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A NEW GENERALIZATION OF THE DOUBLE EXPONENTIAL MODEL: MATHEMATICAL PROPERTIES AND APPLICATIONS

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ABSTRACT

In this work, a new compound lifetime model is presented and studied. The new density can be “asymmetric right skewed”, asymmetric left skewed”, “symmetric” and bimodal”. The failure rate of the new model can be “bathtub”, “monotonically decreasing”, “upside down increasing”, “J-shape” and “bathtub-bathtub”. Statistical properties such as ordinary raw moments, mean deviation, incomplete moments and moment generating function are derived and analyzed. We performed a graphical simulation study to assess the finite sample behavior of the estimators. Finally, two real life applications are analyzed to illustrate the importance of the new model.

KEY WORDS

Double Exponential Model; Zero Truncated Poisson Distribution; Kernel Density Estimation. Modeling; Simulation.

1. INTRODUCTION

In the literature, using the zero-truncated-Poisson (ZTP) distribution, several compound lifetime G families have been defined and studied. However, in the compounding approaches via the ZTP model, there are two different approaches available; one is by using zero truncated power series (ZTPS) distribution and other by using ZTP distribution directly with other continuous distributions. A comprehensive survey regarding the Poisson G models is recently proposed by Maurya and Nadarajah (2020) and El-Morshedy et al. (2021). In this paper we employed the ZTP distribution to propose a new compound version of the double exponential (DE) distribution called the Poisson exponentiated double exponential (PEDE) distribution. Suppose that a device has N (a discrete random variable) sub-components functioning in such independently way at a given time where N has ZTP model with parameter σ and the failure time of i[th] component $Y_i|i = 1,2, \dots$ independent of N. It is the conditional probability distribution of a Poisson-distributed random variable (RV), given that the value of the RV is not zero. The probability mass function (PMF) of N is given by

$$P_\sigma(n|\sigma > 0) = \frac{\sigma^n}{\Gamma(1+n)\mathrm{C}_\sigma} exp(-\sigma),$$

where

$$\mathrm{C}_\sigma = 1 - exp(-\sigma)\,|\sigma > 0.$$

Note that for ZTP RV, the expected value $\mathrm{E}(N|\sigma)$ and variance $\mathrm{V}(N|\sigma)$ are, respectively, given by

$$\mathrm{E}(N|\sigma) = \frac{\sigma}{\mathrm{C}_\sigma},$$

and

$$\mathrm{V}(N|\sigma) = \frac{1}{\mathrm{C}_\sigma}\sigma(1+\sigma) - \frac{1}{\mathrm{C}_\sigma^2}\sigma^2,$$

respectively, (for more details, see Ramos et al. (2015), Aryal and Yousof (2017), Korkmaz et al. (2018) and Alizadeh et al. (2019)). Suppose that the failure time of each subsystem (i^th component) has the exponentiated double exponential (EDE) defined by the cumulative distribution function (CDF)

$$F_{\theta,\beta,\delta}(x) = (1 - exp\{-\beta[exp(\delta x) - 1]\})^\theta \,|\theta,\beta,\delta > 0, x > 0. \tag{1}$$

For $\theta = 1$, the EDE model reduces to the two-parameter DE model. For $\beta = 1$, the EDE model reduces to the two-parameter type I EDE model. For $\delta = 1$, the EDE model reduces to the two-parameter type II EDE model. For $\beta = \delta = 1$, the EDE model reduces to the one-parameter EDE model. For $\theta = \beta = 1$, the EDE model reduces to the one-parameter type I DE model. For $\theta = \delta = 1$, the EDE model reduces to the one-parameter type II DE model.

Let Y_i denote the failure time of the i^th subsystem and let

$$X = min\{Y_1, Y_2, \cdots, Y_N\}.$$

Therefore, the unconditional CDF of the Poisson exponentiated double exponential (PEDE) density function can be expressed as described by Ramos et al. (2015), Aryal and Yousof (2017), Korkmaz et al. (2018) and Alizadeh et al. (2019) as

$$F_\sigma(x) = \mathrm{C}_\sigma^{-1} exp[-\sigma \mathrm{H}(x)],$$

where refers to the CDF of the base line model. For $\sigma = 1$, the Poisson G family reduces to the quasi Poisson G family. Based on the Poisson G family, the PEDE model can then be expressed as

$$F_{\underline{\Lambda}}(x) = \mathrm{C}_\sigma^{-1}\left(1 - exp\left\{-\sigma(1 - exp\{-\beta[exp(\delta x) - 1]\})^\theta\right\}\right)|x > 0, \tag{2}$$

where $\underline{\Lambda} = (\sigma,\theta,\beta,\delta)$ is the parameter vector of the PEDE model. Therefore, the PDF corresponding to (3) can be simplifies as

$$\begin{aligned} f_{\underline{\Lambda}}(x) = \sigma\beta\theta\delta \mathrm{C}_\sigma & \frac{exp(\delta x)\, exp\{-\beta[exp(\delta x) - 1]\}}{exp\{\sigma(1 - exp\{-\beta[exp(\delta x) - 1]\})^\theta\}} \\ & \times (1 - exp\{-\beta[exp(\delta x) - 1]\})^{\theta-1} \,|x > 0. \end{aligned} \tag{3}$$

A RV X having PDF (3) is denoted by $X \sim \mathrm{PEDE}(\underline{\Lambda})$. For $\theta = 1$, the PEDE model reduces to the PDE model. For $\beta = 1$, the PEDE model reduces to the three-parameter type I PDE model. For $\delta = 1$, the PEDE model reduces to the three-parameter type II PDE model. For $\theta = \beta = 1$, the PEDE model reduces to the two-parameter type I PDE model. For $\theta = \delta = 1$, the PEDE model reduces to the two-parameter type II PDE model. For

$\sigma = 1$, the PEDE model reduces to the quasi PEDE (QPEDE) model. For $\sigma = \beta = 1$, the PEDE model reduces to the two-parameter type I QPEDE model. For $\sigma = \delta = 1$, the PEDE model reduces to the two-parameter type II QPEDE model. For $\sigma = \theta = 1$, the PEDE model reduces to the quasi PDE (QPDE) model. For $\sigma = \theta = \beta = 1$, the PEDE model reduces to the one-parameter type I quasi PE (QPE) model. For $\sigma = \theta = \delta = 1$, the PEDE model reduces to the one-parameter type II quasi PE (QPE) model. Figure 1 gives some plots for the PEDE PDF.

In the statistical literature there are many versions of the exponential distribution such as Marshall-Olkin exponential (MOE) distribution (Ghitany et al. (2005)), Beta exponential (BE) distribution (Lee et al. (2007)), Kumaraswamy exponential (KE) distribution (Cordeiro et al. (2010)), Poisson-exponential (PE) distribution (Cancho et al. (2011)), Moment exponential (ME) distribution (Dara and Ahmad (2012)), Generalized Marshall-Olkin exponential (GMOE) distribution (Chakraborty and Handique (2017)), transmuted exponentiated generalized exponential (TEGE) distribution (Yousof et al. (2017a)), Marshall-Olkin Kumaraswamy exponential (MOKE) distribution (Chakraborty and Handique (2017)), Burr XII exponential (BXIIE) distribution (Cordeiro et al. (2018)), odd Lindley exponential (OLE) distribution (Almamy et al. (2018)), Topp Leone zero truncated Poisson exponential(TLZTPE) distribution (Refaie (2018a)), Burr X exponentiated exponential (BXEE) distribution (Refaie (2018b) and Khalil et al. (2019)), Poisson Topp Leone exponentiated exponential (PTLEE) distribution (Refaie (2018c)), Burr–Hatke exponential (BHE) distribution (Yousof et al. (2018)), the odd Lindley exponentiated exponential (OLEE) distribution (Refaie (2019)), Kumaraswamy Marshall-Olkin exponential (KMOE) distribution (George and Thobias (2019)), quasi Poisson Burr X exponentiated exponential (QPBXEE) distribution (Mansour et al. (2020b)), generalized odd log-logistic exponentiated exponential (GOLLEE) distribution (Mansour et al. (2020b)), Marshall-Olkin Lehmann exponential (MOLE) (Elgohari and Yousof (2020)) and the Burr X exponential (BXE) distribution (Yousof et al. (2017b), Mansour et al. (2020c)) and the Burr–Hatke exponential distribution (Yadav et al. (2021)), among others.

Figure 2 gives some plots of the PEDE HRF for some selected parameter values. Based on Figure 1, we note that the new PDF can be “asymmetric right skewed shape”, asymmetric left skewed shape”, “symmetric shape” and bimodal”. Based on Figure 2, it is seen that the new HRF can be “bathtub”, “decreasing”, “upside down increasing (reversed bathtub)”, “J shape” and “bathtub- bathtub (W shape)”.

