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ON THE PARAMETERS IDENTIFICATION PROBLEM FOR THE INVERTED POWER TOPP-LEONE DISTRIBUTION: A THEORETICAL AND PRACTICAL STUDY

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ABSTRACT

In this paper, for the first time, the impact of identification problem on probability distribution parameters is studied, effects of the identification problem are investigated in a new empirical distribution naming the inverted power Topp-Leone (IPTL) distribution, some mathematical properties are derived for the identified distribution. A simulation study is performed to study the behavior of identified estimators using the maximum likelihood method (MLE) and to illustrate the impact of ignoring the identification problem, a real data set is applied to investigate the identified distribution flexibility and a practical comparison between the identified and the non-identified IPTL distribution is performed.

KEYWORDS

The Topp-Leone distribution, parameters identification problem, order statistics, moments, maximum likelihood estimation.

1. INTRODUCTION

Topp and Leone (1955) proposed the bounded Topp-Leone (*TL*) distribution for automatic calculating machine failure and empirical data with J-shaped histogram as powered band tool failures. The Topp-Leone distribution has been studied by Nadarajah and Kotz (2003), Ghitany *et al.* (2005), Zhou *et al.* (2006), van Dorp and Kotz (2006), Kotz and Seier (2007), Nadarajah (2009), and Genç (2012).

The cumulative distribution function (*CDF*) and probability density function (*PDF*) of the *TL* distribution, Nadarajah and Kotz (2003), are

$$\underset{TL}{F}(y)=\left[y\left(2-y\right)\right]^{\alpha};0<y<1;\alpha>0, \tag{1}$$

and

$$\underset{TL}{f}(y)=2\alpha y^{\alpha-1}\left(2-y\right)^{\alpha-1}\left(1-y\right). \tag{2}$$

Generally, when parameter values cannot be determined or known perfectly, even if the true distribution $f(x;\cdot)$ is known, it is defined as the identification problem and $f(x;\cdot)$ is called a non-identified distribution. Obviously, any nested distribution by a non-identified distribution is non-identified, on the other hand, a parametric distribution is assumed to be identified if all its parameters values are identified, so that imposing constraints on the parameters can solve several problems, such constraints are said to be identifying.

The main goal of this manuscript is to present the new *IPTL* distribution and to study the impact of parameters identification problem on the new *IPTL* distribution, also it objects to impose constraints on parameters to solve that problem.

This manuscript is constructed as follows: In section 1, the introduction of the paper is presented. In section 2, the *IPTL* distribution will be proposed, its special cases are presented and its asymptotes are given. In section 3, some properties are obtained. In section 4, the Hazard function is obtained. In section 5, the Rényi entropy is obtained. In section 6, the stress strength model is proposed. In section 7, order statistics are presented. In Section 8, the distribution parameters are estimated by the *MLE* method. In Section 9, a simulation study is presented. Finally, in Section 10, an application is used to show the features of the identified distribution.

2. THE NEW IDENTIFIED *IPTL* DISTRIBUTION

In this section, the *IPTL* distribution will be presented as follows: setting $x = \frac{1}{y^{\frac{1}{\beta}}} - 1,$

then substituting it into (1) and using the complement gives

$$F(x) = 1 - (x+1)^{-\alpha\beta}\left[2 - (x+1)^{-\beta}\right]^{\alpha}; 0 < x < \infty; \alpha, \beta > 0.$$

One can see that when $\alpha\,\beta = 1$ the *IPTL* distribution will be non-identified, on the other hand, to avoid identification problem, in the *IPTL* distribution, the joint product of $\alpha\,\beta$ must be constrained in the following *CDF* of the identified *IPTL* distribution

$$\underset{IPTL}{F(x)} = 1 - (x+1)^{-\alpha\beta}\left[2 - (x+1)^{-\beta}\right]^{\alpha}; 0 < x < \infty; \alpha, \beta > 0; \alpha\beta \neq 1, \tag{3}$$

differentiating (3), w.r.t. x, leads to the following *PDF* of the identified *IPTL* distribution

$$\underset{PTL}{f(x)} = 2\alpha\beta(x+1)^{-\alpha\beta-1}\left[1 - (x+1)^{-\beta}\right]\left[2 - (x+1)^{-\beta}\right]^{\alpha-1}, \tag{4}$$

when $\beta = 1$, the *IPTL* distribution reduces to the new inverted Topp-Leone (*ITL)* distribution, presented for the first time, some density functions shapes for the identified *IPTL* distribution are indicated in Figure 1.

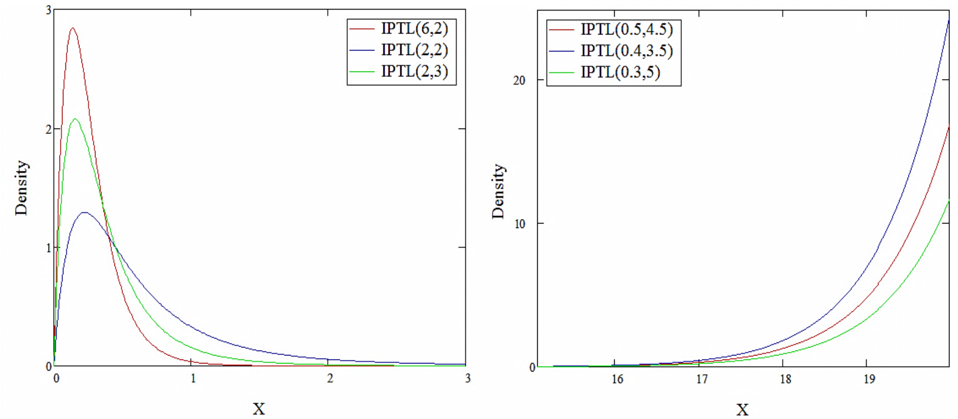

Figure 1: Some Identified *IPTL* Density Functions

2.1 Expansions for the *CDF* and *PDF*

In this section, expansions for the *CDF* and *PDF* of the *IPTL* distribution will be obtained

