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BAYES ESTIMATION OF THE ALPHA POWER INVERTED EXPONENTIAL PARAMETERS UNDER VARIOUS APPROXIMATION TECHNIQUES

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ABSTRACT

Bayes estimation methods are proposed in this paper to estimate the unknown parameters of the Alpha Power Inverted Exponential (APIE) distribution for complete sample and this distribution was introduced by Unal, Cakmakyapan and Ozel (2018). Lindley's approximation, Tierney and Kadane's (T-K) approximation and Markov Chain Monte Carlo (MCMC) method are used for obtaining bayes estimators with squared error loss function (SELF) and Gamma prior for the unknown parameters of this distribution. The performance of these estimators are compared with maximum likelihood estimators by simulation study and the comparison is based on MSE's. Also two real life data sets are used to illustrate the usefulness of the Alpha Power Inverted Exponential (APIE) distribution. Different criterions are used to access the goodness of fit of APIE distribution as compared to other distributions.

KEYWORDS

Alpha Power Inverted Exponential Distribution, Maximum likelihood estimation, Lindley's Approximation, T-K Approximation, MCMC.

1. INTRODUCTION

There are various methods to develop a flexible distributions. In recent years, a new transformation, Alpha Power Transformation (APT) was developed by Mahdavi and Kundu (2017). This transformation introduces an extra parameter to the family of distributions and this parameter makes the distribution more flexible. Then Mahdavi and Kundu (2017) used this APT method to exponential distribution and obtain Alpha Power Exponential distribution. And like this many authors Dey et al. (2017) and Dey, Sharma and Mesfioui (2017) worked on this transformation and developed flexible distributions. Similarly, Alpha Power Inverted Exponential distribution was also proposed by Unal, Cakmakyapan and Ozel (2018) using this transformation.

The probability mass function of the Alpha Power Inverted Exponential distribution is:

$$g(x) = \frac{\log\alpha}{\alpha-1}\,\frac{\gamma}{x^2}\,\exp\left(-\frac{\gamma}{x}\right)\alpha^{\exp\left(-\frac{\gamma}{x}\right)}$$

And the distribution function of the APIE distribution is:

$$G(x) = \frac{\alpha^{\exp\left(-\frac{\gamma}{x}\right)} - 1}{\alpha - 1}$$

where $\alpha > 0$ and it is shape parameter. $\gamma > 0$ and it is scale parameter. Alpha Power Inverted Exponential distribution is useful in modeling lifetime data that have increasing and decreasing failure rates and also commonly used in biology, medicine and engineering. The density function and hazard function is versatile for different values of the parameters.

Exponential distribution is widely used for describing the time between events in Poisson process. Exponential distribution has a constant failure rate and for this type of failure rate exponential distribution is not suitable for modeling real life situations in mechanical systems, business, insurance etc. Because of this disadvantage, Keller, Kamath and Perera (1982) developed Inverted Exponential (IE) distribution and this distribution have the inverted bathtub hazard rate and as a lifetime model this distribution was studied by Lin, Duran and Lewis (1989) in detail. Inverted Exponential distribution has been widely used for engineering, biology and medicine Oguntunde et al. (2017). After that many authors have developed distributions using IE distribution. Singh, Singh and Kumar (2013) obtained the Bayes estimates of the unknown parameters and reliability function of the Inverted Exponential distribution. Bayes estimates of the IE were derived by Dey (2017). Using IE distribution, Generalized Inverted Exponential (GIE) distribution was proposed by Aboummoh and Alshingiti (2009). Further, Exponentiated Generalized Inverse Exponential (EGIE) distribution was developed by Oguntunde and Adejumo (2014). Maximum likelihood estimates and Bayes estimates of GIE distribution were discussed by Singh, Singh and Kumar (2013). Other studies on IE distribution are Kumaraswamy Inverse Exponential (KIE) distribution by Oguntunde, Babatunde and Ogunmola (2014), Transmuted Inverse Exponential (TIE) distribution by Oguntunde and Adejumo (2014) and Weibull Inverse Exponential (WIE) distribution by Oguntunde (2017).

Recently, Bayesian method has become more powerful and attractive to make inferences about research and studied by many authors such as Noor et al. (2019), Taufiq et al. (2020) and Alotaiba et al. (2021). When we face difficulty in analyzing the data then Bayesian approach is used. Beside this, classical approach plays a wide role in field of statistics. For making inferences of unknown parameters in the model this approach is used. This approach is also useful in field of health economics. Bayesian approach is used for prediction and uncertainty of future data. Bayesian study usually needs the information about prior. In this study, unknown parameters of the distribution are treated as random and data is treated as fixed. In Bayesian procedure, a major difficulty is of obtaining the posterior distribution. The posterior density often involves the integration, which is not easily solvable not only for high dimensional complex models as well as for dealing with low dimensional models.

