Pak. J. Statist.
2022 Vol. 38(1), 129-144

BASED ON THE GOODNESS OF FIT APPROACH, A NEW TEST STATISTICS FOR TESTING NBUC$_{mgf}$ CLASS OF LIFE DISTRIBUTIONS

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ABSTRACT

A new non-parametric test statistic is considered based on the goodness of fit methodology. This test statistic is used to test exponentiality versus the class of new better than used convex order at the moment generating function ($NBUC_{mgf}$). Test statistic's percentiles and the power for $NBUC_{mgf}$ are estimated and tabulated. The pitman asymptotic efficiency of the proposed test statistic is given and discussed. The use of the proposed test statistic in the reliability study is demonstrated using real data. Finally, the right-censored data problem is given in some medical science applications to evaluate the performances of our test.

KEYWORDS

$NBUC_{mgf}$ class of life distributions; Goodness of fit methodology; The U-statistic; Asymptotic efficiency of Pitman; Null distribution critical values from the Monte Carlo method; non-censored and censored data.

AMS Subject Classification: 60K10, 62E10, 62N05.

1. INTRODUCTION

The theory of reliability is widely used in many applications such as engineering, medicine and biological sciences. In the analysis of reliability, it is beneficial to know the importance of which the classes of new better than used aging are commonly used. So, A new unit has a longer lifespan than an old used one. Also, using the concept of stochastic ordering to define the classes of life distributions is very beneficial. Increasing the failure rate (IFR), New better than used (NBU), New better than used in expectation (NBUE), and New better than used in convex ordering are some of these classes of life distributions. It is detailed discussions with their properties and also application in Bryson and Siddiqui (1969), Rolski (1975), Barlow and Proschan (1981) and Cao and Wang (1991). The definitions of these classes of life distributions help statistician to find new test statistics and the main goal to present new test statistics to gain higher efficiency.

A positive random variable ($X \geq 0$) with a cumulative distribution function (C.D.F) $F(x)$ and a survival function is commonly used to describe the lifetime of a unit.

$$\bar{F}(x) = P\ (X > x) = 1 - F(x).$$

The concept of residual life is the foundation of one of the good ageing life methodologies. For the random variable $(X \geq 0)$, assume that $X_t = [X - t \mid X > t]$, $t \in \{x: F(x) < 1\}$ be a random variable with survival function

$$\bar{F}_t(x) = \frac{\bar{F}(\mathrm{t} + x)}{\bar{F}(t)}, \bar{F}(t) > 0,$$

the distribution function of X_t is the same of the conditional of $x - t$ given that $x > t$.

Now we can remind the following definitions:

Definition (1):

For Two random variables X and Y, it is said to be:

(i) New better than used ($x \in NBU$) if

$$\bar{F}(t + y) \leq \bar{F}(t)\bar{F}(y), \quad for\ all \quad y, t > 0; \tag{1}$$

(ii) New better than used in the convex ordering ($x \in NBUC$) if

$$\int_x^\infty \bar{F}(y + t)dy \leq \bar{F}(t) \int_x^\infty \bar{F}(y)dy\ , for\ all\ x, t \geq 0. \tag{2}$$

Let

$$(\mathrm{z}) = \int_z^\infty \bar{F}(u)\ du,$$

then, the inequality (2) becomes as the following

$$\gamma\ (x + t)\ \leq\ \bar{F}(t)\gamma(x). \tag{3}$$

Definition (2):

Let X be a random variable which is smaller than the random variable Y with respect to the moment generating function order $(X \leq_{mgf} Y)$

$$\int_0^\infty e^{\beta x} dF(x) \geq \int_0^\infty e^{\beta x} dG(x)\ , \beta\ \geq 0$$

or

$$\int_0^\infty e^{\beta x} \bar{F}(x)dx \leq \int_0^\infty e^{\beta x} \bar{G}(x)dx.$$

Multiplying both sides of the Eq. (3) in $e^{\beta x}$, and integrating, we have the following definition:

Definition (3):

New better than convex order based on moment generating function $NBUC_{mgf}$

$$\int_0^\infty e^{\beta x} \gamma(x\ +\ t)dx\ \ \leq\ \bar{F}(t) \int_0^\infty e^{\beta x} \gamma(x)dx\ \geq\ 0. \tag{4}$$

or

$$X \leq_{st}\ Y \Rightarrow X \leq_{icx} Y \Rightarrow X \leq_{mgf} Y$$

then we have

$$NBU \subset NBUC \subset NBUC_{mgf}.$$

The two classes NBU and NBUC of life distributions have obtained very importance in the analysis of life length as well as usability in many replacement policies, so a lot of related results have been given in the literature see Klar and Müller (2003), Utkin and Gurov (2002) and Barlow (1999). This has motivated use to propose a class $NBUC_{mgf}$. Our class of life distributions is the largest of the NBU and NBUC classes of life distributions.