2.1.1 An Expansion for the *CDF*

Since,

$$(2-z)^c = \sum_{j=0}^{\infty} (-1)^j \, 2^{c-j} \binom{c}{j} z^j, \tag{5}$$

then, using (5) into (3) gives

$$\underset{IPTL}{F}(x) = 1 - \sum_{i=0}^{\infty} w_i \,(x+1)^{-\alpha\beta - i\beta},$$

(6)

where,

$$w_i = (-1)^i \, 2^{\alpha - i} \binom{\alpha}{i}.$$

2.1.2 An Expansion for the *PDF*

Differentiating (6) with respect to x gives

$$\underset{PTL}{f}(x) = \sum_{i=1}^{\infty} w_i \,(\alpha + i)\beta (x+1)^{-\alpha\beta - i\beta - 1},$$

shifting i leads to

$$\underset{PTL}{f}(x) = \sum_{i=0}^{\infty} w_{i+1} (\alpha + i + 1)\beta (x+1)^{-\alpha\beta - \beta(i+1) - 1},$$

where,

$$w_{i+1} = (-1)^{i+1}\, 2^{\alpha-i-1} \binom{\alpha}{i+1},$$

then,

$$\underset{IPTL}{f}(x) = \sum_{i=0}^{\infty} m_i\, \beta \left(x+1\right)^{-\alpha\beta - i\beta - \beta - 1}, \tag{7}$$

where,

$$m_i = (\alpha + i + 1)(-1)^{i+1}\, 2^{\alpha-i-1} \binom{\alpha}{i+1}.$$

The Expansion Condition for the *PDF*

Since,

$$\sum_{i=0}^{\infty} m_i\, \beta \int_0^{\infty} \left(x+1\right)^{-\alpha\beta - i\beta - \beta - 1} dx = 1,$$

based on the following integration, Gradshteyn and Ryzhik (2000),

$$\int_0^{\infty} x^{s-1} (1+x)^{-a} dx = B\left(s, a-s\right), \tag{8}$$

using (8) gives

$$\sum_{i=0}^{\infty} m_i \beta\, B\left(1, \alpha\beta + i\beta + \beta\right) = 1, \tag{9}$$

where $B(.,.)$ is the beta function.

2.2 The Asymptotes of the *CDF* and *PDF*

In this section, the *CDF* and *PDF* asymptotes of the *IPTL* distribution will be given.

2.2.1 The *CDF* Asymptotes

First: as x converges to zero

Since,

$$1 - \underset{IPTL}{F}(x) = \left(x+1\right)^{-\alpha\beta} \left[2 - \left(x+1\right)^{-\beta}\right]^{\alpha},$$

then, using binominal expansion first and second terms yields

$$1 - \underset{IPTL}{F}(x) \sim \left(x+1\right)^{-\alpha\beta} \left[2 - \alpha\left(x+1\right)^{-\beta}\right],$$

since,

$$\lim_{x\to 0} (x+1)^{-\alpha\beta} = 1,$$

then,

$$1-\underset{IPTL}{F(x)} \sim \left[2-\alpha(x+1)^{-\beta}\right].$$

Second: as x converges to ∞

Since,

$$\lim_{x\to\infty} \left[2-(x+1)^{-\beta}\right]^{\alpha} = 2^{\alpha},$$

then,

$$1-\underset{IPTL}{F(x)} \sim (x+1)^{-\alpha\beta} 2^{\alpha}.$$

2.2.2 The *PDF* Asymptotes

First: as x converges to zero

Since, $\lim_{x\to 0} (x+1)^{-\alpha\beta-1} = 1,$ and using binomial expansions first and second terms in (4) gives

$$\underset{IPTL}{f(x)} \sim 2\alpha\beta\left[1-(1+x)^{-\beta}\right]\left[2-(\alpha-1)(1+x)^{-\beta}\right].$$

Second: as x converges to ∞

Since,

$$\lim_{x\to\infty}\left(1-(x+1)^{-\beta}\right)=1,\ \lim_{x\to\infty}\left(2-(x+1)^{-\beta}\right)^{\alpha-1} = 2^{\alpha-1},$$

then,

$$\underset{IPTL}{f(x)} \sim 2\alpha\beta(x+1)^{-\alpha\beta-1} 2^{\alpha-1}.$$

3. SOME PROPERTIES OF THE *IPTL* DISTRIBUTION

In this section: the *IPTL* distribution some properties will be obtained as follows:

3.1 The r-th Moment

Basically, the continuous random variable $X's$ *r-th* moment, Johnson *et al*. (1995), is given by $E\left(X^{r}\right)=\int_{x} x^{r} f(x) dx,$ substituting (7) into last equation gives

$$E\left(X^r\right)=\sum_{i=0}^{\infty} m_i\,\beta\int_0^{\infty} x^r\left(x+1\right)^{-\alpha\beta-i\beta-\beta-1}dx,$$

then, using (8) in last equation yields

$$E\left(X^r\right)=\sum_{i=0}^{\infty} m_i\,\beta\,B\left(r+1,\alpha\beta+i\beta+\beta-r\right),$$

one can see that, setting $r=0$ leads to

$$E\left(X^0\right)=\sum_{i=0}^{\infty} m_i\,\beta\,B\left(1,\alpha\beta+i\beta+\beta\right),$$

substituting (9) into last equation gives

$$E\left(X^0\right)=1.$$

Different values of mean, variance, coefficient of variation, skewness, and kurtosis of the *IPTL* distribution, numerically, can be calculated for α and β in Table 1 for the non-identified case and in Table 2 for the identified case.