It is to be noted that for Alpha Power Inverted Exponential distribution Unal, Cakmakyapan and Ozel (2018) only discussed the statistical properties and classical method of estimation i.e. maximum likelihood estimation of the Inverted Weighted Exponential (IWE) distribution. The novelty of this paper is that we present Bayesian estimation for the Alpha Power Inverted Exponential distribution. We used informative prior i.e. Gamma Prior and Squared Error Loss Function (SELF) for obtaining Bayes estimates. It has been seen that Bayes estimates of the unknown parameters of the Alpha Power Inverted Exponential distribution cannot be possible to have a unique solution. Thus, in order to obtain a unique solution of Bayes estimates we will use different numerical approximation methods. We will use Lindley's Approximation method, Tierney and Kadane's (T-K) Approximation method and also Markov Chain Monte Carlo (MCMC) method for obtaining Bayes estimators. We develop the algorithm to generate MCMC samples from posterior density function using Gibbs sampling technique. Real life data is also been considered to show the usefulness of the entire distribution.

The rest of the article is structured as follows: In section 2 maximum likelihood estimation of the parameters are discussed. Bayes estimation is described in section 3 and different approximation techniques of Bayes estimators are presented in section 3.1, 3.2 and 3.3. Further, in section 4 comparison of the proposed estimators are obtained and in section 5 application is also discussed. In the end, in section 6 conclusion about the whole article is given.

2. MAXIMUM LIKELIHOOD ESTIMATION

Suppose, we take a random sample of size 'n' from APIE distribution which is described in equation (1). Thus, the likelihood function for the whole sample is:

$$\prod_{i=1}^{n} g(x) = L = \frac{(\log\alpha)^n}{(\alpha-1)^n} \frac{\gamma^n}{\prod_{i=1}^{n} x^2} \exp\left[-\gamma\sum_{i=1}^{n} \left(\frac{1}{x}\right)\right] \alpha^{\sum_{i=1}^{n} \exp\left(\frac{-\gamma}{x}\right)} \tag{1}$$

where, $x > 0$, $\alpha > 0$ and $\gamma > 0$.

By differentiating the log of (1) w.r.t. parameters and equating them to zero we obtain maximum likelihood estimates of the parameters. Thus, the two normal equations are:

$$\frac{\partial \ln L}{\partial\alpha} = \frac{n}{\alpha\log\alpha} - \frac{n}{\alpha-1} + \frac{1}{\alpha}\sum_{i=1}^{n} \exp\left(-\frac{\gamma}{x}\right) = 0 \tag{2}$$

$$\frac{\partial \ln L}{\partial\gamma} = \frac{n}{\gamma} - \sum_{i=1}^{n}\left(\frac{1}{x}\right) - \log\alpha\sum_{i=1}^{n} \left[\exp\left(-\frac{\gamma}{x}\right)\left(\frac{1}{x}\right)\right] = 0 \tag{3}$$

Above normal equations are non-linear and cannot be solved easily for θ and φ, so they can be solved using Newton-Raphson method.

3. BAYES ESTIMATION

The Bayesian approach for the parameters are widely used for many lifetime models and this procedure has been discussed in detail by many authors. Different loss functions can be used for obtaining Bayes estimates but it can be observed that most of the Bayes estimators are developed through square error loss function. Square error loss function is a symmetrical loss function. This loss function is defined as:

$$L\left(\delta,\hat{\delta}\right) \propto \left(\hat{\delta}-\delta\right)^2. \tag{4}$$

where $\hat{\delta}$ is the estimator of the parameter δ.

Using the Squared error loss function which is given in (4), posterior mean will be the Bayes estimators. In Bayesian approach, if information about prior is available then one can select prior distribution. But in the situations when the information is not given then it is not easy to select the prior distribution. In this case, we make a choice for a prior that parameters have independent gamma priors i.e. gamma(e, f) and gamma(g, h). When the values of hyperparameters e, f, g and h are assumed to be zero then gamma prior has flexible nature. Thus, the prior for α and γ can be considered as:

$$\prod 1(\alpha) \propto \alpha e - 1e - f\alpha \text{ and } \prod 2(\gamma) \propto \gamma g - 1e - h\gamma \tag{5}$$

respectively. Here e, f, g and h are hyper-parameters of the prior distributions. Now the joint prior density for α and γ is obtained as:

$$\prod(\alpha, \gamma) \propto \alpha e - 1\gamma g - 1e - f\alpha - h\gamma \tag{6}$$

hence, posterior distribution of α and γ i.e. $h(\alpha, \gamma|x)$ can be obtained by substituting $L(x|\alpha, \gamma)$ and $\Pi(\alpha, \gamma)$ from equations (1) and (6) and it is given by:

