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MULTIFACTOR COMPONENT-AMOUNT MIXTURE DESIGNS

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ABSTRACT

Multifactor component-amount designs are introduced. A method for the construction of multifactor component-amount design is devised. The method uses the kronecker product of matrices obtained from a single factor D-optimal component-amount design and an adjacency/incidence matrix of an undirected graph over a field F_2. Further G-efficiency of designs is computed for two and three factors quadratic mixture component-amount models. This method can be used equally when factors have same or different number of components.

KEY WORDS

Component-amounts, G-optimality, Mixture Designs, Incidence matrix, Adjacency matrix, Kronecker product.

1. INTRODUCTION

A mixture-amount experiment is a type of mixture experiment which we perform at two or more levels of the total amount. The response is supposed to be dependent on the proportion of each component in the mixture and also on the amount of the mixture. The effects on the response of varying mixture component proportions and the total amount of mixture are measured by fitting mixture-amount model to the design. The design which is used to fit mixture-amount model is called mixture-amount design, developed by Piepel and Cornell (1987). The quadratic mixture-amount model for q components is,

$$Y = \sum_{i=1}^{q} \gamma_i^0 x_i + \sum_{i<j}^{q} \gamma_{ij}^0 x_i x_j + \sum_{k=1}^{q} \left(\sum_{i=1}^{q} \gamma_i^k x_i + \sum_{i<j}^{q} \gamma_{ij}^k x_i x_j \right) A^k + \varepsilon \tag{1}$$

where x_i is the ith component proportion, A is the total amount variable and ε is the random error component which is independently normally distributed with zero mean and common variance σ^2.

Piepel (1988) modified this model to accommodate zero-amount condition. The alternative model uses the original amounts of mixture components denoted by

$a_1, a_2, \ldots, a_q$ such that $a_1 + a_2 + \cdots + a_q = A$. The proportion x_i is related to the amount a_i through $x_i = a_i / A$. This is called mixture component-amount model,

$$Y = \alpha_0 + \sum_{i=1}^{q} \alpha_i a_i + \sum_{i=1}^{q} \left(\alpha_{ii} a_i^2 + \sum_{i \langle j}^{q} \alpha_{ij} a_i a_j \right) + \varepsilon \tag{2}$$

Nigam (1973) introduced designs and models for mixture experiments which use more than one factors mixture at a time. Such experiments are called multifactor mixture experiments. Many examples of multifactor mixture experiment can be found in biological, agricultural and industrial experiments.

No work, so far has been done for multifactor component-amount designs e.g. in a two factor mixture component-amount experiment one factor is crop with different crops (cotton, wheat etc.) as its components and another factor is fertilizer with different fertilizers (nitrogen, phosphate etc.) as its components. Each factor has blends of different amounts. The goal of such multifactor component-amount experiment is to study the effects on the response by changing the amounts in a blend and the total amount of mixture blends.

In this research work we construct two and three factor component-amount designs and find G-efficiency of designs by fitting two and three factor quadratic mixture component-amount models. This work can also be extended for more than two factors mixture where each factor may have equal or different number of components.

2. COMPONENT-AMOUNT MODEL FOR MULTIFACTOR MIXTURE EXPERIMENT

The component-amount quadratic mixture model for a single factor is given in equation (1). Here we consider a two factor and three factor component amount models in subsequent examples.

$$Y = \alpha_0 + \sum_{i}^{q_1} \alpha_i a_{iu} + \sum_{i}^{q_1} \alpha_{ii} a_{iu}^2 + \sum_{i<i'}^{q_1} \alpha_{ii'} a_{iu} a_{i'u} + \sum_{j}^{q_2} \alpha_j a_{ju} + \sum_{j}^{q_2} \alpha_{jj} a_{ju}^2 + \sum_{j<j'}^{q_2} \alpha_{jj'} a_{ju} a_{j'u} + \sum_{i,j}^{q_1,q_2} \alpha_{ij} a_{iu} a_{ju} + \varepsilon \tag{3}$$

