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A CLASSICAL AND BAYESIAN ESTIMATION TECHNIQUES FOR GOMPERTZ INVERSE RAYLEIGH DISTRIBUTION: PROPERTIES AND APPLICATION

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ABSTRACT

We propose a new three-parameter probability distribution called the Gompertz Inverse Rayleigh distribution with Inverse Rayleigh as the baseline distribution. The mathematical statistical properties and order statistics of the new distribution were also derived. Maximum likelihood estimation method and Bayesian estimation method are adopted for simulation in this work. Various characteristics and properties of the novel Gompertz Inverse Rayleigh (GoIR) distribution would be obtained such as survival, hazard rate, cumulative hazard, reversed hazard functions, limiting behaviour of its probability density and hazard rate functions. Quantile functions, median, moments and other properties would also be obtained. The shape of the distribution would be discussed via its skewness and kurtosis. Sufficient conditions for failure rate functions would be derived. A simulation study would be presented to check the performance of the proposed estimators. Finally, the application of the Gompertz Inverse Rayleigh distribution was demonstrated using a real life data and its performance was compared with other models. R programing was used to perform the analysis and plot the graphs.

KEY WORDS

Gompertz inverse, Inverse Rayleigh distribution, Monotone Likelihood Ratio, Order Statistic, Gompertz Distribution

1. INTRODUCTION

In probability and statistics, continuous probability distributions are commonly applied to describe measurable real world phenomena. Due to the usefulness of these distributions, their theory is widely studied and applied to emerging data of interest; and new ones are developed as the need arises. The interest in developing more flexible continuous distributions to handle bimodal, highly skewed and highly dispersed data remains strong in the field of probability and statistics. Many generalized distributions have been proposed and applied to describe various real life phenomena. Alzaatreh et al. (2013), noted that a common feature of these generalized distributions is that they have more parameters with higher flexibility than their baseline distributions to handle some features of the underlined phenomenon. Johnson et al. (1994) stated that distributions with four parameters should be sufficient for most practical applications but at least three parameters are needed. Adding a fifth or the sixth parameter may not show any noticeable improvement. The two distributions combined in this work are Gompertz and Inverse Rayleigh distributions with the effect of exponential distribution characteristics.

The Gompertz distribution is a continuous probability distribution, named after Benjamin Gompertz and it is often applied to describe the distribution of adult lifespans by demographers [Vaupel (1986) and Preston et al. (2001)] and actuaries (Willemse and Koppelaar, 2000); and can also be applied in biology, medicine, gerontology and related sciences [Economos (1982); Brown and Forbes (1974)].

The article is outlined as follows In Section 2, the derivation of the new distribution GoIR is explained. Explanations of the mathematical properties of the new distribution is described in Section 3. In Section 4, the parameters simulation using maximum likelihood estimation of the distribution is obtained. The maximum likelihood estimation of parameters is discussed. We described the real life application of the new distribution, it is presented and discussed in Section 5. Lastly, Section 6 shows the concluding remarks.

2. DERIVATION OF GOMPERTZ INVERSE RAYLEIGH (GOIR) DISTRIBUTION

In the T-X framework, the random variable T is a $'transformer'$ that is used to ‘transform’ the random variable R into a new family of generalized distributions of R. The cumulative distribution function (CDF) of the generalized family of distributions as defined by Alizadeh (2017) is given by

$$F_X(x) = \int_a^{[-log[1-G_R(x)]]} f_T(t)dt = F_T\{-log[1-G_R(x)]\} \tag{1}$$

and the probability distribution function (PDF) of the CDF function given in (1) is given by

$$f_X(x) = f_R(x)\frac{f_T\{Q_Y[F_R(x)]\}}{f_Y\{Q_Y[F_R(x)]\}}. \tag{2}$$

Alternatively, the PDF in (2) can be written as

$$f_X(x) = f_T\{Q_Y[F_R(x)]\} \times f_Y\{Q'_Y[F_R(x)]\} \times f_R(x). \tag{3}$$

Let x be a random variable that follows the Inverse Rayleigh distribution with CDF given by

$$G_X(x) = exp(-(\xi/x)^2). \tag{4}$$

The corresponding PDF is given by

$$g_X(x) = \frac{2\xi^2}{x^3}exp(-(\xi/x)^2). \tag{5}$$

By using the Alizadeh (2017), we derived the corresponding CDF of GoIR distribution as

$$F_X(x) = 1 - e^{\left(\frac{\varphi}{\eta}\right)[1-(1-G(x))^{-\eta}]} \tag{6}$$

and the corresponding PDF to (6) is given by

$$f_X(x) = \varphi g(x)[1 - G(x)]^{-\eta-1} e^{\left(\frac{\varphi}{\eta}\{1-[1-G(x)]^{-\eta}\}\right)} \tag{7}$$

where $G(X)$ as in (4) and $g(x)$ as in (5).

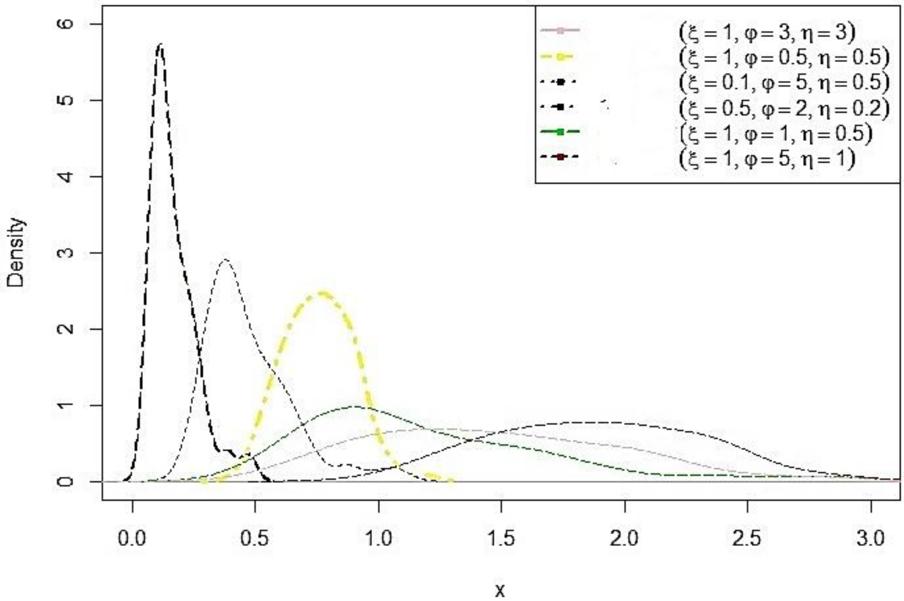

Figure 1: Density Plot of GoIR Distribution

Putting equation (4) into (6), we have the CDF of GoIR distribution,

$$F_X(x) = 1 - e^{\left(\frac{\varphi}{\eta}\right)\left[1-(1-exp(-(\xi/x)^2))^{-\eta}\right]} \tag{8}$$

Also, putting (4)and (5) into (7), we have the PDF of the proposed GoIR Distribution given by

$$f_X(x) = 2\varphi\xi^2 x^{-3} e^{-\left(\frac{\xi}{x}\right)^2} \left[1 - e^{-\left(\frac{\xi}{x}\right)^2}\right]^{-\eta-1} e^{\frac{\varphi}{\eta}\left\{1-\left[1-e^{-\left(\frac{\xi}{x}\right)^2}\right]^{-\eta}\right\}} \tag{9}$$

where η is a shape parameter, φ and ξ are scale parameters.

3. PROPERTIES OF GOIR DISTRIBUTION

3.1 Survival Function

The Survival function, $S_X(x)$ of a continuous random variable X that follows the GoIR distribution is derived from this definition

$$S_X(x) = e^{\frac{\varphi}{\eta}\left\{1-\left[1-e^{-\left(\frac{\xi}{x}\right)^2}\right]^{-\eta}\right\}} \tag{10}$$

3.2 Hazard Function

$$h_X(x) = \frac{2\varphi\xi^2 x^{-3} e^{-\left(\frac{\xi}{x}\right)^2}\left[1-e^{-\left(\frac{\xi}{x}\right)^2}\right]^{-\eta-1} e^{\frac{\varphi}{\eta}\left\{1-\left[1-e^{-\left(\frac{\xi}{x}\right)^2}\right]^{-\eta}\right\}}}{e^{\frac{\varphi}{\eta}\left\{1-\left[1-e^{-\left(\frac{\xi}{x}\right)^2}\right]^{-\eta}\right\}}} \tag{11}$$

$$h_X(x) = 2\varphi\xi^2 x^{-3} e^{-\left(\frac{\xi}{x}\right)^2}\left[1-e^{-\left(\frac{\xi}{x}\right)^2}\right]^{-\eta-1} \tag{12}$$