$$h(\alpha,\gamma|x) = K\alpha^{e+\sum_{i=1}^{n} exp\left(-\frac{\gamma}{x}\right)-1}\gamma^{n+g-1}exp\left[-f\alpha-\gamma\left(h+\sum_{i=1}^{n}\left(\frac{1}{x}\right)\right)\right]\prod_{i=1}^{n}\left[\frac{\log\alpha}{(\alpha-1)x^2}\right] \tag{7}$$

where,

$$K^{-1} = \int_\alpha\int_\gamma \alpha^{e+\sum_{i=1}^{n} exp\left(-\frac{\gamma}{x}\right)-1}\gamma^{n+g-1}exp\left[-f\alpha-\gamma\left(h+\sum_{i=1}^{n}\left(\frac{1}{x}\right)\right)\right]\prod_{i=1}^{n}\left[\frac{\log\alpha}{(\alpha-1)x^2}\right]d\alpha d\gamma$$

Here, K is a normalizing constant. Here we have used square error loss function so the Bayes estimates of the parameters is the mean of posterior density.

$$\hat{\alpha}_{Bayes} = K\int_0^\infty\int_0^\infty \alpha^{e+\sum_{i=1}^{n} exp\left(-\frac{\gamma}{x}\right)}\gamma^{n+g-1}exp\left[-f\alpha-\gamma\left(h+\sum_{i=1}^{n}\left(\frac{1}{x}\right)\right)\right]\prod_{i=1}^{n}\left[\frac{\log\alpha}{(\alpha-1)x^2}\right]d\alpha d\gamma \tag{8}$$

$$\hat{\gamma}_{Bayes} = K\int_0^\infty\int_0^\infty \alpha^{e+\sum_{i=1}^{n} exp\left(-\frac{\gamma}{x}\right)-1}\gamma^{n+g}exp\left[-f\alpha-\gamma\left(h+\sum_{i=1}^{n}\left(\frac{1}{x}\right)\right)\right]\prod_{i=1}^{n}\left[\frac{\log\alpha}{(\alpha-1)x^2}\right]d\alpha d\gamma \tag{9}$$

Now from the above equations (8) and (9) we can observe that integral is involved in numerator and denominator of the Bayes estimates of the parameters and these expressions cannot be extractable easily. Hence, these estimates of α and γ are difficult to find so for solving this type of integrals there are several approximation techniques available in literature. Among those, we consider T-K approximation (Tierney and Kadanes's), Lindley's approximation method and Markov Chain Monte Carlo (MCMC) approximation method. These methods are used to solve integral problems and a single numerical result is obtained from these techniques.

3.1 Bayes Estimation through Lindley's Technique

Lindley's approximation method is used to develop the Bayes estimates of the unknown parameters, which states that in the ratio of integral the expectation of the posterior density can be expressed in the form as follows:

$$V(x) = E(\alpha,\gamma|x) = \frac{\int\int v(\alpha,\gamma)exp[L(\alpha,\gamma)+H(\alpha,\gamma)]d\alpha d\gamma}{\int\int exp[L(\alpha,\gamma)+H(\alpha,\gamma)]d\alpha d\gamma} \tag{10}$$

where,

$v(\alpha,\gamma)$: function of α and γ only $L(\alpha,\gamma)$: log of the joint likelihood
$H(\alpha,\gamma)$: log of joint prior density

Lindley (1980) states that, if maximum likelihood estimates of the parameters exist then we can approximate equation (10) as:

$$V(x) \approx v + \frac{1}{2}\sum_{i=1}^{n}\sum_{i=1}^{n}(v_{ij}+2v_i\rho_j)\sigma_{ij} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} L_{ijk}v_l\sigma_{ij}\sigma_{kl} \tag{11}$$

If $v(\alpha,\gamma)=\alpha$, then $v_\alpha = 1$ and $v_{\alpha\alpha} = v_\gamma = v_{\alpha\gamma} = v_{\gamma\alpha} = v_{\gamma\gamma} = 0$. Thus, the Bayes estimate of α is then obtained as:

$$\begin{aligned}\hat{\alpha}_{LD} = \hat{\alpha}_{ML} + \hat{\sigma}_{\alpha\alpha}\hat{\rho}_\alpha + \hat{\sigma}_{\alpha\gamma}\hat{\rho}_\gamma \\ + \frac{1}{2}\big(\hat{L}_{\alpha\alpha\alpha}\hat{\sigma}^2_{\alpha\alpha} + 3\hat{L}_{\alpha\gamma\alpha}\hat{\sigma}_{\alpha\alpha}\hat{\sigma}_{\alpha\gamma} + \hat{L}_{\gamma\gamma\alpha}\hat{\sigma}_{\gamma\gamma}\hat{\sigma}_{\alpha\alpha} \\ + 2\hat{L}_{\alpha\gamma\gamma}\hat{\sigma}^2_{\alpha\gamma} + \hat{L}_{\gamma\gamma\gamma}\hat{\sigma}_{\gamma\alpha}\hat{\sigma}_{\gamma\gamma}\big)\end{aligned} \tag{12}$$