$$Y = \alpha_0 + \sum_{i}^{q_1} \alpha_i a_{iu} + \sum_{i}^{q_1} \alpha_{ii} a_{iu}^2 + \sum_{i<i'}^{q_1} \alpha_{ii'} a_{iu} a_{i'u} + \sum_{j}^{q_2} \alpha_j a_{ju} + \sum_{j}^{q_2} \alpha_{jj} a_{ju}^2 + \sum_{j<j'}^{q_2} \alpha_{jj'} a_{ju} a_{j'u} + \sum_{i}^{q_3} \alpha_i a_{iu} + \sum_{i}^{q_3} \alpha_{ii} a_{iu}^2 + \sum_{i<i'}^{q_3} \alpha_{ii'} a_{iu} a_{i'u} + \sum_{i,j}^{q_1,q_2} \alpha_{ij} a_{iu} a_{ju} + \sum_{i,j}^{q_1,q_3} \alpha_{ij} a_{iu} a_{ju} + \sum_{i,j}^{q_2,q_3} \alpha_{ij} a_{iu} a_{ju} + \varepsilon . \tag{4}$$

3. CONSTRUCTION OF DESIGNS FOR MULTIFACTOR COMPONENT-AMOUNT MIXTURE EXPERIMENT

Prescott and Draper (2004, 2008) constructed D-optimal component-amount designs via projection of standard mixture designs (Simplex Lattice, Simplex Centroid) and orthogonal Latin squares designs in two or more orthogonal blocks, in lower dimensions. Further Aggarwal et al. (2012) used F-squares for the construction of D-, A- and E-optimal component-amount designs in two orthogonal blocks by their projections into lower dimensions. We can use these optimal designs for the construction of multifactor component-amount designs.

In construction of multifactor component-amount designs we have used Incidence and adjacency matrices obtained from the undirected graphs.

A graph G with the vertex set $V(G)=\{x_1, x_2, ..., x_v\}$ can be described by means of matrices. The adjacency matrix of G is $v \times v$ matrix $A(G)=(a_{ij})$ where

$$a_{ij} = \mu(x_i, x_j) = \left| E_G(x_i, x_j) \right|$$

$$a_{ij} = \begin{cases} 1, & \text{if } \{e_i, e_j\} \text{ is an edge of } G \\ 0, & \text{otherwise.} \end{cases}$$

The incidence matrix of a graph G is $v \times \varepsilon$ matrix $M(G) = \left(m_{x_i}(e_j) \right)$, $x \in V(G)$ and $e_j \in E(G)$, where, if G is undirected graph then

$$m_{x_i}(e_j) = \begin{cases} 1, & \text{when edge } e_j \text{ is incident with } x_i \\ 0, & \text{otherwise.} \end{cases}$$

An account of undirected graph can be found in Rosen (1999).

3.1 Construction Methodology

1. Take any undirected graph and obtain a corresponding incidence matrix D_1 of order $n \times q$. Selection of undirected graph is arbitrary and should be selected in such a way as to get sufficient number of runs to fit the model.
2. Use any D-optimal component-amount design D_2 of order $m \times p$. Such designs are available in Prescott and Draper (2004, 2008) and Aggarwal et al. (2012).
3. Find the Kronecker product D of order $mn \times pq$.

$$D = D_1 \otimes D_2$$

When all component amount factors have equal number of mixture amount components then the product of p and q is equal to the total number of mixture amount components in all factors. The value of m and n depends upon the number of runs to be chosen which must be greater than or equal to the total number of parameters to be estimated in the model. For unequal number of mixture amount components in factors, design can be generated by projection of Kronecker product D in lower dimensions. This generates multi-factor component-amount design.

3.2 Design Evaluation Criteria

We have used Average Prediction Variance (APV) and G-efficiency Criteria to evaluate proposed optimal designs for quadratic multifactor component-amount mixture model.

The APV criterion requires the prediction variance which is averaged over some region of interest χ. This average is executed by means of integration over χ.

$$\text{APV} = \frac{\int_\chi f'(x)(X'X)^{-1} f(x)dx}{\int_\chi dx}$$

where X is extended design matrix.