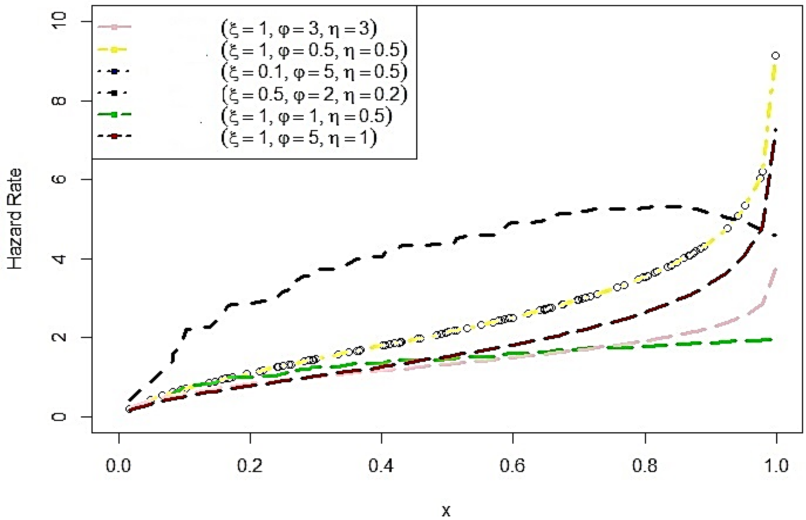

Figure 2: Hazard Function of GoIR Distribution

3.3 Cumulative Hazard Function

The cumulative hazard function, $H_X(x)$ of a continuous random variable X that follows the GoIR distribution is derived from this definition (Sule et al., 2021)

$$H_X(x) = -log[S_X(x)] \tag{13}$$

$$H_X(x) = -log\left[e^{\frac{\varphi}{\eta}\left\{1-\left[1-e^{-\left(\frac{\xi}{x}\right)^2}\right]^{-\eta}\right\}}\right] \quad (14)$$

Put equation (10) into (14) to have

$$H_X(x) = -\frac{\varphi}{\eta}\left\{1-\left[1-e^{-\left(\frac{\xi}{x}\right)^2}\right]^{-\eta}\right\}. \quad (15)$$

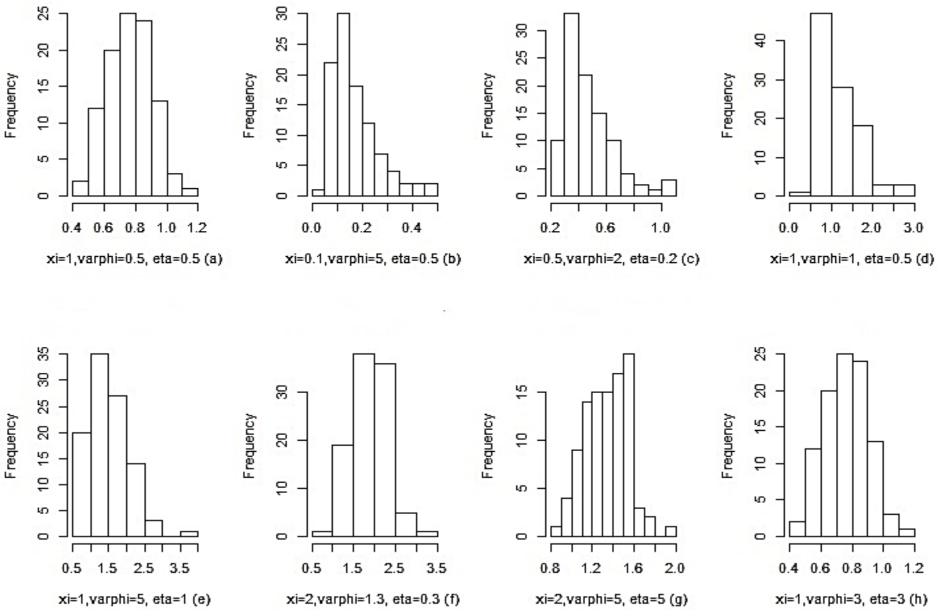

Figure 3: Histogram Plot of GoIR Distribution

3.4 Quantile Function and other definitions

Quantile function is used in simulation study and the p^{th} quantile of the GoIR distribution is derived as

$$Q_X(p) = \frac{\xi}{\sqrt{-ln\{1-[1-\eta/\varphi\; ln(1-p)]^{-1/\eta}\}}} \quad (16)$$

We have the first three, $Q_1 1 = Q(1/4)$ and $Q_3 = Q(3/4)$, that is by substituting value of $p = 0.25$ and $p = 0.75$ in X_p, respectively. Also Quantile is also used in finding the skewness and kurtosis of the distribution.

3.4.1 Median

Substitute $p = 0.5$ in (16), we have the median as:

$$M_e = Q_X(0.5) = \frac{\xi}{\sqrt{-ln\{1-[1-\eta/\varphi\ ln(0.5)]^{-1/\eta}\}}} \tag{17}$$

3.4.2 Reverse Hazard Function of GoIR

$$\tau_X = \frac{f(x)}{F(x)} \tag{18}$$

$$\tau_X = \frac{2\varphi\xi^2 x^{-3} e^{-\left(\frac{\xi}{x}\right)^2}\left[1-e^{-\left(\frac{\xi}{x}\right)^2}\right]^{-\eta-1} e^{\frac{\varphi}{\eta}\left\{1-\left[1-e^{-\left(\frac{\xi}{x}\right)^2}\right]^{-\eta}\right\}}}{1-e^{\frac{\varphi}{\eta}\left\{1-\left[1-e^{-\left(\frac{\xi}{x}\right)^2}\right]^{-\eta}\right\}}} \tag{19}$$

3.4.3 Linear Representation

By using the CDF as in (8) and PDF as in (9) of GoIR distribution, we have the following theorem.

Theorem 1:

Let T be a random variable that follows a GoIR distribution with parameters φ, η and ξ, then the linear representation is the GoIR distribution with parameter $(k+1)$ is given as

$$f_X(x) = 2\varphi\xi^2 \sum_{k=0}^{\infty} \tau_{i,j,k} x^{-3} e^{-(k+1)\left(\frac{\xi}{x}\right)^2} \tag{20}$$

where $(k+1) > 0$ and

$$\tau_{i,j,k} = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty} \frac{(-1)^{i+k}}{i!}\binom{i}{j}\binom{-\eta(j+1)-1}{k}\left(\frac{\varphi}{\eta}\right)^i \tag{21}$$

Proof:

Using Taylor series expansion,

$$exp\{-\alpha[y]^\beta\} = \sum_{k=0}^{\infty} \frac{(-1)^k\alpha^k}{k!}[\mathrm{y}]^{k\beta} \tag{22}$$

expanding the exponential expression

$$e^{\frac{\varphi}{\eta}\left\{1-\left[1-e^{-\left(\frac{\xi}{x}\right)^2}\right]^{-\eta}\right\}}$$

$$exp\left\{\frac{\varphi}{\eta}[y]\right\} = \sum_{i=0}^{\infty} \frac{\left(\frac{\varphi}{\eta}\right)^i}{i!}[\mathrm{y}]^i \tag{23}$$

then, using binomial expansion, we have

$$[\mathrm{y}]^{i} = \left[1 - \left[1 - e^{-\left(\frac{\xi}{x}\right)^{2}}\right]^{-\eta}\right]^{i} = \sum_{j=0}^{\infty} \binom{i}{j} (-1)^{j} \left[1 - e^{-\left(\frac{\xi}{x}\right)^{2}}\right]^{-\eta j} \tag{24}$$

$$\left[1 - \mathrm{e}^{-\left(\frac{\xi}{x}\right)^{2}}\right]^{-\eta j} = \sum_{k=0}^{\infty} (-1)^{k} \binom{-\eta j}{k} \left[\mathrm{e}^{-\left(\frac{\xi}{x}\right)^{2}}\right]^{k} \tag{25}$$

combining all the expansions, we have

$$e^{\frac{\varphi}{\eta}\left\{1-\left[1-e^{-\left(\frac{\xi}{x}\right)^{2}}\right]^{-\eta}\right\}} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{i} \left(\frac{\varphi}{\eta}\right)^{i} \binom{i}{j}}{i!} (-1)^{i+j+k} \binom{-\eta j}{k} \left[e^{-\left(\frac{\xi}{x}\right)^{2}}\right]^{k} \tag{26}$$

$$e^{\frac{\varphi}{\eta}\left\{1-\left[1-e^{-\left(\frac{\xi}{x}\right)^{2}}\right]^{-\eta}\right\}} = \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{k=0}^{\infty} \frac{(-1)^{i+k}}{i!} \binom{i}{j} \binom{-\eta j}{k} \left(\frac{\varphi}{\eta}\right)^{i} \left[e^{-\left(\frac{\xi}{x}\right)^{2}}\right]^{k} \tag{27}$$

then, the linear function of the CDF in (8) can be expressed as

$$F_{X}(x) = 1 - \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{k=0}^{\infty} \frac{(-1)^{i+k}}{i!} \binom{i}{j} \binom{-\eta j}{k} \left(\frac{\varphi}{\eta}\right)^{i} \left[e^{-\left(\frac{\xi}{x}\right)^{2}}\right]^{k} \tag{28}$$

For the linear function of the PDF in (9), it can be derived by expanding the exponential function

$$e^{\frac{\varphi}{\eta}\left\{1-\left[1-e^{-\left(\frac{\xi}{x}\right)^{2}}\right]^{-\eta}\right\}}$$

$$exp\left\{\frac{\varphi}{\eta}[y]\right\} = \sum_{i=0}^{\infty} \frac{\left(\frac{\varphi}{\eta}\right)^{i}}{i!} [y]^{i} \tag{29}$$

$$\left[1 - \left[1 - e^{-\left(\frac{\xi}{x}\right)^{2}}\right]^{-\eta}\right]^{i} = \sum_{j=0}^{\infty} \binom{i}{j} (-1)^{j} \left[1 - e^{-\left(\frac{\xi}{x}\right)^{2}}\right]^{-\eta j} \tag{30}$$

$$= 2\varphi\xi^{2}x^{-3}e^{-\left(\frac{\xi}{x}\right)^{2}} \left[1 - e^{-\left(\frac{\xi}{x}\right)^{2}}\right]^{-\eta-1} \sum_{i=0}^{\infty} \frac{\left(\frac{\varphi}{\eta}\right)^{i} \binom{i}{j} (-1)^{j}}{i!} \left[1 - e^{-\left(\frac{\xi}{x}\right)^{2}}\right]^{-\eta j} \tag{31}$$

$$\left[1-e^{-\left(\frac{\xi}{x}\right)^2}\right]^{-\eta j}\left[1-e^{-\left(\frac{\xi}{x}\right)^2}\right]^{-\eta-1}=\left[1-e^{-\left(\frac{\xi}{x}\right)^2}\right]^{-(\eta(j+1)+1)} \tag{32}$$