$$\begin{aligned}\hat{\gamma}_{LD} = \hat{\gamma}_{ML} + \hat{\sigma}_{\gamma\alpha}\hat{\rho}_\alpha + \hat{\sigma}_{\gamma\gamma}\hat{\rho}_\gamma \\ + \frac{1}{2}\Big(\hat{L}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha}\hat{\sigma}_{\gamma\alpha} + 2\hat{L}_{\alpha\gamma\alpha}\hat{\sigma}^2_{\alpha\gamma} + 3\hat{L}_{\gamma\gamma\alpha}\hat{\sigma}_{\gamma\gamma}\hat{\sigma}_{\alpha\gamma} \\ + \hat{L}_{\alpha\alpha\gamma}\hat{\sigma}_{\alpha\alpha}\hat{\sigma}_{\gamma\gamma} + \hat{L}_{\gamma\gamma\gamma}\hat{\sigma}^2_{\gamma\gamma}\Big)\end{aligned} \tag{13}$$

3.2 Bayes Estimation through Tierney and Kadane's (T-K) Approximation

The above mentioned approximation technique is accurate enough to solve the ratio of integrals but sometimes there is a problem that this method consists of third partial derivatives and in the case of 'm' parameters, the total derivatives are $\frac{m(m+1)(m+2)}{6}$ then this approximation technique will be quite complicated. So in this case, one can think to use T-K approximation technique which is the alternative of Lindley's approximation. Tierney and Kadane's (T-K) approximation states that any integral of the form:

$$\hat{u}(\alpha,\gamma) = E_{p(\alpha,\gamma|\underline{x})}[u(\alpha,\gamma|\underline{x})] = \frac{\int_\alpha\int_\gamma e^{nL_*(\alpha,\gamma)}d(\alpha,\gamma)}{\int_\alpha\int_\gamma e^{nL_o(\alpha,\gamma)}d(\alpha,\gamma)} \tag{14}$$

where,

$$L_o(\alpha,\gamma) = \tfrac{1}{n}\big[L(\alpha,\gamma)+\ln(\text{constant}\,t)\big] \text{ and } L_*(\alpha,\gamma) = L_o(\alpha,\gamma)\tfrac{1}{n}\ln u(\alpha,\gamma)$$

can be approximated as:

$$\hat{u}(\alpha,\gamma) = \sqrt{\frac{|\Sigma_*|}{|\Sigma_o|}} e^{n[L_*(\alpha_*,\gamma_*)-L_o(\alpha_o,\gamma_o)]} \tag{15}$$

where (α_*,γ_*) and (α_o,γ_o) maximize $L_*(\alpha,\gamma)$ and $L_o(\alpha,\gamma)$ respectively and Σ_* and Σ_o are the inverse of matrices of second derivatives of $L_*(\alpha,\gamma)$ and $L_o(\alpha,\gamma)$ at the point (α_*,γ_*) and (α_o,γ_o) respectively. Taking the log of (7) we will obtain the function $L_o(\alpha,\gamma)$ as:

$$L_o(\alpha,\gamma) = \frac{1}{n}\left[\left(e+\sum_{i=1}^{n} exp\left(-\frac{\gamma}{x}\right)-1\right)\ln\alpha - f\alpha + (n+g-1)\ln\gamma - \gamma \left[h+\sum_{i=1}^{n}\left(\frac{1}{x}\right)\right] + n\ln(\ln\alpha) - n\ln(\alpha-1) - 2\sum_{i=1}^{n}\ln x\right] \tag{16}$$

and thus using the approximation the Bayes estimators of α and γ using square error loss function (SELF) can be written as:

$$\hat{\alpha}_{T-K} = \sqrt{\frac{|\Sigma_*|}{|\Sigma_o|}} e^{n[L_*^{\alpha}(\alpha_*,\gamma_*)-L_o(\alpha_o,\gamma_o)]} \tag{17}$$

$$\hat{\gamma}_{T-K} = \sqrt{\frac{|\Sigma_*|}{|\Sigma_o|}} e^{n[L_*^{\gamma}(\alpha_*,\gamma_*)-L_o(\alpha_o,\gamma_o)]} \tag{18}$$

where $L_*^{\alpha}(\alpha,\gamma) = L_o^{\alpha}(\alpha,\gamma) + \frac{1}{n}\ln\alpha$ and $L_*^{\gamma}(\alpha,\gamma) = L_o^{\gamma}(\alpha,\gamma) + \frac{1}{n}\ln\gamma$

