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A NOVEL FAMILY OF GENERATING DISTRIBUTIONS BASED ON MARSHALL OLKIN TRANSFORMATION WITH AN APPLICATION TO EXPONENTIAL DISTRIBUTION

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ABSTRACT

A new method has been introduced based on the idea of Marshall and Olkin. The proposed method can be applied to any distribution by inverting its quantile function as a function of Marshall Olkin method. We applied this method to the exponential distribution to obtain a two parameter Marshall Olkin within exponential quantile function. Several properties of the introduced distribution including mode, quantile function, moments, moment generating function, mean residual lifetime, entropy, order statistic, stress strength parameter and maximum likelihood estimation were highlighted. We illustrate the applicability of the introduced model to two different real data sets and it is observed that the proposed model leads to a better fit than all other competitive models.

KEYWORDS

Exponential distribution; hazard rate function; survival function; mean residual life; maximum likelihood estimation.

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1. INTRODUCTION

Developing new distributions always remains trending topic in the literature of distribution theory. Researchers have introduced various methods for developing new distributions to obtain more flexible models which can be used for complex data structures. Among them (Eugene, Lee and Famoye, 2002) proposed the beta generalized method, Mudholkar and Srivastava (1993) proposed a method to introduce an extra parameter to a two parameter Weibull distribution. Aldeni, Lee and Famoye (2017) developed a new family of distribution arising from the quantile of the generalized lambda distribution. Alzaatreh, Lee and Famoye (2014) acted on the T-normal family of distributions. Alzaatreh et al. (2016) introduced the generalized Cauchy family of distributions. Mahdavi and Kundu (2017) proposed the Alpha Power Transformation (APT) family of distributions. Cordeiro et al., (2017) presented the half Cauchy family of distributions. Ijaz et al. (2020) worked on the Gull Alpha Power Weibull distribution. Recently, Ijaz et al. (2021) proposed class of New Alpha Power Transformed family (NAPT) of distributions. They employed exponential distribution in NAPT family and derived a new distribution called New Alpha Power Transformed exponential (NAPTE) distribution.

Marshall and Olkin (1997) discussed a procedure of adding a new parameter to a baseline distribution. One of the important properties of this family is that Marshall Olkin family of distributions possesses stability property in the sense that if the method is applied twice, it returns to the same distribution. Also this family satisfies geometric extreme stability property.

Marshall and Olkin (1997) started with a parent survival function $\bar{F}(y)$ and considered a family of survival functions given by

$$\bar{G}(y) = \frac{\alpha \bar{F}(y)}{1 - \bar{\alpha}\bar{F}(y)}; \quad y \epsilon \mathbb{R}, \alpha \epsilon \mathbb{R}^+,$$

where $\bar{\alpha} = 1 - \alpha$ and $\bar{F}(y) = 1 - F(y)$ is the survival function of the random variable Y.

The corresponding cumulative distribution function (cdf) and probability density function (pdf) are respectively given by

$$G(y) = \frac{F(y)}{1 - \bar{\alpha}\bar{F}(y)} \tag{1.1}$$

$$g(y) = \frac{\alpha f(y)}{(1 - \bar{\alpha}\bar{F}(y))^2}$$

Marshall and Olkin applied the proposed method to the one parametric exponential distribution and obtained two parametric Weibull distribution and studied their various desirable properties.

Nassar et al. (2018) proposed a method based on the idea of Alpha power transformation introduced by Mahdavi and Kundu (2017). The proposed method can be employed to any distribution by inverting its quantile function as a function of alpha power transformation.

The main goal of this manuscript is to introduce and study a new model called Marshall Olkin within Exponential Quantile (MOEQ) distribution based on the idea of Marshall and Olkin (1997). The rest of the paper is organized as follows: In Section 2, we introduce a new method for generating continuous distributions based on the family of distributions in (1.1). In Section 3, a member of the introduced family namely, MOEQ distribution, is introduced and its general properties are studied including, quantile, moments, moment generating function, mean residual life, Rényi entropy, order statistic and stress strength parameter. The maximum likelihood estimation and simulation study are carried out in Section 4. In Section 5 two data sets have been analysed. Finally, the paper is concluded in the Section 6.

2. PROPERTIES OF THE NEW METHOD

Let $g(y)$ and $G(y)$ be the pdf and cdf of a random variable Y, respectively. Then the cdf, $F(y)$ of the new method can be obtained by inverting the following equation

$$\frac{F(y)}{1-\bar{\alpha}\bar{F}(y)} = G(y);\; y\epsilon\mathbb{R}, \qquad \alpha\epsilon\mathbb{R}^+,$$

clearly,

$$F(y) = \frac{\alpha G(y)}{\left(\alpha+\bar{\alpha}\bar{G}(y)\right)^2},$$

the corresponding pdf is

$$f(y) = \frac{\alpha g(y)}{\left(\alpha+\bar{\alpha}\bar{G}(y)\right)^2}$$

clearly, $f(y)$ is the weighted version of $g(y)$, where the weight function is

$$w(y) = \frac{1}{\left(\alpha+\bar{\alpha}\bar{G}(y)\right)^2}$$

and $f(y)$ can be written as

$$f(y) = \frac{g(y)w(y)}{c},$$

here the normalizing constant $c = E\big(w(y)\big)$.