Using binomial expansion

$$\left[1-e^{-\left(\frac{\xi}{x}\right)^2}\right]^{-(\eta(j+1)+1)}=\sum_{k=0}^{\infty}(-1)^k\binom{-(\eta(j+1)+1)}{k}\left[e^{-\left(\frac{\xi}{x}\right)^2}\right]^k \tag{33}$$

$$=2\varphi\xi^2x^{-3}e^{-\left(\frac{\xi}{x}\right)^2}\sum_{i=j=k=0}^{\infty}\frac{\binom{i}{j}(-1)^j}{i!}(-1)^k \binom{-(\eta(j+1)+1)}{k}\left(\frac{\varphi}{\eta}\right)^i\left[e^{-\left(\frac{\xi}{x}\right)^2}\right]^k \tag{34}$$

and the corresponding PDF in (9) is given by

$$f_X(x)=2\varphi\xi^2x^{-3}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty}\frac{(-1)^{i+k}}{i!}\binom{i}{j} \binom{-\eta(j+1)-1}{k}\left(\frac{\varphi}{\eta}\right)^i\left[e^{-\left(\frac{\xi}{x}\right)^2}\right]^{(k+1)} \tag{35}$$

$$=2\varphi\xi^2x^{-3}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty}\frac{(-1)^{i+k}}{i!}\binom{i}{j} \binom{-\eta(j+1)-1}{k}\left(\frac{\varphi}{\eta}\right)^i\left[e^{-(k+1)\left(\frac{\xi}{x}\right)^2}\right] \tag{36}$$

Now, putting (21) into (36), we have (20).

3.4.4 Moment about Origin

Moments function is used to study many important properties of distribution such as dispersion, tendency, skewness and kurtosis (Sule and Ibrahim, 2021). The rth moments of the GoIR distribution is obtained as follows:

$$E(X^r)=\int_0^{\infty}x^r f_X(x) \tag{37}$$

Theorem 2:

Let X be a random variable that follows a GoIR distribution with parameters φ and η,ξ then the moment about the origin can be given as

$$E(X^r)=\varphi\sum_{k=0}^{\infty}\tau_{i,j,k}\frac{\xi^r\Gamma(1-r/2)}{(k+1)^{1-r/2}} \tag{38}$$

where $(k+1)>0$.

Proof:

Using linear representation of this new proposed distribution given in (20) and putting in (37) we have

$$E(X^r) = \int_0^\infty x^r 2\varphi\xi^2 \sum_{k=0}^{\infty} \tau_{i,j,k} x^{-3} e^{-(k+1)\left(\frac{\xi}{x}\right)^2} dx \tag{39}$$

$$= 2\varphi\xi^2 \sum_{k=0}^{\infty} \tau_{i,j,k} \int_0^\infty x^{r-3} e^{-(k+1)\left(\frac{\xi}{x}\right)^2} dx \tag{40}$$

Let,

$$t = (k+1)\left(\frac{\xi}{x}\right)^2 \tag{41}$$

then,

$$x = \xi(k+1)^{1/2} t^{-1/2} \tag{42}$$

hence

$$dx = (-1/2)\xi(k+1)^{1/2} t^{-3/2} \tag{43}$$

Now, using (41) and (42) the gamma function on (40) gives (38)

$$E(X^r) = 2\varphi\xi^2 \sum_{k=0}^{\infty} \tau_{i,j,k} \frac{\xi^r \Gamma(1-r/2)}{(k+1)^{1-r/2}} \tag{44}$$

Equation (38) completes the proof and it is the rth moment of GoIR distribution with parameters ξ, φ, and η.

3.4.5 Skewness and Kurtosis

Skewness is the measure of the asymmetry of the probability density function and kurtosis is the measure of peakedness of the probability density function (Sharqa et al., 2019). Both measures are the descriptive measures of the shape of the probability distribution (Sharqa, et al. 2019). The measure of skewness (S) and kurtosis (K) defined by Galton (1983) and Moors (1988) are based on quantile function and they are respectively defined in (45) and (48).

$$S = \frac{Q(6/8) - 2Q(4/8) + Q(2/8)}{Q(6/8) - Q(2/8)} \tag{45}$$

The skewness of the GoIR distribution is:

$$S=\frac{\frac{\xi}{\sqrt{-l\,n\left\{1-\left[1-\eta/\varphi l\,n\left(2/8\right)\right]^{-1/\eta}\right\}}}-2\frac{\xi}{\sqrt{-l\,n\left\{1-\left[1-\eta/\varphi l\,n\left(4/8\right)\right]^{-1/\eta}\right\}}}+\frac{\xi}{\sqrt{-\ln\left\{1-\left[1-\eta/\varphi\ln\left(6/8\right)\right]^{-1/\eta}\right\}}}}{\frac{\xi}{\sqrt{-l\,n\left\{1-\left[1-\eta/\varphi l\,n\left(2/8\right)\right]^{-1/\eta}\right\}}}-\frac{\xi}{\sqrt{-l\,n\left\{1-\left[1-\eta/\varphi l\,n\left(6/8\right)\right]^{-1/\eta}\right\}}}} \tag{46}$$

dividing the numerator and denominator by ξ

$$S=\frac{\sqrt{-l\,n\left\{1-\left[1-\eta/\varphi ln\left(2/8\right)\right]^{-1/\eta}\right\}}-\sqrt{-l\,n\left\{1-\left[1-\eta/\varphi ln\left(6/8\right)\right]^{-1/\eta}\right\}}}{\sqrt{-l\,n\left\{1-\left[1-\eta/\varphi ln\left(2/8\right)\right]^{-1/\eta}\right\}}-2\sqrt{-l\,n\left\{1-\left[1-\eta/\varphi ln\left(4/8\right)\right]^{-1/\eta}\right\}+\sqrt{-l\,n\left\{1-\left[1-\eta/\varphi ln\left(6/8\right)\right]^{-1/\eta}\right\}}}} \tag{47}$$

and

$$K=\frac{Q(7/8)-Q(5/8)+Q(3/8)-Q(1/8)}{Q(6/8)-Q(2/8)} \tag{48}$$

The kurtosis of the Goir distribution is:

$$K=\frac{\frac{\xi}{\sqrt{-l\,n\left\{1-\left[1-\eta/\varphi l\,n\left(3/8\right)\right]^{-1/\eta}\right\}}}-\frac{\xi}{\sqrt{-l\,n\left\{1-\left[1-\eta/\varphi l\,n\left(5/8\right)\right]^{-1/\eta}\right\}}}+\frac{\xi}{\sqrt{-l\,n\left\{1-\left[1-\eta/\varphi l\,n\left(7/8\right)\right]^{-1/\eta}\right\}}}}{\frac{\xi}{\sqrt{-l\,n\left\{1-\left[1-\eta/\varphi l\,n\left(2/8\right)\right]^{-1/\eta}\right\}}}-\frac{\xi}{\sqrt{-l\,n\left\{1-\left[1-\eta/\varphi l\,n\left(6/8\right)\right]^{-1/\eta}\right\}}}} \tag{49}$$

$$K=\frac{\sqrt{-l\,n\left\{1-\left[1-\eta/\varphi ln\left(2/8\right)\right]^{-1/\eta}\right\}}-\sqrt{-l\,n\left\{1-\left[1-\eta/\varphi ln\left(6/8\right)\right]^{-1/\eta}\right\}}}{\sqrt{-l\,n\left\{1-\left[1-\eta/\varphi ln\left(2/8\right)\right]^{-1/\eta}\right\}}-5\sqrt{-l\,n\left\{1-\left[1-\eta/\varphi ln\left(4/8\right)\right]^{-1/\eta}\right\}+\sqrt{-l\,n\left\{1-\left[1-\eta/\varphi ln\left(4/8\right)\right]^{-1/\eta}\right\}}}} \tag{50}$$

Table 1
Galton's Skewness and Moors' Kurtosis for the GoIR Distribution for Some Values of φ, ξ and η