3.3 Bayes Estimation through Markov Chain Monte Carlo (MCMC) Method

In this section, we obtain Bayes estimates through Markov Chain Monte Carlo (MCMC) Method. This technique contains two algorithms, one is Gibbs sampler and the other is Metropolis Hastings. These algorithms are used to generate samples from posterior density and then Bayes estimates are computed. When marginal densities of the parameters does not exist in explicit forms but the conditional densities given all the other parameters exist in nice forms, then the Gibbs sampler is applied. But generating samples from full conditional densities is not easily manageable, for this reason we consider (M-H) Metropolis Hastings algorithm. Metropolis is a step which is used to generate samples from full conditional density. Metropolis Hastings algorithm. Metropolis is a step which is used to generate samples from full conditional density. More details about MCMC technique are discussed by Gelfand and Smith (1990) and Upaghaya and Gupta (2010). Thus, using this concept we generate samples from the posterior density (7). We assume that parameters α and γ consists of independent gamma distribution with hyper-parameters e, f, g and h respectively. We consider full conditional posterior densities of α and γ which can be written as:

$$\Pi(\alpha|\gamma,\underline{x}) \propto \alpha^{e+\sum_{i=1}^{n} exp\left(-\frac{\gamma}{x}\right)-1} exp(-f\alpha) \prod_{i=1}^{n} \left[\frac{\log\alpha}{(\alpha-1)x^2}\right] \quad (19)$$

$$\Pi(\alpha|\gamma,\underline{x}) \propto \gamma^{n+g-1} exp\left[-\gamma\left(h+\sum_{i=1}^{n}\left(\frac{1}{x}\right)\right)\right] \prod_{i=1}^{n} \left[\frac{\log\alpha}{(\alpha-1)x^2}\right] \quad (20)$$

The Gibbs algorithm consists of these steps:

- Start with $m = 1$ and initial values $\left(\alpha^o, \gamma^o\right)$
- Use Metropolis-Hasting algorithm to generate samples from posterior density for α and γ
- Repeat steps 1-2, for all $m = 1,2,3,\ldots\ldots\ldots,M$ and obtaion (α_1,γ_1), (α_2,γ_2),.....,(α_M,γ_M)
- After we obtain the posterior samples, the Bayes estimates of α and γ under square error loss function are as follows:

$$\hat{\alpha}_{MC} = [E_\Pi(\alpha|x)] \approx \left(\frac{1}{M-M_o}\sum_{i=1}^{M-M_o}\alpha_i\right) \quad (21)$$

$$\hat{\gamma}_{MC} = [E_\Pi(\gamma|x)] \approx \left(\frac{1}{M-M_o}\sum_{i=1}^{M-M_o}\gamma_i\right) \quad (22)$$

where, M_o is the Markov Chain burn period.

4. COMPARISON OF THE PROPOSED ESTIMATORS

In this section of the paper, we have compared the performance of different estimation methods which we described in the above sections. And this comparison is executed through simulation study. We computed Mean Square Errors (MSE's) and on the basis of MSE's, the proposed estimates are compared. It is not easy to calculate MSE's of estimators so for this purpose we have obtained MSE's on the basis of simulated samples. We generate 1000 Monte Carlo runs with different sample sizes from the Alpha Power Inverted Exponential distribution. In order to calculate Bayes estimators from MCMC method we generated 40000 mcmc iterations for the parameters α and γ using the algorithm discussed in section (3.3). First 5000 iterations are discarded from the generated samples. We used different starting values to check the convergence of sequence of parameters.

We have fixed the values of parameters i.e. $\alpha = 1.5$ and $\gamma = 2.9$, then we firstly assume the hyper-parameters as: $e = f = h = 1$ and $g = 0.5$ and secondly we assume the values of hyper-parameters to zero and when hyper-parameters are assume to be zero the prior is taken as non-informative prior.

Table 1
Estimates and their MSE's for Unknown Parameters of APIE Distribution obtained by using various Estimation Methods under Informative Priors