The survival function $S(y)$ for MOEQ distribution is given by

$$S(y) = \frac{\bar{G}(y)}{\alpha+\bar{\alpha}\bar{G}(y)}.$$

The hazard rate function $\lambda(y)$ is given by

$$\begin{aligned}\lambda(y) &= \frac{\alpha g(y)}{\left(\alpha+\bar{\alpha}\bar{G}(y)\right)\bar{G}(y)} \\ &= \lambda_g(y)\frac{\alpha}{\alpha+\bar{\alpha}\bar{G}(y)},\end{aligned} \tag{2.1}$$

where $\lambda_g(y)$ is the hazard rate function of the base model.

Thus

$$\lim_{y\to-\infty}\lambda(y) = \alpha \lim_{y\to-\infty}\lambda_g(y)$$

and

$$\lim_{y\to\infty}\lambda(y) = \lim_{y\to\infty}\lambda_g(y)$$

clearly from 2.1 we have

$$\lambda_g(y) \le \lambda(y) \le \alpha\lambda_g(y) \quad ; \quad y\epsilon\mathbb{R}, \quad \alpha \ge 1$$

$$\lambda_g(y) \ge \lambda(y) \ge \alpha\lambda_g(y) \quad ; \quad y\epsilon\mathbb{R}, \quad \alpha \le 1$$

$$\frac{\bar{F}(y)}{\alpha} \le S(y) \le \bar{F}(y) \qquad ; \quad y \epsilon \mathbb{R}, \quad \alpha \ge 1$$

$$\frac{\bar{F}(y)}{\alpha} \ge S(y) \ge \bar{F}(y) \qquad ; \quad y \epsilon \mathbb{R}, \quad \alpha \le 1.$$

Obviously, $\frac{\lambda(y)}{\lambda_g(y)}$ is increasing in y for $0 < \alpha \le 1$ and decreasing in y for $\alpha > 1$.

The qth quantile x_q of $F(y)$ is given by

$$x_q = F^{-1}\left(\frac{q}{\alpha + \bar{\alpha} q}\right). \tag{2.2}$$

If y_q denotes the qth quantile for $G(y)$, then it follows that

$$x_q \le y_q \quad if \quad \frac{q}{\alpha + \bar{\alpha} q} \le q.$$

Thus it is possible to determine for what values of $\alpha, F(y)$ will be heavier tail than $G(y)$.

$$x_q \le y_q \quad if \quad \alpha \ge 1 \;\text{ and }\; x_q \ge y_q \quad if \quad \alpha \le 1.$$

Therefore, if $\alpha > 1$ then $G(y)$ has a heavier tail than $F(y)$, and for $\alpha < 1$ it is the other way.

Theorem 1:
If $g(y)$ is a decreasing function, and $\alpha > 1$, then $f(y)$ is a decreasing function.

Proof:
We have,

$$\frac{d}{dy} logf(y) = \frac{g'(y)}{g(y)} + \frac{2\bar{\alpha} g(y)}{\alpha + \bar{\alpha}\bar{G}(y)}.$$

Since, both the terms on the RHS are negative. Therefore, $f(y)$ is a decreasing function.

Theorem 2:
If $g(y)$ is a decreasing function and $g(y)$ is log-convex, then for $\alpha > 1$, the hazard function $\lambda(y)$ is a decreasing function.

Proof:
We have,

$$\frac{d^2}{dy^2} logf(y) = \frac{d^2}{dy^2} logg(y) + 2\left\{\frac{\left(\alpha + \bar{\alpha}\bar{G}(y)\right)\bar{\alpha} g'(y) + \bar{\alpha}^2 g^2(y)}{\left(\alpha + \bar{\alpha}\bar{G}(y)\right)^2}\right\}$$

Since, both the terms on the RHS are positive, it implies that $f(y)$ is log-convex. Hence the result follows from (Barlow and Proschan, 1975).

3. MARSHALL OLKIN WITHIN EXPONENTIAL QUANTILE (MOEQ) DISTRIBUTION AND ITS PROPERTIES

Let Y be a random variable follows the exponential distribution with cdf $G(y) = 1 - e^{-\theta y}$; $y, \theta > 0$, then the cdf of the MOEQ distribution is defined as

$$F(y) = \frac{\alpha\left(1 - e^{-\theta y}\right)}{\alpha + \bar{\alpha} e^{-\theta y}} \;\; ; \;\; y > 0, \qquad \alpha > 0$$

The corresponding pdf is

$$f(y) = \frac{\alpha \theta e^{-\theta y}}{(\alpha + \bar{\alpha} e^{-\theta y})^2} \;\; ; \;\; y > 0, \qquad \alpha > 0$$