The problem of testing exponentiality against various classes of life distributions (such as IFR, IFRA, NBU, NBUE, NBUC and $NBUC_{mgf}$ has received a lot of attention in the literature, for example see Proschan and Pyke (1967), Styan (1983), Ahmad (1995), Abouammoh and Ahmed (1988), Abouammoh et al. (1994), Abouammoh and Newby (1989), Hendi et al. (1998) and Klar and Müller (2003). Most of the methods given to discuss life testing problems over the last few decades are vastly different from those used in the related but wider field of the goodness of fit problems. It is shown in this paper that using the goodness of fit method to solve life testing problems makes sense and the results in simplified procedures that are asymptotically equal to or better than standard procedures.

This paper is organized as the following:

In Section (2), Based on goodness of fit approach, a new test statistic is presented and the Pitman's asymptotic efficiency is shown. In Section (3), the critical points for the current test statistics' lower and upper percentile values are tabulated, and the test's Powers are examined. The right-censored data problem is also considered in Section (4). Finally in Section (5), some medical science applications are given to evaluate the performances of the proposed test statistics.

2. TESTING EXPONENTIALITY VERSUS $NBUC_{mgf}$ CLASS OF LIFE DISTRIBUTION

In this section, we discuss the construction of the proposed test statistic as a U-Statistic, discuss its asymptotic βnormality, and evaluate the asymptotic efficiencies for three alternatives in the class $NBUC_{mgf}$ to appreciate the quality of this method.

2.1 Test Statistics

Using a sample of $X_1, X_2, \dots, X_n$ from a population with the distribution function F. We test the null hypothesis $H_0\colon F$ is exponential, to the alternative hypothesis that $H_1\colon F$ is $NBUC_{mgf}$, and is not exponential.

The measure of the departure from $H_0\colon F$ according to Eq. (4) is

$$\delta_\beta = \int_0^\infty \left[\bar{F}(t)\int_0^\infty e^{\beta x}\gamma(x)dx - \int_0^\infty e^{\beta x}\gamma(x+t)dx\right]dF_0(t), \qquad (5)$$

where

$$F_0(t) = 1 - e^{-T}, t \geq 0.$$

The following theorem is fundamental for the evolution of our proposed test statistic.

Theorem 1.

If X is a random variable with the distribution F from $NBUC_{mgf}$ class of life distributions, then

$$\delta_\beta = \int_0^\infty e^{-t}\, dF(t) \left[\frac{1}{\beta+1} + \frac{1}{\beta^2} + \frac{\mu}{\beta} - \frac{M(\beta)}{\beta^2}\right] + \frac{M(\beta)}{\beta(\beta+1)} - \frac{1}{\beta}. \quad (6)$$

where

$$\mu = \int_0^\infty \bar{F}(t)dt \text{ and } M(\beta) = 1 + \beta \int_0^\infty e^{\beta x}\, \bar{F}(x)dx.$$

Proof:

From Eq. (5) we have

$$\delta_\beta = \int_0^\infty \bar{F}(t)e^{-t} \int_0^\infty e^{\beta x}\, \gamma(x)dxdt - \int_0^\infty \int_0^\infty e^{\beta x}\, \gamma(x+t)e^{-t}dxdt. \quad (7)$$

But the first term of Eq. (7) is given by

$$\begin{aligned}\int_0^\infty \bar{F}(t)e^{-t} \int_0^\infty e^{\beta x}\, \gamma(x)dxdt &= \left[\frac{1}{\beta}\left(\int_0^\infty e^{\beta x}\, \bar{F}(x)dx - \mu\right)\right]\left[1 - \int_0^\infty e^{-t}\, F(t)dt\right] \\ &= \frac{1}{\beta}\left[\frac{M(\beta)-1}{\beta} - \mu\right]\left[1 - \int_0^\infty e^{-t}\, dF(t)\right], \quad (8)\end{aligned}$$

and the second term of Eq. (7) is given by

$$\begin{aligned}\int_0^\infty \int_0^\infty e^{\beta x}\, \gamma(x+t)e^{-t}dxdt &= \int_0^\infty e^{\beta x}\, \gamma(t) \int_0^t e^{-y(1+\beta)}\, dydt \\ &= \frac{1}{\beta(1+\beta)} \int_0^\infty \gamma(t)\, de^{\beta t} - \frac{1}{1+\beta} \int_0^\infty e^{-t}\, \gamma(t)dt \\ &= \frac{1}{1+\beta}\left[\frac{1}{\beta}\left(\frac{M(\beta)-1}{\beta} - \mu\right) - \mu + 1 - \int_0^\infty e^{-t} dF(t)\right]. \quad (9)\end{aligned}$$

In view of Eq. (8), (9) and (7), the result is obtained.