N	Parameters	MLE	MSE	T-K	MSE	Lindley	MSE	MCMC	MSE
10	$\hat{\alpha}$	21.7650	1.9802	29.0871	1.9087	38.9016	0.7145	5.0981	1.0871
	$\hat{\gamma}$	2.9876	1.8907	4.2139	1.6543	3.8761	0.6098	3.5619	1.4529
20	$\hat{\alpha}$	29.8761	1.8123	31.9018	1.8890	35.9187	0.5123	10.8071	1.0089
	$\hat{\gamma}$	3.0078	1.7890	4.1679	1.5543	3.7123	0.5213	3.3129	1.3901
25	$\hat{\alpha}$	31.6667	1.6723	32.4321	1.8763	33.0532	0.3864	12.5571	0.9643
	$\hat{\gamma}$	3.0576	1.4321	4.0129	1.4321	3.5044	0.4366	3.2136	1.2346
35	$\hat{\alpha}$	32.095	1.4621	34.0129	1.6431	40.7673	0.1206	21.9618	0.7643
	$\hat{\gamma}$	3.7841	1.3210	3.9621	1.2121	3.3455	0.1580	3.6381	1.1329
45	$\hat{\alpha}$	40.6123	1.3901	36.9087	1.4213	49.8076	0.0345	29.0071	0.6102
	$\hat{\gamma}$	2.6178	1.2190	2.6154	1.1298	3.2789	0.0917	2.9160	1.0214
50	$\hat{\alpha}$	51.628	1.2619	42.6934	1.2321	61.2982	0.0257	32.1243	0.5429
	$\hat{\gamma}$	2.931	1.1218	3.7934	1.0963	3.1713	0.0755	2.6155	1.0129
60	$\hat{\alpha}$	60.8971	1.0876	53.1089	1.1180	69.1087	0.0083	40.1091	0.4120
	$\hat{\gamma}$	2.6981	1.0687	2.9081	1.0076	2.9087	0.0214	3.0087	0.2156
80	$\hat{\alpha}$	71.9018	0.9612	66.1067	1.0213	74.1890	0.0021	45.1078	0.2130
	$\hat{\gamma}$	2.4321	1.0123	2.5541	0.8721	2.5190	0.0081	2.8651	0.1078
100	$\hat{\alpha}$	80.7123	0.3219	75.9180	1.0021	84.9167	0.0007	49.9108	0.1120
	$\hat{\gamma}$	2.3321	1.0056	2.2134	0.5641	2.2139	0.0012	2.5410	0.0967

It can be observed from Table 1 that when we use informative prior, for both parameters bayes estimators attained by Lindley's method and MCMC method have least MSE as compared to MSE's of ML estimators and from these both bayes estimators Lindley's technique has smaller MSE as contrast to MSE's of MCMC technique. While only for parameter γ, bayes estimator using T-K method has less MSE than MSE's of maximum likelihood estimators and for parameter α, MSE's of T-K method are smaller as compared to that of ML estimators when sample size is very small say $n \leq 10$.

Table 2
Estimates and their MSE's for Unknown Parameters of APIE Distribution obtained by using Various Estimation Methods under Non-Informative Priors

N	Parameters	MLE	MSE	T-K	MSE	Lindley	MSE	MCMC	MSE
10	$\hat{\alpha}$	21.7650	1.9802	79.8123	1.7019	21.9087	1.9180	19.9081	1.9081
	$\hat{\gamma}$	2.9876	1.8907	2.9180	1.6091	3.0187	0.7123	2.7123	1.7651
20	$\hat{\alpha}$	29.8761	1.8123	75.1290	1.6971	29.8901	1.5091	22.1790	1.8009
	$\hat{\gamma}$	3.0078	1.7890	2.8912	1.5010	2.9180	0.5189	2.8191	1.6123
25	$\hat{\alpha}$	31.6667	1.6723	72.9321	1.5432	31.6786	0.2039	25.6922	1.7432
	$\hat{\gamma}$	3.0576	1.4321	2.8023	1.3231	3.0349	0.4919	2.9845	1.4213
35	$\hat{\alpha}$	32.095	1.4621	81.1218	1.4629	33.1183	0.1324	28.2084	1.6743
	$\hat{\gamma}$	3.7841	1.3210	2.9696	1.2921	3.3733	0.3964	3.1873	1.2131
45	$\hat{\alpha}$	40.6123	1.3901	86.9019	1.3921	41.9180	0.0871	30.9170	1.5091
	$\hat{\gamma}$	2.6178	1.2190	2.6701	1.1180	2.5123	0.0412	3.2980	1.1569
50	$\hat{\alpha}$	51.6281	1.2619	62.5432	1.3629	51.6199	0.0152	32.9329	1.4769
	$\hat{\gamma}$	2.931	1.1218	2.6939	1.0121	3.4161	0.0229	2.6931	1.1178
60	$\hat{\alpha}$	60.8971	1.0876	31.9180	1.1871	61.0345	0.0091	37.9180	1.3120
	$\hat{\gamma}$	2.6981	1.0687	2.6150	1.0070	2.6912	0.0035	2.9981	1.0532
80	$\hat{\alpha}$	71.9018	0.9612	29.0871	1.0431	71.9120	0.0051	29.8016	1.1121
	$\hat{\gamma}$	2.4321	1.0123	2.5180	0.9871	2.4219	0.0009	2.7651	1.0065
100	$\hat{\alpha}$	80.7123	0.3219	21.1082	1.0064	80.9180	0.0009	22.9180	1.0761
	$\hat{\gamma}$	2.3321	1.0056	2.1280	0.5132	2.3291	0.0002	2.1089	0.0876

It can be seen from Table 2 that when we use non-informative prior, for small sample size, only for parameter α, the bayes estimators using T-K approximation and MCMC method have smaller MSE than that of maximum likelihood estimator. But for large sample size, all three methods for bayes estimator give smaller MSE than MSE's for ML estimators. And from the above two tables we also observed that MSE's of all techniques decrease by increasing sample size.

