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THE EXTENDED LAGUERRE POLYNOMIALS $A_{3,m}^{(\alpha)}(y)$ INVOLVING ${}_3F_3$

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ABSTRACT

In this paper, for the proposed extended Laguerre polynomials $A_{3,m}^{(\alpha)}(y)$, the generalised hypergeometric function of the type ${}_3F_3$ and extension of the Laguerre polynomial is introducing and similar to those related to the Laguerre polynomial, generating function, recurrence relations and Rodrigue's formula will determined. Some Corollaries are also discussed at the end.

KEYWORDS

Polynomials, Generating Functions, Recurrence relations and Rodrigue’s Formula.

1. INTRODUCTION AND BACKGROUND

Due to wide applications, the study of orthogonal polynomials has been a popular research topic for many years. Many of these polynomials are generated by hypergeometric functions. Indeed, the orthogonal polynomials have numerous properties of interest e.g. recurrence relations and differential equations. Based on their Rodrigues formulae, generating functions and solutions of integral equations with orthogonal polynomials as kernels have been extensively investigated.

Laguerre polynomials are described by many different well known methods. One such definition is based upon confluent hypergeometric functions that we are more interested in.

This note concerns polynomials based on ${}_3F_3$ the polynomials $A_{3,m}^{(\alpha)}(y)$ dealt with in Andrews et al. (1999) and Rainville (1965), similar to the Laguerre polynomials. By using relationships involving hypergeometric functions, several of the classical results of the Laguerre polynomials can be generalised immediately. In books and papers, related properties of the Laguerre orthogonal polynomials, their extensions and their applications are available. We may refer to many recent works in this respect, e.g. Akbary et al. (2009), Alam and Chongdar (2007), Chen and Srivastava (2005), Doha et al. (2009), Khan et al. (2017), Khan et al. (2019), Kim and Kim (2012), Lee (2007), Marinkovic et al. (2012), Nisar and Khan (2011), Radulescu (2008), Wang et al. (2012) and Zilina (1988).

There are a large range of applications of the Laguerre polynomials in many areas, including permutation statistics. The generating functions for permutations are the moments of calculation for these polynomials. Gurland et al. (1983) was considered a discrete distribution in which the probabilities are expressible by the polynomials of Laguerre. It is formulated in terms of a function generating likelihood involving three parameters. Multiple authors (see Aksoy (2009), Krasikov and Zarkh (2010) and Wang, (2009)).

Khan and Habibullah (2012) have introduced $A_{2,n}(x) = {}_2F_2\left(\frac{-n}{2}, \frac{-n+1}{2}; \frac{1}{2}, 1; x^2\right)$.

The work of Khan and Habibullah (2012) was generalised here. Extended Laguerre polynomials. $A_{3,m}^{(\alpha)}(y)$ defined by

$$A_{3,m}^{(\alpha)}(y) = \frac{(1+\alpha)_m}{m!}\, {}_3F_3\left(\frac{-m}{3}, \frac{-m+1}{3}, \frac{-m+2}{3}; \frac{1+\alpha}{3}, \frac{2+\alpha}{3}, \frac{3+\alpha}{3}; y^3\right),$$

where $\alpha > 0$ and m is any non-negative integer, and the factor $\frac{(1+\alpha)_m}{m!}$ is inserted for convenience only.

$$A_{3,m}^{(\alpha)}(y) = \frac{(1+\alpha)_m}{m!} \sum_{k=0}^{\left[\frac{m}{3}\right]} \left[\frac{\left(\frac{-m}{3}\right)_k \left(\frac{-m+1}{3}\right)_k \left(\frac{-m+2}{3}\right)_k}{\left(\frac{1+\alpha}{3}\right)_k \left(\frac{2+\alpha}{3}\right)_k \left(\frac{3+\alpha}{3}\right)_k} \right] \frac{y^{3k}}{(3k)!}.$$

$$= (1+\alpha)_m \sum_{k=0}^{\left[\frac{m}{3}\right]} \left[\frac{(-1)^{3k}}{(m-3k)!(1+\alpha)_{3k}} \right] \frac{y^{3k}}{(3k)!}. \tag{1.1}$$