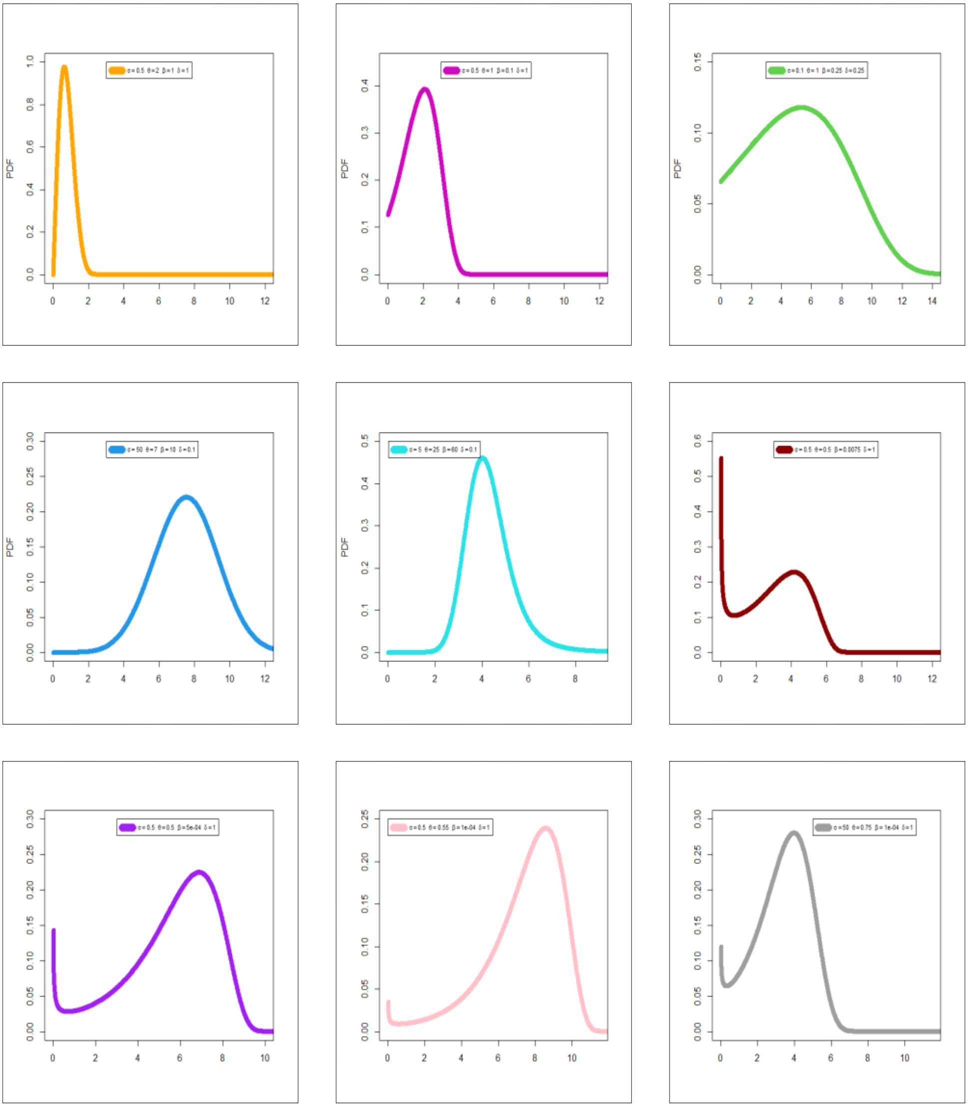

Figure 1: Plots for the PDFs for Some Selected Parameter Values

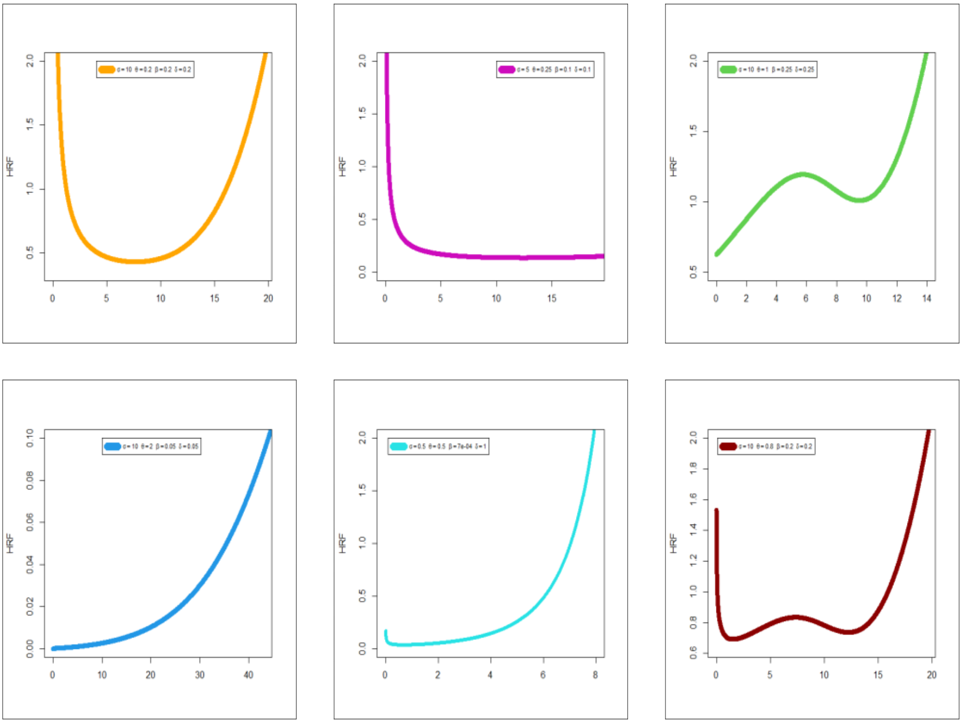

Figure 2: Plots for the HRF for Some Selected Parameter Values

The new PEDE model could be useful in modeling

I. The real-life datasets which have "monotonically increasing failure rate " as illustrated in Figures 6 and (bottom left plots).

II. The real-life data sets which have some extreme values as shown in Figures 6 and (bottom right plots).

III. The real-life datasets which their nonparametric Kernel density are bimodal and asymmetric with right leavy tail as shown in Figures 6 and (top left plots).

IV. The real-life datasets which their PDF can be “asymmetric right skewed shape”, asymmetric left skewed shape”, “symmetric shape” and bimodal” (see Figure 1).

V. The real-life datasets which their HRF can be “bathtub”, “decreasing”, “upside down increasing (reversed bathtub)”, “J shape” and “bathtub- bathtub (W shape)” (see Figure 2).

The PEDE model proved its applicability and superiority against many well-known exponential extensions as shown below:

I. In modeling the relief times data, the PEDE model is better than the odd Lindley exponential model, Marshall-Olkin exponential model, Moment exponential model, the Logarithmic Burr-Hatke exponential model, Generalized Marshall-

Olkin exponential model, Beta exponential model, Marshall-Olkin Kumaraswamy exponential model, Kumaraswamy exponential model, the Burr X exponential model, Kumaraswamy Marshall-Olkin exponential model and standard exponential model under the Cramér-Von Mises Criteria, Anderson-Darling Criteria, Akaike Information Criteria, Consistent Akaike Information Criteria, Bayesian Information Criteria, Hannan-Quinn Information Criteria, Kolmogorov-Smirnov (KS) and its corresponding P-value.

II. In modeling the survival times of the aircraft windshield, the PGE-G family is better than the odd Lindley exponential model, Marshall-Olkin exponential model, Moment exponential model, the Logarithmic Burr-Hatke exponential model, Generalized Marshall-Olkin exponential model, Beta exponential model, Marshall-Olkin Kumaraswamy exponential model, Kumaraswamy exponential model, the Burr X exponential model, Kumaraswamy Marshall-Olkin exponential model and standard exponential model under the Cramér-Von Mises Criteria, Anderson-Darling Criteria, Akaike Information Criteria, Consistent Akaike Information Criteria, Bayesian Information Criteria, Hannan-Quinn Information Criteria, Kolmogorov-Smirnov (KS) and its corresponding P-value.

2. MATHEMATICAL PROPERTIES

2.1 Representation

In this section, we provide a useful linear representation for the PEDE density function. Using the power series, we expand the quantity $A(x)$ where

$$A(x) = exp\left\{-\sigma(1-exp\{-\beta[exp(\delta x)-1]\})^{\theta}\right\}$$
$$= \sum_{p=0}^{+\infty}\frac{(-\sigma)^p}{\Gamma(1+p)}(1-exp\{-\beta[exp(\delta x)-1]\})^{\theta p}. \qquad (4)$$

Then, the PDF in (4) can be expressed as

$$f_{\underline{\Lambda}}(x) = \mathrm{C}_{\sigma}^{-1}\beta\theta\delta\sum_{p=0}^{+\infty}\frac{(-1)^p\sigma^{1+p}exp(\delta x)}{\Gamma(1+p)\,exp\{\beta[exp(\delta x)-1]\}}$$
$$\underbrace{[1-exp\{-\beta[exp(\delta x)-1]\}]^{\theta(p+1)-1}}_{B(x)}. \qquad (5)$$

Then, consider the power series

$$\left(1-\frac{a_1}{a_2}\right)^{a_3} = \sum_{u=0}^{+\infty}\frac{\Gamma(a_3+1)}{u!\,\Gamma(a_3-u+1)}\left(-\frac{a_1}{a_2}\right)^u \Big|_{\left|\frac{a_1}{a_2}\right|<1 \text{ and } a_3>0}. \qquad (6)$$

Applying (6) to the quantity $B(x)$ in (5), we get

$$f_{\underline{\Lambda}}(x) = \mathrm{C}_{\sigma}^{-1}\beta\theta\delta\sum_{p,u=0}^{+\infty}\frac{\sigma^{1+p}(-1)^{p+u}\Gamma\big(\theta(p+1)\big)exp(\delta x)}{\Gamma(1+u)\Gamma(1+p)\Gamma(\theta(p+1)-u)}$$
$$\underbrace{exp\{-(u+1)\beta[exp(\delta x)-1]\}}_{C(x)}. \qquad (7)$$

Expanding the quantity $C(x)$ in power series, we can write

$$C(x) = \sum_{j=0}^{+\infty} \frac{(-1)^j (u+1)^j}{\Gamma(1+j)} [exp(\delta x) - 1]^j. \tag{8}$$

Inserting the above expression of $C(x)$ in (9), the PEDE density reduces to

$$f_{\underline{\Delta}}(x) = \theta\beta \mathrm{C}_{\sigma}^{-1}\delta \sum_{p,u,j=0}^{+\infty} \sigma^{1+p} \frac{(-1)^{p+j+u}\Gamma\big(\theta(p+1)\big)(u+1)^j exp(-\delta x)[1-exp(-\delta x)]^j}{\Gamma(1+p)\Gamma(1+u)\Gamma(1+j)\Gamma(\theta(p+1)-u)[exp(-\delta x)]^{j+2}}. \tag{9}$$

Using the generalized binomial expansion to $[exp(-\delta x)]^{j+2}$, we can write

$$\{1-[1-exp(-\delta x)]\}^{-j-2} = \sum_{v=0}^{+\infty} \frac{\Gamma(1+j^{\bullet})}{\Gamma(1+v)\Gamma(j+2)} [1-exp(-\delta x)]^v|_{(j^{\bullet}=j+v+1)}. \tag{10}$$

Inserting (10) in (9), the PEDE density can be expressed as an infinite linear combination of exponentiated exponential (exp-E) density functions

$$f_{\underline{\Delta}}(x) = \sum_{j,v=0}^{+\infty} \Delta_{j,v}\, \mathrm{g}_{j^{\bullet},\delta}(x), \tag{11}$$

where

$$\mathrm{g}_{j^{\bullet},\delta}(x) = \delta j^{\bullet} exp(-\delta x)[1-exp(-\delta x)]^{j^{\bullet}-1},$$

is the PDF of the exponentiated exponential (exp-E) with power parameter $j^{\bullet}$ and

$$\Delta_{j,v} = \sum_{p,u=0}^{+\infty} \sigma^{1+p}\theta\beta\mathrm{C}_{\sigma}^{-1} \frac{(-1)^{p+j+u}(u+1)^j\Gamma\big(\theta(p+1)\big)\Gamma(1+j^{\bullet})}{\Gamma(1+p)\Gamma(1+u)\Gamma(1+v)\Gamma(1+j)\, j^{\bullet}\Gamma(\theta(p+1)-u)\Gamma(j+2)}.$$

From Equation (11), it is seen that the PDF of PEDE model can be expressed as a linear combination of exp-E PDFs. So, several mathematical properties of the new family can be obtained by knowing those of the exp-E distribution. Similarly, the CDF of the PEDE model can also be expressed as a linear combination of exp-E CDFs given by

$$F_{\underline{\Delta}}(x) = \sum_{j,v=0}^{+\infty} \Delta_{j,v}\ \mathrm{G}_{j^{\bullet},\delta}(x), \tag{12}$$

where

$$\mathrm{G}_{j^{\bullet},\delta}(x) = [1-exp(-\delta x)]^{j^{\bullet}},$$

is the CDF of the exp-E model with power parameter $j^{\bullet}$.

2.2 Moments and Moment Generating Function (MGF)

Based on Theorem 1, the rth moment of X, say $\mu'_{r,X}$, follows from equation (11) as

$$\mu'_{r,X} = E(X^{r}) = \sum_{j,v=0}^{+\infty} \Delta_{j,v}\, E\left(Z_{j^{\bullet}}^{r}\right).$$

Then, the rth moment of X can then be expressed as

$$\mu'_{r,X}|_{r>-1} = E(X^{r}) = \Gamma(r+1) \sum_{j,v,\hbar=0}^{+\infty} \Delta_{j,v,\hbar}^{(r,j^{\bullet})},$$

where

$$\Delta_{j,v,\hbar}^{(r,j^{\bullet})} = \Delta_{j,v}\Delta_{\hbar}^{(r,j^{\bullet})}.$$

The nth central moment of X, say $M_{n,X}$ can be derived directly from $M_{n,X} = E\left(X - \mu'_{1,X}\right)^{n}$. Using (11) and Theorem 1, the MGF $M_X(t)$ of X can be expressed as

$$M_X(t)|_{r>-1} = \sum_{j,v,\hbar,r=0}^{+\infty} \frac{t^{r}}{r!}\mu'_{r,X},$$

2.3 Incomplete Moments

Using Theorem 2, the sth incomplete moments of X can be expressed as

$$\mathrm{I}_{s,X}(t)|_{s>-1} = \gamma(s+1,\delta t) \sum_{\hbar=0}^{+\infty} \Delta_{j,v,\hbar}^{(s,j^{\bullet})}.$$

Therefore, the first incomplete moment of X can be derived from $\mathrm{I}_{s,X}(t)$ when $s = 1$.