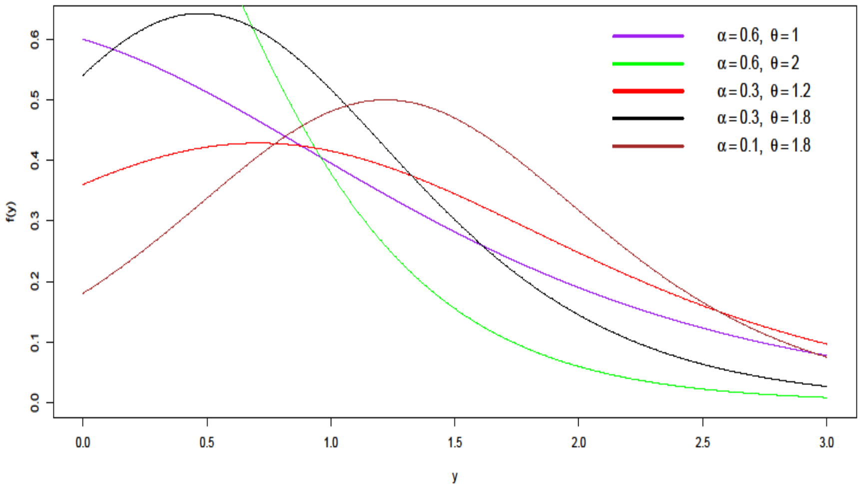

Figure 1: Plots of the MOEQ Density for Different Values of α and θ

The survival and hazard rate functions are respectively, given by

$$S(y) = \frac{e^{-\theta y}}{(\alpha + \bar{\alpha} e^{-\theta y})}, \qquad ; \;\; y > 0, \qquad \alpha > 0$$

and

$$\lambda(y) = \frac{\alpha \theta}{(\alpha + \bar{\alpha} e^{-\theta y})} \qquad ; \;\; y > 0, \qquad \alpha > 0$$

Since for $\alpha > 1$, $\lambda(y)$ decreases from $\alpha\theta$ to θ and for $\alpha < 1$ it increases from $\alpha\theta$ to θ.

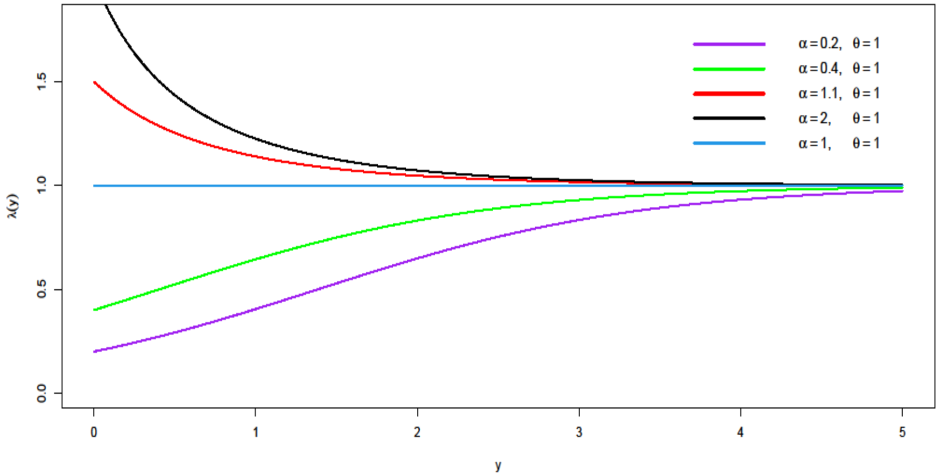

Figure 2: Plots of the MOEQ Hazard Rate Function for Different Values of α and θ

3.1 Mode

Let Y be a positive random variable follows MOEQ distribution. Then the mode of MOEQ is obtained by solving the equation $f'(y) = 0$. That is,

$$f'(y) = \frac{d}{dy}\left[\frac{\alpha\theta e^{-\theta y}}{(\alpha + \bar{\alpha}e^{-\theta y})^2}\right] \tag{3.1}$$

After solving the equation (3.1) we get the following result:

$$y = \frac{1}{\theta}log(2\bar{\alpha})\,; \quad \alpha < \frac{1}{2} \tag{3.2}$$

When $\alpha \geq \frac{1}{2}$ then $f(y)$ is a decreasing function of $y > 0$ with mode at 0 and when $\alpha < \frac{1}{2}$ then $f(y)$ has unique mode given by equation (3.2).

Theorem 3:

If $\alpha > 1$, then $\lambda(y)$ is a decreasing function of $y > 0$, and if $\alpha < 1$, then $\lambda(y)$ is a increasing function of $y > 0$.

Proof:

We have,

$$\lambda'(y) = \frac{\alpha\bar{\alpha}\theta^2 e^{-\theta y}}{(\alpha + \bar{\alpha}e^{-\theta y})^2}$$

Clearly, $\lambda'(y) < 0$, if $\alpha > 1$ and $\lambda'(y) > 0$, if $\alpha < 1$.

Hence the proof.

3.2 Quantile Function

The quantile function of MOEQ is given by

$$Y = \frac{1}{\theta} log \left[\frac{\alpha + \bar{\alpha}U}{\alpha(1-U)}\right]$$

where U follows uniform (0,1) distribution. The qth quantile function of MOEQ distribution is given by

$$y_q = \frac{1}{\theta} log \left[\frac{\alpha + \bar{\alpha}U}{\alpha(1-U)}\right]$$

The median is obtained as

$$y_{0.5} = \frac{1}{\theta} log \left(\frac{1+\alpha}{\alpha}\right).$$

3.3 Moments

Applying the below mentioned series representations

$$(1-x)^{-2} = \sum_{k=0}^{\infty} (k+1)\, x^k \, ; \qquad |x| < 1, \tag{3.3}$$

$$(1-x)^{-n} = \sum_{k=0}^{\infty} \binom{n}{k} x^k \, ; \qquad |x| < 1, \tag{3.4}$$

and

$$(1-x)^{n} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k x^k \, ; \quad |x| < 1, \tag{3.5}$$

The rth moment of the MOEQ distribution is given by

$$E(Y^r) = \int_0^{\infty} y^r f(y) dy$$

$$E(Y^r) = \frac{1}{\alpha\theta^r} \sum_{k=0}^{\infty} \left(\frac{\alpha-1}{\alpha}\right)^k \frac{\Gamma(r+1)}{(k+1)^r}.$$