Remarks:-

(i) If F is exponential then $\delta_\beta = 0$,

(ii) $\delta_\beta > 0$ under H_1.

To estimate δ_β, based on the random sample $X_1, X_2, \ldots, X_n$ with the distribution F, let the empirical form of δ_β in Eq. (6) is the following:

$$\hat{\delta}_\beta = \frac{1}{n^2} \sum_i \sum_j \left[e^{-X_i}\left(\frac{1}{1+\beta} + \frac{1}{\beta^2} + \frac{X_i}{\beta} - \frac{e^{\beta X_j}}{\beta^2}\right) + \frac{e^{\beta X_j}}{\beta(1+\beta)} - \frac{1}{\beta}\right]. \quad (10)$$

Now, we can set

$$\phi_1(X_1, X_2) = e^{-X_1}\left[\frac{1}{1+\beta} + \frac{1}{\beta^2} + \frac{X_2}{\beta} - \frac{e^{\beta X_2}}{\beta^2}\right] + \frac{e^{\beta X_2}}{\beta(1+\beta)} - \frac{1}{\beta}.$$

Thus, it is easy to show that

$$\begin{aligned}\phi_{1,\beta}(X_1) &= E\left[\phi_\beta(X_1, X_2)\middle|X_1\right] \\ &= e^{-X_1}\left[\frac{1}{1+\beta} + \frac{1}{\beta^2} + \frac{1}{\beta} - \frac{1}{\beta^2(1-\beta)}\right] + \frac{1}{\beta(1+\beta)(1-\beta)} - \frac{1}{\beta},\end{aligned} \quad (11)$$

and

$$\begin{aligned}\phi_{2,\beta}(X_1) &= E\left[\phi_\beta(X_2, X_1)\middle|X_1\right] \\ &= \frac{1}{2}\left[\frac{1}{1+\beta} + \frac{1}{\beta^2} + \frac{X_1}{\beta} - \frac{e^{\beta X_1}}{\beta^2}\right] + \frac{e^{\beta X_1}}{\beta(1+\beta)} - \frac{1}{\beta}.\end{aligned} \quad (12)$$

Therefore,

$$\begin{aligned}\psi_\beta(X_1) &= \phi_{1,\beta}(X_1) + \phi_{2,\beta}(X_1) \\ &= \frac{-2\beta e^{-X_1}}{(1-\beta)(1+\beta)} + \frac{X_1}{2\beta} + \frac{e^{\beta X_1}(\beta-1)}{2\beta^2(1+\beta)} \\ &+ \frac{3\beta^3 - 2\beta + 1}{2\beta^2(1-\beta)(1+\beta)}, \beta \neq -1,1,0.5,2\,.\end{aligned} \quad (13)$$

The mean and the variance of test statistic $\hat{\delta}_\beta$ is given by

$$E(\psi_\beta(X_1)) = 0,$$

and

$$\sigma_\beta^2 = Var(\psi_\beta(X_1)) = \left[E(\psi_\beta(X_1))\right]^2 \quad (14)$$

Now $\hat{\delta}_\beta$ is equivalent U-statistics shown blew (see Lee (1990)

$$U_n = \frac{1}{\binom{n}{2}}\sum_{i<j}\phi(X_i, X_j)\,. \quad (15)$$

The broad sample properties of $\hat{\delta}_\beta$ are summarized in the following theorem.

Theorem 2.

(i) *As $n \to \infty$ and with mean of 0 and variance of σ_β^2 which is in Eq. (14), ($\sqrt{n}(\hat{\delta}_\beta - \delta_\beta)$, is asymptotically normal).*

(ii) *Under H_0, the variance of $\hat{\delta}_\beta$ is given by*

$$\sigma^2{}_{0,\beta} = \frac{\beta(2\beta-9)+6}{12(\beta-2)(2\beta-1)(\beta-1)^2}, \beta \neq 0.5,1,2. \quad (16)$$

Proof:

(i) and (ii) follow by using the standard theorem of U-statistic (Lee 1990) and direct calculations, respectively.