Table 1
The Non-Identified *IPTL* Distribution Mean, Variance, Coefficient of Variation, Skewness and Kurtosis

Measure	***β=0.1, α=10***	***β =0.3, α=3.33***	***β =0.5, α=2***	***β =0.75, α=1.33***	***β =0.9, α=1.11***	***β =1.5, α=0.66***	***β =2, α=0.5***
Mean	*0.019*	*0.057*	*0.086*	*0.112*	*0.125*	*0.162*	*0.186*
Variance	0.0004	0.0024	0.0048	0.0078	0.009	0.016	0.021
Coefficient of variation	1.062	0.861	0.812	0.79	0.783	0.777	0.776
Skewness	0.83	0.407	0.295	0.241	0.223	0.209	0.208
Kurtosis	-0.743	-1.315	-1.398	-1.428	-1.436	-1.443	-1.443

Table 2
The Identified *IPTL* Distribution Mean, Variance, Coefficient of Variation, Skewness and Kurtosis

Measure	***β=4, α=10***	***β=4, α=3.33***	***β=4, α=2***	***β=4, α=1.33***	***β=4, α=1.11***	***β=4, α=0.66***	***β=4, α=0.5***
Mean	*5.673*	*2.746*	*1.855*	*1.325*	*1.127*	*0.692*	*0.52*
Variance	7.955	1.507	0.783	0.449	0.353	0.166	0.109
Coefficient of variation	0.497	0.447	0.477	0.506	0.527	0.589	0.634
Skewness	-1.318	-0.658	-0.561	-0.468	-0.406	-0.231	-0.112
Kurtosis	1.441	-0.835	-0.998	-1.127	-1.205	-1.362	-1.427

From these two tables: As β increases with fixed α, mean, variance and kurtosis increase but skewness decreases. Also, as α increases with fixed β, mean, variance and kurtosis increase but skewness decreases. An impact of identification problem appears in these two tables, as one can see that, at the same α, the identified distribution coefficient of variation is smaller than the non-identified distribution coefficient of variation.

3.2 Moment Generating Function

Generally, the continuous random variable $X's$ moment generating function (*MGF*) is given by

$$M_x(t) = E\left(e^{tx}\right) = \int_x e^{tx} f\left(x\right) dx,$$

a first representation, can be given by substituting (7) into last equation, gives

$$M_x(t) = \beta \sum_{i=0}^{\infty} m_i \int_0^{\infty} e^{tx} \left(x+1\right)^{-\alpha\beta - i\beta - \beta - 1} dx,$$

then, based on the integration, Gradshteyn and Ryzhik (2000),

$$\int_0^{\infty} e^{tx} (1+x)^{-a-1} dx = t^a e^t \, \Gamma\left(t; a\right), \tag{10}$$

using (10) gives

$$M_x(t) = \beta \sum_{i=0}^{\infty} m_i \, t^{\alpha\beta + i\beta + \beta} e^t \, \Gamma\left(t; \alpha\beta + i\beta + \beta\right),$$

where, $\Gamma\left(t; a\right)$ is the incomplete gamma function.

A second representation for *MGF*, based on exponential expansion, can be obtained as follows:

Since,

$$M_x(t) = E\left(e^{tx}\right),$$

using exponential expansion, in last equation, gives

$$M_x(t) = E\left(\sum_{k=0}^{\infty} \frac{(tx)^k}{k!}\right),$$

then,

$$M_x(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} E\left(x^k\right).$$

3.3 The Quantile Function and the Median

The *100 u-th* well-known definition is

$$u = P(X \le x_u) = F(x_u)\ ;\ \ x_u > 0,\ 0 < u < 1,$$

equating (3) to u gives

$$1 - u = \left(x+1\right)^{-\alpha\beta}\left[2 - \left(x+1\right)^{-\beta}\right]^{\alpha},$$

obviously, the last equation is a nonlinear function w.r.t. x and needs to be solved numerically.

3.4 The Mean Deviation

Generally, the random variable $X's$, respectively, mean deviation about mean and about median can be given by

$$S_1\left(x\right) = \int_x \left|x - \mu\right|\ f\left(x\right)dx \ \text{ and } \ S_2\left(x\right) = \int_x \left|x - M\right| f\left(x\right)dx,$$

it can be given by, Ali Ahmed (2019), Ali Ahmed (2020), the proof is included in appendix (I),

$$S_1\left(x\right) = 2\mu F\left(\mu\right) - 2t\left(\mu\right) \ \text{ and } \ S_2\left(x\right) = \mu - 2t\left(M\right),$$

where $T\left(q\right) = \int_{-\infty}^{q} x\ f\left(x\right)dx$ is the linear incomplete moment.

Substituting (7) into $T(.)$ gives

$$T\left(q\right) = \sum_{i=0}^{\infty} m_i\ \beta \int_0^q x\left(x+1\right)^{-\alpha\beta - i\beta - \beta - 1}dx,$$

using (8) in last equation yields

$$T\left(q\right) = \sum_{i=0}^{\infty} m_i\ \beta\, B\left(q; 2, \alpha\beta + i\beta + \beta - 1\right),$$

where $B\left(.;.,.\right)$ is the incomplete beta function.