A G-optimal design looks for minimizing the maximum variance of the prediction over the experimental region where the prediction variance at the point $x(1 \times k$ row vector) is given by $\sigma^2 v$ where $v = x(X'X)^{-1}x'$. The G-efficiency of the design is given by:

$$G - efficiency = (p/n \times d) \times 100.$$

where, p is the number of model parameters, n is the number of design points and d is the maximum value of v over the experimental region.

4. EXAMPLES SHOWING CONSTRUCTION OF DESIGNS FOR MULTIFACTOR COMPONENT-AMOUNT MIXTURE EXPERIMENTS

In this section, we illustrate the construction of designs for multifactor component amount mixture experiments. In first and third example each factor has equal number of components whereas in second example the two factors have unequal number of components.

Example 4.1

Consider a multi-factor component-amount experiment with two factors each with three components (a_{11}, a_{12}, a_{13}) and (a_{21}, a_{22}, a_{23}), respectively. We select a graph with four vertices and two edges given in Figure 4.1,

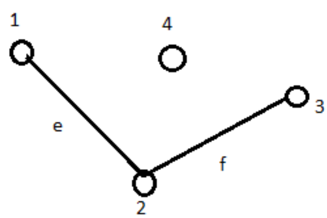

Figure 4.1: G(4, 2)

The incidence matrix of the above graph is of order 4×2. It is obtained as follows:

$$V(G) = \{x_1, x_2, x_3, x_4\} = \{1, 2, 3, 4\},\ E(G) = \{e_1, e_2\} = \{e, f\}$$

$m_{x_1}(e)=1$, $m_{x_1}(f)=0$, $m_{x_2}(e)=1$, $m_{x_2}(f)=1$, $m_{x_3}(e)=0$, $m_{x_3}(f)=1$,
$m_{x_4}(e)=0$, $m_{x_4}(f)=0$

$$D_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Next we consider a three component D-optimal orthogonally blocked component-amount design given by Prescott and Draper (2004, p.424).

Table 4.1
D-optimal Orthogonally Blocked Component-amount Design with Three Components

Block I					Block II				
Run	a_1	a_2	a_3	**A**	**Run**	a_1	a_2	a_3	**A**
1	0	0	0.24	0.24	10	0	0.24	0	0.24
2	0	0.76	0	0.76	11	0	0	0.76	0.76
3	0.24	0	0.76	1.00	12	0.24	0.76	0	1
4	0.76	0.24	0	1.00	13	0.76	0	0.24	1
5	0	0	0.24	0.24	14	0	0.76	0.24	1
6	0	0.24	0.76	1.00	15	0	0	0.76	0.76
7	0.24	0.76	0	1.00	16	0.24	0	0	0.24
8	0.76	0	0	0.76	17	0.76	0.24	0	1
9	0.25	0.25	0.25	0.75	18	0.25	0.25	0.25	0.75

We can use either one or two blocks depending upon the required runs. The design matrix is denoted by D_2. Here we choose D_2 as a matrix with the runs in Block I.

$$D_2 = \begin{pmatrix} 0 & 0 & 0.24 \\ 0 & 0.76 & 0 \\ 0.24 & 0 & 0.76 \\ 0.76 & 0.24 & 0 \\ 0 & 0 & 0.24 \\ 0 & 0.24 & 0.76 \\ 0.24 & 0.76 & 0 \\ 0.76 & 0 & 0 \\ 0.25 & 0.25 & 0.25 \end{pmatrix}$$

