.0136	0.5084	0.0131
			β	0.5561	0.1367	0.5151	0.0145	0.5129	0.0141	0.5084	0.0135
		32	α	0.5579	0.0688	0.5040	0.0050	0.5032	0.0050	0.5016	0.0049
			β	0.5620	0.1053	0.5030	0.0055	0.5022	0.0055	0.5006	0.0055
	0.7	25	α	0.5927	0.0688	0.5133	0.0152	0.5110	0.0149	0.5065	0.0143
			β	0.5462	0.1024	0.5236	0.0159	0.5212	0.0155	0.5165	0.0148
		32	α	0.5494	0.0391	0.5059	0.0053	0.5051	0.0053	0.5036	0.0052
			β	0.4773	0.0764	0.4984	0.0048	0.4976	0.0048	0.4962	0.0048
60	0.25	36	α	0.5625	0.0574	0.5180	0.0135	0.5159	0.0132	0.5117	0.0126
			β	0.5629	0.1031	0.5147	0.0135	0.5126	0.0132	0.5084	0.0127
		48	α	0.5526	0.0547	0.5025	0.0047	0.5017	0.0047	0.5002	0.0047
			β	0.5789	0.0990	0.5057	0.0052	0.5050	0.0052	0.5035	0.0051
	0.7	36	α	0.5790	0.0539	0.5121	0.0136	0.5100	0.0133	0.5059	0.0129
			β	0.5221	0.0694	0.5213	0.0152	0.5191	0.0148	0.5147	0.0141
		48	α	0.5638	0.0490	0.5028	0.0045	0.5021	0.0045	0.5006	0.0045
			β	0.4820	0.0528	0.5045	0.0047	0.5038	0.0047	0.5024	0.0046
80	0.25	48	α	0.5371	0.0434	0.5176	0.0134	0.5157	0.0131	0.5118	0.0125
			β	0.5550	0.0748	0.5145	0.0134	0.5125	0.0131	0.5086	0.0126
		64	α	0.5252	0.0367	0.5075	0.0046	0.5068	0.0046	0.5054	0.0045
			β	0.5733	0.0942	0.5010	0.0044	0.5003	0.0044	0.4990	0.0043
	0.7	48	α	0.5463	0.0454	0.5178	0.0139	0.5159	0.0136	0.5119	0.0130
			β	0.5453	0.0693	0.5145	0.0124	0.5126	0.0121	0.5086	0.0116
		64	α	0.5362	0.0378	0.5053	0.0041	0.5047	0.0041	0.5033	0.0041
			β	0.4867	0.0465	0.5030	0.0040	0.5023	0.0040	0.5010	0.0040

6. CONCLUSION

In this paper we consider the analysis of Type-II hybrid censored data for EC distribution. We carry out both the most frequent and Bayesian analyses of the unknown parameters, and it is observed that the performances of the Bayes estimates are better than the MLEs. MCMC approximation is used in the Bayesian procedure to solve the hard integration. Numerical computations and comparisons are presented to illustrate the methods of inference developed here. The Application of real data from Vaccination of COVID-19 data for different countries of the region of the Americas is discussed. The Bayesian estimation is the best estimation method to estimate parameters of EC distribution based on Type-II hybrid censored data. By using confidence intervals, the width of the HPD credible intervals is smaller than the width of the asymptotic CIs. The EC distribution is compared with other related models such as Weibull, inverse Weibull, inverse Kumaraswamy, generalized Rayleigh, Lomax, inverse Lomax, Kumaraswamy inverse Topp–Leone.