3.5 The Mode

The natural logarithm of (4) is

$$\log \underset{IPTL}{f(x)} = \log\left(2\alpha\beta\right) - \left(\alpha\beta + 1\right)\log\left(x+1\right)$$
$$+ \log\left[1 - \left(x+1\right)^{-\beta}\right] + \left(\alpha - 1\right)\log\left[2 - \left(x+1\right)^{-\beta}\right],$$

differentiating the last equation, w.r.t. x, and equating it to zero gives

$$-\frac{\left(\alpha\beta + 1\right)}{x+1} + \frac{\beta\left(x+1\right)^{-\beta-1}}{1 - \left(x+1\right)^{-\beta}} + \frac{\left(\alpha - 1\right)\beta\left(x+1\right)^{-\beta-1}}{2 - \left(x+1\right)^{-\beta}} = 0.$$

The last equation is a nonlinear equation and needs to be solved numerically w.r.t. x, if x_0 is a root then it must be $f''\left[\log(x_0)\right]<0.$

4. THE *IPTL* DISTRIBUTION HAZARD FUNCTION

Basically, the random variable $X's$ survival function, Meeker and Escobar (1998), can be given by

$$S(x)=1-F(x),$$

substituting (3) into last equation yields

$$\underset{IPTL}{S(x)}=(x+1)^{-\alpha\beta}\left[2-(x+1)^{-\beta}\right]^{\alpha};0<x<\infty;\alpha>0,\beta>0;\alpha\beta\neq 1. \tag{11}$$

On the other hand, the Hazard function can be given by, Meeker and Escobar (1998),

$$H(x)=\frac{f(x)}{S(x)},$$

substituting (4) and (11) into last equation gives

$$\underset{IPTL}{H(x)}=\frac{2\alpha\beta(x+1)^{-\alpha\beta-1}\left[1-(x+1)^{-\beta}\right]\left[2-(x+1)^{-\beta}\right]^{\alpha-1}}{(x+1)^{-\alpha\beta}\left[2-(x+1)^{-\beta}\right]^{\alpha}},$$

then,

$$\underset{IPTL}{H(x)}=\frac{2\alpha\beta\left[1-(x+1)^{-\beta}\right]}{(x+1)\left[2-(x+1)^{-\beta}\right]}.$$

Some Hazard function shapes for the identified *IPTL* distribution are indicated in Figure 2.

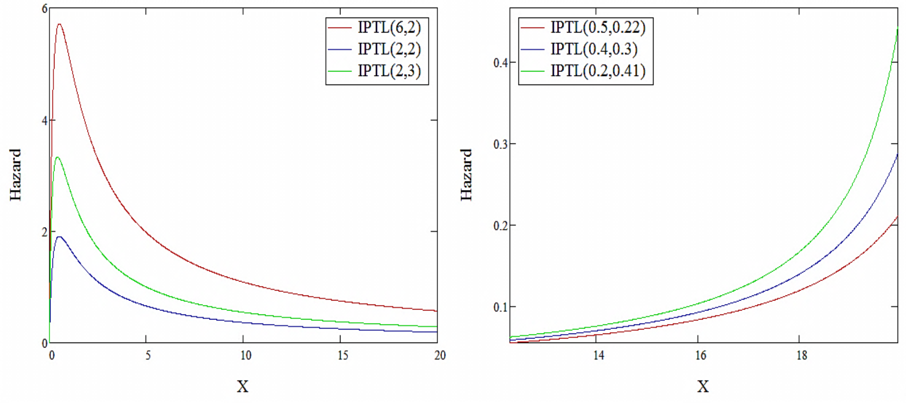

Figure 2: The Identified *IPTL* Hazard Functions

One can see, in Figure 2, two types of Hazard functions curves of the *IPTL* distribution are described as follows: An increasing then decreasing Hazard curve and an increasing Hazard curve.

5. THE *IPTL* DISTRIBUTION RÉNYI ENTROPY

The random variable $X's$ Rényi entropy is given by, Meeker and Escobar (1998),

$$e_R(\rho)=\frac{1}{1-\rho}\log\left[\int_x\left[f(x)\right]^{\rho}dx\right],$$

substituting (7) into last equation gives

$$\underset{IPTL}{e_R}(\rho)=\frac{1}{1-\rho}\log\left\{\beta^{\rho}\int_0^{\infty}(x+1)^{(-\alpha\beta-\beta-1)\rho}\left[\sum_{i=0}^{\infty}m_i(x+1)^{-i\beta}\right]^{\rho}dx\right\},$$

since, $\left[\sum_{i=0}^{\infty}m_i(x+1)^{-i\beta}\right]^{\rho}=\sum_{i=0}^{\infty}n_i(x+1)^{-i\beta}$, Gradshteyn and Ryzhik (2000),

where $n_0=m_0^{\rho}$, $n_t=\frac{1}{t\,m_0}\sum_{i=1}^{t}(i\rho-t+i)m_i\,n_{t-i};t\geq 1,$

then,

$$\underset{IPTL}{e_R}(\rho)=\frac{1}{1-\rho}\log\left\{\beta^{\rho}\sum_{i=0}^{\infty}n_i\int_0^{\infty}(x+1)^{(-\alpha\beta-\beta-1)\rho-i\beta}dx\right\},$$

hence,

$$\underset{IPTL}{e_R}(\rho)=\frac{1}{1-\rho}\log\left\{\beta^{\rho}\sum_{i=0}^{\infty}\frac{n_i}{(\alpha\beta+\beta+1)\rho+i\,\beta-1}\right\}.$$