The Kronecker product of D_1 and D_2 is of order 36×6

$$D = D_1 \otimes D_2 .$$

Table 4.2
Two Factor Component-Amount Design each with Three Components

Run	a_{11}	a_{12}	a_{13}	a_{21}	a_{22}	a_{23}	Run	a_{11}	a_{12}	a_{13}	a_{21}	a_{22}	a_{23}
1	0	0	0.24	0	0	0	19	0	0	0	0	0	0.24
2	0	0.76	0	0	0	0	20	0	0	0	0	0.76	0
3	0.24	0	0.76	0	0	0	21	0	0	0	0.24	0	0.76
4	0.76	0.24	0	0	0	0	22	0	0	0	0.76	0.24	0
5	0	0	0.24	0	0	0	23	0	0	0	0	0	0.24
6	0	0.24	0.76	0	0	0	24	0	0	0	0	0.24	0.76
7	0.24	0.76	0	0	0	0	25	0	0	0	0.24	0.76	0
8	0.76	0	0	0	0	0	26	0	0	0	0.76	0	0
9	0.25	0.25	0.25	0	0	0	27	0	0	0	0.25	0.25	0.25
10	0	0	0.24	0	0	0.24	28	0	0	0	0	0	0
11	0	0.76	0	0	0.76	0	29	0	0	0	0	0	0
12	0.24	0	0.76	0.24	0	0.76	30	0	0	0	0	0	0
13	0.76	0.24	0	0.76	0.24	0	31	0	0	0	0	0	0
14	0	0	0.24	0	0	0.24	32	0	0	0	0	0	0
15	0	0.24	0.76	0	0.24	0.76	33	0	0	0	0	0	0
16	0.24	0.76	0	0.24	0.76	0	34	0	0	0	0	0	0
17	0.76	0	0	0.76	0	0	35	0	0	0	0	0	0
18	0.25	0.25	0.25	0.25	0.25	0.25	36	0	0	0	0	0	0

This is a two factor mixture component-amount design each with three components. Both factors have five different levels of the total amount, namely 0, 0.24, 0.75, 0.76 and 1. It can be used for fitting a two factor component-amount mixture model given in Equation 2. The design has G-efficiency 64% at design points. The Average prediction variance is 0.639.

Example 4.2

Consider another example of a multi-factor component-amount experiment with two factors where on factor has three components (a_{11}, a_{12}, a_{13}) and other has two components (a_{21}, a_{22}). We select the graph with three vertices and two edges given in Figure 4.2.

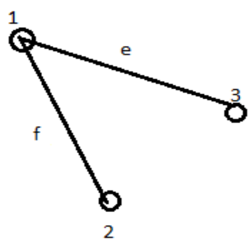

Figure 4.2: G(3, 2)

The incidence matrix of order 3×2 is obtained as follows:

$$V(G)=\{x_1,x_2,x_3\}=\{1,2,3\},\ E(G)=\{e_1,e_2\}=\{e,f\}$$

$$m_{x_1}(e)=1,\ m_{x_1}(f)=1,\ m_{x_2}(e)=1,\ m_{x_2}(f)=0,\ m_{x_3}(e)=0,\ m_{x_3}(f)=1$$

$$D_1=\begin{pmatrix}1 & 1\\ 1 & 0\\ 0 & 1\end{pmatrix}.$$

Next we consider a three component D-optimal component-amount design in two orthogonal blocks constructed by Aggarwal et al. (2012), via projection of orthogonally blocked F-square design.

Table 4.3
D-optimal Orthogonally Blocked Component-Amount Design with Three Components

Block I					**Block II**				
Run	a_1	a_2	a_3	**A**	**Run**	a_1	a_2	a_3	**A**
1	0.13	0.74	0	0.87	10	0.13	0.13	0	0.26
2	0.74	0	0.13	0.87	11	0.74	0.13	0.13	1.00
3	0	0.13	0.13	0.26	12	0	0.74	0.13	0.87
4	0.13	0.13	0.74	1.00	13	0.13	0	0.74	0.87
5	0.13	0	0.13	0.26	14	0.13	0	0.74	0.87
6	0.74	0.13	0.13	1.00	15	0.74	0.13	0	0.87
7	0	0.13	0.74	0.87	16	0	0.13	0.13	0.26
8	0.13	0.74	0	0.87	17	0.13	0.74	0.13	1.00
9	0.25	0.25	0.25	0.75	18	0.25	0.25	0.25	0.75

We can choose either one or two blocks depending upon the required runs. The design matrix is denoted by D_2. Here we choose D_2 as a matrix with the runs in Block II.

$$D_2=\begin{pmatrix}0.13 & 0.13 & 0\\ 0.74 & 0.13 & 0.13\\ 0 & 0.74 & 0.13\\ 0.13 & 0 & 0.74\\ 0.13 & 0 & 0.74\\ 0.74 & 0.13 & 0\\ 0 & 0.13 & 0.13\\ 0.13 & 0.74 & 0.13\\ 0.25 & 0.25 & 0.25\end{pmatrix}$$