Actual Values			Skewness	Kurtosis
$\hat{\varphi}$	$\hat{\xi}$	$\hat{\eta}$		
0.1	1	40	-0.1575	1.3213
0.2	1	40	-0.1575	1.3213
0.5	1	40	-0.1575	1.3213
1	1	40	-0.1575	1.3213
2	1	40	-0.1575	1.3213
5	1	40	-0.1575	1.3213
0.1	1	25	-0.1496	1.3065
0.2	1	25	-0.1496	1.3065
0.5	1	25	-0.1496	1.3065
1	1	25	-0.1496	1.3065
2	1	25	-0.1496	1.3065
5	1	25	-0.1496	1.3065
10	1	25	-0.1496	1.3065
0.2	0.2	40	-0.1479	1.3149
0.5	0.2	40	-0.1479	1.3149
1	0.2	40	-0.1479	1.3149
2	0.2	40	-0.1479	1.3149
5	0.2	40	-0.1479	1.3149
10	0.5	25	-0.1478	1.3109
15	0.5	25	-0.1478	1.3109
20	0.5	25	-0.1478	1.3109
50	0.5	25	-0.1478	1.3109
0.1	7	40	-0.1472	1.2763
0.2	7	40	-0.1472	1.2763
0.5	7	40	-0.1472	1.2763
1	7	40	-0.1472	1.2763
2	7	40	-0.1472	1.2763
5	7	40	-0.1472	1.2763
10	5	25	-0.1333	1.2595
15	5	25	-0.1333	1.2595
20	5	25	-0.1333	1.2595
50	5	25	-0.1333	1.2595
0.1	0.1	15	-0.1311	1.2954
0.2	0.1	15	-0.1311	1.2954
0.5	0.1	15	-0.1311	1.2954
1	0.1	15	-0.1311	1.2954
50	0.2	10	-0.1235	1.2836
0.1	0.5	10	-0.123	1.2743

3.5 Order Statistic

The s^{th} order statistic of a random variable X (Sule et al., 2021) is given by

$$f_{s:n} = \frac{n!}{(s-1)!\,(n-s)!} f_X(x;\xi,\varphi,\eta)[F_X(x)]^{(s-1)}[1-F_X(x)]^{(n-s)} \tag{51}$$

If $X \sim \text{GoIR}(\xi,\varphi,\eta)$, then the $f_{s:n}(x_s)$ of X is derived thus;

$$f_{s:n}(\xi,\varphi,\eta) = \frac{n!}{(s-1)!\,(n-s)!}\sum_{p}^{\infty}(-1)^p\binom{n-s}{p} f_X(x)[F_X(x)]^{(p+s-1)} \tag{52}$$

from (6) we have

$$[F_X(x)]^{(p+s-1)} = \left[1-e^{\frac{\varphi}{\eta}\left\{1-\left[1-e^{-\left(\frac{\xi}{x}\right)^2}\right]^{-\eta}\right\}}\right]^{(p+s-1)} \tag{53}$$

and from (28), we have

$$[F_X(x)]^{(p+s-1)} = \left[1-\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty}\frac{(-1)^{i+k}}{i!}\binom{i}{j}\binom{-\eta j}{k}\left(\frac{\varphi}{\eta}\right)^i\left[e^{-\left(\frac{\xi}{x}\right)^2}\right]^k\right]^{(p+s-1)} \tag{54}$$

then, the CDF of the order statistics is given by

$$[F_X(x)]^{(p+s-1)} = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty}\sum_{p=q=0}^{\infty}\frac{(-1)^{i+k+q}}{i!}\binom{i}{j}\binom{-\eta j}{k}\binom{p+s-1}{q}\left(\frac{q\varphi}{\eta}\right)^i\left[e^{-\left(\frac{\xi}{x}\right)^2}\right]^{qk} \tag{55}$$

Also the corresponding PDF of the order statistics can be expressed as

$$f_{s:n} = \frac{n!}{(s-1)!\,(n-s)!}2\varphi\xi^2x^{-3}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty}\sum_{p=q=0}^{\infty}\frac{(-1)^{i+k+p+q}}{i!}\binom{i}{j}\binom{-\eta j}{k}\binom{n-s}{p}\binom{p+s-1}{q}\left(\frac{\varphi}{\eta}(q+1)\right)^i\left[e^{-\left(\frac{\xi}{x}\right)^2}\right]^{qk+1} \tag{56}$$

4. PARAMETER ESTIMATION

4.1 Maximum Likelihood Estimation

Maximum likelihood estimation (MLE) method is mostly used for the estimation of an unknown parameter(s) because it satisfies the properties of a good estimator such as consistency, asymptotic efficiency, and invariance property (Sule, 2021). Let

$x_1, x_2, \ldots, x_n$ be random sample of size n drawn from GoIR distribution, then the MLE can be obtained as follows:

$$L = 2^n \varphi^n \xi^{2n} \sum x^{-3} e^{-\xi^2 \sum x^{-2}} \prod_{i=1}^{n} \left[1 - e^{-\left(\frac{\xi}{x}\right)^2}\right]^{-\eta-1} e^{\frac{\varphi}{\eta}\sum\left(1-\left[1-e^{-\left(\frac{\xi}{x}\right)^2}\right]^{-\eta}\right)} \tag{57}$$

Then the log-likelihood function of (57) is given by

$$\begin{aligned} l = nln(2) + nln(\varphi) + 2nln(\xi) - 3\sum (\ln(x)) - \xi^2 \sum \left(\frac{1}{x}\right)^2 \\ - (\eta+1)\sum \ln\left(1 - e^{-(\frac{\xi}{x})^2}\right) \\ + \frac{\varphi}{\eta}\sum\left(1 - \left[1 - e^{-\left(\frac{\xi}{x}\right)^2}\right]^{-\eta}\right) \end{aligned} \tag{58}$$

Differentiating (58) with respect to φ, ξ and η, then equating it to zero, we obtain the following estimating equations

$$\frac{\partial l}{\partial \varphi} = \frac{n}{\varphi} + \frac{1}{\eta}\sum\left(1 - \left[1 - e^{-\left(\frac{\xi}{x}\right)^2}\right]^{-\eta}\right) \tag{59}$$

$$\begin{aligned} \frac{\partial l}{\partial \xi} = \frac{2n}{\xi} - 2\xi \sum x^{-2} - 2(\eta+1)\sum \frac{\xi e^{-(\frac{\xi}{x})^2}}{x^2\left(1 - e^{-(\frac{\xi}{x})^2}\right)} \\ + 2\varphi\xi \sum \frac{\left(1 - e^{-(\frac{\xi}{x})^2}\right)^{-\eta} e^{-(\frac{\xi}{x})^2}}{x^2\left(1 - e^{-(\frac{\xi}{x})^2}\right)} \end{aligned} \tag{60}$$

$$\begin{aligned} \frac{\partial l}{\partial \eta} = -\sum \ln\left(1 - e^{-(\frac{\xi}{x})^2}\right) - \frac{\varphi}{\eta^2}\sum\left(1 - \left[1 - e^{-\left(\frac{\xi}{x}\right)^2}\right]^{-\eta}\right) \\ + \frac{\varphi}{\eta}\sum\left(1 - e^{-(\frac{\xi}{x})^2}\right)^{-\eta} \ln\left(1 - e^{-(\frac{\xi}{x})^2}\right) \end{aligned} \tag{61}$$

The maximum likelihood estimator $\hat{\theta} = (\hat{\varphi}, \hat{\xi}, \hat{\eta})$ of $\theta = (\varphi, \xi, \eta)$ is obtained by solving the nonlinear system of equations (59) - (61). In this article, we used a nonlinear optimization algorithm known as Newton Raphson method to numerically maximize the log-likelihood function given in (58) Obisesan et al. (2015); Yahaya et al. (2016); Bakari et al. (2016). The asymptotic distribution of the element of the 3×3 observed information matrix of GoIR distribution can be expressed as

$$\sqrt{n}(\hat{\theta} - \theta) \sim N_3(0, \Sigma^{-1}) \tag{62}$$

where Σ is the expected information matrix. Thus, the expected information matrix is expressed as

$$\Sigma^{-1} = -E\begin{bmatrix} \frac{\partial^2 l}{\partial \varphi^2} & \frac{\partial^2 l}{\partial \varphi\, \partial \xi} & \frac{\partial^2 l}{\partial \varphi\, \partial \eta} \\ \frac{\partial^2 l}{\partial \varphi\, \partial \xi} & \frac{\partial^2 l}{\partial \xi^2} & \frac{\partial^2 l}{\partial \eta\, \partial \xi} \\ \frac{\partial^2 l}{\partial \varphi\, \partial \eta} & \frac{\partial^2 l}{\partial \eta\, \partial \xi} & \frac{\partial^2 l}{\partial \eta^2} \end{bmatrix} \tag{63}$$

where

$$L_{11} = \frac{\partial^2 l}{\partial \varphi^2} = -\frac{n}{\varphi^2} \tag{64}$$

$$\begin{aligned} L_{22} = \frac{\partial^2 l}{\partial \xi^2} = &-\frac{2n}{\xi^2} - 2\sum \frac{1}{x^2} - 2(\eta+1)\sum\left(\frac{e^{-\left(\frac{\xi}{x}\right)^2}}{x^2\left(1-e^{-\left(\frac{\xi}{x}\right)^2}\right)}\right) \\ &+4(\eta+1)\xi^2\sum\left(\frac{e^{-\left(\frac{\xi}{x}\right)^2}}{x^4\left(1-e^{-\left(\frac{\xi}{x}\right)^2}\right)}\right) + 4(\eta+1)\xi^2\sum\left(\frac{\left(e^{-\left(\frac{\xi}{x}\right)^2}\right)^2}{x^4\left(1-e^{-\left(\frac{\xi}{x}\right)^2}\right)^2}\right) \\ &-4\varphi\eta\xi^2\sum\left(\frac{\left(e^{-\left(\frac{\xi}{x}\right)^2}\right)^2\left(1-e^{-\left(\frac{\xi}{x}\right)^2}\right)^{-\eta}}{x^4\left(1-e^{-\left(\frac{\xi}{x}\right)^2}\right)^2}\right) \\ &+2\varphi\sum\left(\frac{e^{-\left(\frac{\xi}{x}\right)^2}\left(1-e^{-\left(\frac{\xi}{x}\right)^2}\right)^{-\eta}}{x^2\left(1-e^{-\left(\frac{\xi}{x}\right)^2}\right)^2}\right) - 4\varphi\xi^2\sum\left(\frac{e^{-\left(\frac{\xi}{x}\right)^2}\left(1-e^{-\left(\frac{\xi}{x}\right)^2}\right)^{-\eta}}{x^4\left(1-e^{-\left(\frac{\xi}{x}\right)^2}\right)}\right) \\ &-4\varphi\xi^2\sum\left(\frac{\left(e^{-\left(\frac{\xi}{x}\right)^2}\right)^2\left(1-e^{-\left(\frac{\xi}{x}\right)^2}\right)^{-\eta}}{x^4\left(1-e^{-\left(\frac{\xi}{x}\right)^2}\right)^2}\right) \end{aligned} \tag{65}$$