6. RELIABILITY: THE *IPTL* DISTRIBUTION STRESS STRENGTH MODEL

Basically, the random variable $X's$ stress strength model can be given by, Meeker and Escobar (1998),

$$R=\int_x f_1(x;\lambda_1)F_2(x;\lambda_2)dx,$$

substituting (6) and (7) into last equation, β is common parameter, leads to

$$R=\beta\int_0^{\infty}\sum_{i=0}^{\infty}m_i(x+1)^{-\alpha_1\beta-i\beta-\beta-1}\left[1-\sum_{i=0}^{\infty}w_i(x+1)^{-\alpha_2\beta-i\beta}\right]dx,$$

then,

$$R = I_1 - I_2,$$

moreover,

$$I_1 = \beta \int_0^{\infty} \sum_{i=0}^{\infty} m_i \left(x+1\right)^{-\alpha_1\beta - i\beta - \beta - 1} dx = 1,$$

furthermore,

$$I_2 = \beta \int_0^{\infty} \left(x+1\right)^{-\alpha_1\beta - \alpha_2\beta - \beta - 1} \left[\sum_{i=0}^{\infty} m_i \left(x+1\right)^{-i\beta} \right] \left[\sum_{i=0}^{\infty} w_i \left(x+1\right)^{-i\beta} \right] dx,$$

since, $\left[\sum_{i=0}^{\infty} m_i \left(x+1\right)^{-i\beta} \right] \left[\sum_{i=0}^{\infty} w_i \left(x+1\right)^{-i\beta} \right] = \sum_{i=0}^{\infty} p_i \left(x+1\right)^{-i\beta}$, Gradshteyn and Ryzhik (2000),

where $p_u = \sum_{i=0}^{u} m_i \, w_{u-i}$,

then,

$$I_2 = \beta \sum_{i=0}^{\infty} p_i \int_0^{\infty} \left(x+1\right)^{-i\beta - \alpha_1\beta - \alpha_2\beta - \beta - 1} dx,$$

hence,

$$I_2 = \sum_{i=0}^{\infty} \frac{p_i}{i + \alpha_1 + \alpha_2 + 1}.$$

7. THE *IPTL* DISTRIBUTION ORDER STATISTICS

The *u-th* order statistics density function $f\left(x_{u:v}\right)$ for $u = 1,2,\dots,v$ from *iid* random variables $X_1, X_2, \dots, X_v$ following the *IPTL* distribution is given by, Arnold *et al.*(1992),

$$f\left(x_{u:v}\right) = \frac{f\left(x_u\right)}{B\left(u, v-u+1\right)} F\left(x_u\right)^{u-1} \left\{1 - F\left(x_u\right)\right\}^{v-u},$$

applying binomial expansion in last equation leads to

$$f\left(x_{u:v}\right) = \frac{f\left(x_u\right)}{B\left(u, v-u+1\right)} \sum_{j=o}^{v-u} (-1)^j \binom{v-u}{j} \left[F\left(x_u\right)\right]^{u+j-1}, \tag{12}$$

substituting (6) and (7) into (12) leads to

$$f\left(x_{u:v}\right)=\frac{\beta\sum_{j=o}^{v-u}\left(-1\right)^{j}\binom{v-u}{j}}{B\left(u,v-u+1\right)}\sum_{i=0}^{\infty}m_{i}\ \left(x_{u}+1\right)^{-\alpha\beta-i\beta-\beta-1}$$
$$\left[1-\sum_{i=0}^{\infty}w_{i}\left(x_{u}+1\right)^{-\alpha\beta-i\beta}\right]^{u+j-1},$$

applying binomial expansion in last equation gives

$$f\left(x_{u:v}\right)=\frac{\beta\sum_{j=o}^{v-u}\left(-1\right)^{j}\binom{v-u}{j}\sum_{k=0}^{u+j-1}\left(-1\right)^{k}\binom{u+j-1}{k}}{B\left(u,v-u+1\right)}$$
$$\sum_{i=0}^{\infty}m_{i}\ \left(x_{u}+1\right)^{-\alpha\beta-i\beta-\beta-1-\alpha\beta k}\left[\sum_{i=0}^{\infty}w_{i}\ \left(x_{u}+1\right)^{-i\beta}\right]^{k},$$

since, $\left[\sum_{i=0}^{\infty}w_{i}\left(x_{u}+1\right)^{-i\beta}\right]^{k}=\sum_{i=0}^{\infty}q_{i}\left(x_{u}+1\right)^{-i\beta}$, Gradshteyn and Ryzhik (2000),

where $q_{0}=w_{0}^{k}$, $q_{t}=\frac{1}{t\,w_{0}}\sum_{i=1}^{t}\left(ik-t+i\right)w_{i}\,q_{t-i};t\geq 1$,

then,

$$f\left(x_{u:v}\right)=\frac{\beta\sum_{j=o}^{v-u}\left(-1\right)^{j}\binom{v-u}{j}\sum_{k=0}^{u+j-1}\left(-1\right)^{k}\binom{u+j-1}{k}}{B\left(u,v-u+1\right)}$$
$$\left(x_{u}+1\right)^{-\alpha\beta-\beta-1-\alpha\beta k}\left[\sum_{i=0}^{\infty}m_{i}\ \left(x_{u}+1\right)^{-i\beta}\right]\times\left[\sum_{i=0}^{\infty}q_{i}\left(x_{u}+1\right)^{-i\beta}\right],$$