The Kronecker product of D_1 and D_2 is the matrix D

$$D = D_1 \otimes D_2$$

$$D = \begin{pmatrix} 0.13 & 0.13 & 0 & 0.13 & 0.13 & 0 \\ 0.74 & 0.13 & 0.13 & 0.74 & 0.13 & 0.13 \\ 0 & 0.74 & 0.13 & 0 & 0.74 & 0.13 \\ 0.13 & 0 & 0.74 & 0.13 & 0 & 0.74 \\ 0.13 & 0 & 0.74 & 0.13 & 0 & 0.74 \\ 0.74 & 0.13 & 0 & 0.74 & 0.13 & 0 \\ 0 & 0.13 & 0.13 & 0 & 0.13 & 0.13 \\ 0.13 & 0.74 & 0.13 & 0.13 & 0.74 & 0.13 \\ 0.25 & 0.25 & 0.25 & 0.25 & 0.25 & 0.25 \\ 0.13 & 0.13 & 0 & 0 & 0 & 0 \\ 0.74 & 0.13 & 0.13 & 0 & 0 & 0 \\ 0 & 0.74 & 0.13 & 0 & 0 & 0 \\ 0.13 & 0 & 0.74 & 0 & 0 & 0 \\ 0.13 & 0 & 0.74 & 0 & 0 & 0 \\ 0.74 & 0.13 & 0 & 0 & 0 & 0 \\ 0 & 0.13 & 0.13 & 0 & 0 & 0 \\ 0.13 & 0.74 & 0.13 & 0 & 0 & 0 \\ 0.25 & 0.25 & 0.25 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.13 & 0.13 & 0 \\ 0 & 0 & 0 & 0.74 & 0.13 & 0.13 \\ 0 & 0 & 0 & 0 & 0.74 & 0.13 \\ 0 & 0 & 0 & 0.13 & 0 & 0.74 \\ 0 & 0 & 0 & 0.13 & 0 & 0.74 \\ 0 & 0 & 0 & 0.74 & 0.13 & 0 \\ 0 & 0 & 0 & 0 & 0.13 & 0.13 \\ 0 & 0 & 0 & 0.13 & 0.74 & 0.13 \\ 0 & 0 & 0 & 0.25 & 0.25 & 0.25 \end{pmatrix}$$

Prescott and Draper (2004) obtained mixture component-amount designs via projections of standard mixture designs and of orthogonally blocked designs. We can also obtain the design by projection of Kronecker product matrix D i.e. by deletion of one or more columns of the matrix. Since the first factor has three components and the second factor has two components, therefore we delete only one column. The last column 6 is deleted. As a result we get two factors component-amount design with the first factor having three components (a_{11}, a_{12}, a_{13}) and the second factor having two components (a_{21}, a_{22}).

Table 4.4
Two Factor Component-Amount Design when One Factor has Three Components and Other has Two Components

Run	a_{11}	a_{12}	a_{13}	a_{21}	a_{22}	Run	a_{11}	a_{12}	a_{13}	a_{21}	a_{22}
1	0.13	0.13	0	0.13	0.13	15	0.74	0.13	0	0	0
2	0.74	0.13	0.13	0.74	0.13	16	0	0.13	0.13	0	0
3	0	0.74	0.13	0	0.74	17	0.13	0.74	0.13	0	0
4	0.13	0	0.74	0.13	0	18	0.25	0.25	0.25	0	0
5	0.13	0	0.74	0.13	0	19	0	0	0	0.13	0.13
6	0.74	0.13	0	0.74	0.13	20	0	0	0	0.74	0.13
7	0	0.13	0.13	0	0.13	21	0	0	0	0	0.74
8	0.13	0.74	0.13	0.13	0.74	22	0	0	0	0.13	0
9	0.25	0.25	0.25	0.25	0.25	23	0	0	0	0.13	0
10	0.13	0.13	0	0	0	24	0	0	0	0.74	0.13
11	0.74	0.13	0.13	0	0	25	0	0	0	0	0.13
12	0	0.74	0.13	0	0	26	0	0	0	0.13	0.74
13	0.13	0	0.74	0	0	27	0	0	0	0.25	0.25
14	0.13	0	0.74	0	0	--	--	--	--	--	--

The above design can be used for fitting a quadratic component-amount model given in Equation 2. The first factor has five different level of total amount, namely 0, 0.26, 0.75, 0.87 and 1. The second factor has six different levels of the total amount i.e. 0, 0.13, 0.26, 0.50, 0.74 and 0.87. This design has G-efficiency 71.5% at design points and APV 0.704.