Setting $\beta = 0.9$, which maximize the efficiency of our test statistics $\hat{\delta}_\beta$ we found $\sigma_{0,\beta}^2 = 4.54545$.

2.2 Pitman Asymptotic Efficiency (PAE) of the Test

Here in this subsection, we calculated the PAE for the following three alternatives of our class of life distributions, where

$$PAE(\delta_\beta) = \frac{\left|\frac{d}{d\theta}\delta_\theta\right|_{\theta\to\theta_0}}{\sigma_{0,\beta}}.$$

Therefore,

$$PAE(\delta) = \frac{1}{\sigma_{0,\beta}}\Bigg[\left[\int_0^\infty e^{-x}\acute{f}_\theta(x)\,dx\right]\left[\frac{1}{1+\beta}+\frac{1}{\beta^2}+\frac{\mu_\theta}{\beta}-\frac{M_\theta(\beta)}{\beta^2}\right] + \left[\int_0^\infty e^{-x}f_\theta(x)\,dx\right]\left[\frac{1}{\beta}\frac{d}{d\theta}\mu_\theta - \frac{1}{\beta^2}\frac{d}{d\theta}M_\theta(\beta)\right] + \frac{1}{\beta(1+\beta)}\frac{d}{d\theta}M_\theta(\beta)\Bigg]_{\theta\to\theta_0}. \quad (17)$$

We used three alternative families, they are:

(i) Linear failure rate family:

$$\bar{F}_1(x) = exp\left(-x-\frac{\theta x^2}{2}\right), x > 0, \theta \ge 0$$

(ii) Makeham family:

$$\bar{F}_2(x) = exp\,(-x-\theta\,(x+e^{-x}-1)), x > 0, \theta \ge 0$$

(iii) Weibull family:

$$\bar{F}_3(x) = exp\,(-x^\theta), x > 0, \theta \ge 0$$

So, at $\theta = 0$, the null hypothesis H_0 is obtained in (i), (ii) and $\theta = 1$ the null hypothesis H_0 is obtained in (iii).

(i) PAE (δ_β) for linear failure rate family:

$$PAE(\hat{\delta}_\beta) = \frac{1}{\sigma_{0,\beta}}\left|\frac{2-\beta}{4(1-\beta)^2}\right|, \quad \beta \ne 1.$$

(ii) PAE (δ_β) for Makeham family:

$$PAE(\hat{\delta}_\beta) = \frac{1}{\sigma_{0,\beta}}\left|\frac{3-\beta}{12(2-\beta)(1-\beta)}\right|, \quad \beta \ne 1, \beta \ne 2.$$

(iii) At $\beta = 0.9$, PAE (δ_β) for Weibull family:

$$PAE(\hat{\delta}_\beta) = \frac{1}{\sigma_{0,\beta}}\left|\phi_1(\beta)[-0.067589] + \frac{1}{2\beta}[-0.577216] + \phi_2(\beta) + \phi_3(\beta)\right|, \beta \ne 0,1,-1,$$

where,

$$\phi_1(\beta) = \frac{-2\beta}{1-\beta^2}, \quad \phi_2(\beta) = \frac{1}{2\beta(1+\beta)}\int_0^\infty e^{-x(1-\beta)}\, ln(x)dx \quad and$$

$$\phi_3(\beta) = \frac{2-\beta^2}{2\beta(1-\beta^2)}.$$

So,

$$PAE\left(\hat{\delta}_\beta\right) = \frac{1}{\sigma_{0,\beta}}(8.844).$$

In Table (1), we show that the Pittman asymptotic efficiency (PAE) of the proposed test statistics $\hat{\delta}_\beta$ versus other test statistics.

Table 1
PAE for LFR, Makeham and Weibull families

Test	LFR	Makeham	Weibull
Mugdadi and Ahmad (2005)	0.408	0.0395	0.170
Mahmoud and Alim (2008)	0.217	0.144	0.050
Mahmoud et al. (2019)	1.0103	0.2504	0.9999
Abu-Youssef et al. (2020)	1.38	0.73	0.25
(our test $\hat{\delta}_\beta$ (0.9))	12.8986	0.7462	1.9457

Table (2) shows that asymptotic relative efficiency of Pittman (PARE) for our test statistics $\hat{\delta}_\beta$ comparing with other test statistics. Note that:

$$PARE(T_1, T_2) = \frac{PAE(T_1)}{PAE(T_2)}.$$

Table 2
PARE's of $\hat{\delta}_\beta$ with respect to , Mugdadi and Ahmad (2005),
Mahmoud and Alim (2008), Mahmoud et al. (2019) and Abu-Youssef et al. (2020)