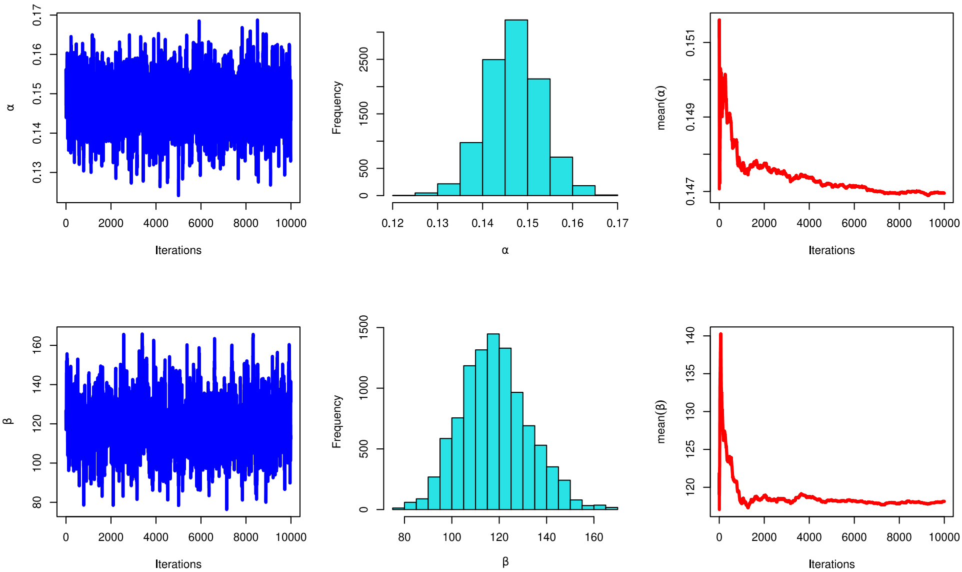

Figure 2: The MCMC Plots for Parameters of EC Distribution

Table 2
The Average and the MSE of MLE, BS and BL for $\alpha = 0.5, \beta = 2$

α = 0.5, β = 2				MLE		Bayesian					
						SE		LINEX (c=0.5)		LINEX (c=1.5)	
n	T	R		Av.	MSE	Av.	MSE	Av.	MSE	Av.	MSE
40	0.7	25	α	0.6769	0.2592	0.5144	0.0089	0.5125	0.0088	0.5087	0.0084
			β	1.9615	0.6051	1.9887	0.0281	1.9849	0.0282	1.9776	0.0284
		32	α	0.6254	0.1955	0.4992	0.0050	0.4985	0.0050	0.4970	0.0050
			β	2.0171	0.5935	1.9968	0.0078	1.9958	0.0078	1.9938	0.0078
	1.5	25	α	0.6992	0.2445	0.5165	0.0072	0.5147	0.0070	0.5111	0.0068
			β	1.8377	0.4507	1.9949	0.0254	1.9913	0.0254	1.9842	0.0257
		32	α	0.5769	0.1021	0.5059	0.0039	0.5052	0.0038	0.5039	0.0038
			β	1.9590	0.2709	2.0004	0.0082	1.9994	0.0082	1.9974	0.0083
60	0.7	36	α	0.6571	0.3176	0.5126	0.0060	0.5111	0.0059	0.5079	0.0057
			β	1.9685	0.4735	1.9967	0.0277	1.9929	0.0278	1.9855	0.0280
		48	α	0.5988	0.1635	0.5014	0.0042	0.5007	0.0041	0.4994	0.0041
			β	2.0030	0.4292	1.9992	0.0083	1.9982	0.0084	1.9962	0.0084
	1.5	36	α	0.7112	0.4606	0.5176	0.0061	0.5161	0.0060	0.5131	0.0057
			β	1.8673	0.3863	1.9922	0.0272	1.9887	0.0272	1.9818	0.0272
		48	α	0.5449	0.0194	0.5058	0.0030	0.5052	0.0029	0.5041	0.0029
			β	1.9780	0.1691	1.9990	0.0075	1.9980	0.0075	1.9961	0.0075
80	0.7	48	α	0.6047	0.2028	0.5132	0.0055	0.5118	0.0054	0.5091	0.0052
			β	1.9586	0.3351	2.0005	0.0261	1.9971	0.0260	1.9903	0.0260
		64	α	0.5562	0.0705	0.5039	0.0032	0.5033	0.0032	0.5021	0.0032
			β	2.0140	0.2816	2.0024	0.0075	2.0014	0.0075	1.9995	0.0075
	1.5	48	α	0.6167	0.1741	0.5133	0.0044	0.5121	0.0043	0.5097	0.0042
			β	1.9077	0.2735	1.9968	0.0238	1.9935	0.0238	1.9870	0.0239
		64	α	0.5282	0.0110	0.5039	0.0022	0.5034	0.0021	0.5024	0.0021
			β	1.9833	0.1220	2.0017	0.0078	2.0007	0.0078	1.9987	0.0077