2.4 Residual Life Function (RLf) and the Life Expectation (LfE)

The q^{th} moment of the RLf of the RV X can be obtined from

$$A_{q,X}(t) = \mathbb{E}[(X-t)^{q}]|_{X>t\ anp\ q\in\mathbb{N}},$$

or from

$$A_{q,X}(t) = \frac{1}{1 - F_{\underline{\Lambda}}(t)} \int_{t}^{\infty} (-t+x)^{q} f_{\underline{\Lambda}}(x)dx,$$

which can also be written as

$$A_{q,X}(t)|_{q>-1} = \frac{\Gamma(q+1,\delta t)}{\left[1 - F_{\underline{\Lambda}}(t)\right]} \sum_{j,v,\hbar=0}^{+\infty} \Delta_{j,v,\hbar}^{(q,j^{\bullet})}(A,X),$$

where

$$\Delta_{j,v,\hbar}^{(q,j^{\bullet})}(A,X) = \Delta_{j,v}\Delta_{\hbar}^{(q,j^{\bullet})} \sum_{l=0}^{q} \binom{q}{l} (-t)^{q-l},$$

where $\Gamma(1+n,t)$ refers to upper incomplete gamma function. For $q=1$, we obtain the LfE which can be drived as

$$A_{1,X}(t)=\frac{\Gamma(2,\delta t)}{\left[1-F_{\underline{\Lambda}}(t)\right]}\sum_{j,v,\hbar=0}^{+\infty}\Delta_{j,v,\hbar}^{(2,j^{\bullet})}(A,X),$$

and represents the additional expected life for a certin system or component which is already alive at the age t.

2.5 Reversed Residual Life (RRLf) and Mean Waiting Time (MWT) Functions

The q^{th} moment of the RRLf is

$$B_{q,X}(t)=\mathbb{E}[(t-X)^{q}]|_{X\leq t,t>0 \text{ and } q\in\mathbb{N}},$$

or

$$B_{q,X}(t)=\frac{1}{F_{\underline{\Lambda}}(t)}\int_{0}^{t}(-x+t)^{q}f_{\underline{\Lambda}}(x)dx,$$

which can also be expressed as

$$B_{q,X}(t)|_{q>-1}=\frac{1}{F_{\underline{\Lambda}}(t)}\gamma(q+1,\delta t)\sum_{j,v,\hbar=0}^{+\infty}\Delta_{j,v,\hbar}^{(q,j^{\bullet})}(B,X),$$

where

$$\Delta_{j,v,\hbar}^{(q,j^{\bullet})}(B,X)=\Delta_{j,v}\Delta_{\hbar}^{(q,j^{\bullet})}\sum\nolimits_{l=0}^{q}(-1)^{h}\binom{q}{l}t^{q-l}.$$

For $q=1$, we obtain the MWT

$$B_{1,X}(t)=\frac{1}{F_{\underline{\Lambda}}(t)}\gamma(2,\delta t)\sum_{j,v,\hbar=0}^{+\infty}\Delta_{j,v,\hbar}^{(1,j^{\bullet})}(B,X),$$

which also can be called the mean inactivity time (MIT).

2.5 Numerical and Graphical Analysis for Some Measures

Table 1 below gives numerical analysis for the mean ($E(X)$), variance ($\mathrm{V}(X)$), skewness ($\mathrm{S}(X)$) and kurtosis ($\mathrm{K}(X)$). Based on results listed in Table 1, it is noted that $E(X)$ decreases as σ,β and δ increases, $E(X)$ increases as θ increases, $\mathrm{S}(X)\in(-3.88,\infty)$ and $\mathrm{K}(X)$ ranging from 1.445 to ∞. Figure 3 gives three-dimensional skewness plots for the for some selected parameter values. Figure 4 displays three-dimensional kurtosis plots for the for some selected parameter values.

Table 1
Numerical Results for $E(X)$, $V(X)$, $S(X)$ and $K(X)$

σ	θ	β	δ	$E(X)$	$V(X)$	$S(X)$	$K(X)$
−500	0.5	0.5	1	2.364448	0.5066787	−2.791627	9.599033
−100				2.036134	0.5451138	−2.090994	6.259912
−50				1.856956	0.5646022	−1.718185	4.860248
−10				1.292527	0.6170022	−0.597689	2.147473
10				0.006475	0.0026303	13.31246	258.5861
50				1.5×10^{-05}	4.9×10^{-07}	49.93754	3343.939
100				4.9×10^{-08}	1.6×10^{-09}	811.5032	658556.7
500				1.4×10^{-29}	4.4×10^{-31}	∞	∞
10	0.001	0.75	0.75	4.0×10^{-07}	5.4×10^{-07}	2250.770	5769684
	0.1			6.9×10^{-05}	7.2×10^{-05}	172.4978	35739.09
	5			0.579327	0.3320642	0.1998015	1.444928
	50			1.699491	0.7590853	−1.374691	3.042766
	200			2.172982	0.8340872	−1.919511	4.794361
	1000			2.589280	0.8566511	−2.413575	6.908853
	50000			3.281533	0.8237015	−3.324846	12.11024
	1000000			3.650380	0.7801610	−3.880359	16.10315
−100	10	0.15	0.15	25.53609	15.57769	−5.846234	38.12432
		0.5		17.21736	21.29620	−3.274437	12.46377
		5		3.636574	9.254214	−0.296626	1.221108
		50		0.123158	0.104939	2.351518	6.879394
		200		0.009057	0.002266	5.256961	29.85848
		1000		1.1×10^{-05}	3.5×10^{-07}	55.06121	3033.787
10	2	10	0.001	0.1840655	4.1620260	11.6253	141.458
			0.05	0.0328407	0.0418182	8.367325	105.2007
			0.15	0.0109469	0.0046465	8.367325	105.2007
			0.25	0.0065681	0.0016727	8.367324	105.2077
			0.35	0.0046917	0.0008534	8.367331	105.1457