5. REAL DATA APPLICATION

In this section, we consider two real data sets to identify the validity of Alpha Power Inverted Exponential distribution. To identify this validity we have discussed different criterions like, criteria of log-likelihood, Akaike Information criteria (AIC) and Bayesian Information criteria (BIC).

The first data consists of 55 patients and these patients suffer from the disease of head and neck cancer. This data was analyzed by Efron (1988). The observations are as follows:

6.537, 10.42, 14.48, 16.10, 22.70, 3441.55, 4245.28, 49.40, 53.62, 63, 64, 83, 84, 91, 108, 112, 129, 133, 133, 139, 140, 140, 146, 149, 154, 157, 160, 160, 165, 146, 149, 154, 157, 160, 160, 165, 173, 176, 218, 225, 241, 248, 273, 277, 297, 405, 417, 420, 440, 523, 583, 594, 1101, 1146, 1417.

Table 3
Values of -2L, AIC, BIC, K-S statistic and P-value for APIE Distribution and Competitive Distributions for Real Data Set 1

Distributions	-2L	AIC	BIC	K-S	P-value
Alpha Power Inverted Exponential (α, γ)	753.226	757.2254	761.3463	0.1315	0.0191
Inverse Exponential (α)	771.374	773.3742	775.4346	0.4128	0.00081
Generalized Inverse Exponential (α, θ)	769.182	773.1815	777.3024	0.6123	0.000007

The summary of the fitted model is given in Table 3. It shows that the Alpha Power Inverted Exponential distribution fits the data better than Inverse Exponential and Generalized Inverse Exponential distribution. The least values of AIC and BIC for the Alpha Power Inverted Exponential distribution show that it provides the good fit. Based on the value of K-S test statistic along with its p-value, we are unable to reject the null hypothesis and conclude that this data follows the Alpha Power Inverted Exponential distribution.

Table 4
Estimates of Unknown Parameters of APIE Distribution under Different Approximation Methods using Non-Informative Prior for Real Data Set 1

Size (n)	Parameters	MLE	T-K	Lindley's	MCMC
55	$\hat{\alpha}$	87.653	76.621	87.532	32.165
	$\hat{\gamma}$	2.6794	3.1213	2.5493	3.4607

Furthermore, we calculated the maximum likelihood estimates and also bayes estimates using three different techniques in Table 4. Bayes estimates are obtained on the assumption that prior are non-informative. It can be noticed that for this data set, Lindley's approximation behaves almost the same as ML estimation technique.

The second data consists of 44 survival times of patients and they have head and neck cancer disease. Patients are treated using a combination of radiotherapy and chemotherapy. The observations for this data was given by Efron (1988) and the observations are:

12.20, 23.56, 23.74, 25.87, 31.98, 37, 41.35, 47.38, 55.46, 58.36, 63.47, 68.46, 78.26, 74.47, 81.43, 84, 92, 94, 110, 112, 119, 127, 130, 133, 140, 146, 155, 159, 173, 179, 194, 195, 209, 249, 281, 319, 339, 432, 469, 519, 633, 725, 817, 1776.

Table 5
Values of -2L, AIC, BIC, K-S Statistic and P-value for APIE Distribution and Competitive Distributions for Real Data Set 2

Distributions	-2L	AIC	BIC	K-S	P-value
Alpha Power Inverted Exponential (α,γ)	558.8460	562.84	566.4137	0.2349	0.0129
Inverse Exponential (α)	568.4380	571.6622	572.8669	0.5123	0.0018
Generalized Inverse Exponential (α,θ)	569.062	572.4309	566.0443	0.8124	0.00004

In Table 5, the Alpha Power Inverted Exponential distribution is compared with Inverse Exponential and Generalized Inverse Exponential Distribution on the basis of Akaike information criteria(AIC) and Bayesian information criteria (BIC) and Kolmogrov-Smirnov(K-S) test. As the AIC and BIC is small as compared to the other distributions so the Alpha Power Inverted Exponential distribution provides better fit. The p-value of K.S test shows that the Alpha Power Inverted Exponential distribution better fit at significance level 0.05. Thus it can be concluded that this data follow the Alpha Power Inverted Exponential distribution.