Table 3
The Average and the MSE of MLE, BS and BL for $\alpha = 2, \beta = 2$

α = 2, β = 2				MLE		Bayesian					
						SE		LINEX (c=0.5)		LINEX (c=1.5)	
n	T	r		Av.	MSE	Av.	MSE	Av.	MSE	Av.	MSE
40	0.7	25	α	2.2687	1.5477	1.9999	0.0306	1.9960	0.0306	1.9882	0.0306
			B	2.4468	2.4816	1.9978	0.0295	1.9940	0.0295	1.9864	0.0296
		32	α	2.1296	1.3609	2.0023	0.0082	2.0012	0.0082	1.9992	0.0082
			B	2.3453	2.0454	2.0005	0.0080	1.9995	0.0080	1.9976	0.0080
	1.5	25	α	2.2562	1.2546	1.9992	0.0275	1.9955	0.0275	1.9881	0.0276
			B	1.8512	0.3982	1.9913	0.0260	1.9878	0.0260	1.9807	0.0261
		32	α	2.2483	0.5328	2.0050	0.0079	2.0041	0.0079	2.0021	0.0079
			B	1.9540	0.2159	1.9967	0.0090	1.9957	0.0090	1.9936	0.0090
60	0.7	36	α	2.1546	0.9275	1.9919	0.0285	1.9881	0.0285	1.9805	0.0286
			B	2.4562	1.5149	1.9996	0.0271	1.9959	0.0270	1.9886	0.0270
		48	α	2.1208	0.8228	2.0011	0.0080	2.0000	0.0080	1.9980	0.0080
			B	2.3606	0.9778	2.0010	0.0081	1.9999	0.0081	1.9979	0.0081
	1.5	36	α	2.3923	1.0447	2.0005	0.0281	1.9968	0.0280	1.9897	0.0280
			B	1.9041	0.3037	2.0010	0.0239	1.9976	0.0238	1.9908	0.0239
		48	α	2.1701	0.2596	2.0016	0.0076	2.0006	0.0076	1.9986	0.0076
			B	1.9668	0.1473	2.0023	0.0077	2.0013	0.0077	1.9994	0.0077
80	0.7	48	α	2.1670	0.9725	1.9991	0.0271	1.9955	0.0270	1.9882	0.0270
			B	2.2891	1.3321	2.0036	0.0264	2.0000	0.0262	1.9928	0.0259
		64	α	2.1634	0.9513	2.0032	0.0083	2.0022	0.0083	2.0002	0.0083
			B	2.2702	1.2922	1.9981	0.0081	1.9971	0.0081	1.9951	0.0081
	1.5	48	α	2.3430	0.7855	2.0001	0.0263	1.9966	0.0263	1.9896	0.0263
			B	1.8861	0.2352	2.0003	0.0237	1.9971	0.0236	1.9905	0.0236
		64	α	2.1285	0.1799	1.9984	0.0073	1.9974	0.0073	1.9956	0.0073
			B	1.9576	0.1148	2.0000	0.0076	1.9990	0.0076	1.9971	0.0076

Table 4
The Average 95% CIs of the MLE and Credible Lengths for Bayesian Estimation based on Different Loss Functions for $\alpha = 0.5, \beta = 0.5$

α = 0.5, β = 0.5				MLE		Bayesian		
						SE	LINEX (c=0.5)	LINEX (c=1.5)
n	T	t		L.CI	Co.Pro	L.CI	L.CI	L.CI
40	0.25	25	α	1.0001	99.10%	0.4305	0.4302	0.4281
			β	1.4330	93.60%	0.4624	0.4600	0.4510
		32	α	1.1428	98.10%	0.2749	0.2747	0.2758
			β	1.4589	95.60%	0.2847	0.2843	0.2840
	0.7	25	α	0.9727	99.00%	0.4649	0.4621	0.4582
			β	1.2688	93.30%	0.4689	0.4672	0.4564
		32	α	1.2947	97.60%	0.2818	0.2791	0.2764
			β	1.0719	94.10%	0.2609	0.2613	0.2619
60	0.25	36	α	0.9964	98.20%	0.4209	0.4179	0.4115
			β	1.2347	94.30%	0.4313	0.4288	0.4241
		48	α	0.9070	99.40%	0.2671	0.2661	0.2652
			β	1.1945	95.10%	0.2742	0.2729	0.2719
	0.7	36	α	0.8563	98.10%	0.4419	0.4354	0.4312
			β	1.0297	93.90%	0.4450	0.4404	0.4381
		48	α	0.8043	98.10%	0.2580	0.2569	0.2572
			β	0.8981	95.10%	0.2554	0.2546	0.2558
80	0.25	48	α	0.8040	97.60%	0.4454	0.4427	0.4372
			β	1.0509	94.10%	0.4376	0.4339	0.4241
		64	α	0.7991	97.70%	0.2589	0.2585	0.2573
			β	1.1688	95.50%	0.2568	0.2564	0.2552
	0.7	48	α	0.8156	97.60%	0.4271	0.4253	0.4230
			β	1.0167	93.60%	0.4163	0.4117	0.4039
		64	α	0.7992	97.20%	0.2515	0.2514	0.2492
			β	0.8442	95.00%	0.2420	0.2414	0.2405