$$\begin{aligned} L_{33} = \frac{\partial^2 l}{\partial \eta^2} = \frac{2\varphi}{\eta^3}\sum\left(1-\left(1-e^{-\left(\frac{\xi}{x}\right)^2}\right)^{-\eta}\right) \\ -\frac{2\varphi}{\eta^2}\sum\left(\left(1-e^{-\left(\frac{\xi}{x}\right)^2}\right)^{-\eta}\ln\left(1-e^{-\left(\frac{\xi}{x}\right)^2}\right)\right) \\ -\frac{\varphi}{\eta}\sum\left(\left(1-e^{-\left(\frac{\xi}{x}\right)^2}\right)^{-\eta}\left(\ln\left(1-e^{-\left(\frac{\xi}{x}\right)^2}\right)\right)^2\right) \end{aligned} \tag{66}$$

$$L_{12} = L_{21} = \frac{\partial^2 l}{\partial\varphi\,\partial\xi} = 2\xi \sum \left(\frac{e^{-\left(\frac{\xi}{x}\right)^2}\left(1 - e^{-\left(\frac{\xi}{x}\right)^2}\right)^{-\eta}}{x^2\left(1 - e^{-\left(\frac{\xi}{x}\right)^2}\right)} \right) \tag{67}$$

$$L_{13} = L_{31} = \frac{\partial^2 l}{\partial\varphi\,\partial\eta} = -\frac{1}{\eta}\sum\left(1 - \left(1 - e^{-\left(\frac{\xi}{x}\right)^2}\right)^{-\eta}\right) + \frac{1}{\eta}\sum\left(\left(1 - e^{-\left(\frac{\xi}{x}\right)^2}\right)^{-\eta} \ln\left(1 - e^{-\left(\frac{\xi}{x}\right)^2}\right)^2\right) \tag{68}$$

$$L_{23} = L_{32} = \frac{\partial^2 l}{\partial\eta\,\partial\xi} = -2\xi\sum\left(\frac{e^{-\left(\frac{\xi}{x}\right)^2}}{x^2\left(1 - e^{-\left(\frac{\xi}{x}\right)^2}\right)}\right) - 2\varphi\xi\sum\left(\frac{e^{-\left(\frac{\xi}{x}\right)^2}\left(1 - e^{-\left(\frac{\xi}{x}\right)^2}\right)^{-\eta} ln\left(1 - e^{-\left(\frac{\xi}{x}\right)^2}\right)}{x^2\left(1 - e^{-\left(\frac{\xi}{x}\right)^2}\right)}\right) \tag{69}$$

The solutions to be obtained by solving (63) will yield the asymptotic variance and covariances of the parameters $\hat{\varphi}, \hat{\xi}$ and $\hat{\eta}$. Using (63), the approximate $100(1-\alpha)\%$ confidence intervals for φ, ξ and η can be expressed as

$$\hat{\varphi} \pm Z_{\frac{\alpha}{2}}\sqrt{\hat{\Sigma}}_{11}, \hat{\xi} \pm Z_{\frac{\alpha}{2}}\sqrt{\hat{\Sigma}}_{22}, \hat{\eta} \pm Z_{\frac{\alpha}{2}}\sqrt{\hat{\Sigma}}_{33}$$

where $Z_{\frac{\alpha}{2}}$ is the upper α^{th} percentile of the standard normal distribution.

4.2 Bayesian Estimation

Let $x = (x_1, x_2, \dots x_n)$ be a random variable with parameters φ, η and ξ having a size n. From the bayes' the posterior probability density function of the parameters φ, ξ and η given x can be expressed as

$$\Pr(\varphi, \xi, \eta | x) = \frac{\pi(\varphi, \xi, \eta) l(\varphi, \xi, \eta)}{\int\int\int \pi(\varphi, \xi, \eta) l(\varphi, \xi, \eta)\,\partial(\varphi, \xi, \eta)} \tag{70}$$

where $l(\varphi, \xi, \eta)$ is the likelihood and $\varphi, \xi, \eta)$ is the prior probability distribution.

4.2.1 Prior Assumption

Observe that if the shape parameter η is known, the scale parameters ξ and φ have a conjugate prior Weibull prior with parameter $W(c,d)$ and inverse Gamma prior with parameters $IG(a,b)$. But when the three parameters are unknown they do not have conjugate prior distribution. So, we consider the following priors: $\pi_2(\xi) \sim W(c,d)$, $\pi_1(\varphi) \sim IG(a,b)$ and a non-infomative prior for $\pi_3(\eta) \sim U(0,\eta)$. The PDFs of the prior are expressed as

$$\pi_1(\varphi) = \frac{b^a}{\Gamma(a)}\varphi^{a-1}e^{-b\varphi} \quad a > 0, b > 0, \varphi > 0 \tag{71}$$

$$\pi_2(\xi) = \frac{c}{d}\left(\frac{\xi}{d}\right)^{c-1} e^{-\left(\frac{\xi}{d}\right)^c} \qquad \xi > 0, c > 0, d > 0 \tag{72}$$

for conjugate of ξ we set c=2 then we have

$$\pi_2(\xi) = \frac{2\xi}{d^2}e^{-\left(\frac{\xi}{d}\right)^2} \qquad d > 0, \xi > 0 \tag{73}$$

$$\pi_3(\eta) = \frac{1}{\eta} \qquad \eta > 0 \tag{74}$$

So, therefore the joint prior distribution of parameters φ, η and ξ is stated as

$$\pi(\varphi, \xi, \eta) = \frac{\xi\varphi^{a-1}}{\eta}e^{-\left(b\varphi+\frac{\xi^2}{d^2}\right)}\left(\frac{2b^2}{d^2\Gamma(a)}\right)$$
$$\varphi > 0, \xi > 0, \eta > 0, a > 0, d > 0, b > 0 \tag{75}$$

4.2.2 Posterior Distribution

Following (70), the posterior distribution for the parameters φ, η and ξ was obtain by combining (57) and (75) together and substituting the combination into (70) to form the expression below

$$\Pr(\varphi, \xi, \eta|x) = \frac{2^n\varphi^n\xi^{2n}\sum x^{-3}e^{-\xi^2\sum x^{-2}}\prod_{i=1}^n\left[1-e^{-\left(\frac{\xi}{x}\right)^2}\right]^{-\eta-1} e^{\frac{\varphi}{\eta}\sum\left(1-\left[1-e^{-\left(\frac{\xi}{x}\right)^2}\right]^{-\eta}\right)}\frac{\xi\varphi^{a-1}}{\eta}e^{-\left(b\varphi+\frac{\xi^2}{d^2}\right)}}{\int\int\int 2^n\varphi^n\xi^{2n}\sum x^{-3}e^{-\xi^2\sum x^{-2}}\prod_{i=1}^n\left[1-e^{-\left(\frac{\xi}{x}\right)^2}\right]^{-\eta-1} e^{\frac{\varphi}{\eta}\sum\left(1-\left[1-e^{-\left(\frac{\xi}{x}\right)^2}\right]^{-\eta}\right)}\frac{\xi\varphi^{a-1}}{\eta}e^{-\left(b\varphi+\frac{\xi^2}{d^2}\right)}\partial(\varphi, \xi, \eta)} \tag{76}$$

We can now derive the Bayes estimators for the unknown parameters φ, η and ξ under the Linex loss function.

4.2.3 Linex Loss Function

The Linex (linear-exponential) loss function is an asymmetric loss function which was introduced by Varian (1975). The loss function have been adopted by several authors; among of them are Sule and Adegoke (2020), Rojo (1987), Basu and Ebrahimi (1991), Pandey (1997), Soliman (2002, 2005), Soliman et al. (2006) and Nassar and Eissa (2004), Sule et al. (2021). The function rises approximately exponentially on one side of zero and approximately linearly on the other side. The Linex loss function is stated as

$$\theta_L \propto \kappa(e^{\tau(\hat{\theta}-\theta)} - \tau(\hat{\theta}-\theta) - 1) \qquad \kappa > 0, \tau \neq 0 \tag{77}$$

where τ and κ are known as the scale and shape parameters of the Linex loss function. We assume that $\kappa = 1$ in this study. Bayes estimator of the linex function is the value $\hat{\theta}$ that minimizes (77) which is stated as Zellner(1986)

$$\hat{\theta} = -\frac{1}{\tau}\ln\left(E_\theta\left[e^{-\tau\theta}\right]\right) \tag{78}$$

provided that $E_\theta\left[e^{-\tau\theta}\right]$ exits. We can note here that the posterior distribution of (φ, ξ, η) takes a ratio form which involves an integration in the denominator and cannot be reduced to a closed form. Hence, evaluating the posterior quantities for the parameters (φ, ξ, η) will be tedious. So we adopt the Lindley's approximation introduced by Lindley (1980) which approaches the posterior ratio of the integrals as a whole process and results to a single numerical result.