Test	LFR	Makeham	Weibull
e($\hat{\delta}_\beta$, Mugdadi and Ahmad 2005)	31.614	18.891	11.445
e($\hat{\delta}_\beta$, Mahmoud and Alim 2008)	59.441	5.182	38.914
e($\hat{\delta}_\beta$, Mahmoud et al. 2019)	12.767	2.98	1.946
e($\hat{\delta}_\beta$, Abu-Youssef et al. 2020)	9.347	1.022	7.783

From Table 2, it is obvious that the proposed test statistic $\hat{\delta}_\beta$ performs well for $\bar{F}_1$, $\bar{F}_2$ and $\bar{F}_3$ and it is more efficient than Mugdadi and Ahmad (2005), Mahmoud and Alim (2008), Mahmoud et al. (2019) and Abu-Youssef et al. (2020) for the cases, Linear failure rate family, Makeham family and Weibull family.

3. MONTE CARLO NULL DISTRIBUTION CRITICAL POINTS

In Table 3, Based on 10000 simulated samples of size $n = 5(2)50$ and 40 we simulated and tabulated the lower and upper percentile points for 1%, 5%, 10%, 90%, 95% and 99% of the statistics $\hat{\delta}_\beta$ (0.9).

Table 3
The Lower and Upper Percentile Points of $\hat{\delta}_\beta$

n	1%	5%	10%	90%	95%	99%
5	-1.78964	-0.408219	-0.241639	0.965623	1.87076	7.52936
7	-1.60419	-0.450609	-0.268661	0.712021	1.4648	5.95606
9	-2.28813	-0.528415	-0.295934	0.599604	1.11469	3.98938
11	-1.9459	-0.517361	-0.302901	0.522994	0.978154	3.48249
13	-1.74242	-0.517891	-0.309149	0.4548	0.847792	2.81202
15	-1.58091	-0.486819	-0.301649	0.397737	0.774874	2.71292
17	-1.64574	-0.506998	-0.309301	0.376067	0.68407	2.44695
19	-1.61829	-0.509959	-0.321801	0.355995	0.644355	2.22277
21	-1.68245	-0.53185	-0.329479	0.311967	0.592684	1.7712
23	-1.85876	-0.54005	-0.320723	0.305808	0.583827	1.7953
25	-1.77811	-0.534662	-0.331343	0.2778	0.513482	1.73992
27	-1.59972	-0.498064	-0.314445	0.263543	0.471997	1.49574
29	-1.7858	-0.529699	-0.327176	0.250709	0.452753	1.48101
31	-1.68659	-0.52841	-0.328922	0.235633	0.429771	1.36359
33	-1.87831	-0.519037	-0.333284	0.230122	0.405493	1.21148
35	-1.95043	-0.524733	-0.327682	0.215288	0.411379	1.20146
37	-1.84123	-0.529408	-0.337972	0.199784	0.364199	1.10035
39	-1.66463	-0.50416	-0.325131	0.199365	0.367604	1.11797
40	-1.83409	-0.519128	-0.329157	0.193404	0.3671	1.17391
41	-1.72265	-0.514073	-0.330904	0.183113	0.335484	1.07317
43	-2.09183	-0.5113	-0.324054	0.173136	0.324231	0.95199
45	-1.53985	-0.525363	-0.339173	0.176799	0.324327	1.0584
47	-1.70681	-0.51177	-0.322406	0.165353	0.317495	0.836444
49	-1.59419	-0.494875	-0.320257	0.150997	0.284397	0.80065

From Table 3, it is clear that the critical values decrease slowly as the sample sizes increase and also they increase as the confidence levels increase and it is shown in the following figure.