Table 5
The Average 95% CIs of the MLE and Credible Lengths for Bayesian Estimation based on Different Loss Functions for $\alpha = 0.5, \beta = 2$

$\alpha = 0.5, \beta = 2$				MLE		Bayesian		
						SE	LINEX (c=0.5)	LINEX (c=1.5)
n	T	t		L.CI	Co.Pro	L.CI	L.CI	L.CI
40	0.7	25	*A*	1.8723	92.40%	0.3479	0.3462	0.3421
			B	3.0472	95.90%	0.6348	0.6351	0.6344
		32	*A*	1.6628	95.60%	0.2650	0.2650	0.2639
			B	3.0208	95.40%	0.3369	0.3373	0.3385
	1.5	25	*A*	1.7749	92.70%	0.3102	0.3092	0.3045
			B	2.5548	93.70%	0.6135	0.6159	0.6166
		32	*A*	1.2162	98.20%	0.2344	0.2337	0.2307
			B	2.0349	94.60%	0.3538	0.3533	0.3544
60	0.7	36	*A*	2.1227	95.30%	0.2857	0.2843	0.2822
			B	2.6960	92.90%	0.6329	0.6345	0.6300
		48	*A*	1.5377	97.30%	0.2514	0.2514	0.2502
			B	2.5694	95.00%	0.3618	0.3610	0.3596
	1.5	36	*A*	2.5295	94.50%	0.2827	0.2807	0.2787
			B	2.3815	92.40%	0.6425	0.6347	0.6278
		48	*A*	0.5176	96.10%	0.2060	0.2049	0.2026
			B	1.6107	94.80%	0.3302	0.3302	0.3313
80	0.7	48	*A*	1.7177	97.30%	0.2722	0.2685	0.2648
			B	2.2646	93.70%	0.6100	0.6098	0.6109
		64	*A*	1.0179	97.80%	0.2194	0.2191	0.2191
			B	2.0805	95.70%	0.3393	0.3387	0.3368
	1.5	48	*A*	1.5709	97.60%	0.2444	0.2438	0.2432
			β	2.0190	94.70%	0.6153	0.6193	0.6141
		64	α	0.3971	95.30%	0.1810	0.1808	0.1807
			β	1.3682	95.20%	0.3288	0.3285	0.3290

Table 6
The Average 95% CIs of the MLE and Credible Lengths for Bayesian Estimation based on Different Loss Functions for $\alpha = 2$, $\beta = 2$

$\alpha = 2, \beta = 2$				MLE		Bayesian		
						SE	LINEX (c=0.5)	LINEX (c=1.5)
n	T	t		L.CI	Co.Pro	L.CI	L.CI	L.CI
40	0.7	25	α	4.7641	98.70%	0.6807	0.6797	0.6786
			β	5.9245	94.80%	0.6535	0.6538	0.6501
		32	α	4.1837	98.93%	0.3483	0.3481	0.3499
			β	5.0908	94.90%	0.3460	0.3480	0.3494
	1.5	25	α	4.3491	93.70%	0.6422	0.6424	0.6454
			β	2.4051	97.60%	0.6336	0.6381	0.6356
		32	α	2.6919	95.70%	0.3417	0.3412	0.3410
			β	1.8134	94.70%	0.3657	0.3649	0.3617
60	0.7	36	α	4.3872	98.10%	0.6543	0.6557	0.6564
			β	5.4634	94.10%	0.6362	0.6313	0.6296
		48	α	4.2680	98.90%	0.3384	0.3372	0.3359
			β	5.0341	94.30%	0.3454	0.3471	0.3476
	1.5	36	α	3.7017	93.60%	0.6599	0.6610	0.6572
			β	2.1285	95.00%	0.6129	0.6116	0.6096
		48	α	1.8836	95.20%	0.3466	0.3469	0.3431
			β	1.4996	94.40%	0.3317	0.3320	0.3360
80	0.7	48	α	3.8118	98.00%	0.6311	0.6291	0.6287
			β	4.3823	95.00%	0.6141	0.6098	0.6096
		64	α	3.7712	98.00%	0.3577	0.3595	0.3589
			β	4.3304	93.80%	0.3428	0.3423	0.3447
	1.5	48	α	3.2050	94.60%	0.6432	0.6387	0.6301
			β	1.8488	93.80%	0.5787	0.5803	0.5871
		64	α	1.5852	96.20%	0.3419	0.3432	0.3436
			β	1.3185	94.80%	0.3392	0.3393	0.3396