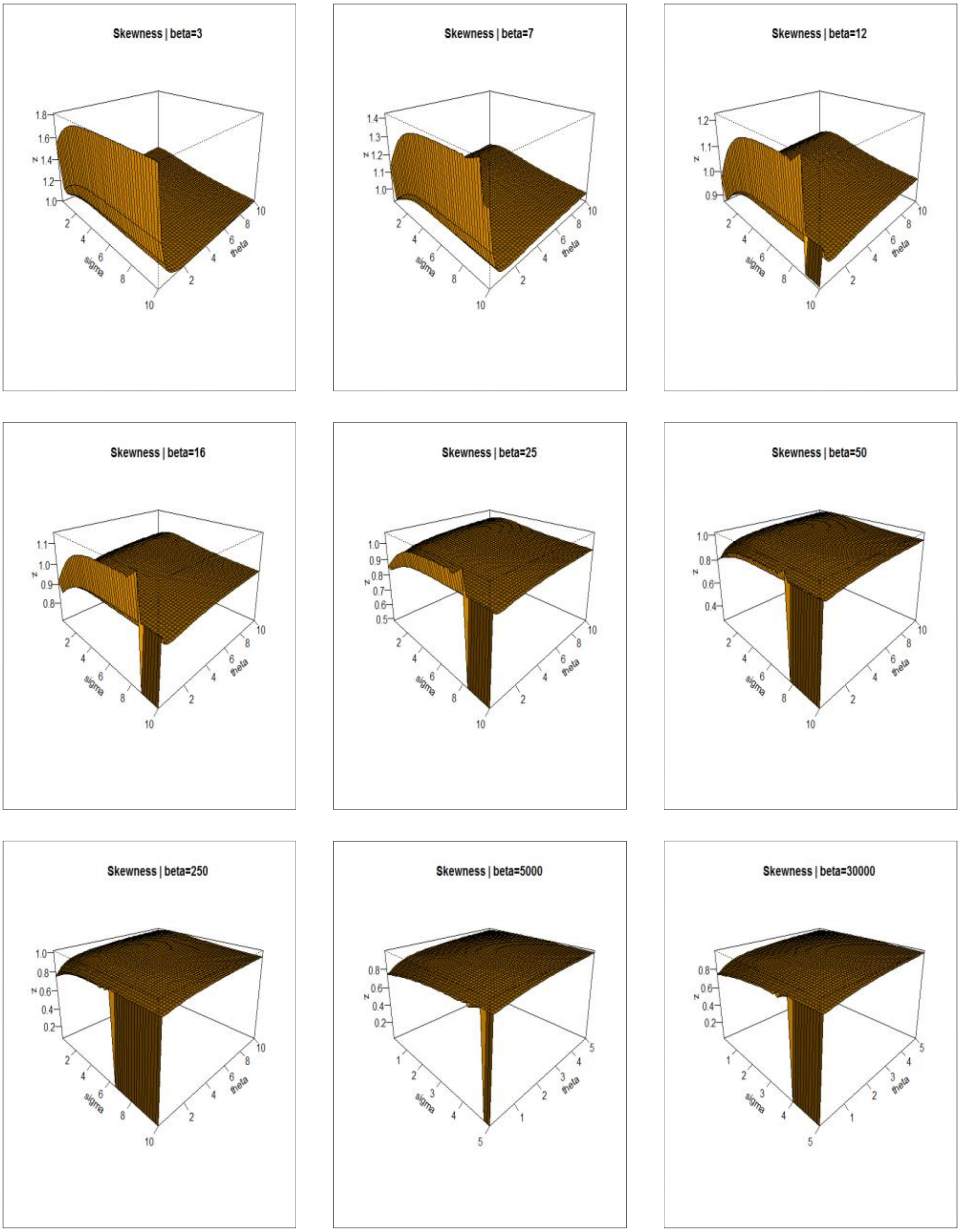

Figure 3: Three-Dimensional Skewness Plots

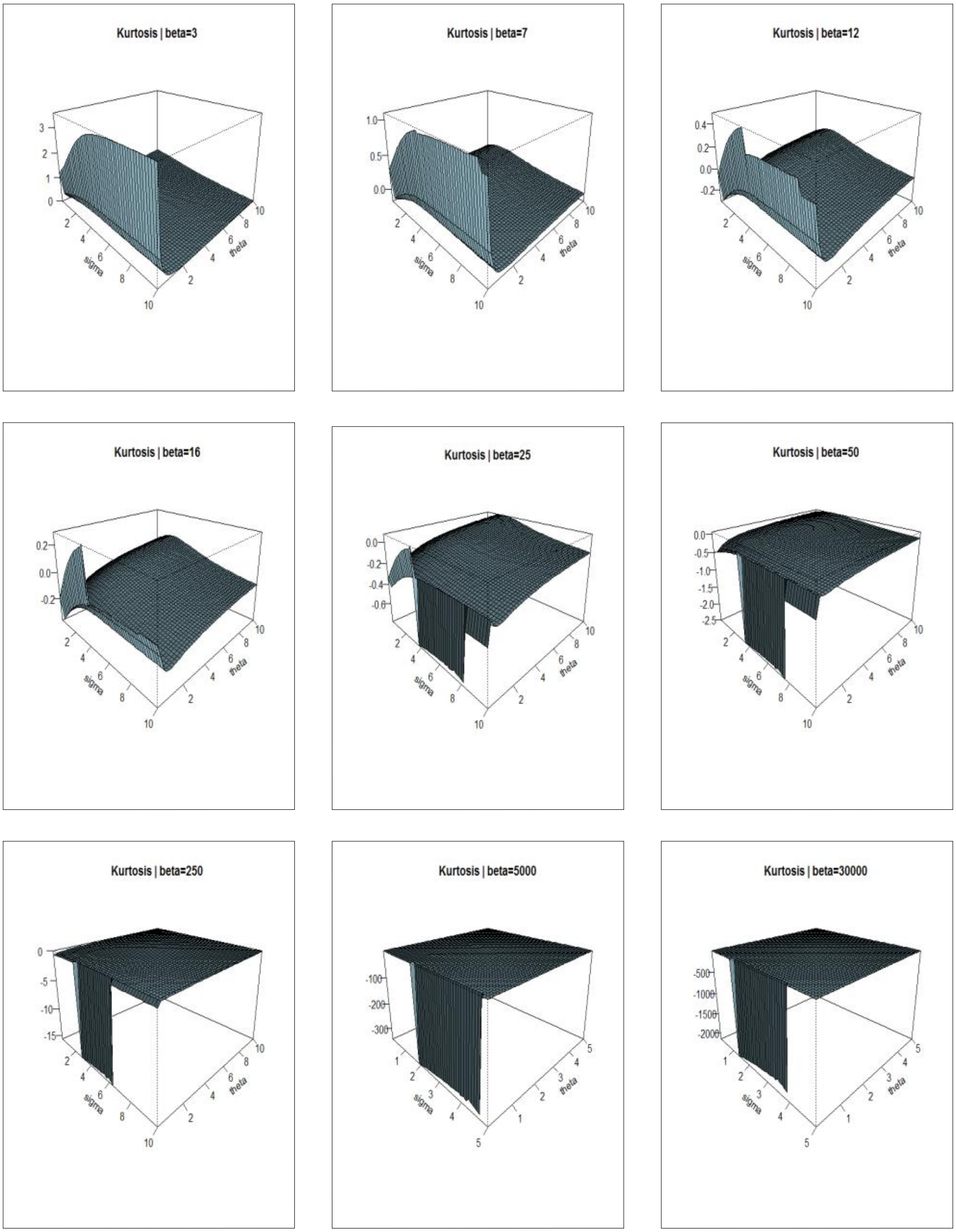

Figure 4: Three-Dimensional Kurtosis Plots

3. COPULAS

For the propose of statistical modeling of the bivariate real-life datasets, we present many new bivariate PEDE (Biv-PEDE) type distributions using the theorems of the "Farlie-Gumbel-Morgenstern copula" (FGMC) copula, modified FGMC, "Clayton copula (CyC)", "Renyi's entropy copula (REC)" and Ali-Mikhail-Haq copula (AMHC) (see Farlie (1960), Morgenstern (1956), Gumbel (1960), Gumbel (1961), Johnson and Kotz (1977), Nelsen (2007), Ali (1978), Al-babtain et al. (2020), Mansour et al. (2020a-f), Salah et al. (2020), Yousof et al. (2021c,d), Ali et al. (2021a,b)), Shehata and Yousof (2021) and Elgohari et al. (2021). The multivariate PEDE (Mv PEDE) type can be easily derived based on the Clayton copula. However, some future articles may be devoted for studying these new Biv-PEDE extension.

3.1 Biv-PEDE Type via CyC

Let us assume that $X_1 \sim \text{PEDE}(\underline{\Lambda}_1)$ and $X_2 \sim \text{PEDE}(\underline{\Lambda}_2)$. The CyC depending on the continuous marginal functions $\overline{\mathcal{W}} = 1 - \mathcal{W}$ and $\overline{\mathcal{K}} = 1 - \mathcal{K}$ can be considered as

$$C_{\delta}(\overline{\mathcal{W}},\overline{\mathcal{K}}) = \left[\max\left(\overline{\mathcal{W}}^{-\delta} + \overline{\mathcal{K}}^{-\delta} - 1\right);0\right]^{-\frac{1}{\delta}}, \delta \in [-1,\infty) - \{0\}, \overline{\mathcal{W}} \in (0,1) \; and \; \overline{\mathcal{K}} \in (0,1).$$

Let

$$\overline{\mathcal{W}} = 1 - F_{\underline{\Lambda}_1}(\mathcal{X}_1)|_{\underline{\Lambda}_1}, \overline{\mathcal{K}} = 1 - F_{\underline{\Lambda}_2}(\mathcal{X}_2)|_{\underline{\Lambda}_2},$$

and

$$F_{\underline{\Lambda}_i}(\mathcal{X}_i)|_{i=1,2} = \mathcal{C}_{\lambda_i}^{-1}\left(1 - e\mathcal{X}p\left\{-\lambda_i\left[1 - \varsigma_{\beta_i,\underline{\xi}_i}(\mathcal{X}_i)\right]^{\theta_i}\right\}\right).$$

Then, the Biv-PEDE type distribution can be obtained from $C_{\delta}(\overline{\mathcal{W}},\overline{\mathcal{K}})$.

3.2 Biv-PEDE Type via REC

The REC can express as

$$C(\mathcal{W},\mathcal{K}) = \mathcal{X}_2\mathcal{W} + \mathcal{X}_1\mathcal{K} - \mathcal{X}_1\mathcal{X}_2,$$

with the continuous marginal functions $\mathcal{W} = 1 - \overline{\mathcal{W}} = F_{\underline{\Lambda}_1}(\mathcal{X}_1) \in (0,1)$ and $\mathcal{K} = 1 - \overline{\mathcal{K}} = F_{\underline{\Lambda}_1}(\mathcal{X}_2) \in (0,1)$, where the values $\mathcal{X}_1$ and $\mathcal{X}_2$ are in order to guarantee that $C(\mathcal{W},\mathcal{K})$ of is a copula. Then, the associated CDF of the Biv-PEDE will be

$$F(\mathcal{X}_1,\mathcal{X}_2) = C\left(F_{\underline{\Lambda}_1}(\mathcal{X}_1), F_{\underline{\Lambda}_1}(\mathcal{X}_2)\right),$$

where $F_{\underline{\Lambda}_i}(\mathcal{X}_i)$ is defined above. It is worth mentioning that this copula does not show a closed shape and numerical approaches become necessary.

3.3 Biv-PEDE Type via FGMC

Considering the FGMC, then the joint CDF (J-CDF) can be expressed as

$$C_{\delta}(\mathcal{W},\mathcal{K}) = \mathcal{W}\mathcal{K}\left(1 + \delta\overline{\mathcal{W}\mathcal{K}}\right),$$

where the continuous marginal function $\mathcal{W} \in (0,1)$, $\mathcal{K} \in (0,1)$ and $\delta \in [-1,1]$ where $C_{\delta}(\mathcal{W},0) = C_{\delta}(0,\mathcal{K}) = 0|_{(\mathcal{W},\mathcal{K}\in(0,1))}$, which called " condition of the grounded minimum" and $C_{\Delta}(\mathcal{W},1) = \mathcal{W}$ and $C_{\Delta}(1,\mathcal{K}) = \mathcal{K}$ which called " condition of the grounded maximum". The conditions of the grounded minimum/maximum always valid for any copula.

Setting $\overline{\mathcal{W}} = \overline{\mathcal{W}}_{\underline{\Delta}_1}|_{\underline{\Delta}_1>0}$ and $\overline{\mathcal{K}} = \overline{\mathcal{K}}_{\underline{\Delta}_2}|_{\underline{\Delta}_2>0}$. Then, we have

$$F(\mathcal{X}_1,\mathcal{X}_2) = C\left(F_{\underline{\Delta}_1}(\mathcal{X}_1), F_{\underline{\Delta}_2}(\mathcal{X}_2)\right) = \mathcal{W}\mathcal{K}\left(1 + \delta\overline{\mathcal{W}\mathcal{K}}\right).$$

The J-CDF can be derived from

$$c_{\delta}(\mathcal{W},\mathcal{K}) = 1 + \delta\mathcal{W}^*\mathcal{K}^*, (\mathcal{W}^* = 1 - 2\mathcal{W} \text{ and } \mathcal{K}^* = 1 - 2\mathcal{K}).$$

or from

$$f_{\delta}(\mathcal{X}_1,\mathcal{X}_2) = f_{\underline{\Delta}_1}(\mathcal{X}_1) f_{\underline{\Delta}_2}(\mathcal{X}_2) c\left(F_{\underline{\Delta}_1}(\mathcal{X}_1), F_{\underline{\Delta}_2}(\mathcal{X}_2)\right),$$

where the two function $c_{\delta}(\mathcal{W},\mathcal{K})$ and $f_{\delta}(\mathcal{X}_1,\mathcal{X}_2)$ are densities corresponding to the J-CDFs $C_{\delta}(\mathcal{W},\mathcal{K})$ and $F_{\delta}(\mathcal{X}_1,\mathcal{X}_2)$.

3.4 Biv-PEDE Type via Modified FGMC

The modified formula of the modified FGMC can written as

$$C_{\delta}(\mathcal{W},\mathcal{K}) = \mathcal{W}\mathcal{K} + \delta\mathcal{O}(\mathcal{W})^{\bullet}\mathcal{U}(\mathcal{K})^{\bullet},$$

with $\mathcal{O}(\mathcal{W})^{\bullet} = \mathcal{W}\overline{\mathcal{O}(\mathcal{W})}$ and $\mathcal{U}(\mathcal{K})^{\bullet} = \mathcal{K}\overline{\mathcal{U}(\mathcal{K})}$ where $\mathcal{O}(\mathcal{W}) \in (0,1)$ and $\mathcal{U}(\mathcal{K}) \in (0,1)$ are two continuous functions where $\mathcal{O}(\mathcal{W}=0) = \mathcal{O}(\mathcal{W}=1) = \mathcal{U}(\mathcal{K}=0) = \mathcal{U}(\mathcal{K}=1) = 0$. Let

$$A(\mathcal{D}_1(\mathcal{W})) = inf\left\{\mathcal{O}(\mathcal{W})^{\bullet}: \frac{\partial}{\partial\mathcal{W}}\mathcal{O}(\mathcal{W})^{\bullet}, \forall\mathcal{D}_1(\mathcal{W})\right\} < 0,$$

$$B(\mathcal{D}_1(\mathcal{W})) = sup\left\{\mathcal{O}(\mathcal{W})^{\bullet}: \frac{\partial}{\partial\mathcal{W}}\mathcal{O}(\mathcal{W})^{\bullet}, \forall\mathcal{D}_1(\mathcal{W})\right\} < 0,$$

$$A(\mathcal{D}_2(\mathcal{K})) = inf\left\{\mathcal{U}(\mathcal{K})^{\bullet}: \frac{\partial}{\partial\mathcal{K}}\mathcal{U}(\mathcal{K})^{\bullet}, \forall\mathcal{D}_2(\mathcal{K})\right\} > 0,$$

and

$$B(\mathcal{D}_2(\mathcal{K})) = sup\left\{\mathcal{U}(\mathcal{K})^{\bullet}: \frac{\partial}{\partial\mathcal{K}}\mathcal{U}(\mathcal{K})^{\bullet}, \forall\mathcal{D}_2(\mathcal{K})\right\} > 0.$$