Example 4.3

Consider a multi-factor component-amount experiment with two factors where each factor has two components (a_{11}, a_{12}) and (a_{21}, a_{22}) respectively. We select a graph with two vertices and two edges given in Figure 4.3:

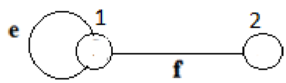

Figure 4.3: G(2, 2)

The incidence matrix of order 2×2 is obtained as follows:

$$V(G) = \{x_1, x_2\} = \{1, 2\}, E(G) = \{e_1, e_2\} = \{e, f\}$$

$$m_{x_1}(e) = 1,\ m_{x_1}(f) = 1,\ m_{x_2}(e) = 0,\ m_{x_2}(f) = 1$$

$$D_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

We consider a two component D-optimal component-amount design obtained from projection of Simplex-Lattice design {4, 2} given in Prescott and Draper (2008).

Table 4.5
D-optimal Component-Amount Design in Three Components

Run	a_1	a_2	**A**
1	1	0	1
2	0	1	1
3	0	0	0
4	0	0	0
5	0.5	0.5	1
6	0.5	0	0.5
7	0	0.5	0.5
8	0.5	0	0.5
9	0	0.5	0.5
10	0	0	0

The design matrix D_2 is

$$D_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0.5 & 0 \\ 0 & 0.5 \\ 0 & 0 \\ 0.5 & 0 \\ 0 & 0.5 \\ 0 & 0 \end{pmatrix}$$

The Kronecker product of D_1 and D_2 is the matrix D

$$D = D_1 \otimes D_2$$

$$D = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Table 4.6
Two Factor Component-Amount Design when each Factor has Two Components

Run	a_{11}	a_{12}	a_{21}	a_{22}	**Run**	a_{11}	a_{12}	a_{21}	a_{22}
1	1	0	1	0	11	0	0	1	0
2	0	1	0	1	12	0	0	0	1
3	0	0	0	0	13	0	0	0	0
4	0	0	0	0	14	0	0	0	0
5	0.5	0.5	0.5	0.5	15	0	0	0.5	0.5
6	0.5	0	0.5	0	16	0	0	0.5	0
7	0	0.5	0	0.5	17	0	0	0	0.5
8	0.5	0	0.5	0	18	0	0	0.5	0
9	0	0.5	0	0.5	19	0	0	0	0.5
10	0	0	0	0	20	0	0	0	0

We get a two factor component-amount design with both factors having two components. Each factor has three different levels of the total amount, namely 0, 0.5 and 1. The design has G-efficiency 55% at design points and the APV 0.55.

Example 4.4

Here we consider a multi-factor component-amount experiment with three factors where each factor has two components i.e. (a_{11}, a_{12}), (a_{21}, a_{22}), (a_{31}, a_{32}). We select a graph with three vertices and three edges, given in Figure 4.4.

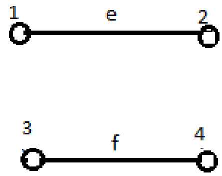

Figure 4.4: G(4, 2)

The incidence matrix of order 4×2 is obtained as follows:

$$V(G) = \{x_1, x_2, x_3, x_4\} = \{1, 2, 3, 4\}, E(G) = \{e_1, e_2\} = \{e, f\}$$

$$m_{x_1}(e) = 1,\ m_{x_1}(f) = 0,\ m_{x_2}(e) = 1,\ m_{x_2}(f) = 0$$

$$m_{x_3}(e) = 0,\ m_{x_3}(f) = 1,\ m_{x_4}(e) = 0,\ m_{x_4}(f) = 1$$

$$D_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$$

Again we consider a three component D-optimal component-amount design in two orthogonal blocks constructed by Prescott and Draper (2004, p.424), given in Table 4.1. We choose D_2 as a matrix with the runs in Block II.