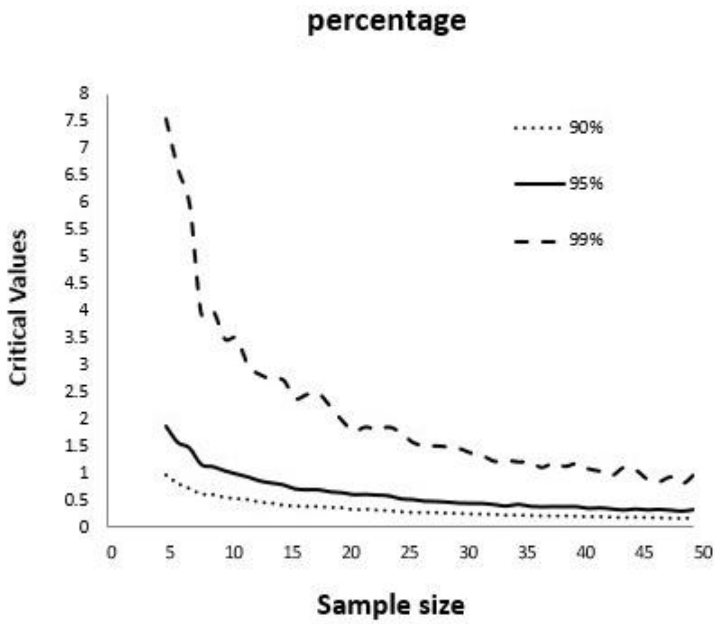

Figure 1: The Relation between Sample Size and Critical Values of $\hat{\delta}_\beta$

3.1 The Power Estimate of $\hat{\delta}_\beta$

The power estimate of our test $\hat{\delta}_\beta$ is evaluated for 5% percentiles using three of the most commonly used alternatives distributions which are:

(i) Linear Failure Rate:

$$\bar{F}_1(x) = e^{-x-\frac{\theta x^2}{2}}, \qquad x > 0, \theta \geq 0$$

(ii) Gamma:

$$\overline{F_2}(x) = 1 - \frac{1}{\Gamma(\lambda)}\gamma(\lambda, \theta x), \qquad x > 0, \lambda > 0, \theta > 0$$

(iii) Weibull:

$$\bar{F}_3 = e^{-x^\theta}, \qquad x > 0, \theta \geq 0$$

At $n = 10$, 20 and 30, Table 4 gives the power estimate with parameter $\theta = 0.5$, 1, 1.5, 2 and 3.

Table 4
The Power Estimate of $\hat{\delta}_\beta$ (0.9)

n	Θ	LFR	Gamma	Weibull
10	0.5	0.9969	0.8908	0.4553
	1	0.9999	0.9495	0.9533
	1.5	1.0000	0.9956	0.9998
	2	1.0000	0.9999	1.0000
	3	1.0000	1.0000	1.0000
20	0.5	0.9999	0.8959	0.2599
	1	1.0000	0.9538	0.9521
	1.5	1.0000	0.9997	1.0000
	2	1.0000	1.0000	1.0000
	3	1.0000	1.0000	1.0000
30	0.5	1.0000	0.8716	0.1614
	1	1.0000	0.9534	0.953
	1.5	1.0000	1.0000	1.0000
	2	1.0000	1.0000	1.0000
	3	1.0000	1.0000	1.0000

It is shown that the power estimates of the proposed test statistic increase when the sample size n and the parameter θ are increase. This indicates that our test is more power than other tests.

4. TESTING FOR CENSORED DATA

In a life model of research or a clinical trial, where patients may be lost (censored) before the study is completed, the censored data is usually the only information available. In this case, we will test $H_0: F$ is exponential against $H_1: F$ is $NBUC_{mgf}$ with randomly right censored. The experimental situations is described as the following: Let $X_1, X_2, \ldots, X_n$ be independent and identically distribution according to the continuous life distribution F. Let $Y_1, Y_2, \ldots, Y_n$ be independent identically distribution according to the continuous life distribution G. Then we assume that the Xs and Ys are independent. In a randomly right censored model, we observe the pairs (Z_j, δ_j),

$j = 1, 2, \ldots, n$, where $Z_j = min\ (X_j, Y_j)$ and

$$\delta_j = \begin{cases} 1, if\ Z_j = X_j\ (jth\ observed\ is\ uncensored) \\ 0, if\ Z_j = Y_j\ (jth\ observed\ is\ censored) \end{cases}$$

Let $Z_{(0)} = 0 < Z_{(1)} < Z_{(2)} < \ldots < Z_{(n)}$ denote the order $Z's$ and is $\delta_{(j)}$ is δ_j corresponding to is $Z_{(j)}$. Kaplan and Meier (1958) proposed a product limit estimator based on the censored data is Z_j, is $\delta_j, j = 1, 2, \ldots, n$, as the following

$$\bar{F}_n(X) = \prod_{[j:Z_{(j)} \le X]} (n-j)/(n-j+1)^{\delta(j)}, X \in [0, Z_{(n)}].$$