Then for

$$1 \leq min(A(\mathcal{D}_1(\mathcal{W}))B(\mathcal{D}_1(\mathcal{W})), A(\mathcal{D}_2(\mathcal{K}))B(\mathcal{D}_2(\mathcal{K})))$$

we have

$$\frac{\partial}{\partial\mathcal{W}}\mathcal{O}(\mathcal{W}) + \mathcal{O}(\mathcal{W}) = \frac{\partial}{\partial\mathcal{W}}\mathcal{O}(\mathcal{W})^{\bullet},$$

where

$$\mathcal{D}_1(\mathcal{W}) = \left\{\frac{\partial}{\partial\mathcal{W}}\mathcal{O}(\mathcal{W})^{\bullet} \text{ exists}\right\},$$

and

$$\mathcal{D}_2(\mathcal{K}) = \left\{\frac{\partial}{\partial\mathcal{K}}\mathcal{U}(\mathcal{K})^{\bullet} \text{ exists}\right\}.$$

The following four types can be derived and considered:

I. **Type I Modified FGMC**
Let $\mathcal{H}_1(\mathcal{W}) = \lambda_1 H_{\theta_1,\beta_1,\underline{\xi}}(\mathcal{W})$ and $\mathcal{H}_2(\mathcal{K}) = \lambda_2 H_{\theta_2,\beta_2,\underline{\xi}}(\mathcal{K})$. Then, the new bivariate version via modified FGMC type I can written using

$$C_{\S}(\mathcal{W},\mathcal{K}) = \mathcal{W}\mathcal{K} + \S\mathcal{O}(\mathcal{W})^{\bullet}\mathcal{U}(\mathcal{K})^{\bullet},$$

II. **Type II Modified FGMC**
Consider $\mathcal{P}(\mathcal{W};\S_1)$ and $\mathcal{G}(\mathcal{K};\S_2)$ which satisfy the above conditions where

$$\mathcal{P}(\mathcal{W};\S_1)|_{(\S_1>0)} = \mathcal{W}^{\S_1}(1-\mathcal{W})^{1-\S_1} \text{ and } \mathcal{G}(\mathcal{K};\S_2)|_{(\S_2>0)} = \mathcal{K}^{\S_2}(1-\mathcal{K})^{1-\S_2}.$$

Then, the corresponding bivariate version (modified FGMC Type II) can be derived from

$$C_{\S_0,\S_1,\S_2}(\mathcal{W},\mathcal{K}) = \mathcal{W}\mathcal{K}[1+\S_0\mathcal{P}(\mathcal{W};\S_1)\mathcal{G}(\mathcal{K};\S_2)].$$

III. **Type III Modified FGMC**
Let $\widetilde{\mathcal{P}(\mathcal{W})} = \mathcal{W}\left[log\left(1+\overline{\mathcal{W}}\right)\right]|_{(\overline{\mathcal{W}}=1-\mathcal{W})}$ and $\widetilde{\mathcal{G}(\mathcal{K})} = \mathcal{K}\left[log\left(1+\overline{\mathcal{K}}\right)\right]|_{(\overline{\mathcal{K}}=1-\mathcal{K})}$. Then, the associated CDF of the Biv-PEDE-FGM (modified FGMC type III) as

$$C_{\S}(\mathcal{W},\mathcal{K}) = \mathcal{W}\mathcal{K}\left[1+\S\widetilde{\mathcal{P}(\mathcal{W})}\widetilde{\mathcal{G}(\mathcal{K})}\right].$$

IV. **Type IV Modified FGMC**
Using the quantile concept, the CDF of the Biv-PEDE-FGM (modified FGMC type IV) model can be obtained using

$$C(\mathcal{W},\mathcal{K}) = \mathcal{W}F^{-1}(\mathcal{W}) + \mathcal{K}F^{-1}(\mathcal{K}) - F^{-1}(\mathcal{W})F^{-1}(\mathcal{K})$$

where $F^{-1}(\mathcal{W}) = Q(\mathcal{W})$and $F^{-1}(\mathcal{K}) = Q(\mathcal{K})$.

3.5 Biv-PEDE Type via AMHC

Under the "stronger Lipschitz condition", the J-CDF of the Archimedean AMHC can written as

$$C_{\S}(\mathcal{W},\mathcal{K}) = \frac{\mathcal{W}\mathcal{K}}{1-\S\overline{\mathcal{W}\mathcal{K}}}|_{\S\in(-1,1)},$$

the corresponding J-CDF of the Archimedean Ali-Mikhail-Haq copula can be express as

$$c_{\S}(\mathcal{W},\mathcal{K}) = \frac{1-\S+2\S\frac{\mathcal{W}\mathcal{K}}{1-\S\overline{\mathcal{W}\mathcal{K}}}}{\left[1-\S\overline{\mathcal{W}\mathcal{K}}\right]^2}|_{\S\in(-1,1)},$$

then for any $\overline{\mathcal{W}} = 1 - F_{\underline{\Delta}_1}(\mathcal{X}_1) = |_{[\overline{\mathcal{W}}=(1-\mathcal{W})\in(0,1)]}$ and $\overline{\mathcal{K}} = 1 - F_{\underline{\Delta}_2}(\mathcal{X}_2)|_{[\overline{\mathcal{K}}=(1-\mathcal{K})\in(0,1)]}$ we easily derive the copula $C_{\$}(\mathcal{W},\mathcal{K})$.

4. THE MAXIMUM LIKELIHOOD METHOD

Let $x_1, x_2, \dots, x_n$ be an observed random sample (RS) from the PEDE model with parameters σ, θ, β and δ. The log-likelihood function ($\ell_{\sigma,\theta,\beta,\delta}$) can be derived and maximized either directly by using the Ox program (via the "MaxBFGS sub-routine"), R (via the "optim" function) and MATH-CAD or by solving the nonlinear likelihood equations obtained by differentiating $\ell_{\sigma,\theta,\beta,\delta}$. The score vector components are given by

$$\mathrm{U}_\sigma = \frac{\partial}{\partial\sigma}\ell_{\sigma,\theta,\beta,\delta}, \mathrm{U}_\theta = \frac{\partial}{\partial\theta}\ell_{\sigma,\theta,\beta,\delta}, \mathrm{U}_\beta = \frac{\partial}{\partial\beta}\ell_{\sigma,\theta,\beta,\delta} \text{ and } \mathrm{U}_\delta = \frac{\partial}{\partial\delta}\ell_{\sigma,\theta,\beta,\delta}.$$

Setting the nonlinear system of equations $\mathrm{U}_\sigma = \mathrm{U}_\theta = \mathrm{U}_\beta = \mathrm{U}_\delta = 0$ and solving them simultaneously yields the maximum likelihood estimations (MLEs) of $\sigma, \theta, \beta, \delta$. These equations can be solved numerically using iterative methods such as the "Newton-Raphson" type algorithms. For confidence interval (C.I) estimation of the model parameters, we require the observed information matrix $\mathrm{J}(\hat{\sigma}, \hat{\theta}, \hat{\beta}, \hat{\delta})$ which can be obtained from the authors upon request.

5. SIMULATION

In this Section, we can present a two simulations studies, the first is graphical simulation and the second is numerical. First, the graphical simulation study is performed for assessing the finite sample behavior of the MLEs. The graphical assessment was based on the following algorithm:

I. Using the QF of the PEDE distribution: we generate 1000 samples of size n from the PEDE distribution where

$$Q_U = \frac{1}{\delta}\ln\left(1 - \frac{1}{\beta}\ln\left\{1 - \left[-\frac{1}{\sigma}ln(1 - U(1 - exp(-\sigma)))\right]^{\frac{1}{\theta}}\right\}\right).$$

II. Compute the MLEs for the 1000 samples.

III. Compute the SEs of the MLEs for the 1000 samples, the standard errors (SEs) were computed by inverting the observed information matrix.

IV. Compute the biases and mean squared errors given for $\underline{\Lambda} = (\sigma, \theta, \beta, \delta)$.

V. Repeat these steps for $n = 10, 20, \dots, 500$ with $\sigma = 1,2, \dots, 100$; $\theta = 1,2, \dots, 100$; $\beta = 1,2, \dots, 100$ and $\delta = 1,2, \dots, 100$ and computing the biases and mean squared errors (MSEs) for the model parameters.

Figure 5 (left panels) show how the biases vary with respect to the sample size$n$. Figure 5 (right panels) show how the MSEs vary with respect to sample size$n$. From Figure 5 the biases for each parameter are generally negative and decrease to zero as $n \rightarrow \infty$ and the MSEs for each parameter decrease to zero as $n \rightarrow \infty$.

Second, we perform a Monte Carlo numerical simulation study to verify the finite sample behavior of the MLEs numerically. All simulation results are obtained from $N =\simeq 000$ Monte Carlo replications carried out using MATHCAD V15. In each replication, a random sample of size n is drawn from $X \sim$ PEDE distribution $(\sigma, \theta, \beta, \delta)$. Clearly, the conjugate gradient method is used for maximizing the "total log-likelihood function". The random number generation of the PEDE model is performed using the inversion method via the Q_U given above. Six different combinations of initial values of the four parameters are considered in Table 3. Table 3 lists the mean square errors (MSEs) of the MLEs of the PEDE model parameters by taking sample sizes *n* = 50, 100, 200, 300 and 500. The values of the MSEs decrease when the sample size increases as expected under first-order asymptotic theory.