4.2.4 Lindley's Approximation

The Lindley approximation techniques have been adopted by many authors such as Sule and Adegoke (2020), Preda (2014) and Mohammed et al. (2018) to obtain the bayes estimates of some parameters of a lifetime distributions. According to Lindley (1980), if n is sufficiently large, any ratio of the integral of the form

$$I(x) = E[u(\varphi,\xi,\eta)] = \frac{\int\int\int u(\varphi,\xi,\eta)e^{l(\varphi,\xi,\eta)+\rho(\varphi,\xi,\eta)}\,\partial(\varphi,\xi,\eta)}{\int\int\int e^{l(\varphi,\xi,\eta)+\rho(\varphi,\xi,\eta)}\,\partial(\varphi,\xi,\eta)} \tag{79}$$

where $u(\varphi,\xi,\eta)$ is a function of φ, ξ and η only, $l(\varphi,\xi,\eta)$ is the log-likelihood and $\rho(\varphi,\xi,\eta)$ is the log of prior distribution $\pi(\varphi,\xi,\eta)$. Equation (79) can be evaluated as

$$\begin{aligned} I(x) = u(\hat{\varphi},\hat{\xi},\hat{\eta}) + (u_1a_1 + u_2a_2 + u_3 + a_3 + a_4 + a_5) + \frac{1}{2}A(u_1\sigma_{11} \\ + u_2\sigma_{12} + u_3\sigma_{13}) + \frac{1}{2}B(u_1\sigma_{21} + u_2\sigma_{22} + u_3\sigma_{23}) \\ + \frac{1}{2}C(u_1\sigma_{31} + u_2\sigma_{32} + u_3\sigma_{33}) \end{aligned} \tag{80}$$

where $\hat{\varphi}, \hat{\xi}$ and $\hat{\eta}$ are the maximum likelihood estimators of φ, ξ and η.

$$a_i = \rho_1\sigma_{i1} + \rho_2\sigma_{i2} + \rho_3\sigma_{i3} \qquad i = 1,2,3 \tag{81}$$

$$a_4 = u_{12}\sigma_{12} + u_{13}\sigma_{13} + u_{23}\sigma_{23} \tag{82}$$

$$a_5 = \frac{1}{2}(u_{11}\sigma_{11} + u_{22}\sigma_{22} + u_{33}\sigma_{33}) \tag{83}$$

$$A = \sigma_{11}Ł_{111} + 2\sigma_{12}L_{121} + 2\sigma_{13}L_{131} + 2\sigma_{23}L_{231} + \sigma_{22}L_{221} + \sigma_{33}L_{331} \tag{84}$$

$$B = \sigma_{11}L_{112} + 2\sigma_{12}L_{122} + 2\sigma_{13}L_{132} + 2\sigma_{23}L_{232} + \sigma_{22}L_{222} + \sigma_{33}L_{332} \tag{85}$$

$$C = \sigma_{11}L_{113} + 2\sigma_{12}L_{123} + 2\sigma_{13}L_{133} + 2\sigma_{23}L_{233} + \sigma_{22}L_{223} + \sigma_{33}L_{333} \tag{86}$$

$$\rho_i = \frac{\partial\rho}{\partial\theta_i} \qquad i = 1,2,3 \tag{87}$$

$$u_i = \frac{\partial u(\theta_1, \theta_2, \theta_3)}{\partial \theta_i} \qquad i = 1,2,3 \tag{88}$$

$$u_{i,j} = \frac{\partial^2 u(\theta_1, \theta_2, \theta_3)}{\partial \theta_i \, \partial \theta_j} \qquad i,j = 1,2,3 \tag{89}$$

$$L_{i,j,k} = \frac{\partial^3 l(\theta_1, \theta_2, \theta_3)}{\partial \theta_i \, \partial \theta_j \, \partial \theta_k} \qquad i,j,k = 1,2,3 \tag{90}$$

where $\theta_1 = \alpha$, $\theta_2 = \beta$ and $\theta_3 = \delta$. $\sigma_{i,j}$ is the $(i,j)^{th}$ element of the inverse of the matrix $L_{i,j}$ all evaluated at the maximum likelihood estimates. For the prior distribution (75) we have

$$\rho = \ln\big(\pi(\varphi, \xi, \eta)\big) = \ln\xi + (a-1)\ln\varphi - \ln\eta - b\varphi - \frac{\xi^2}{d^2} \tag{91}$$

$$\frac{\partial \rho}{\partial \varphi} = \frac{a-1}{\varphi} - b \tag{92}$$

$$\frac{\partial \rho}{\partial \xi} = \frac{1}{\xi} - \frac{2\xi}{d^2} \tag{93}$$

$$\frac{\partial \rho}{\partial \eta} = -\frac{1}{\eta} \tag{94}$$

Also, the values of L_{ijk} can be obtained as follows for $i,j,k = 1,2,3$.

$$L_{111} = \frac{2n}{\varphi^3} \tag{95}$$

$$\begin{aligned} L_{333} = &-\frac{6\varphi}{\eta^4} \sum \left(1 - e^{-\left(\frac{\xi}{x}\right)^2}\right)^{-\eta} \\ &+ \frac{6\varphi}{\eta^3} \sum \left(1 - e^{-\left(\frac{\xi}{x}\right)^2}\right)^{-\eta} \ln\left(1 - e^{-\left(\frac{\xi}{x}\right)^2}\right) \\ &+ \frac{3\varphi}{\eta^2} \sum \left(1 - e^{-\left(\frac{\xi}{x}\right)^2}\right)^{-\eta} \left(\ln\left(1 - e^{-\left(\frac{\xi}{x}\right)^2}\right)\right)^2 \\ &+ \frac{\varphi}{\eta} \sum \left(1 - e^{-\left(\frac{\xi}{x}\right)^2}\right)^{\eta} \left(\ln\left(1 - e^{-\left(\frac{\xi}{x}\right)^2}\right)\right)^3 \end{aligned} \tag{96}$$

$$L_{113} = L_{131} = L_{311} = L_{112} = L_{121} = L_{211} = 0 \tag{97}$$

$$L_{223} = L_{232} = L_{322} = 2\varphi\xi \sum \left(\frac{e^{-\left(\frac{\xi}{x}\right)^2} \left(1 - e^{-\left(\frac{\xi}{x}\right)^2}\right)^{-\eta} \left(\ln\left(1 - e^{-\left(\frac{\xi}{x}\right)^2}\right)\right)^2}{x^2 \left(1 - e^{-\left(\frac{\xi}{x}\right)^2}\right)} \right) \tag{98}$$

$$\begin{aligned} L_{122} = L_{212} = L_{221} = &-4\xi^2 \sum \left(\frac{\left(e^{-\left(\frac{\xi}{x}\right)^2}\right)^2 \left(1 - e^{-\left(\frac{\xi}{x}\right)^2}\right)^{-\eta}}{x^4 \left(1 - e^{-\left(\frac{\xi}{x}\right)^2}\right)^2} \right) \\ &+2 \sum \left(\frac{e^{-\left(\frac{\xi}{x}\right)^2} \left(1 - e^{-\left(\frac{\xi}{x}\right)^2}\right)^{-\eta}}{x^2 \left(1 - e^{-\left(\frac{\xi}{x}\right)^2}\right)} \right) \\ &-4\xi^2 \sum \left(\frac{e^{-\left(\frac{\xi}{x}\right)^2} \left(1 - e^{-\left(\frac{\xi}{x}\right)^2}\right)^{-\eta}}{x^4 \left(1 - e^{-\left(\frac{\xi}{x}\right)^2}\right)} \right) \\ &-4\xi^2 \sum \left(\frac{\left(e^{-\left(\frac{\xi}{x}\right)^2}\right)^2 \left(1 - e^{-\left(\frac{\xi}{x}\right)^2}\right)^{-\eta}}{x^4 \left(1 - e^{-\left(\frac{\xi}{x}\right)^2}\right)^2} \right) \end{aligned} \tag{99}$$

$$L_{123} = L_{132} = L_{213} = L_{231} = L_{312} = L_{321} = -2\xi \sum \left(\frac{e^{-\left(\frac{\xi}{x}\right)^2} \left(1 - e^{-\left(\frac{\xi}{x}\right)^2}\right)^{-\eta} \left(\ln\left(1 - e^{-\left(\frac{\xi}{x}\right)^2}\right)\right)}{x^2 \left(1 - e^{-\left(\frac{\xi}{x}\right)^2}\right)} \right) \tag{100}$$