$$\sum_{n=0}^{\infty} \left[\frac{A_{3,m}^{(\alpha)}(y)}{(1+\alpha)_m} \right] s^m = \sum_{m=0}^{\infty} \left[\sum_{k=0}^{\left[\frac{m}{3}\right]} \left[\frac{1}{(m-3k)!(1+\alpha)_{3k}} \right] \frac{(-y)^{3k}}{(3k)!} \right] s^m, \tag{1.2}$$

which leads to the generating function

$$\sum_{n=0}^{\infty} \frac{A_{3,m}^{(\alpha)}(y) s^n}{(1+\alpha)_m} = e^s\, {}_0F_3\left(--; \frac{1+\alpha}{3}, \frac{2+\alpha}{3}, \frac{3+\alpha}{3}; \left(\frac{-ys}{3}\right)^3 \right). \tag{1.3}$$

2. MAIN RESULTS

I explain key findings in this section and evaluate recurrence relationships for the extended Laguerre polynomials $A_{3,m}^{(\alpha)}(y)$.

Theorem 1:

If $\alpha > 0$, $m \in 0 \cup Z^+$, and $d \in Z^+$, then

$$\sum_{m=0}^{\infty} \frac{(d)_m A_{3,m}^{(\alpha)}(y) s^m}{(1+\alpha)_m}$$

$$= \frac{1}{(1-s)^d} \, {}_3F_3\left(\frac{d}{3}, \frac{d+1}{3}, \frac{d+2}{3}; \frac{1+\alpha}{3}, \frac{2+\alpha}{3}, \frac{3+\alpha}{3}; \left(\frac{-ys}{1-s} \right)^3 \right). \quad (1.4)$$

Proof:

From Equation (1.2),

$$\sum_{m=0}^{\infty} (d)_m \left[\frac{A_{3,m}^{(\alpha)}(y)}{(1+\alpha)_m} \right] s^m = \sum_{m=0}^{\infty} (d)_m \left[\sum_{k=0}^{\left[\frac{m}{3}\right]} \left[\frac{(-1)^{3k}}{(m-3k)!(1+\alpha)_{3k}} \right] \frac{y^{3k}}{(3k)!} \right] s^n$$

$$= \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{(d)_{m+3k} \, s^{m+3k}}{m!(1+\alpha)_{3k}} \frac{(-y)^{3k}}{(3k)!}$$

$$= \sum_{k=0}^{\infty} \left[\sum_{m=0}^{\infty} \frac{(d+3k)_m \, s^m}{m!} \right] \left[\frac{(d)_{3k}}{(1+\alpha)_{3k}} \right] \frac{(-ys)^{3k}}{(3k)!}$$

$$= \frac{1}{(1-s)^d} \sum_{k=0}^{\infty} \left[\frac{(d)_{3k}}{(1+\alpha)_{3k}} \right] \frac{1}{(3k)!} \left(\frac{-ys}{1-s} \right)^{3k} .$$

So, that

$$\sum_{m=0}^{\infty} \frac{(d)_m A_{3,m}^{(\alpha)}(y) s^m}{(1+\alpha)_m} = \frac{1}{(1-s)^d} \, {}_3F_3\left(\frac{d}{3}, \frac{d+1}{3}, \frac{d+2}{3}; \frac{1+\alpha}{3}, \frac{2+\alpha}{3}, \frac{3+\alpha}{3}; \left(\frac{-ys}{1-s} \right)^3 \right).$$

Corollary 1:

If $\alpha > 0$ and m is any non-negative integer, then

$$\sum_{m=0}^{\infty} A_{3,m}^{(\alpha)}(y) s^m = \frac{1}{(1-s)^{1+\alpha}} \exp\left(\frac{-ys}{1-s} \right)^3 . \quad (1.5)$$

Proof:

Put $d = 1+\alpha$ in Equation (1.4), we obtain our result.

RECURRENCE RELATIONS:

We, now, determine recurrence relations for the extended Laguerre polynomials $A_{3,m}^{(\alpha)}(y)$.