since, $\left[\sum_{i=0}^{\infty}m_{i}\left(x_{u}+1\right)^{-i\beta}\right]\left[\sum_{i=0}^{\infty}q_{i}\left(x_{u}+1\right)^{-i\beta}\right]=\sum_{i=0}^{\infty}s_{i}\left(x_{u}+1\right)^{-i\beta}$, Gradshteyn and Ryzhik(2000),

where $s_{u}=\sum_{i=0}^{u}m_{i}\,q_{u-i}$, then,

$$f\left(x_{u:v}\right)=\frac{\beta}{B\left(u,v-u+1\right)}\sum_{k=0}^{u+j-1}\sum_{j=0}^{v-u}\sum_{i=0}^{\infty}t_{i,j,k}\left(x_{u}+1\right)^{-i\beta-\alpha\beta-\beta-\alpha\beta k-1}, \tag{13}$$

where,

$$t_{i,j,k}=\left(-1\right)^{j}\binom{v-u}{j}\left(-1\right)^{k}\binom{u+j-1}{k}s_{i}.$$

Order Statistics r-th Moment

The *IPTL* distribution r-th moment of order statistics can be got by

$$E\left(X^{r}_{u:v}\right)=\int_{x} x_u^{\,r}\, f\left(x_u\right)dx_u,$$

substituting (13) into last equation gives

$$E\left(X^{r}_{u:v}\right)=\frac{\beta}{B\left(u,v-u+1\right)}\sum_{k=0}^{u+j-1}\sum_{j=0}^{v-u}\sum_{i=0}^{\infty}t_{i,j,k}\int_{0}^{\infty}x^{r}\left(x_u+1\right)^{-i\beta-\alpha\beta-\beta-\alpha\beta k-1}dx,$$

then,

$$E\left(X^{r}_{u:v}\right)=\frac{\beta}{B\left(u,v-u+1\right)}\sum_{k=0}^{u+j-1}\sum_{j=0}^{v-u}\sum_{i=0}^{\infty}t_{i,j,k}B\left(r+1,i\beta+\alpha\beta+\beta+\alpha\beta k-r\right).$$

8. THE *IPTL* DISTRIBUTION PARAMETERS ESTIMATION USING *MLE* METHOD

Let $X_1, X_2, \dots, X_n$ be the *iid* random variables from the *IPTL* $(x;\Lambda)$ distribution, where $\Lambda=\left(\alpha,\beta\right)$, then the likelihood function for the vector of parameter $\Lambda=\left(\alpha,\beta\right)$, can be written as, Garthwait *et al*. (2002),

$$L=\left(2\alpha\beta\right)^{n}\prod_{i=1}^{n}\left(x_i+1\right)^{-\alpha\beta-1}\prod_{i=1}^{n}\left[1-\left(x_i+1\right)^{-\beta}\right]\prod_{i=1}^{n}\left[2-\left(x_i+1\right)^{-\beta}\right]^{\alpha-1},$$

the log likelihood function is given by

$$\ell=n\log\left(2\alpha\beta\right)-\left(\alpha\beta+1\right)\sum_{i=1}^{n}\log\left(x_i+1\right)$$
$$+\sum_{i=1}^{n}\log\left[1-\left(x_i+1\right)^{-\beta}\right]+\left(\alpha-1\right)\sum_{i=1}^{n}\log\left[2-\left(x_i+1\right)^{-\beta}\right].$$

Parameters α and β score functions are given by

$$\frac{\partial\ell}{\partial\alpha}=\frac{n}{\alpha}-\beta\sum_{i=1}^{n}\log\left(x_i+1\right)+\sum_{i=1}^{n}\log\left[2-\left(x_i+1\right)^{-\beta}\right], \tag{14}$$

and

$$\frac{\partial\ell}{\partial\beta}=\frac{n}{\beta}-\alpha\sum_{i=1}^{n}\log\left(x_i+1\right)+\sum_{i=1}^{n}\frac{\left(x_i+1\right)^{-\beta}\log\left(x_i+1\right)}{1-\left(x_i+1\right)^{-\beta}}$$
$$+\left(\alpha-1\right)\sum_{i=1}^{n}\frac{\left(x_i+1\right)^{-\beta}\log\left(x_i+1\right)}{2-\left(x_i+1\right)^{-\beta}}. \tag{15}$$

Maximum likelihood estimators *(MLEs)* unknown parameters are got by solving the nonlinear likelihood (14) and (15), numerically, using statistical software. Obtaining the estimates is performed via an iterative technique as Newton–Raphson algorithm.

Let Λ be the vector of the unknown parameters (α,β), so that elements of the 2×2 information matrix $I(\alpha,\beta)$ are approximated by

$$I_{ij}(\hat{\Lambda}) = E\left[-\frac{\partial^2 \ell(\Lambda)}{\partial\Lambda_i\,\partial\Lambda_j}\Big|_{\Lambda=\hat{\Lambda}}\right],$$

where $I_{ij}^{-1}(\hat{\Lambda})$ is the unknown parameters variance covariance matrix, the asymptotic distributions of the *IPTL* parameters is

$$\sqrt{n}\left(\hat{\Lambda}_i - \Lambda_i\right) \approx N_2\left(0, I^{-1}\left(\hat{\Lambda}_i\right)\right),\ i = 1,2,$$

and the approximation $100\,(1-\gamma)\,\%$ the unknown parameters confidence intervals based on the *IPTL* (α,β) distribution asymptotic distribution are given by

$$\hat{\Lambda}_i \pm Z_{\frac{\gamma}{2}} \sqrt{I^{-1}(\hat{\Lambda}_i)}\ ; i = 1,2,$$

where $z_{\frac{\gamma}{2}}$ is the upper $\frac{\gamma}{2}$ th percentile of a standard normal distribution.