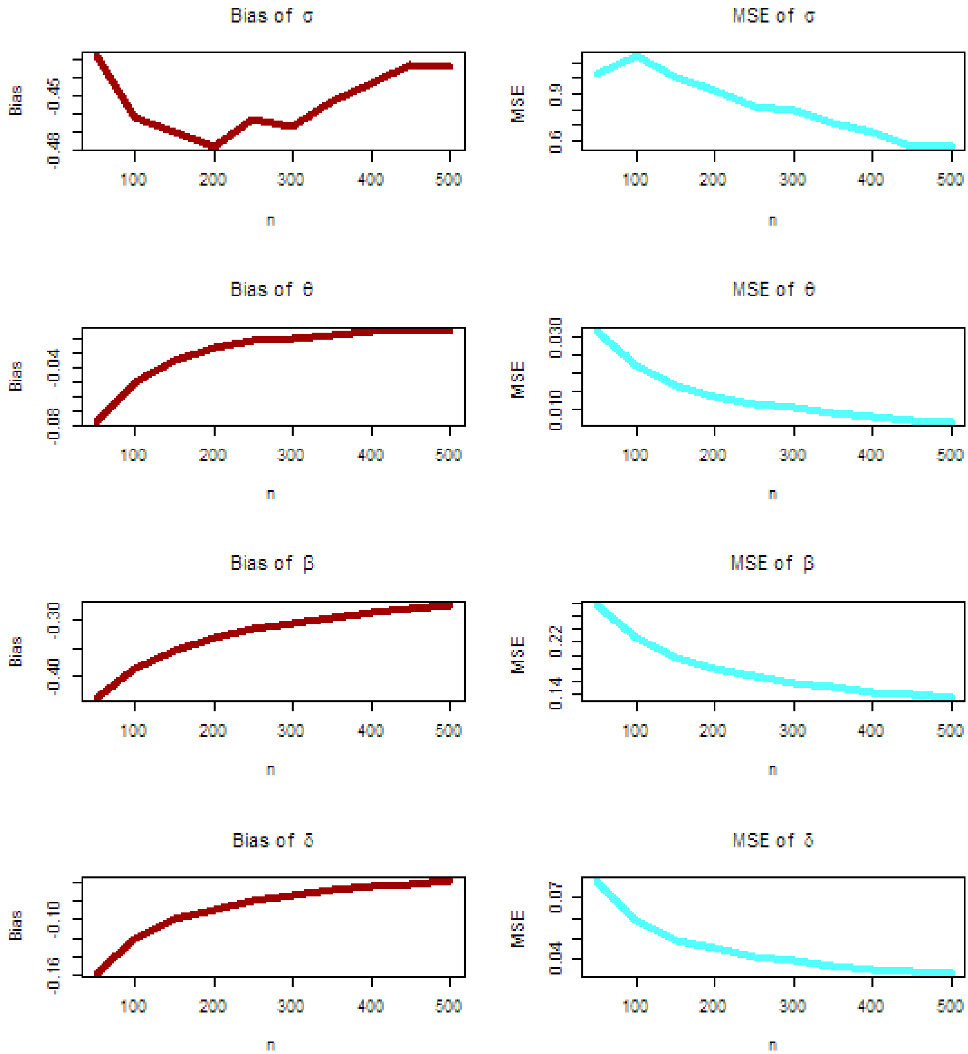

Figure 5: Biases and MSEs for the Parameter σ, θ, β and δ

Table 2
Results of the Numerical Simulation

		Λ				MSE		
n	***σ***	***β***	***θ***	***δ***	***σ***	***β***	***θ***	***δ***
50	0.6	1.5	1.2	0.5	0.252943	0.046923	0.028109	0.002539
	0.9	1.2	0.6	0.3	0.258263	0.070622	0.006463	0.002109
	0.3	0.9	1.8	0.6	0.250053	0.084786	0.017171	0.004901
	0.5	1.1	1.4	0.7	0.251712	0.021200	0.039335	0.003494
	0.8	1.2	0.9	0.3	0.234719	0.036026	0.014766	0.001013
	0.2	0.8	1.6	0.5	0.230866	0.061989	0.013785	0.002926
100	0.6	1.5	1.2	0.5	0.122645	0.022676	0.012611	0.001233
	0.9	1.2	0.6	0.3	0.125459	0.033081	0.002905	0.00101
	0.3	0.9	1.8	0.6	0.121087	0.040814	0.007689	0.002401
	0.5	1.1	1.4	0.7	0. 12200	0.010279	0.017636	0.001728
	0.8	1.2	0.9	0.3	0.132609	0.021667	0.007138	0.000623
	0.2	0.8	1.6	0.5	0.128956	0.037367	0.00661	0.001798
200	0.6	1.5	1.2	0.5	0.060885	0.010924	0.00641	0.000583
	0.9	1.2	0.6	0.3	0.059612	0.015473	0.001386	0.000462
	0.3	0.9	1.8	0.6	0.057599	0.019332	0.003656	0.00113
	0.5	1.1	1.4	0.7	0.060577	0.004981	0.008962	0.000811
	0.8	1.2	0.9	0.3	0.061358	0.010102	0.003302	0.000291
	0.2	0.8	1.6	0.5	0.059224	0.017497	0.003068	0.000843
300	0.6	1.5	1.2	0.5	0.040422	0.006845	0.004252	0.000355
	0.9	1.2	0.6	0.3	0.041277	0.010158	0.000981	0.000297
	0.3	0.9	1.8	0.6	0.039864	0.012892	0.00259	0.00074
	0.5	1.1	1.4	0.7	0.04024	0.003137	0.005943	0.000497
	0.8	1.2	0.9	0.3	0.041168	0.006648	0.002216	0.000189
	0.2	0.8	1.6	0.5	0.039772	0.011472	0.002059	0.000545
500	0.6	1.5	1.2	0.5	0.023874	0.004279	0.00239	0.000227
	0.9	1.2	0.6	0.3	0.023520	0.005494	0.000557	0.000155
	0.3	0.9	1.8	0.6	0.022670	0.007093	0.01471	0.000397
	0.5	1.1	1.4	0.7	0.023730	0.001953	0.003341	0.000317
	0.8	1.2	0.9	0.3	0.023612	0.003545	0.001353	0.000099
	0.2	0.8	1.6	0.5	0.023068	0.006090	0.00125	0.000286

6. MODELING RELIEF AND SURVIVAL TIMES

In this section, we analyze another two different real-life sets to illustrate the importance, applicability and flexibility of the proposed model. We compare the fit of the PEDE with other competitive E models. The fist data set represents the lifetime data relating to relief times (in minutes) of patients receiving an analgesic (see Gross, J. and Clark (1975)). The second data set represents the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli (see Bjerkedal (1960)). These two real data sets are recently analyzed by Goual et al. (2019) and Ibrahim et al. (2020).

In the statistical literature there are many extensions of the exponential model which can be used in comparison such as Marshall-Olkin exponential (MOE) model (see Ghitany et al. (2005)), Beta exponential (BE) model (see Lee et al. (2007)), Kumaraswamy exponential (KE) model (Cordeiro et al. (2010)), Poisson-exponential (PE) model (see Cancho et al. (2011)), Moment exponential (ME) model (Dara and Ahmad (2012)), Generalized Marshall-Olkin exponential (GMOE) model (see Chakraborty and Handique (2017)), transmuted exponentiated generalized exponential (TEGE) distribution (see Yousof et al. (2017a)), Marshall-Olkin Kumaraswamy exponential (MOKE) model (see Chakraborty and Handique (2017)), Burr XII exponential (BXIIE) distribution (see Cordeiro et al. (2018)), odd Lindley exponential (OLE) model (Almamy et al. (2018)), Burr-Hatke exponential (BHE) model (see Yousof et al. (2018b)), Kumaraswamy Marshall-Olkin exponential (KMOE) model (see George and Thobias (2019)), quasi Poisson Burr X exponentiated exponential (QPBXEE) model (see Mansour et al. (2020b)), generalized odd log-logistic exponentiated exponential (GOLLEE) model (see Mansour et al. (2020b)) and the Burr X exponential (BXE) model (see Yousof et al. (2017a)) and Mansour et al. (2020c)), among others. In this section some competitive models are selected as competitive exponential extensions such as the odd Lindley exponential (OLE) model, Marshall-Olkin exponential (MOE) model, Moment exponential (ME) model, The Logarithmic Burr-Hatke exponential (LBHE) model, Generalized Marshall-Olkin exponential (GMOE)model, Beta exponential (BE)model, Marshall-Olkin Kumaraswamy exponential (MOKE)model, Kumaraswamy exponential (KE), the Burr X exponential (BXE)model, Kumaraswamy Marshall-Olkin exponential (KMOE) model and standard exponential (E) model. Some details related to these competitive models are available in Ibrahim et al. (2020).

The following are the CDFS of the competitive models:

I. The standard exponential distribution

$$F_{\delta}(x) = 1 - exp(-\delta x) \,|\delta > 0, x > 0;$$

II. Burr type-X exponential distribution

$$F_{a,\delta}(x) = (1 - exp\{-[exp(\delta x) - 1]^2\})^a \,|a, \delta > 0, x > 0;$$

III. odd Lindley exponential distribution

$$F_{\delta}(x) = 1 - \{exp(-\delta x)\}^{-1}[1 + exp(-\delta x)] \times \frac{1}{2} exp\left(\frac{-[1 - exp(-\delta x)]}{1 - [1 - exp(-\delta x)]}\right) |\delta > 0, x > 0;$$

IV. Kumaraswamy Marshall-Olkin exponential distribution

$$F_{a,b,\lambda,\delta}(x) = 1 - \left\{1 - \left[\frac{exp(-\delta x)}{1-(1-\lambda)1-exp(-\delta x)}\right]^a\right\}^b \, |a,b,\delta > 0, x > 0;$$

V. Moment exponential distribution

$$F_{\delta}(x) = 1 - \left(1 + \frac{x}{\delta}\right) e^{-\frac{x}{\delta}}, |_{\ \delta>0, x\geq 0};$$

VI. Marshall-Olkin Kumaraswamy exponential distribution

$$F_{a,b,\lambda,\delta}(x) = \frac{\{1-[1-exp(-\delta x)]^a\}^b}{1-(1-\lambda)(1-\{1-[1-exp(-\delta x)]^a\}^b)} |a,b,\lambda,\delta > 0, x > 0;$$

VII. Burr–Hatke exponential distribution

$$F_{\delta}(x) \frac{1-exp(-\delta x)}{1-\delta x} |\delta > 0, x > 0;$$

VIII. Beta exponential distribution

$$F_{a,b,\delta}(x) = I_{1-exp(-\delta x)}(a,b) | a,b,\lambda,\delta > 0, x > 0;$$

IX. Marshall-Olkin exponential distribution

$$F_{\lambda,\delta}(x) = \frac{exp(-\delta x)}{1-(1-\lambda)[1-exp(-\delta x)]} |\lambda,\delta > 0, x > 0;$$

X. Kumaraswamy exponential distribution

$$F_{a,b,\delta}(x) = 1 - \{1-[1-exp(-\delta x)]^a\}^b | a,b,\delta > 0, x > 0;$$

XI. Generalized Marshall-Olkin exponential distribution.

$$F_{\lambda,\delta}(x) = \frac{1-[1-exp(-\delta x)]^a}{1-(1-\lambda)[1-exp(-\delta x)]^a} |a,\delta > 0, x > 0.$$

For exploring the initial shape of real data, the nonparametric Kernel density estimation "NKDE" is presented in Figure 6 and 7 (top left panels). For checking the "normality" condition, the normal quantile- quantile "Q-Q plot" is presented in Figure 6 and 7 (top right panels). For discovering the shape of the empirical HRFs, the total time in test "TTT" plot is provided in Figure 6 and 7 (bottom left panels). To explore the extreme observations, the "box plot" is sketched in Figure 6 and 7 (bottom right panels). Based on Figure 6 and 7 (top left panels), it is noted that the NKDEs are "symmetric with right skewed heavy tail". Based on Figure 6 and 7 (top right panels), we see that the "normality" could be not exists. Based on Figure 6 and 7 (bottom left panels), we note that the HRF is "asymmetric monotonically increasing HRF" for the two data sets. Based on Figure 6 and 7 (bottom right panels), we note that no extreme values were spotted.

The following goodness-of-fit (GOF) are used in comparing the competitive models:

I. Cramér-Von Mises (C^*).
II. Anderson-Darling (A^*).

III. Akaike information (AI-Cr).
IV. Consistent-AIC (CAI-Cr).
V. Bayesian-IC (BI-Cr).
VI. Hannan-Quinn-IC (HQI-Cr).
VII. Kolmogorov-Smirnov (KS).
VIII. P-value.

For relief times, relevant results are listed in Tables 2 and 3. Table 3 gives the MLEs, SEs, 95%- lower C.I (95%-L.C.I) and 95%- upper C.I (95%-U.C.I). Table 4 provides the GOF test statistics for relief times data. For survival times: the analysis results are listed in Tables 4 and 5. Table 5 gives the MLEs, SEs, 95%-L.C.I and 95%-U.C.I. Table 6 gives the GOFs test statistics for survival times data. Figures 8 and 9 give estimated HRF (EHRF), estimated PDF (EPDF), Kaplan Meier survival (KMS) plot and probability- probability (P-P) plot and for the two data set respectively. Based on Tables 3 and 5, we note that the PEDE model gives the lowest values for all test statistics where $A^{*}=0.33, C^{*}=0.056$, p-value=0.88, KS=0.13, AI-Cr=40.9, BI-Cr=44.8, CAI-Cr=43.2 and HQI-Cr=41.5 for the relief times data and $A^{*}=0.39$, $C^{*}=0.06$, p-value=0.73, KS = 0.080, AI-Cr=205.5, BI-Cr=214.6, CAI-Cr=206.3 and HQI-Cr=209.2 for the survival times data.