Theorem 2:

If $\alpha > 0$ and $m \geq 1$, then

$$y\,DA_{3,m}^{(\alpha)}(y) = mA_{3,m}^{(\alpha)}(y) - (\alpha+m)\,A_{3,m-1}^{(\alpha)}(y),\ D = \frac{d}{dy}. \tag{1.6}$$

Proof:

From Equation (1.3)

$$\sum_{m=0}^{\infty} \frac{A_{3,m}^{(\alpha)}(y)\,s^m}{(1+\alpha)_m} = e^s\,{}_0F_3\left(---;\frac{1+\alpha}{3},\frac{2+\alpha}{3},\frac{3+\alpha}{3};\left(\frac{-ys}{3}\right)^3\right).$$

Let

$$\sigma_{3,m}(y) = \frac{A_{3,m}^{(\alpha)}(y)}{(1+\alpha)_m}.$$

Suppose that

$$\psi\left(\frac{y^3 s^3}{3}\right) = {}_0F_3\left(--;\frac{1+\alpha}{3},\frac{2+\alpha}{3},\frac{3+\alpha}{3};\left(\frac{-ys}{3}\right)^3\right).$$

Then

$$F = e^s \psi\left(\frac{y^3 s^3}{3}\right) = \sum_{n=0}^{\infty} \sigma_{3,m}(y)\,s^m, \tag{1.7}$$

$$y\frac{\partial F}{\partial y} - s\frac{\partial F}{\partial s} = -sF. \tag{1.8}$$

Now, since $F = \sum_{m=0}^{\infty} \sigma_{3,m}(y)\,s^m$

$$\frac{\partial F}{\partial y} = \sum_{m=0}^{\infty} \sigma'_{3,m}(y)\,s^m,$$

and

$$s\frac{\partial F}{\partial s} = \sum_{m=0}^{\infty} m\sigma_{3,m}(y)\,s^m.$$

Equation (1.8), than yields

$$y\sum_{m=0}^{\infty}\sigma'_{3,m}(y)s^n-\sum_{m=0}^{\infty}m\sigma_{3,m}(y)s^m=-\sum_{m=1}^{\infty}\sigma_{3,m-1}(y)s^m.$$

It follows that $\sigma'_{3,0}(y)=0$, and for $m>1$, we get our result.

Theorem 3:

If $\alpha>0$ and $m\geq 2$, then

$$DA_{3,m}^{(\alpha)}(y)=3DA_{3,m-1}^{(\alpha)}(y)-3DA_{3,m-2}^{(\alpha)}(y)+DA_{3,m-3}^{(\alpha)}(y)-3y^2A_{3,m-3}^{(\alpha)}(y). \quad (1.9)$$

Proof:

Let

$$F=A(s)\exp\left[y^3\left(\frac{-s}{1-s}\right)^3\right]=\sum_{m=0}^{\infty}t_{3,m}(y)s^m. \quad (1.10)$$

$$\frac{\partial F}{\partial y}=3y^2\left(\frac{-s}{1-s}\right)^3A(s)\exp\left[y^3\left(\frac{-s}{1-s}\right)^3\right]=\sum_{m=0}^{\infty}t'_{3,m}(y)s^m. \quad (1.11)$$

$$(1-s)^3\frac{\partial F}{\partial y}=-3y^2s^3A(s)\exp\left[y^3\left(\frac{-s}{1-s}\right)^3\right]=-3y^2s^3F. \quad (1.12)$$

Consequently

$$\sum_{m=0}^{\infty}t'_{3,m}(y)s^m-3\sum_{m=1}^{\infty}t'_{3,m-1}(y)s^m+3\sum_{m=2}^{\infty}t'_{3,m-2}(y)s^m-\sum_{m=3}^{\infty}t'_{3,m-3}(y)s^m$$
$$=-3y^2\sum_{m=3}^{\infty}t_{3,n-3}(y)s^m.$$

It follows that $t'_{3,0}(y)=0$, $t'_{3,1}(y)=0$, $t'_{3,2}(y)=0$, and for $m>3$, get the result.