From Eq. (6) We propose the following test statistic

$$\hat{\delta}_\beta = \theta\left(\frac{1}{1+\beta} + \frac{1}{\beta^2} + \frac{\varphi}{\beta} - \frac{\Omega}{\beta^2}\right) + \frac{\Omega}{\beta(1+\beta)} - \frac{1}{\beta}. \tag{18}$$

where,

$$\varphi = \sum_{k=1}^{n} \prod_{m=1}^{k-1} C_m^{\delta(m)} (Z_k - Z_{k-1}),$$

$$\Omega = 1 + \beta \sum_{i=1}^{n} \prod_{v=1}^{i-1} e^{\beta Z_i} C_v^{\delta(v)} (Z_i - Z_{i-1}),$$

$$\theta = \sum_{j=1}^{n} e^{-Z_j} \left[\prod_{p=1}^{j-2} C_p^{\delta(p)} - \prod_{p=1}^{j-1} C_p^{\delta(p)} \right],$$

and

$$dF_n(Z_j) = \bar{F}_n(Z_{j-1}) - \bar{F}_n(Z_j),$$

$$C_k = \frac{n-k}{n-k+1}.$$

Here also we will present the simulation of the Monte Carlo null distribution critical points for is $\hat{\delta}_\beta$ in Eq. (18) based on 10000 simulated 40(5)60 ,10,20,30,51,81 and 86 from the standard exponential distribution by using Mathematica program (12). Table 5 gives the upper and the lower percentile points for 1%, 5%, 10%, 90%, 95%, 99% of the statistic $\hat{\delta}_\beta$.

Table 5
The Critical Values of $\hat{\delta}_\beta$ (0.9) with 10000 Replications

n	1%	5%	10%	90%	95%	99%
10	0.00118451	0.00167983	0.00191022	0.0206773	0.199106	19.0757
20	0.000268991	0.000447864	0.000493122	0.00991683	1.2245	6546.61
30	-0.24831	0.000209683	0.000227871	0.00976347	11.4908	$8.77747*10^{6}$
40	-593.222	0.000123344	0.000132217	0.011735	126.397	$9.38002*10^{8}$
45	-30961.5	0.0000988267	0.000105939	0.010419	476.652	$8.69878*10^{10}$
50	$-1.13951*10^{6}$	0.0000804376	0.0000864171	0.00606897	1402.43	$1.62544*10^{12}$
51	-622810.	0.0000782835	0.0000836313	0.0113427	761.926	$3.3175*10^{12}$
55	$-3.57091*10^{7}$	0.0000671366	0.0000722605	0.00302699	589.985	$1.15876*10^{13}$
60	$-5.1544*10^{8}$	0.0000570383	0.0000610899	0.00805955	9115.4	$3.81467*10^{14}$
81	$-2.91629*10^{18}$	0.0000314813	0.000034233	0.00178816	170996.	$1.49293*10^{20}$
86	$-1.17137*10^{19}$	0.0000282713	0.000030473	0.000273928	239197.	$8.06432*10^{21}$

4.1 Power Estimates

At significance level $\beta = 0.05$, The power of the proposed test $\hat{\delta}_\beta$ (0.9) is calculated with respect to three alternatives distributions (Linear failure rate (LFR), Gamma and Weibull). Using Mathematica (12) program and that based on 10000 samples. Table 6 gives the power estimates with parameter $\theta = 1{,}2$ and 3 at $n = 10$, 20 and 30.

Table 6

The Power Estimate of $\hat{\delta}(0.9)$ of Censored

n	θ	LFR	Gamma	Weibull
10	1	0.943	0.9461	0.9528
	2	0.9444	0.9999	0.991
	3	0.9462	1.0000	0.9994
20	1	0.978	0.9433	0.9464
	2	0.9816	0.98	0.9895
	3	0.9848	0.9999	0.9992
30	1	0.9869	0.9491	0.9509
	2	0.9944	0.8963	0.9863
	3	0.9942	0.9997	0.9986

It is shown that the estimated powers of our test increasing as the sample size n and the parameter θ increase.

5. APPLICATIONS

We present some real-world examples to demonstrate the utility of our test at a 95% confidence level.

5.1 Case of Complete Data

Example 1:

Consider the data in Abouammoh et al. (1994), which represent a group of 40 patients with blood cancer (leukemia) from one of Saudi Arabia's ministry of health hospitals, with the following ordered values in years:

0.315	0.496	0.616	1.145	1.208	1.263	1.414	2.025	2.036	2.162
2.211	2.370	2.532	2.693	2.805	2.910	2.912	3.192	3.263	3.348
3.348	3.427	3.499	3.534	3.767	3.751	3.858	3.986	4.049	4.244
4.323	4.381	4.392	4.397	4.647	4.753	4.929	4.973	5.074	4.381

It was found $\hat{\delta}_\beta$ = 11.9923 which is greater than the critical value of Table 3. Then we reject the null hypotheses which states that the data set have $NBUC_{mgf}$ and not exponential.