Table 3
MLEs, SEs 95%-L.C.I and 95%-U.C.I for Relief Times Data

Models		**MLEs, SE, 95%-L.C.I and 95%-U.C.I**			
PEDE $(\sigma,\theta,\beta,\delta)$	MLE	2.624	13.61	5.739	0.189
	SE	(2.017)	(10.05)	(3.37)	(0.07)
	95%-L.C.I	0	0	0	0.49
	95%-U.C.I	6.6	34.6	12.4	0.329
KwMOE(a,b,λ,δ)	MLE	8.868	34.826	0.299	4.899
	SE	(9.15)	(22.31)	(0.24)	(3.18)
	95%-L.C.I	0	0	0	0
	95%-U.C.I	29	78.6	0.76	11
MOKE(a,b,λ,δ)	MLE	0.133	33.232	0.571	1.669
	SE	(0.332)	(57.84)	(0.72)	(1.81)
	95%-L.C.I	0	0	0	0
	95%-U.C.I	0.8	146.6	2	5.2
GMOE(a,λ,δ)	MLE	0.519	89.462	3.169	
	SE	(0.256)	(66.278)	(0.77)	
	95%-L.C.I	0.02	0	1.66	
	95%-U.C.I	1.02	219.4	4.7	
KE(a,b,δ)	MLE	83.756	0.568	3.330	
	SE	(42.361)	(0.326)	(1.188)	
	95%-L.C.I	0.7	0	1.00	
	95%-U.C.I	167	1.2	5.75	

Models		**MLEs, SE, 95%-L.C.I and 95%-U.C.I**		
BE(a,b,δ)	MLE	81.633	0.542	3.514
	SE	(120.41)	(0.327)	(1.410)
	95%-L.C.I	0	0	0.750
	95%-U.C.I	317.63	1.18	6.344
MOE(λ,δ)	MLE	54.47	2.32	
	SE	(35.58)	(0.37)	
	95%-L.C.I	0	1.58	
	95%-U.C.I	124.2	3.00	
BXE(a,δ)	MLE	1.1635	0.3207	
	SE	(0.33)	(0.03)	
	95%-L.C.I	0.522	0.261	
	95%-U.C.I	1.825	0.384	
E(δ)	MLE	0.5261		
	SE	(0.117)		
	95%-L.C.I	0.321		
	95%-U.C.I	0.856		
ME(δ)	MLE	0.950		
	SE	(0.150)		
	95%-L.C.I	0.721		
	95%-U.C.I	1.288		
BHE (δ)	MLE	0.5263		
	SE	(0.118)		
	95%-L.C.I	0.432		
	95%-U.C.I	0.666		
OLE(δ)	MLE	0.604		
	SE	(0.054)		
	95%-L.C.I	0.501		
	95%-U.C.I	0.744		

Table 4
GOF Statistics for Relief Times Data

Models	$\boldsymbol{A^*}$	$\boldsymbol{C^*}$	**BI-Cr**	**AI-Cr**	**HQI-Cr**	**CAI-Cr**	**p-value**	**KS**
PEDE	0.33	0.056	44.8	40.9	41.5	43.2	0.88	0.13
OLE	1.30	0.22	50.1	49.1	49.3	49.3	<0.1%	0.90
E	4.60	0.96	68.7	68.3	68.0	67.9	0.004	0.42
BXE	1.32	0.24	50.1	48.1	48.5	49.2	0.17	0.25
MOE	0.80	0.14	45.5	43.5	43.9	44.2	0.55	0.18
KE	0.45	0.07	44.8	42.5	42.3	43.3	0.86	0.14
KMOE	1.08	0.19	46.8	43.4	43.6	45.6	0.86	0.15
ME	0.70	0.12	55.3	54.3	54.5	54.5	0.07	0.32
GMOE	0.51	0.08	45.7	42.8	43.3	44.3	0.78	0.15
MOKE	0.60	0.11	45.5	41.6	42.3	44.3	0.87	0.14
BHE	0.62	0.11	68.7	67.7	67.8	67.9	<0.1%	0.40
BE	0.70	0.12	44.2	43.5	46.5	44.9	0.80	0.16

Table 5
MLEs, SEs, 95%-L.C.I and 95%-U.C.I for Survival Times Data

Models		MLEs, SE, 95%-L.C.I and 95%-U.C.I			
PEDE $(\sigma, \theta, \beta, \delta)$	MLE	3.649	3.069	60.199	0.009
	SE	(2.748)	(0.748)	(37.28)	(0.003)
	95%-L.C.I	0	2.2	0	0.006
	95%-U.C.I	8.9	4.9	134	14.8
KwMOE (a,b,λ,δ)	MLE	0.37	3.48	3.31	0.30
	SE	(0.14)	(0.86)	(0.78)	(1.11)
	95%-L.C.I	0.11	1.8	1.8	0
	95%-U.C.I	0.6	5	4.8	2.5
MOKE (a,b,λ,δ)	MLE	0.008	2.716	1.986	0.099
	SE	(0.002)	(1.316)	(0.784)	(0.048)
	95%-L.C.I	0.004	0.14	0.4	0
	95%-U.C.I	0.010	5.3	3.5	0.23
GMOE(a,λ,δ)	MLE	0.179	47.635	4.470	
	SE	(0.07)	(44.901)	(1.327)	
	95%-L.C.I	0.041	0	2.1	
	95%-U.C.I	0.33	14	7.2	
BE(a,b,δ)	MLE	0.807	3.461	1.331	
	SE	(0.696)	(1.003)	(0.855)	
	95%-L.C.I	0	1.49	0	
	95%-U.C.I	2.17	5.42	3.01	
KE(a,b,δ)	MLE	3.304	1.100	1.037	
	SE	(1.106)	(0.764)	(0.614)	
	95%-L.C.I	1.13	0	0	
	95%-U.C.I	5.5	2.6	2.2	
MOE(a,δ)	MLE	8.780	1.380		
	SE	(3.555)	(0.193)		
	95%-L.C.I	1.81	1.0		
	95%-U.C.I	15.74	1.80		
BXE(a,δ)	MLE	0.480	0.2060		
	SE	(0.061)	(0.012)		
	95%-L.C.I	0.4	0.18		
	95%-U.C.I	0.5	0.23		
E(δ)	MLE	0.540			
	SE	(0.063)			
	95%-L.C.I	0.4			
	95%-U.C.I	0.7			
OLE(δ)	MLE	0.38145			
	SE	(0.021)			
	95%-L.C.I	0.3			
	95%-U.C.I	0.4			
ME(δ)	MLE	0.9250			
	SE	(0.080)			
	95%-L.C.I	0.62			
	95%-U.C.I	1.08			
BHE (δ)	MLE	0.542			
	SE	(0.06)			
	95%-L.C.I	0.41			
	95%-U.C.I	0.68			

Table 6
GOF Statistics for Survival Times Data

Models	A^*	C^*	BI-Cr	AI-Cr	HQI-Cr	CAI-Cr	p-value	KS
PEDE	0.39	0.06	214.6	205.5	209.2	206.3	0.73	0.080
KE	0.74	0.11	216.2	209.4	212.1	209.8	0.500	0.09
E	6.53	1.25	236.9	234.6	235.5	234.7	0.060	0.27
BXE	2.90	0.52	239.9	235.3	237.1	235.5	0.002	0.22
GMOE	1.02	0.16	217.4	210.5	213.2	211.2	0.510	0.09
OLE	1.94	0.33	231.4	229.1	230.1	229.2	<0.1%	0.49
KMOE	0.61	0.11	217.5	208.3	211.4	208.4	(0.530	0.09
MOKE	0.79	0.12	218.6	209.4	213.3	210.2	0.440	0.10
BHE	0.71	0.12	237.2	235.4	236.6	235.2	<0.1%	0.28
BE	0.98	0.15	214.2	207.3	210.1	207.7	0.340	0.11
MOE	1.20	0.17	215.0	210.2	212.2	210.5	0.430	0.10
ME	1.52	0.25	212.7	210.4	211.3	210.5	0.130	0.14