3.4 Moment Generating Function

Using (3.3), (3.4) and (3.5) moment generating function of MOEQ distribution is obtained by

$$M_Y(t) = \int_0^{\infty} e^{ty} f(y) dy$$

$$M_Y(t) = \frac{1}{\alpha} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{t^j}{j!} \left(\frac{\alpha-1}{\alpha}\right)^k \frac{\Gamma(j+1)}{\theta^j (k+1)^j}.$$

3.5 Mean Residual Life and Mean Waiting Time

Suppose that Y is a continuous random variable with survival function $S(y)$, then the mean residual life function, say $\mu(t)$, is given by

$$\mu(t)=\frac{1}{S(t)}\left(E(t)-\int_{0}^{t}yf(y)dy\right)-t$$

The mean residual life of MOEQ distribution is given by

$$\mu(t)=\frac{\alpha+\bar{\alpha}e^{-\theta t}}{e^{-\theta t}}\frac{1}{\alpha}\sum_{k=0}^{\infty}\left(\frac{\alpha-1}{\alpha}\right)^{k}\left\{\frac{1}{\theta(k+1)}-\gamma(\theta(k+1)t,2)\right\}-t$$

where $\gamma(a,b)=\int_0^a y^{b-1}e^{-y}dy$ is the lower incomplete gamma function.

The mean waiting time of Y, say $\bar{\mu}(t)$ is defined by

$$\bar{\mu}(t)=t-\frac{1}{F(t)}\int_{0}^{t}yf(y)dy$$

$$\bar{\mu}(t)=t-\frac{\alpha+\bar{\alpha}e^{-\theta t}}{\alpha(1-e^{-\theta t})}\left\{\frac{1}{\alpha}\sum_{k=0}^{\infty}\left(\frac{\alpha-1}{\alpha}\right)^{k}\gamma(\theta(k+1)t,2)\right\}$$

3.6 Rényi Entropy

The Rényi entropy, say $\boldsymbol{RE_Y(\beta)}$ is defined as

$$\boldsymbol{RE_Y(\beta)=\frac{1}{1-\beta}log\left(\int_{-\infty}^{\infty}f(y)^{\beta}dy\right)};\quad \boldsymbol{\beta>0},\quad \boldsymbol{\beta\neq 1}.$$

The Rényi entropy of MOEQ distribution is given by

$$RE_Y(\beta)=\frac{\beta}{\beta-1}log\alpha-log\beta+\frac{\beta}{1-\beta}log\left(\sum_{k=0}^{\infty}\binom{2\beta}{k}\left(\frac{\alpha-1}{\alpha}\right)^{k}\frac{1}{(k+\beta)}\right).$$

3.7 Order Statistics

Let $\boldsymbol{Y_1,Y_2,\dots,Y_n}$ be a random sample of size $\boldsymbol{n}$, and let $\boldsymbol{Y_{i:n}}$ denote the ith order statistic, then the pdf of $\boldsymbol{Y_{i:n}}$, say $\boldsymbol{f_{i:n}(y)}$ is given by

$$\boldsymbol{f_{i:n}(y)=\frac{n!}{(i-1)!\,(n-i)!}F(y)^{i-1}f(y)\big(1-F(y)\big)^{n-i}}.$$

We can write $\boldsymbol{f_{i:n}(y)}$ as

$$\boldsymbol{f_{i:n}(y)=\frac{\theta\alpha^{i}}{B(i,n-i+1)}\frac{(1-e^{-\theta y})^{i-1}e^{-\theta(n-i+1)y}}{(\alpha+\bar{\alpha}e^{-\theta y})^{n+1}}}.$$

where $\boldsymbol{B(a,b)}$ is the beta function.

3.8 Stress Strength Parameter

Suppose Y_1 and Y_2 be independent strength and stress random variables respectively, where $Y_1 \sim MOEQ(\alpha_1, \theta)$ and $Y_2 \sim MOEQ(\alpha_2, \theta)$, then the stress strength parameter $P(Y_1 > Y_2)$, say R is defined as

$$R = \int_{-\infty}^{\infty} f_1(y)\, F_2(y) dy$$

The stress strength parameter R, can be obtained as

$$\boldsymbol{R = \frac{1}{\alpha_1} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \left(\frac{\alpha_1 - 1}{\alpha_1}\right)^k \left(\frac{\alpha_2 - 1}{\alpha_2}\right)^j (k+1) \frac{1}{(1+j+k)(2+j+k)}}.$$

4. STATISTICAL INFERENCE

4.1 Maximum Likelihood Estimators

Let $y_1, y_2, \dots, y_n$ be a random sample from MOEQ distribution, then the logarithm of the likelihood function becomes

$$l = nlog\alpha + nlog\theta - \theta \sum_{i=1}^{n} y_i - 2 \sum_{i=1}^{n} log\left(\alpha + \bar{\alpha} e^{-\theta y_i}\right) \tag{4.1}$$

The MLEs of α and θ are obtained by partially differentiating (4.1) with respect to the corresponding parameters and equating them to zero, we get

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} - 2 \sum_{i=1}^{n} \left\{\frac{\left(1 - e^{-\theta y_i}\right)}{\alpha + \bar{\alpha} e^{-\theta y_i}}\right\} = 0$$

$$\frac{\partial l}{\partial \theta} = \frac{n}{\theta} - \sum_{i=1}^{n} y_i + 2 \sum_{i=1}^{n} \left\{\frac{\left(\bar{\alpha} y_i e^{-\theta y_i}\right)}{\alpha + \bar{\alpha} e^{-\theta y_i}}\right\} = 0$$

As the normal equations are not closed in behavior, so they are solved by using R software.