Theorem 4:

If $\alpha>0$ and $m\geq 3$, then

$$DA_{3,m}^{(\alpha)}(y)=-3y^2\sum_{k=0}^{m-3}{}^{m-k-1}C_{m-k-3}A_{3,k}^{(\alpha)}(y). \quad (1.13)$$

Proof:

By using Equation (1.11),

$$\sum_{m=0}^{\infty} t'_{3,m}(y)s^m = -3y^2\left[\sum_{m=0}^{\infty} {}^{m+2}C_m s^{m+3}\right]\left[\sum_{m=0}^{\infty} t_{3,m}(y)s^m\right]$$

$$= -3y^2 \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} {}^{m+2}C_m t_{m,k}(y)s^k s^{m+3}$$

$$= -3y^2 \sum_{m=3}^{\infty}\sum_{k=0}^{m-3} {}^{m-k-1}C_{m-k-3}\, t_{m,k}(x)s^m.$$

By comparing we get

$$t'_{3,m}(y)s^m = -3y^2 \sum_{k=0}^{m-3} {}^{m-k-1}C_{m-k-3}\, t_{m,k}(y).$$

It follows that $t'_{3,0}(y)=0,\ t'_{3,1}(y)=0,\ t'_{3,2}(y)=0,$ and for $m>3,$ get the result

Theorem 5:
If $\alpha>0$ and $m\geq 4$, then

$$mA_{3,m}^{(\alpha)}(y) = -(m-3+\alpha)A_{3,m-4}^{(\alpha)}(y) + \left(4m-9+3\alpha-3y^2\right)A_{3,m-3}^{(\alpha)}(y) + \\ -(6m-9+3\alpha)A_{3,m-2}^{(\alpha)}(y) + (4m-3+\alpha)A_{3,m-1}^{(\alpha)}(y). \tag{1.14}$$

Proof:
After eliminating the derivatives from Equations (1.6) and (1.9) (by multiplying Equation (1.9) by y, and then by subtracting from Equation (1.6)). We get our result

Theorem 6:
If $\alpha>0$ and $m\geq 1$, then

$$A_{3,m-1}^{(1+\alpha)}(y) + A_{3,m}^{(\alpha)}(y) = A_{3,m}^{(1+\alpha)}(y). \tag{1.15}$$

Proof:
Using Equation (1.6), we obtain

$$A_{3,m-1}^{(1+\alpha)}(y) = (2+\alpha)_{m-1} \sum_{k=0}^{\left[\frac{m-1}{3}\right]} \frac{(-1)^{3k}}{(m-1-3k)!(2+\alpha)_{3k}} \frac{y^{3k}}{(3k)!}, \tag{1.16}$$

so that

$$A_{3,m}^{(\alpha)}(y) = (1+\alpha)_m \sum_{k=0}^{\left[\frac{m}{3}\right]} \frac{(-1)^{3k}}{(m-3k)!(1+\alpha)_{3k}} \frac{y^{3k}}{(3k)!}. \tag{1.17}$$

By adding equations (1.16) and (1.17), and after simplification we get the result

RODRIGUE'S FORMULA:

In this section, we determine the Rodrigue's formula of the extended Laguerre polynomials $A_{3,m}^{(\alpha)}(y)$ in terms of the *m*th derivatives of a function.

Theorem 7:

If $\alpha > 0$ and m is any non-negative integer, then

$$A_{3,m}^{(\alpha)}(y) = \frac{y^{-\alpha}e^{2y}}{m!} D^m \left(y^{\alpha+m} e^{-y} \right). \tag{1.18}$$

Proof:

By Theorem (1.1), we have

$$A_{3,m}^{(\alpha)}(y) = \frac{e^y}{m!} \sum_{k=0}^{\left[\frac{m}{3}\right]} \left[\frac{m!}{(m-3k)!(3k)!} \right] \frac{(1+\alpha)_m (-y)^{3k}}{(1+\alpha)_{3k}}$$

$$= \frac{e^y y^{-\alpha}}{m!} \sum_{k=0}^{\left[\frac{m}{3}\right]} \left[\frac{(-1)^{3k} m!}{(m-3k)!(3k)!} \right] \frac{(1+\alpha)_m y^{\alpha+3k}}{(1+\alpha)_{3k}} .$$