Table 7
Data of total Vaccine Doses Administered Population for Region AMRO

C.	Paraguay	VCT	Bahamas	TTO	BOL	Peru	MTQ	GLP
x	14.701	22.494	24.41	25.908	26	26.375	26.731	27.956
C.	Grenada	Belize	Suriname	Mexico	Panama	Colombia	Guyana	El Salvador
x	31.169	34.245	35.271	38.161	38.565	39.696	45.008	48.363
C.	Dominica	Barbados	ATG	Saba	VGB	KNA	BES	DOM
x	55.568	58.375	66.453	67.253	75.527	78.877	79.07	79.601
C.	Anguilla	TCA	Aruba	Uruguay	Chile	FLK	Bermuda	PRI
x	112.992	117.108	121.87	122.26	122.684	126.529	131.429	132.484
C.	Ecuador	BES	LCA	GUF	Brazil	CRI	Argentina	Montserrat
x	28.104	28.441	29.142	29.31	49.122	49.701	52.487	53.911
C.	SXM	USA	CUW	Canada	Bonaire	CYM		
x	96.994	100.991	102.996	110.484	132.493	143.453		

Table 8
MLE with SE, KS test with P-Value and different measures

		Estimate	SE	KS	P-Value	AIC	CAIC	BIC	HQIC
EC	α	0.1500	0.0065	0.1032	0.6735	460.8892	461.1683	464.5465	462.2592
	β	120.8566	30.8476						
KITL	α	11.0116	4.5279	0.1158	0.5302	462.0031	462.5746	467.4891	464.0582
	β	102.6300	220.7092						
	λ	0.2984	0.1903						
IW	α	1.3361	0.1059	0.1662	0.1404	469.1642	469.4432	472.8215	470.5342
	λ	138.3991	51.1193						
GR	α	0.8770	0.1658	0.1307	0.3790	462.0919	462.3709	465.7492	463.4619
	λ	-0.0124	0.0012						
W	α	1.8154	0.2108	0.1228	0.4561	461.8702	462.1492	465.5275	463.2402
	λ	75.3091	6.4734						
IK	α	1.4295	0.1250	0.1581	0.1799	467.2017	467.4807	470.8589	468.5717
	λ	210.3373	93.9490						

Table 9
Estimates with SE and CIs for MLE and Bayesian Estimation Method

m	T		MLE				Bayesian			
			Estimates	SE	Lower	Upper	Mean	SE	Lower	Upper
30	55	α	0.1588	0.0101	0.1390	0.1786	0.1584	0.0066	0.1463	0.1715
		β	167.5831	62.9207	44.2586	290.9076	167.8258	22.6257	126.9465	215.5871
	90	α	0.1493	0.0087	0.1322	0.1664	0.1483	0.0060	0.1359	0.1595
		β	125.3535	40.2856	46.3937	204.3132	120.5701	10.0768	102.0023	137.1154
40	90	α	0.1474	0.0084	0.1310	0.1638	0.1471	0.0066	0.1340	0.1598
		β	118.0604	36.5064	46.5079	189.6129	117.7081	15.9019	86.4897	148.5554
	120	α	0.1471	0.0075	0.1325	0.1617	0.1470	0.0061	0.1351	0.1595
		β	117.0545	33.4486	51.4953	182.6138	118.1463	9.3804	92.7608	144.2825

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