$$
\begin{aligned}
L_{222} = {} & \frac{4n}{\xi^3} + 12(\eta+1)\xi \sum \left(\frac{-\left(\frac{\xi}{x}\right)^2}{x^4\left(1-e^{-\left(\frac{\xi}{x}\right)^2}\right)} \right) \\
& +12(\eta+1)\xi \sum \left(\frac{\left(e^{-\left(\frac{\xi}{x}\right)^2}\right)^2}{x^4\left(1-e^{-\left(\frac{\xi}{x}\right)^2}\right)^2} \right) - 8(\eta+1)\xi^3 \sum \left(\frac{e^{-\left(\frac{\xi}{x}\right)^2}}{x^6\left(1-e^{-\left(\frac{\xi}{x}\right)^2}\right)} \right) \\
& -24(\eta+1)\xi^3 \sum \left(\frac{\left(e^{-\left(\frac{\xi}{x}\right)^2}\right)^2}{x^2\left(1-e^{-\left(\frac{\xi}{x}\right)^2}\right)^2} \right) - 16(\eta+1)\xi^3 \sum \left(\frac{\left(e^{-\left(\frac{\xi}{x}\right)^3}\right)^2}{x^6\left(1-e^{-\left(\frac{\xi}{x}\right)^2}\right)^3} \right) \\
& +8\varphi\eta^2\xi^2 \sum \left(\frac{\left(e^{-\left(\frac{\xi}{x}\right)^2}\right)^3\left(1-e^{-\left(\frac{\xi}{x}\right)^2}\right)^{-\eta}}{x^6\left(1-e^{-\left(\frac{\xi}{x}\right)^2}\right)^2} \right) 12\varphi\eta\xi \sum \left(\frac{\left(e^{-\left(\frac{\xi}{x}\right)^2}\right)^2\left(1-e^{-\left(\frac{\xi}{x}\right)^2}\right)^{-\eta}}{x^4\left(1-e^{-\left(\frac{\xi}{x}\right)^2}\right)^2} \right) \\
& +24\varphi\eta\xi^3 \sum \left(\frac{\left(e^{-\left(\frac{\xi}{x}\right)^2}\right)^2\left(1-e^{-\left(\frac{\xi}{x}\right)^2}\right)^{-\eta}}{x^6\left(1-e^{-\left(\frac{\xi}{x}\right)^2}\right)^2} \right) \\
& +24\varepsilon\xi^3 \sum \left(\frac{\left(e^{-\left(\frac{\xi}{x}\right)^2}\right)^3\left(1-e^{-\left(\frac{\xi}{x}\right)^2}\right)^{-\eta}}{x^6\left(1-e^{-\left(\frac{\xi}{x}\right)^2}\right)^3} \right) - 12\varphi\xi \sum \left(\frac{\left(e^{-\left(\frac{\xi}{x}\right)^2}\right)\left(1-e^{-\left(\frac{\xi}{x}\right)^2}\right)^{-\eta}}{x^4\left(1-e^{-\left(\frac{\xi}{x}\right)^2}\right)} \right) \\
& -12\varphi\xi \sum \left(\frac{\left(e^{-\left(\frac{\xi}{x}\right)^2}\right)^2\left(1-e^{-\left(\frac{\xi}{x}\right)^2}\right)^{-\eta}}{x^4\left(1-e^{-\left(\frac{\xi}{x}\right)^2}\right)^2} \right) + 8\varphi\xi^3 \sum \left(\frac{\left(e^{-\left(\frac{\xi}{x}\right)^2}\right)\left(1-e^{-\left(\frac{\xi}{x}\right)^2}\right)^{-\eta}}{x^6\left(1-e^{-\left(\frac{\xi}{x}\right)^2}\right)} \right) \\
& +24\varphi\xi^3 \sum \left(\frac{\left(e^{-\left(\frac{\xi}{x}\right)^2}\right)\left(1-e^{-\left(\frac{\xi}{x}\right)^2}\right)^{-\eta}}{x^6\left(1-e^{-\left(\frac{\xi}{x}\right)^2}\right)} \right) \\
& +16\varphi\xi^3 \sum \left(\frac{\left(e^{-\left(\frac{\xi}{x}\right)^2}\right)^3\left(1-e^{-\left(\frac{\xi}{x}\right)^2}\right)^{-\eta}}{x^6\left(1-e^{-\left(\frac{\xi}{x}\right)^2}\right)^3} \right) \qquad (101)
\end{aligned}
$$

$$L_{133} = L_{313} = L_{331} = \frac{2}{\eta^3}\sum \left(1 - \left(1 - e^{-\left(\frac{\xi}{x}\right)^2}\right)^{-\eta}\right) - \frac{2}{\eta^2}\sum \left(\left(1 - e^{-\left(\frac{\xi}{x}\right)^2}\right)^{-\eta} \ln\left(1 - e^{-\left(\frac{\xi}{x}\right)^2}\right)\right) - \frac{1}{\eta}\sum \left(\left(1 - e^{-\left(\frac{\xi}{x}\right)^2}\right)^{-\eta} \ln\left(1 - e^{-\left(\frac{\xi}{x}\right)^2}\right)^2\right) \quad (102)$$

Now, we can obtain the Bayes estimate of the parameters φ, ξ and η under the Linex loss function

Remark 3

To obtain the bayes estimate of φ the take $u(\varphi, \xi, \eta) = e^{\tau\hat{\varphi}}$

$$\hat{\varphi}_{Lin} = -\frac{1}{\tau}\ln\left\{e^{-\tau\hat{\varphi}} + u_1 a_1 + \frac{1}{2}u_{11}\sigma_{11} + \frac{1}{2}Au_1\sigma_{11} + \frac{1}{2}Bu_1\sigma_{21} + \frac{1}{2}Cu_1\sigma_{31}\right\} \quad (103)$$

Remark 4

To obtain the bayes estimate of ξ the take $u(\varphi, \xi, \eta) = e^{\tau\hat{\xi}}$

$$\hat{\xi}_{Lin} = -\frac{1}{\tau}\ln\left\{e^{-\tau\hat{\xi}} + u_2 a_2 + \frac{1}{2}u_{22}\sigma_{22} + \frac{1}{2}Au_2\sigma_{12} + \frac{1}{2}Bu_2\sigma_{22} + \frac{1}{2}Cu_3\sigma_{32}\right\} \quad (104)$$

Remark 5

To obtain the bayes estimate of η the take $u(\varphi, \xi, \eta) = e^{\tau\hat{\eta}}$

$$\hat{\eta}_{Lin} = -\frac{1}{\tau}\ln\left\{e^{-\tau\hat{\eta}} + u_3 a_3 + \frac{1}{2}u_3\sigma_{33} + \frac{1}{2}Au_3\sigma_{13} + \frac{1}{2}Bu_3\sigma_{23} + \frac{1}{2}Cu_3\sigma_{33}\right\} \quad (105)$$

where $u_1 = \tau e^{\tau\varphi}$, $u_{11} = \tau^2 e^{\tau\varphi}$,$u_2 = \tau e^{\tau\xi}$, $u_{22} = \tau^2 e^{\tau\xi}$, $u_3 = \tau e^{\tau\eta}$, $u_{33} = \tau^2 e^{\tau\eta}$.

5. APPLICATION

5.1 Simulation Studies

In this section, we simulated data set for sizes $n =$ 30, 100, 200 and 500 that follows Gompertz Inverse Rayleigh distribution using different parameter values for the three parameters φ, ξ and η using the quantile function (inverse transformation method of simulation). We considered the following combinations for the parameters (φ, ξ, η) = ((1,1,2), (1.5,1,1.5), (1,2,3) and (2,3,1))at different sample sizes $n =$ 30, 100, 200, and 500. The results presented in Table 2 displayed the true values of (φ, ξ, η), estimated values of (φ, ξ, η) with the standard errors in brackets. The results are replicated 10,000 times and the average result were presented in the Tables 2 and 3.

Table 2
Estimates and MSE of the Estimators under the Method of Maximum Likelihood Estimation and Bayes Estimation

		Actual Values			MLE Estimates			Bayes Estimates		
		φ	ξ	η	$\hat{\varphi}$	$\hat{\xi}$	$\hat{\eta}$	$\hat{\varphi}_{Lin}$	$\hat{\xi}_{Lin}$	$\hat{\eta}_{Lin}$
30	$\alpha=1$ $\beta=1$ $\tau=2$ $\lambda=1$	1	0.5	0.5	0.8002 (0.5114)	0.4441 (0.0867)	0.5038 (0.3690)	0.5679 (0.1867)	0.5427 (0.0018)	0.5046 (0.00002)
	$\alpha=1.5$ $\beta=1$ $\tau=1.5$ $\lambda=1.5$	1	2	2	0.6797 (0.8028)	1.7414 (0.4915)	1.7964 (0.5712)	0.4284 (0.3267)	2.0507 (0.0023)	1.4880 (0.2621)
	$\alpha=1$ $\beta=2$ $\tau=3$ $\lambda=2$	0.5	0.5	0.5	0.4234 (0.2682)	0.4382 (0.1021)	0.4756 (0.2448)	0.3362 (0.0268)	0.3367 (0.0267)	0.5123 (0.0002)
	$\alpha=2$ $\beta=3$ $\tau=1$ $\lambda=3$	2	1.5	1	1.4371 (1.1552)	1.3392 (0.2539)	1.0086 (0.6926)	0.7549 (1.1550)	1.7915 (0.0847)	0.3856 (0.3775)
100	$\alpha=1$ $\beta=1$ $\tau=2$ $\lambda=1$	1	0.5	0.5	1.0733 (0.3705)	0.5038 (0.0522)	0.4684 (0.2391)	0.9239 (0.0058)	0.5871 (0.0759)	0.3842 (0.0134)
	$\alpha=1.5$ $\beta=1$ $\tau=1.5$ $\lambda=1.5$	1	2	2	1.2133 (0.7661)	2.1516 (0.2898)	2.0372 (0.4320)	1.0621 (0.0038)	2.2356 (0.0555)	1.8814 (0.0141)
	$\alpha=1$ $\beta=2$ $\tau=3$ $\lambda=2$	0.5	0.5	0.5	0.5272 (0.1839)	0.5356 (0.0647)	0.4900 (0.1527)	0.4589 (0.0017)	0.5983 (0.0097)	0.42714 (0.0053)
	$\alpha=2$ $\beta=3$ $\tau=1$ $\lambda=3$	2	1.5	1	2.2721 (0.9720)	1.5862 (0.1458)	0.9278 (0.5049)	1.8316 (0.0283)	1.7356 (0.0555)	0.5555 (0.1976)