Since $D^{m-3k}\left(y^{\alpha+m} \right) = \dfrac{(1+\alpha)_m y^{\alpha+3k}}{(1+\alpha)_{3k}}$, we write it as

$$A_{3,m}^{(\alpha)}(y) = \frac{y^{-\alpha}e^{2y}}{m!} \sum_{k=0}^{\left[\frac{m}{3}\right]} \left[\frac{m!}{(m-3k)!(3k)!} \right] \left[(-1)^{3k} e^{-y} \right] \left[D^{m-2k} \left(y^{\alpha+m} \right) \right].$$

Since $D^{3k}(e^{-y}) = (-1)^{3k} e^{-y}$, ${}^{m}C_{3k} = \dfrac{m!}{(m-3k)!(3k)!}$ we, therefore, conclude that

$$A_{3,m}^{(\alpha)}(y) = \frac{y^{-\alpha}e^{2y}}{m!} \sum_{k=0}^{\left[\frac{m}{3}\right]} {}^{m}C_{3k} D^{m-3k} (y^{\alpha+m}) D^{3k} (e^{-y}) .$$

$$= \frac{y^{-\alpha}e^{2y}}{m!} D^m \left(y^{\alpha+m} e^{-y} \right).$$

Corollary 2:

Put $\alpha = 0$, in theorem 1, we will get

$$\sum_{m=0}^{\infty} \frac{(d)_m A_{3,m}(y) s^m}{m!} = \frac{1}{(1-s)^d} {}_3F_3 \left(\frac{d}{3}, \frac{d+1}{3}, \frac{d+2}{3}; \frac{1}{3}, \frac{2}{3}, \frac{3}{3}; \left(\frac{-ys}{1-s} \right)^3 \right).$$

Corollary 3:

Put $\alpha = 0$, in theorem 2, we will get

$$y\,DA_{3,m}(y) = mA_{3,m}(y) - mA_{3,m-1}(y),\ D = \frac{d}{dy}.$$

Corollary 4:

Put $\alpha = 0$, in Corollary 1, we will get

$$\sum_{m=0}^{\infty} A_{3,m}(y)s^m = \frac{1}{(1-s)}\exp\left(\frac{-ys}{1-s}\right)^3.$$

Corollary 6:

Put $\alpha = 0$, in theorem 3, we will get

$$DA_{3,m}(y) = 3DA_{3,m-1}(y) - 3DA_{3,m-2}(y) + DA_{3,m-3}(y) - 3y^2 A_{3,m-3}(y).$$

Corollary 7:

Put $\alpha = 0$, in theorem 4, we will get

$$DA_{3,m}(y) = -3y^2 \sum_{k=0}^{m-3} {}^{m-k-1}C_{m-k-3} A_{3,k}(y).$$

Corollary 8:

Put $\alpha = 0$, in theorem 5, we will get

$$mA_{3,m}(y) = -(m-3)A_{3,m-4}(y) + \left(4m - 9 - 3y^2\right)A_{3,m-3}(y) + \\ -(6m-9)A_{3,m-2}(y) + (4m-3)A_{3,m-1}(y).$$

Corollary 9:

Put $\alpha = 0$, in theorem 6, we will get

$$A_{3,m-1}^{(1)}(y) + A_{3,m}(y) = A_{3,m}^{(1)}(y).$$

Corollary 10:

Put $\alpha = 0$, in theorem 7, we will get

$$A_{3,m}(y) = \frac{e^{2y}}{m!} D^m \left(y^m e^{-y}\right).$$

CONCLUSION

Finally in conclusion we compromised the extended Laguerre polynomials $A_{3,m}^{(\alpha)}(y)$ based on the ${}_3F_3$. We obtained generating functions, recurrence relations and Rodrigue's formula for these extended Laguerre polynomials. In future work we can extend it and can get more results. We will apply Laplace transformation, Elzaki transformation and same more transformations can apply on the results of extended Laguerre polynomials.

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