Example 2:

Using the data set given in Grubbs (1971), these data have been used in Ebrahimi et al. (1992) and Shapiro (1995), This data set gives the times between arrivals of 25 customers at a facility:

1.80	2.89	2.93	3.03	3.15	3.43	3.48	3.57	3.85	3.92
3.98	4.06	4.11	4.13	4.16	4.23	4.34	4.73	4.53	4.62
4.65	4.84	4.91	4.99	5.17					

It is easy to show that $\hat{\delta}_\beta$ = 23.0299 which is greater than the critical value of Table 3. Then we reject H_0 which states that the data set have $NBUC_{mgf}$ property and not exponential.

Example 3:

The survival times (in years) after diagnosis of 43 patients with a specific type of leukemia are represented in this data set from Styan (1983).

0.019	0.129	0.159	0.203	0.485	0.636	0.748	0.781	0.869	1.175
1.206	1.219	1.219	1.282	1.356	1.362	1.458	1.564	1.586	1.592
1.781	1.923	1.959	2.134	2.413	2.466	2.548	2.652	2.951	3.038
3.600	3.655	3.745	4.203	4.690	4.888	5.143	5.167	5.603	5.633
6.192	6.655	6.874							

It was found that $\hat{\delta}_\beta$ = 13.9862 which is greater than the critical value of Table 3. Then we reject H_0 which states that the data set have $NBUC_{mgf}$ property and not exponential.

Example 4:

Consider the data below, which reflect failure times in hours for a particular form of electrical insulation in an experiment where the insulation was subjected to a continuously increasing voltage stress Lawless (1982).

0.205	0.363	0.407	0.770	0.720	0.782	1.178	1.255	1.592	1.635
2.310									

It was found that $\hat{\delta}_\beta$ = 0.153126 which is less than the critical value of Table (3). Then we accept the null hypothesis.

5.2 Case of Censored Data

Example 5:

Consider the data in Mahmoud et al. (2005), which represents 51 liver cancer patients from Egypt's Ministry of Health's Elminia Cancer Center (1999). The moments of orderly living (in days) the ordered non-censored data:

10	14	14	14	14	14	15	17	18	20
20	20	20	20	23	23	24	26	30	30
31	40	49	51	52	60	61	67	71	74
75	87	96	105	107	107	107	116	150	

The ordered censored data:

30	30	30	30	30	60	150	150	150	150
150	185								

In this case $\hat{\delta}_\beta = 4.11125 * 10^{144}$ which is greater than the critical value of Table 5. Then we reject the null hypothesis which states that the data set have not $NBUC_{mgf}$ property.

Example 6:

Based on right-censored data from Pena (2002) for lung cancer patients. There are 86 survival times (in months) in this data set, with 22 of them censored on the right. The ordered non-censored data:

0.99	1.28	1.77	1.97	2.17	2.63	2.66	2.76	2.79	2.86
2.99	3.06	3.15	3.45	3.71	3.75	3.81	4.11	4.27	4.34
4.40	4.63	4.73	4.93	4.93	5.03	5.16	5.17	5.49	5.68
5.72	5.85	5.98	8.15	8.48	8.61	8.62	9.46	9.53	10.05
10.15	10.94	10.94	11.24	11.63	12.26	12.65	12.78	13.18	13.47
13.96	14.88	15.05	15.31	16.13	16.46	17.45	17.61	18.20	18.37
19.06	20.70	22.54	23.36						

The ordered censored data:

11.04	13.53	14.23	14.65	14.91	15.47	15.47	17.05	17.28	17.88
17.97	18.83	19.55	19.55	19.75	19.78	19.95	20.04	20.24	20.73
21.55	21.98								

In this case $\hat{\delta}_\beta = 5.00309 * 10^{11}$ which is greater than the critical value of Table 5. Then we reject the null hypothesis which states that the data set have not $NBUC_{mgf}$ property.

6. CONCLUSION

Based on the goodness of fit approach, a new test statistic for class ($NBUC_{mgf}$) of life distributions is derived. The proposed test's asymptotic efficiencies are computed for three alternatives and compared. Our test statistic is found to be more efficient than other tests and have good power for other alternative classes of life distributions. Finally, our test's critical values are given. Censored and non-censored real data sets are used to show the test's usefulness.

7. ACKNOWLEDGEMENTS

The authors would like to thank the editors and referee(s) for their valuable comments, which considerably improved the presentation.

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