Theorem 4:

If the parameter θ is known, then the MLE of α exists and is unique.

Proof:

Since,

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} - 2 \sum_{i=1}^{n} \left\{\frac{\left(1 - e^{-\theta y_i}\right)}{\alpha + \bar{\alpha} e^{-\theta y_i}}\right\}$$

$$\lim_{\alpha \to 0} \frac{\partial l}{\partial \alpha} = \infty - 2 \lim_{\alpha \to 0} \sum_{i=1}^{n} \left\{\frac{\left(1 - e^{-\theta y_i}\right)}{\alpha + \bar{\alpha} e^{-\theta y_i}}\right\} = \infty$$

Also,

$$\lim_{\alpha\to\infty}\frac{\partial l}{\partial\alpha}=0-2\lim_{\alpha\to\infty}\sum_{i=1}^{n}\left\{\frac{\left(1-e^{-\theta y_i}\right)}{\alpha+\bar{\alpha}e^{-\theta y_i}}\right\}<0$$

Therefore, there exists at least one root say $\hat{\alpha}(0,\infty)$, such that $\frac{\partial l}{\partial\alpha}=0$

For uniqueness of the root, we have

$$\frac{\partial^2 l}{\partial\alpha^2}=-\frac{n}{\alpha^2}+2\sum_{i=1}^{n}\left\{\frac{\left(1-e^{-\theta y_i}\right)^2}{(\alpha+\bar{\alpha}e^{-\theta y_i})^2}\right\}<0$$

whenever,

$$\frac{n}{\alpha^2}>2\sum_{i=1}^{n}\left\{\frac{\left(1-e^{-\theta y_i}\right)^2}{(\alpha+\bar{\alpha}e^{-\theta y_i})^2}\right\}.$$

Theorem 5:

If the parameter α is known, then the MLE of θ exists and is unique.

Proof:

Since,

$$\frac{\partial l}{\partial\theta}=\frac{n}{\theta}-\sum_{i=1}^{n}y_i+2\sum_{i=1}^{n}\left\{\frac{\left(\bar{\alpha}y_i e^{-\theta y_i}\right)}{\alpha+\bar{\alpha}e^{-\theta y_i}}\right\}$$

$$\lim_{\theta\to 0}\frac{\partial l}{\partial\theta}=\infty-\sum_{i=1}^{n}y_i+2\sum_{i=1}^{n}\bar{\alpha}y_i=\infty$$

Also,

$$\lim_{\theta\to\infty}\frac{\partial l}{\partial\theta}=0-\sum_{i=1}^{n}y_i+2\lim_{\theta\to\infty}\sum_{i=1}^{n}\left\{\frac{\left(\bar{\alpha}y_i e^{-\theta y_i}\right)}{\alpha+\bar{\alpha}e^{-\theta y_i}}\right\}<0$$

Therefore, there exists at least one root say $\hat{\theta}(0,\infty)$, such that $\frac{\partial l}{\partial\theta}=0$

For uniqueness of the root, we have

$$\frac{\partial^2 l}{\partial\theta^2}=-\frac{n}{\theta^2}-2\alpha {y_i}^2 e^{-\theta y_i}\sum_{i=1}^{n}\left\{\frac{\left(\alpha+2\bar{\alpha}e^{-\theta y_i}\right)}{(\alpha+\bar{\alpha}e^{-\theta y_i})^2}\right\}<0.$$

4.2 Simulation Study

Here the simulation study has been performed by using R software to know the consistency of the MLEs in terms of the sample size n. Two sets of sample $(n=50, n=100)$ each replicated 100 times with different values of parameters $\alpha=(0.5,1,1.5,3), \theta=(0.5,1,1.5,2,3,5)$ were generated from MOEQ. In each case, the average values of MLEs and the corresponding empirical mean squared errors (MSEs) and bias were attained. The simulation results are presented in Table 1 and Table 2.