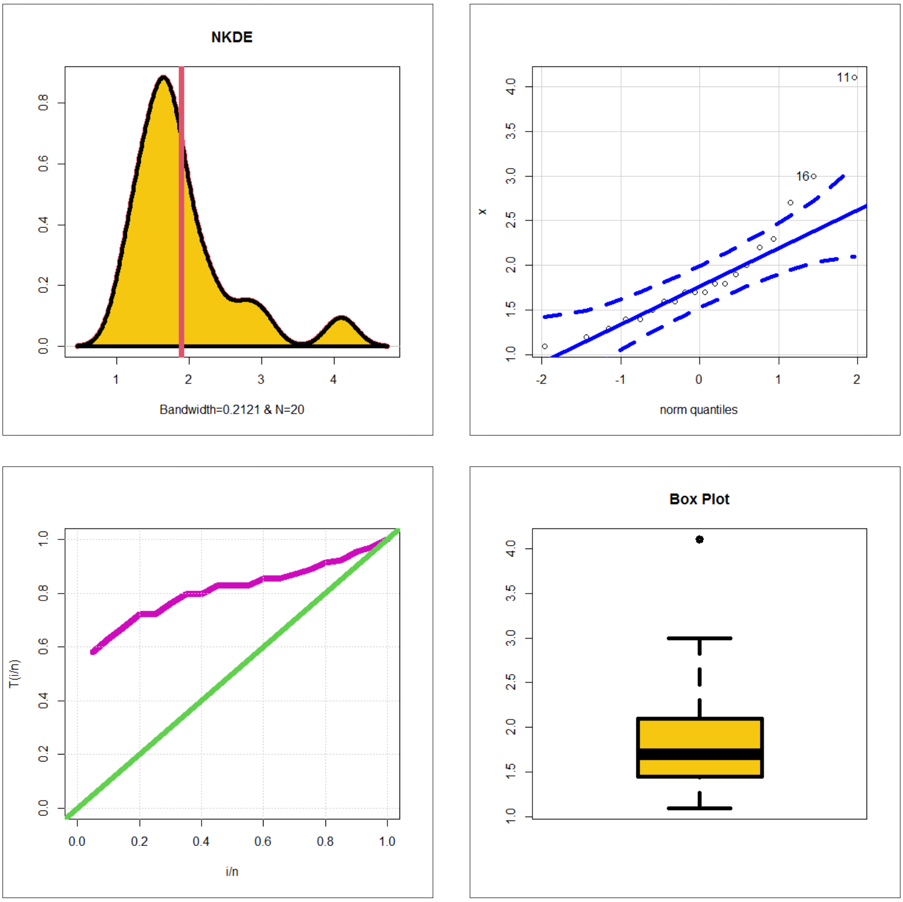

Figure 6: NKDE, Q-Q Plot TTT Plot and Box Plot for Relief Times Data

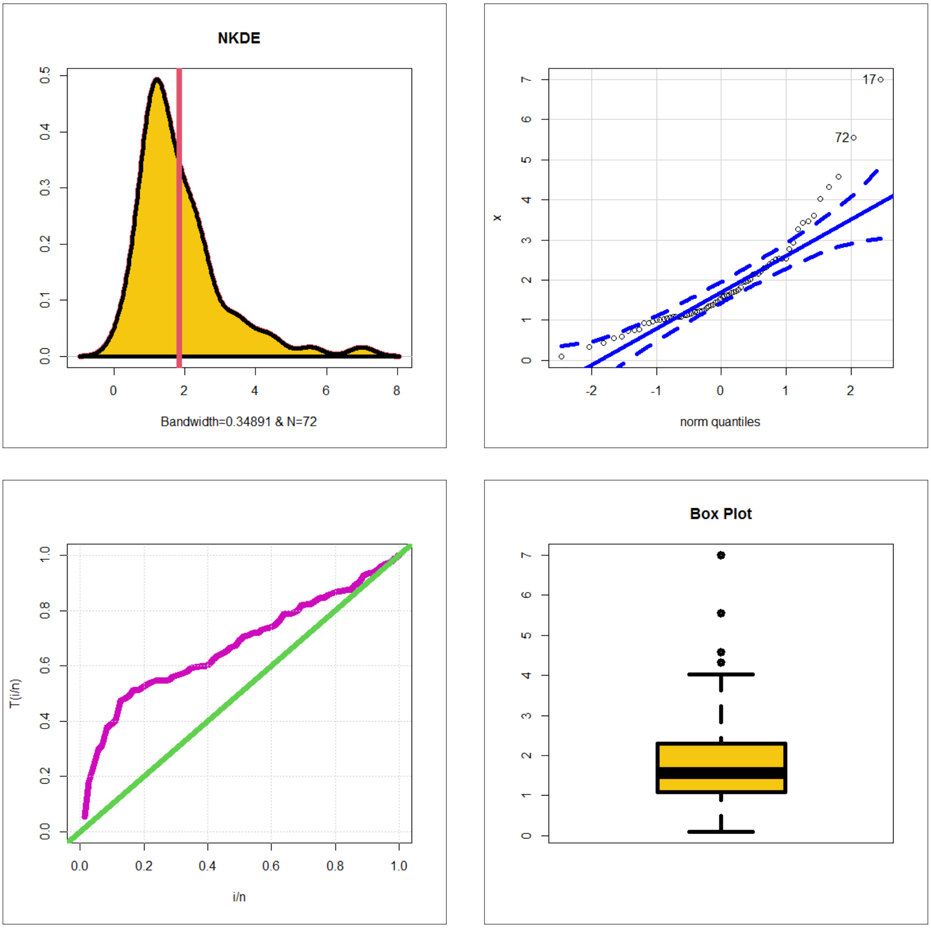

Figure 7: NKDE, Q-Q Plot TTT Plot and Box Plot for Survival Times Data

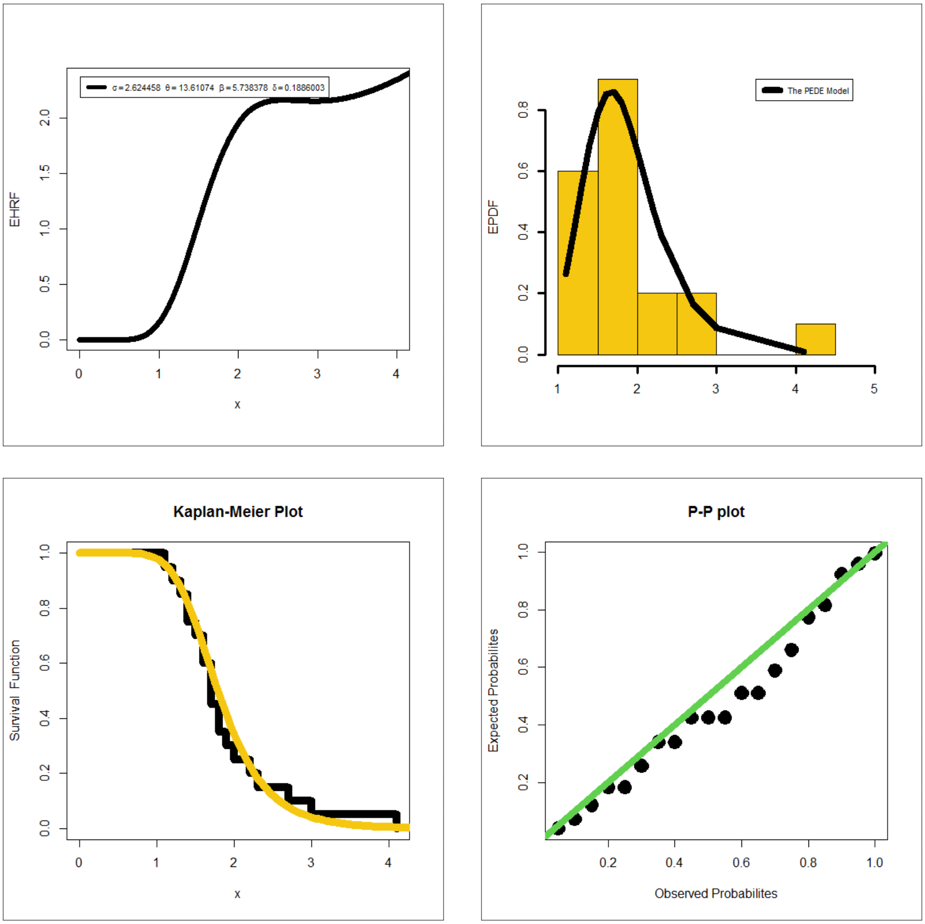

Figure 8: EHRF Plot, EPDF Plot, KMS Plot P-P Plot for Relief Times Data

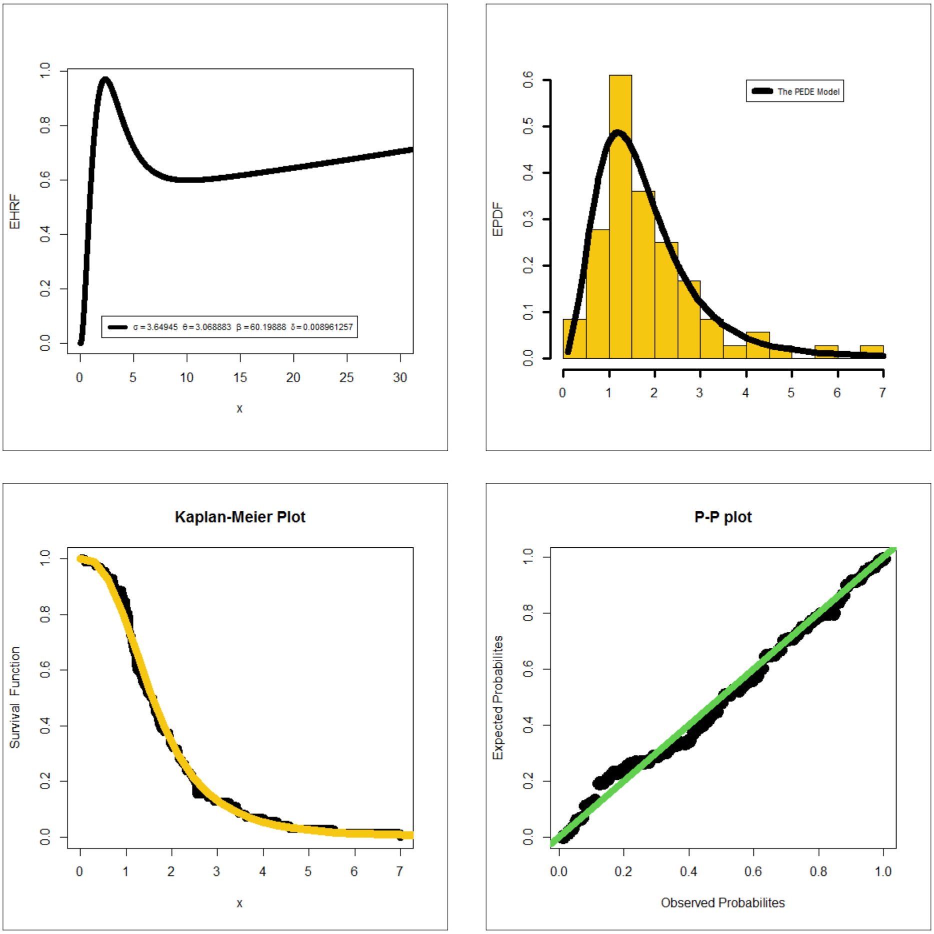

Figure 9: EHRF Plot, EPDF Plot, KMS Plot P-P Plot for Survival Times Data

7. CONCLUSIONS

In this work, a new compound lifetime model called the Poisson exponentiated double exponential distribution is defined and studied. The new density can be "asymmetric right skewed shape", asymmetric left skewed shape", "symmetric shape" and bimodal". The new corresponding failure rate can be "bathtub (U-shape)", "monotonically decreasing", "upside down increasing", "J-shape" and "bathtub- bathtub (W-shape)".Relevant statistical properties such as ordinary raw moments, mean deviation, incomplete moments and moment generating function are derived and analyzed. We performed a graphical simulation study to assess the finite sample behavior of the estimators. Finally, two real life applications are analyzed to illustrate the importance of the new model. For the all real data sets, the Kernel density estimation is presented for exploring the "initial density shape" non parametrically, the "Quantile-Quantile plot" is presented for checking the "normality" condition, the “total time in test” plot is provided for discovering the shape of the empirical failure rates, the “box plot” is sketched for exploring the extreme observations. As a future works, many new useful goodness-of-fit statistic tests for right censored validation such as the Nikulin-Rao-Robson goodness-of-fit statistic test, modified Nikulin-Rao-Robson goodness-of-fit statistic test, Bagdonavicius-Nikulin goodness-of-fit statistic test, modified Bagdonavicius-Nikulin goodness-of-fit statistic test as performed by Ibrahim et al. (2019), Goual et al. (2019, 2020), Mansour et al. (2020d,e,f), Yadav et al. (2020), Yousof et al. (2021a,b), and Goual and Yousof (2020), among others. Characterization results and regression modeling can be derived based on OBEE model (see Altun et al. (2018a,b,c,d) for more details). Saber and Yousof (2021), a stress-strength reliability estimation for the Poisson exponentiated double exponential may be also introduced. Following Yousof et al. (2019), regression modeling and characterizations may be presented.

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APPENDIX

Theorem 1:

Let Z_{ς} be a RV having the exp-E distribution (ς, δ), then the r^{th} moment of Z, say $\mu'_{r,Z}$, follows from

$$\mu'_{r,Z}|_{r>-1} = \Gamma(r+1)\sum_{\hbar=0}^{+\infty} \Delta_{\hbar}^{(r,\varsigma)},$$

where

$$\Delta_{\hbar}^{(r,\varsigma)} = \frac{\varsigma}{\delta^{r}}(-1)^{\hbar}\binom{\varsigma-1}{\hbar}(\hbar+1)^{r+1}.$$

Theorem 2:

Let Z_{ς} be a RV having the exp-E distribution (ς, δ), then the r^{th} incomplete moment of Z, say $\mathrm{I}_{r,Z}(t)$, follows from

$$\mathrm{I}_{r,Z}(t)|_{r>-1} = \gamma(r+1, \delta t)\sum_{\hbar=0}^{+\infty} \Delta_{\hbar}^{(r,\varsigma)},$$

where $\gamma(1+n, t)$ refers to lower incomplete gamma function. Therefore, the first incomplete moment of Z_{ς} can then be expressed as

$$\mathrm{I}_{1,Z}(t)|_{s>-1} = \gamma(2, \delta t)\sum_{\hbar=0}^{+\infty} \Delta_{\hbar}^{(r,\varphi)}.$$