Table 3
Estimates and MSE of the Estimators under the Method of Maximum Likelihood Estimation and Bayes Estimation

		Actual Values			MLE Estimates			Bayes Estimates		
		φ	ξ	η	$\hat{\varphi}$	$\hat{\xi}$	$\hat{\eta}$	$\hat{\varphi}_{Lin}$	$\hat{\xi}_{Lin}$	$\hat{\eta}_{Lin}$
200	$\alpha=1$ $\beta=1$ $\tau=2$ $\lambda=1$	1	0.5	0.5	1.0205 (0.2658)	0.5275 (0.0382)	0.6307 (0.1776)	0.9486 (0.0026)	0.5491 (0.0024)	0.5858 (0.0074)
	$\alpha=1.5$ $\beta=1$ $\tau=1.5$ $\lambda=1.5$	1	2	2	1.1587 (0.5335)	2.1534 (0.2061)	2.3401 (0.3091)	1.0845 (0.0072)	2.1890 (0.0361)	2.258 (0.0666)
	$\alpha=1$ $\beta=2$ $\tau=3$ $\lambda=2$	0.5	0.5	0.5	0.5019 (0.1313)	0.5342 (0.0471)	0.5922 (0.1137)	0.4684 (0.0009)	0.5615 (0.0038)	0.5579 (0.0034)
	$\alpha=2$ $\beta=3$ $\tau=1$ $\lambda=3$	2	1.5	1	2.1394 (0.6933)	1.5761 (0.1072)	1.2671 (0.3642)	1.9235 (0.0059)	1.6344 (0.0181)	1.0754 (0.0057)
500	$\alpha=1$ $\beta=1$ $\tau=2$ $\lambda=1$	1	0.5	0.5	1.0147 (0.1521)	0.5055 (0.0222)	0.4871 (0.0984)	0.9845 (0.0002)	0.5152 (0.0002)	0.4733 (0.0007)
	$\alpha=1.5$ $\beta=1$ $\tau=1.5$ $\lambda=1.5$	1	2	2	1.0056 (0.2880)	2.0144 (0.1275)	2.0101 (0.1709)	0.9785 (0.0005)	2.0296 (0.00009)	1.9810 (0.0004)
	$\alpha=1$ $\beta=2$ $\tau=3$ $\lambda=2$	0.5	0.5	0.5	0.5001 (0.0765)	0.5046 (0.0274)	0.4980 (0.0643)	0.4871 (0.00017)	0.5134 (0.0002)	0.4875 (0.00016)
	$\alpha=2$ $\beta=3$ $\tau=1$ $\lambda=3$	2	1.5	1	2.0531 (0.3855)	1.5161 (0.0632)	0.9722 (0.2012)	1.9614 (0.0015)	1.5424 (0.0018)	0.8963 (0.0107)

5.2 Real Life Application

In this section, we demonstrated the performance of this proposed distribution with a data set. The data set of malignant melanomas was extracted from Boot package in R, it consists of 205 survival times of patients after the surgical removal of malignant melanomas. The performance of goodness of fit of GoIR is compared with Gompertz Log-logistics (Alizadeh et al. (2017), four-parameter betaBirnbaum–Saunders (Cordeiro and Lemonte (2011)), exponentiated Weibull (EW) (Mudholkar and Srivastava (1993)) and other result in Alizadeh et al. (2017)

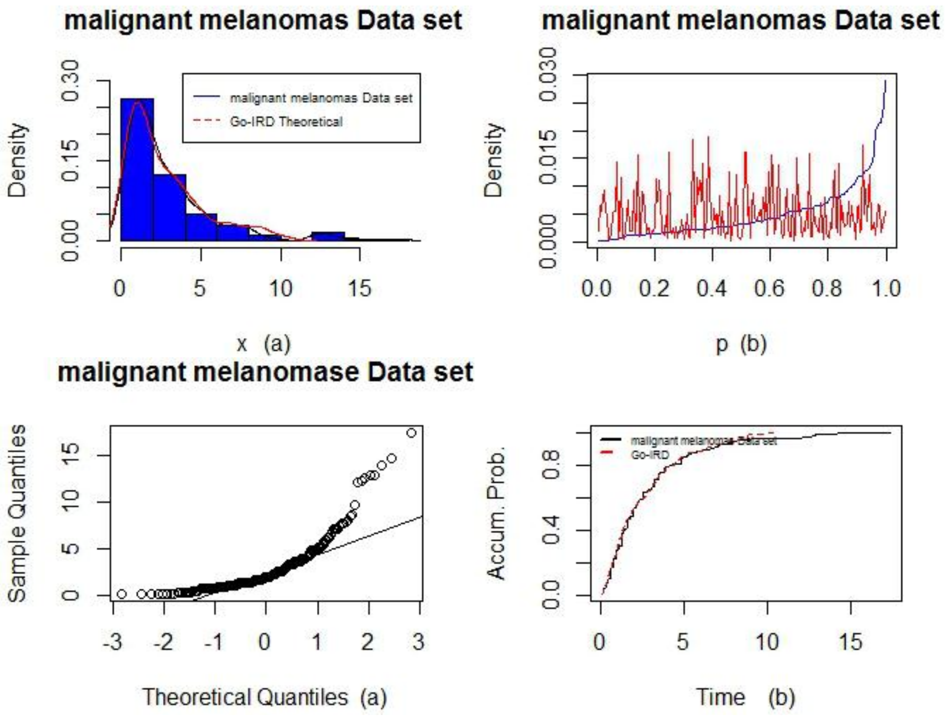

Figure 4: QQPlot of Malignant Melanemas Data Set and Time Plot with GoIR Distribution

Table 4
Summary Statistics of the Measurements Made on Patients with Malignant Melanoma

N	Min	Max	Q1	Q2	Q3	Mean	Var	SD	Kurtosis	Skewness
205	0.10	17.42	0.97	1.94	3.56	2.92	8.758	2.959	5.353	2.1497

Table 5
Fitted Distributions to the Melanoma Data

Distribution	Parameters	Estimates (S.E)	AIC	BIC	AD	P Value
GoIR	φ	0.03319	1124.704	1125.184		0.8301
	ξ	0.16956				
	η	0.48424				
GoLL	θ	1.1503×10^{-3} (6.0127×10^{-4})	3458.91	3472.20	1.4214	0.1964
	γ	4.4224 (1.2445)				
	a	1.0001×10^{2} (2.6393×10)				
	b	5.3164×10^{-1} (1.2397×10^{-1})				
BBS	α	1.4697×10^{2} (5.7688×10)	3466.71	3480.00	2.0247	0.0891
	β	4.8063 (3.8304)				
	a	4.9659×10^{2} (2.1779×10)				
	b	3.9881×10^{2} (1.9559×10)				
BW	*k*	4.7600×10^{-1} (1.9302×10^{-1})	3478.97	3492.27	5.1341	0.0025
	λ	3.2519×10^{2} (7.2713×10)				
	a	9.8401 (6.8299)				
	b	1.4950 (1.5323)				
EW	α	8.5399×10^{-1} (1.3854×10^{-1})	3456.64	3475.61	2.8213	0.0338
	k	2.0533 (2.3234×10^{-1})				
	λ	2.4826×10^{3} (1.5778×10^{2})				
Gamma	α	2.3496 (1.0100×10^{-1})	3492.32	3498.97	4.52	0.0049
	β	1.0908×10^{-3} (2.2708×10^{-5})				
Weibull	α	1.8997 (1.0772×10^{-1})	3466.68	3473.33	1.8721	0.1082
	λ	2.4338×10^{3} (9.4169×10)				
Log-Logistic	*a*	1.9085×10^{3} (8.8491×10)	3520.77	3527.41	4.9939	0.0029
	λ	2.5150 (1.5231×10^{-1})				
Exponential	λ	4.6451×10^{-4} (3.2442×10^{-5})	3558.56	3561.88	19.5269	$< 10^{-4}$

Table 6
MLE and Bayes Estimates of the Measurements Made on Patients with Malignant Melanoma

MLE			Bayes Estimates ($\alpha = 1.5$, $\beta = 1$, $\tau = 1$, $\lambda = 2$)			Bayes Estimates ($\alpha = 0.5$, $\beta = 2$, $\tau = 3$, $\lambda = 4$)		
$\hat{\varphi}$	$\hat{\xi}$	$\hat{\eta}$	φ_{Lin}	ξ_{Lin}	η_{Lin}	φ_{Lin}	ξ_{Lin}	η_{Lin}
0.03319	0.16956	0.48424	0.0263	0.175	0.468	0.0262	0.1764	0.4674

6. CONCLUDING REMARKS

The performance of the proposed Bayes estimators for the GoIR has been compared to the maximum likelihood estimator. The maximum likelihood estimators and Bayes estimators $\hat{\varphi}$ and $\hat{\xi}$ are better for both the small and moderate sample sizes whereas whereas the Bayes estimator of all parameters population perform better than the MLE estimators for a very large sample sizes.

However, it was observed that the performances of the Bayesian estimators were in their best form compared to that of the MLE estimate under which the MSEs of all the estimators were relatively smaller. The application of GoIR distribution was provided and both the MLE estimate and Bayes estimates were presented. The results reveal better fits to real data than other well-known models.

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