Table 6
Estimates of Unknown Parameters of APIE Distribution under different Approximation Methods using Non-Informative Prior for Real Data Set 2

Size (n)	Parameters	MLE	T-K	Lindley's	MCMC
44	$\hat{\alpha}$	38.73	36.2312	37.934	28.5098
	$\hat{\gamma}$	2.63	2.9623	2.512	3.2754

Maximum likelihood estimates and bayes estimates using Lindley, T-K, and MCMC approach are calculated in Table 6. These bayes estimates are obtained on the assumption that prior are non-informative. It can be seen that for this data set, the estimates obtained from Lindley's approximation are very close to those obtained from ML estimation technique.

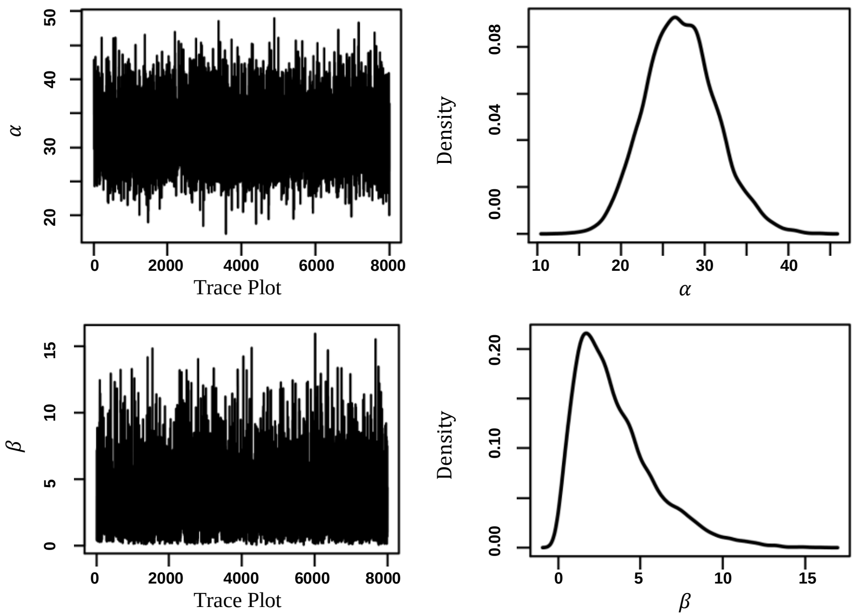

Figure 1: Trace and Density Plots for First Data

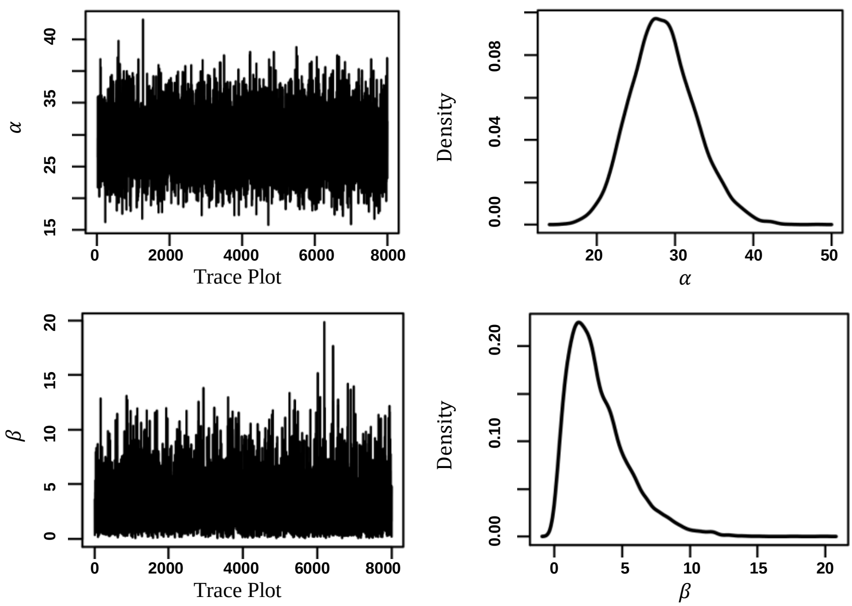

Figure 2: Trace and Density Plots for Second Data

Trace plots for the parameters in Figures 1 and 2 show that MCMC samples are well mixed and we observed from the density plots that the density of parameter γ is symmetrical for both data sets and the density of parameter α is positively skewed and has a long right tail for both real data sets.

6. CONCLUSION

In this paper, classical and also the bayes estimators of the parameters of the APIE distribution are discussed. By using different approximation techniques Bayes estimates are obtained and compared with ML estimates on the base of bias and mean square error. In all cases when sample size increases mean square error of all estimators decrease. Bayes estimators are developed using informative priors as well as non-informative priors. The results of the simulation concluded that Bayes estimates obtained from all techniques give better results than classical estimates.

It is observed that for both given data sets Alpha Power Inverted Exponential distribution is superior than other distributions i.e. Inverse Exponential and Generalized Inverse Exponential distributions.

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