The derivatives in the observed information matrix $I(\alpha,\beta)$ for the unknown parameters are

$$\frac{\partial^2\ell}{\partial\alpha^2} = \frac{-n}{\alpha^2},\ \frac{\partial^2\ell}{\partial\alpha\,\partial\beta} = -\sum_{i=1}^{n}\log\left(x_i+1\right) + \sum_{i=1}^{n}\frac{\left(x_i+1\right)^{-\beta}\log\left(x_i+1\right)}{2-\left(x_i+1\right)^{-\beta}},$$

and

$$\begin{aligned}\frac{\partial^2\ell}{\partial\beta^2} = &-\frac{n}{\beta^2} - (\alpha-1)\sum_{i=1}^{n}\left[\log\left(x_i+1\right)\right]^2\left(x_i+1\right)^{-\beta}\left[2-\left(x_i+1\right)^{-\beta}\right]^{-1} \\ &\times\left\{\left(x_i+1\right)^{-\beta}\left[2-\left(x_i+1\right)^{-\beta}\right]^{-1}+1\right\} - \sum_{i=1}^{n}\left[\log\left(x_i+1\right)\right]^2\left(x_i+1\right)^{-\beta} \\ &\times\left[1-\left(x_i+1\right)^{-\beta}\right]^{-1}\left\{\left[1-\left(x_i+1\right)^{-\beta}\right]^{-1}\left(x_i+1\right)^{-\beta}+1\right\}.\end{aligned}$$

9. A NUMERICAL STUDY

In this experiment, obtaining *MLEs* of the *IPTL* distribution parameters is performed using random numbers to study the *MLEs* finite sample behavior. The algorithm of obtaining parameters estimates is illustrated in the following steps:

Step (1): Generating a random sample $X_1, X_2, \ldots, X_n$ of sizes $n =$ (10, 20, 30, 50, 100, 300) using the IPTL distribution.

Step (2): Parameters six different sets values are selected as: set(1): $(\alpha = 0.5, \beta = 0.5)$, set(2): $(\alpha = 0.5, \beta = 1.5,)$, set(3): $(\alpha = 0.5, \beta = 2.5)$, set(4): $(\alpha = 1.5, \beta = 0.5)$, set(5): $(\alpha = 2.5, \beta = 0.5)$, set(6): $(\alpha = 4, \beta = 0.25)$.

Step (3): Solving (14) and (15) via iteration to compute *MLEs*, *RMSE* (the root of mean squared error), biases and parameters estimators Pearson type, Pearson (1895), of the *IPTL* distribution.

Step (4): Repeating steps, from 1 to 3, 10000 times.

Samples of random numbers are generated via Mathcad package v15 where the conjugate gradient iteration method is performed. All results are included in tables and indicated in appendix II. From study results, one can see that, in appendix II, as sample size increases, biases, estimators and *RMSEs* decrease, as expected. Moreover, $\hat{\beta}$ sampling distribution can be the Pearson type IV distribution in all times, $\hat{\alpha}$ sampling distribution differs according to sample size. As $\hat{\alpha}$ increases, mean, *RMSE* and bias of $\hat{\beta}$ decrease. An impact of identification problem acts here: When sample size increases, in the identified cases (set 1 - set 5), $\hat{\alpha}$ and $\hat{\beta}$ can be consistent, but in the non-identified distribution
(set 6) they cannot be consistent.

10. APPLICATION

A real data set is selected to investigate the identified *IPTL* distribution, practically, using *MLE* method, via the Mathematica package version 10. In this application, different distributions are used as: the *IPTL* distribution, the *ITL* distribution, the Weibull distribution, the gamma (scaled) distribution, and the Singh-Maddala distribution, Singh and Maddala (1976), on the other hand a comparison between the identified and non-identified *IPTL* distribution is performed. The following data represents the strength of 1.5 cm glass fibers for 60 devices, the data are given from the *UK* National Physical Laboratory, and more information can be available at: http://www.npl.co.uk/

0.636, 0.252, 0.157, 0.187, 2.771, 0.209, 0.617, 2.078, 1.013, 0.499, 0.431,
0.642, 0.460, 0.749, 0.205, 0.576, 0.439, 0.471, 0.262, 0.387, 0.324, 0.424,
0.548, 1.794, 1.233, 0.915, 0.702, 0.417, 0.337, 0.435, 0.359, 0.293, 0.147,
0.870, 0.608, 0.153, 0.098, 0.557, 0.415, 0.122, 0.912, 0.341, 0.725, 0.364,
0.240, 0.594, 0.325, 0.416, 0.080, 0.582, 1.257, 1.575, 0.480, 0.909, 0.170,
0.319, 0.090, 0.154, 2.248, 0.292.

Probability density functions for different distributions having similar skewness and kurtosis (the identified and non-identified *IPTL* distribution, the Weibull distribution, the gamma (scaled) distribution, and the Singh-Maddala distribution) are illustrated in Figure 3, probability density functions for nested distribution by identified *IPTL* distribution (the *ITL* distribution) is illustrated in Figure 4.