Table 1
Average Values of MLEs their Corresponding MSEs and Bias $(n = 50)$

Parameter		MLEs		MSE		Bias	
α	θ	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\theta}$
0.5	0.5	0.52740	0.52741	0.07861	0.01671	0.02740	0.02741
	1	0.52101	1.06455	0.05749	0.07149	0.02101	0.05645
	1.5	0.47948	1.62453	0.56331	0.16703	-0.02051	0.12453
	2	0.49051	2.13654	0.04755	0.26070	0.00948	0.13654
	3	0.55738	3.14291	0.09749	0.48546	0.05738	0.14291
	5	0.50185	5.23818	0.06469	1.34852	0.00185	0.34878
1	0.5	1.05076	0.54725	0.38829	0.02431	0.05075	0.04725
	1	1.04097	1.11006	0.26703	0.10527	0.04097	0.11006
	1.5	1.01134	1.61560	0.27918	0.22251	0.01134	0.11560
	2	1.03721	2.16363	0.28633	0.35038	0.03721	0.16363
	3	1.02264	3.40980	0.31410	1.36878	0.02264	0.40980
	5	0.98376	5.44550	0.26406	2.42053	-0.01623	0.44550
1.5	0.5	1.49344	0.57981	0.68621	0.04575	-0.00655	0.79812
	1	1.71439	1.05921	0.99900	0.13176	0.21430	0.05921
	1.5	1.50392	1.69348	0.87723	0.33013	0.33392	0.19348
	2	1.55008	2.21292	0.70994	0.44336	0.05008	0.21392
	3	1.61934	3.23980	0.62430	1.52733	0.11934	0.23987
	5	1.47814	5.72801	0.68808	3.21099	-0.02185	0.72801
2	0.5	2.12307	0.57170	2.44766	0.04065	0.12307	0.07170
	1	2.00620	1.16791	1.07818	0.15515	0.00620	0.16791
	1.5	1.89052	1.75050	0.91144	0.33922	-0.10947	0.25050
	2	2.01638	2.30936	1.02065	0.75756	0.01638	0.30936
	3	1.91835	3.53332	1.03016	2.20462	-0.08164	0.53332
	5	2.14085	5.54582	1.07991	4.33479	0.14085	0.54558
3	0.5	3.03187	0.61040	4.26700	0.09046	0.031874	0.11040
	1	2.94290	1.22256	3.64377	0.28037	-0.05760	0.22256
	1.5	3.02957	1.72143	2.63774	0.47886	0.029957	0.22143
	2	3.21780	2.30601	4.91063	1.42010	0.21780	0.30601
	3	3.09566	3.55044	3.12401	3.05379	0.09566	0.55044
	5	2.94164	5.83940	2.11898	5.17955	-0.05835	0.83940

Table 2
Average Values of MLEs their Corresponding MSEs and Bias($n = 100$)

Parameter		MLEs		MSE		Bias	
α	θ	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\theta}$
0.5	0.5	0.52740	0.52741	0.07861	0.01671	0.02740	0.02741
	1	0.52101	1.06455	0.05749	0.07149	0.02101	0.05645
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	1.5	1.50392	1.69348	0.87723	0.33013	0.33392	0.19348
	2	1.55008	2.21292	0.70994	0.44336	0.05008	0.21392
	3	1.61934	3.23980	0.62430	1.52733	0.11934	0.23987
	5	1.47814	5.72801	0.68808	3.21099	-0.02185	0.72801
2	0.5	2.12307	0.57170	2.44766	0.04065	0.12307	0.07170
	1	2.00620	1.16791	1.07818	0.15515	0.00620	0.16791
	1.5	1.89052	1.75050	0.91144	0.33922	-0.10947	0.25050
	2	2.01638	2.30936	1.02065	0.75756	0.01638	0.30936
	3	1.91835	3.53332	1.03016	2.20462	-0.08164	0.53332
	5	2.14085	5.54582	1.07991	4.33479	0.14085	0.54558
3	0.5	3.03187	0.61040	4.26700	0.09046	0.031874	0.11040
	1	2.94290	1.22256	3.64377	0.28037	-0.05760	0.22256
	1.5	3.02957	1.72143	2.63774	0.47886	0.029957	0.22143
	2	3.21780	2.30601	4.91063	1.42010	0.21780	0.30601
	3	3.09566	3.55044	3.12401	3.05379	0.09566	0.55044
	5	2.94164	5.83940	2.11898	5.17955	-0.05835	0.83940

From Tables 1 and 2, it has been observed that the estimates are stable and very close to the true parameter values. As the sample size increases the MSE decreases in all cases.

5. APPLICATIONS

To justify the applicability of the MOEQ distribution two real data sets have been used. The data set I corresponds to the waiting time (in minutes) of 100 bank customers. The data were taken from Ghitany, Atieh and Nadarajah (2008) and was also reported by Bhat, Mudasir and Ahmad (2018).

The data set II corresponds to the remission time in months of 128 bladder cancer patients. The data were taken from Aldeni, Lee and Famoye (2017) and was recently reported by Ijaz et al. (2021).

For comparison purpose, we have fitted the proposed MOEQ with several other models, namely exponential (E), Rayleigh (R), Weibull exponential (WE) (Oguntunde et al., 2015), modified Weibull (MW) (Sarhan and Zaindin, 2009), inverse Rayleigh (IR) distributions.

Table 3
MLEs and -2l, AIC, AICC, BIC for Data Set I

Model	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\lambda}$	-2l	AIC	AICC	BIC
MOEQ	0.24293 (0.07990)	0.19236 (0.02599)	-	641.4241	645.4241	645.5478	650.6344
E	0.10124 (0.01012)	-	-	658.0418	660.0418	660.0826	662.6469
R	0.00665 (0.00064)	-	-	658.4842	660.4842	660.5250	663.0894
WE	3.13216 (0.13308)	0.01807 (0.01229)	0.80693 (0.13308)	641.6886	647.6886	647.9386	655.5410
MW	0.10124 (0.01012)	0.00100 (0.00703)	0.02597 (0.01287)	658.2426	664.2426	664.4926	672.0581
IR	13.09707 (1.30970)	-	-	759.5627	761.5627	761.6035	764.1678