$$D_2 = \begin{pmatrix} 0 & 0.24 & 0 \\ 0 & 0 & 0.76 \\ 0.24 & 0.76 & 0 \\ 0.76 & 0 & 0.24 \\ 0 & 0.76 & 0.24 \\ 0 & 0 & 0.76 \\ 0.24 & 0 & 0 \\ 0.76 & 0.24 & 0 \\ 0.25 & 0.25 & 0.25 \end{pmatrix}$$

The Kronecker product of D_1 and D_2 is the matrix D

$$D = D_1 \otimes D_2$$

Table 4.7
Three Factor Component-Amount Design each with Two Components

Run	a_{11}	a_{12}	a_{21}	a_{22}	a_{31}	a_{32}	**Run**	a_{11}	a_{12}	a_{21}	a_{22}	a_{31}	a_{32}
1	0	0	0.24	0	0	0	19	0	0	0	0.76	0	0.24
2	0	0	0.24	0	0	0	20	0	0	0	0.76	0	0.24
3	0	0	0	0.24	0	0	21	0	0	0	0	0.76	0
4	0	0	0	0.24	0	0	22	0	0	0	0	0.76	0
5	0	0	0	0	0.76	0	23	0	0	0	0	0	0.76
6	0	0	0	0	0.76	0	24	0	0	0	0	0	0.76
7	0	0	0	0	0	0.76	25	0.24	0	0	0	0	0
8	0	0	0	0	0	0.76	26	0.24	0	0	0	0	0
9	0.24	0	0.76	0	0	0	27	0	0.24	0	0	0	0
10	0.24	0	0.76	0	0	0	28	0	0.24	0	0	0	0
11	0	0.24	0	0.76	0	0	29	0.76	0	0.24	0	0	0
12	0	0.24	0	0.76	0	0	30	0.76	0	0.24	0	0	0
13	0.76	0	0	0	0.24	0	31	0	0.76	0	0.24	0	0
14	0.76	0	0	0	0.24	0	32	0	0.76	0	0.24	0	0
15	0	0.76	0	0	0	0.24	33	0.25	0	0.25	0	0.25	0
16	0	0.76	0	0	0	0.24	34	0.25	0	0.25	0	0.25	0
17	0	0	0.76	0	0.24	0	35	0	0.25	0	0.25	0	0.25
18	0	0	0.76	0	0.24	0	36	0	0.25	0	0.25	0	0.25

It is three factor mixture component-amount design where each factor has two components. All three factors have four different levels of the total amount, namely 0, 0.24, 0.25 and 0.76. The design has G-efficiency 88.9% at design points. The APV value of the design is 0.44. The design is G-efficient for fitting multifactor component-amount model given in Equation 3.

5. DISCUSSION

Most of the literature and application about single or multifactor mixture design is available, but so far no work can be seen on multifactor mixture component-amount design. We have carried out a study for determining optimal G-efficient multifactor mixture component-amount designs. Nigam (1973) and Alam et al. (2014) have given different methods for construction of multifactor mixture designs in which main focus is on Kronecker product of designs. We have used this analogy in construction of multifactor mixture component-amount designs. Existing single factor D-optimal component-amount designs developed by Prescott and Draper (2004, 2008) and Aggarwal et al. (2012) are used for construction of two factor mixture component-amount designs. The Kronecker product of D-optimal component amount designs and adjacency/incidence matrix is calculated. The orders of matrices depend upon the number of components in each factor included and the number of runs to be chosen. The G-efficiencies of above developed designs, given in four examples, are 64%, 71.5%, 55% and 88.9% respectively.

Wheeler (1972) has suggested that a design with G-efficiency greater than 50% is considered as good design for practical purposes. Therefore the above developed designs are equally practicable with reference to their G-efficiencies. This procedure can also be adapted for construction of multifactor mixture component-amount designs with more than three factors.

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