Table 4
MLEs and -2l, AIC, AICC, BIC for Data Set II

Model	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\lambda}$	-2l	AIC	AICC	BIC
MOEQ	0.94699 (0.28844)	0.10987 (0.01986)	-	828.6523	832.6523	832.6840	838.3563
E	0.10676 (0.00943)	-	-	828.6838	830.6838	830.7155	833.5358
R	0.00507 (0.00043)	-	-	982.5318	984.5318	984.5635	987.3838
WE	3.95810 (1.21408)	0.01796 (0.00466)	0.85819 (0.05928)	839.7996	845.7996	845.9931	854.3557
MW	0.10518 (0.05587)	0.00100 (0.03494)	1.16895 (0.98153)	828.6628	834.6628	834.8564	843.2189
IR	0.61733 (0.61733)	-	-	1548.683	1550.683	1550.715	1553.535

From Table 3 it has been observed that MOEQ distribution has the smallest values of the criteria -2l, AIC, AICC and BIC among all the other distributions. Hence, we can say that the proposed model fits best for the data set I. From Table 4, though the exponential distribution fits slightly better than MOEQ for data set II, yet we recommend the use of MOEQ because it incorporates both monotone and non-monotone density and non-constant hazard rate as well.

Figure 3(i) and 4(i) display relative histograms for data set I and II respectively. Also, the Figure 3(ii) and 4(ii) shows the plots of the fitted MOEQ survival function and empirical survival function of the data set I and II, respectively. The Q-Q and P-P plots are presented by 5(i) and (ii) and 6(i) and (ii) for data set I and II, respectively, which permits us to compare the empirical distribution of the data with the MOEQ distribution. These graphical representations also support the results in Tables 3 and 4.

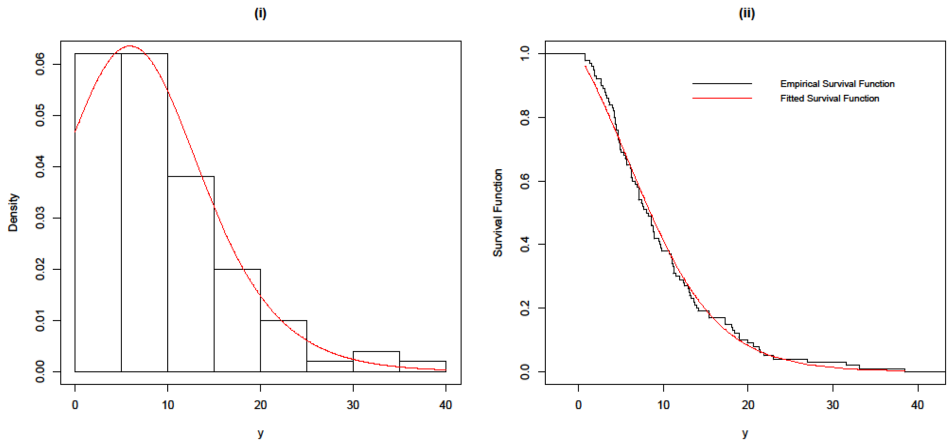

Figure 3: i) The relative histogram and the fitted MOEQ Distribution.
ii) The fitted MOEQ survival function and empirical survival unction for the data set I.

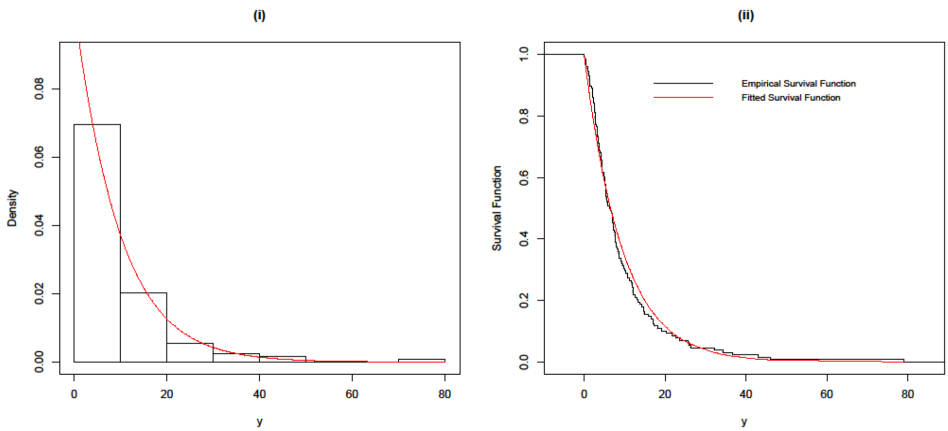

Figure 4: i) The relative histogram and the fitted MOEQ distribution.
ii) The fitted MOEQ survival function and empirical survival function for the data set II.

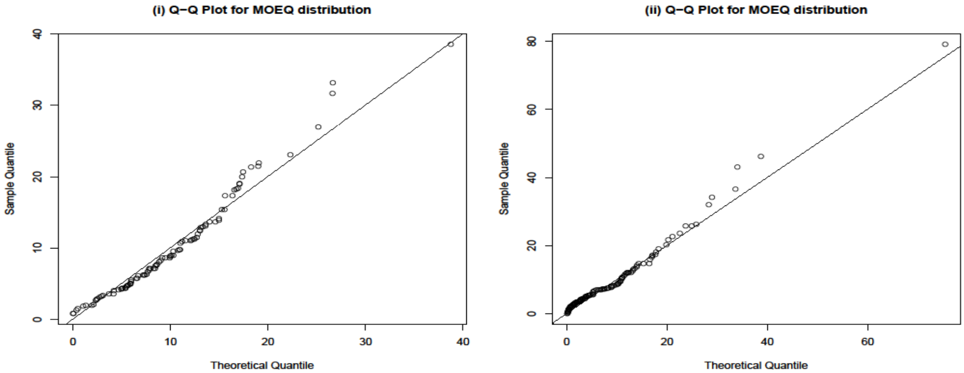

Figure 5: Q-Q plot for the MOEQ distribution for data set I and data set II, respectively

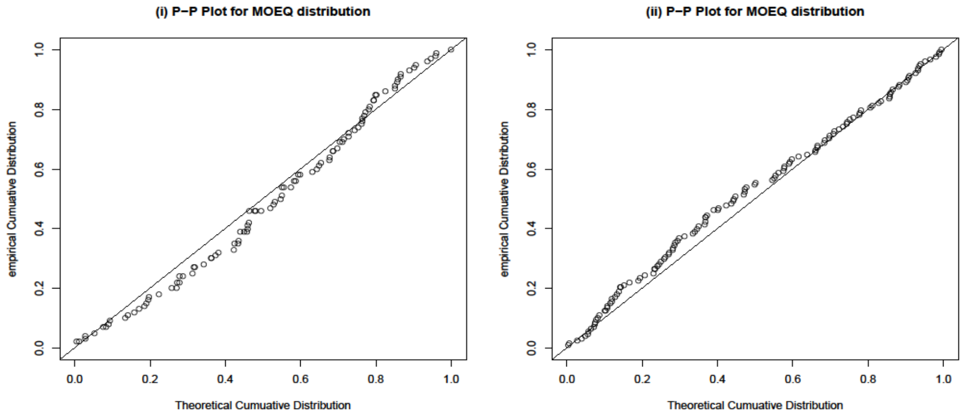

Figure 6: P-P plot for the MOEQ distribution for data set I and data set II, respectively

6. CONCLUSION

A new method for generating family of distributions has been introduced. A member of the introduced family namely, MOEQ distribution has been studied in detail. The density and the hazard rate functions are quite flexible in terms of shape behavior. In fact, the density function can be decreasing, right-skewed and symmetric. The hazard rate function assimilates an IFR, DFR and constant failure rate shapes. For the applicability of the MOEQ distribution two data sets have been discussed, it has been shown that MOEQ distribution fits better than all other competitive models.

REFERENCES

1. Aldeni, M., Lee, C. and Famoye, F. (2017). Families of distributions arising from the quantile of generalized lambda distribution. *Journal of Statistical Distributions and Applications*, 4(1), 1-18.
2. Alzaatreh, A., Lee, C. and Famoye, F. (2014). T-normal family of distributions: A new approach to generalize the normal distribution. *Journal of Statistical Distributions and Applications*, 1(1), 1-18.
3. Alzaatreh, A., Lee, C., Famoye, F. and Ghosh, I. (2016). The generalized Cauchy family of distributions with applications. *Journal of Statistical Distributions and Applications*, 3(1), 1-16.
4. Barlow, R.E. and Proschan, F. (1975). *Statistical theory of reliability and life testing*. Holt, Rinehart and Winston, Inc., New York.
5. Bhat, A.A., Mudasir, S. and Ahmad, S.P. (2018). Mixture of exponential and weighted exponential distribution: Properties and applications. *International Journal of Scientific Research in Mathematical and Statistical Sciences*, 5(6), 38-46.
6. Cordeiro, G.M., Alizadeh, M., Ramires, T.G. and Ortega, E.M. (2017). The generalized odd half-cauchy family of distributions: properties and applications. *Communications in Statistics-Theory and Methods*, 46(11), 5685-5705.
7. Eugene, N., Lee, C. and Famoye, F. (2002). Beta-normal distribution and its applications. *Communications in Statistics-Theory and Methods*, 31(4), 497-512.
8. Ghitany, M.E., Atieh, B. and Nadarajah, S. (2008). Lindley distribution and its application. *Mathematics and Computers in Simulation*, 78(4), 493-506.
9. Ijaz, M., Asim, S.M., Farooq, M., Khan, S.A. and Manzoor, S. (2020). A gull alpha power Weibull distribution with applications to real and simulated data. *PloS one*, 15(6), e0233080.
10. Ijaz, M., Mashwani, W.K., Göktaş, A. and Unvan, Y.A. (2021). A novel alpha power transformed exponential distribution with real-life applications. *Journal of Applied Statistics*, 1-16.
11. Mahdavi, A. and Kundu, D. (2017). A new method for generating distributions with an application to exponential distribution. *Communications in Statistics-Theory and Methods*, 46(13), 6543-6557.
12. Marshall, A.W. and Olkin, I. (1997). A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families. *Biometrika*, 84(3), 641-652.
13. Mudholkar, G.S. and Srivastava, D.K. (1993). Exponentiated Weibull family for analyzing bathtub failure-rate data. *IEEE transactions on Reliability*, 42(2), 299-302.
14. Nassar, M., Alzaatreh, A., Abo-Kasem, O., Mead, M. and Mansoor, M. (2018). A new family of generalized distributions based on alpha power transformation with application to cancer data. *Annals of Data Science*, 5(3), 421-436.
15. Oguntunde, P., Balogun, O., Okagbue, H. and Bishop, S. (2015). The Weibull-exponential distribution: Its properties and applications. *Journal of Applied Sciences*, 15(11), 1305-1311.
16. Sarhan, A.M. and Zaindin, M. (2009). Modified Weibull distribution. *APPS. Applied Sciences*, 11, 123-136.