In Table 3, distributions parameters *MLE*s, parameters standard error (SEs), in parentheses, *CAIC* (the consistent Akaike Information Criterion), Kolmogorov-Smirnov (*KS*) test statistic, *AIC* (Akaike Information Criterion) and *BIC* (Bayesian information criterion), Merovcia and Puka (2014), are calculated for every distribution having similar skewness and kurtosis values (the identified and non-identified *IPTL* distribution, the Weibull distribution, the gamma (scaled) distribution, and the Singh-Maddala distribution). The null hypothesis that the data follow the *IPTL* distribution, can be accepted at significance level $\alpha = 0.05$. One can see that the identified *IPTL* distribution has the smallest *CAIC*, *KS*, *AIC*, *BIC*, *SEs* and the largest log likelihood and p-value, so that, the identified *IPTL* distribution can be the best fitted distribution to the data compared with other distributions having similar skewness and kurtosis. On the other hand, the non-identified *IPTL* distribution has the largest *CAIC*, *KS*, *BIC*, *AIC*, *SEs* and the smallest log likelihood and p-value, all of that reflect some effects of the identification problem.

In Table 4, depending on the likelihood ratio test, the null hypothesis is the data follow the nested distribution and the alternative is the data follow the full distribution, where the *ITL* distribution is nested by the identified *IPTL* distribution. Obviously, null hypothesis can be rejected at significance level $\alpha = 0.05$.

Table 3
Parameters MLEs with the Associated *BIC* and *AIC* Values

Distribution	MLE_ Parameters			Skewness	Kurtosis	KS	P-value	Log Likelihood	AIC	BIC	CAIC
	α	β	θ								
Identified *IPTL*	3.347 (0.137)	1.298 (0.028)	–	2.717	7.497	0.084	0.749	-22.396	46.792	48.886	46.8614
Non-Identified *IPTL*	0.101 (4.799)	10 (0.910)	–	4.261	1.410	0.426	3.308×10^{-10}	-50.067	104.13	108.32	104.34
Weibull	1.275 (0.161)	0.651 (0.039)	–	3.385	5.603	0.124	0.028	-26.193	56.387	60.575	56.597
Gamma	2.156 (0.295)	0.205 (0.066)	–	2.361	8.781	0.138	0.018	-31.636	67.273	71.462	67.483
Singh Maddala	0.983 (3.512)	2.266 (0.792)	0.682 (0.146)	1.210	4.141	0.289	0.015	-31.823	69.647	75.930	70.075

Table 4
The Likelihood Ratio Tests Statistic, the Log-Likelihood Function and p-Values

Distribution	Parameters		ℓ (Log Likelihood)	Λ (Likelihood Ratio Test Statistics)	DF (Degrees of Freedom)	p-value
	α	β				
ITL	6.592 (0.218)	–	-26.213	7.634	1	5.728×10^{-3}

*Note that the identified *IPTL* distribution log likelihood function = -22.396

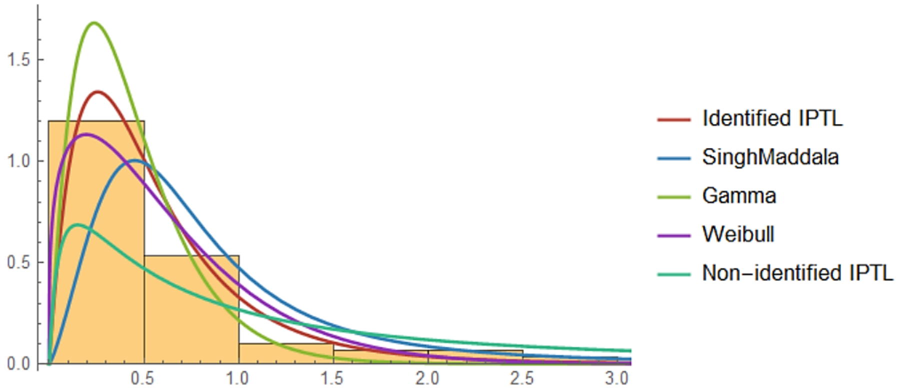

Figure 3: Different Distributions Probability Density Functions having Similar Skewness and Kurtosis

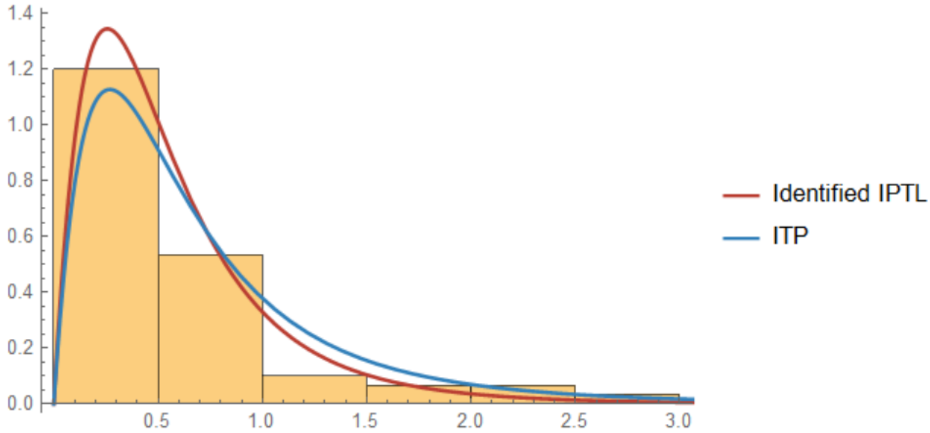

Figure 4: Probability Density Functions for the Nested Distribution by Identified *IPTL* Distribution

11. CONCLUSION

The serious impact of the identification problem affects distributions estimators to be inconsistent causing wrong interpretations which result wrong decisions.

The inverted power Topp-Leone distribution is a useful distribution generalizing the new inverted Topp-Leone distribution (presented for the first time), the inverted power Topp-Leone distribution has flexible properties and many applications but imposing constrains in its parameters is a must to avoid the identification problem.

The author encourages researchers to do more researches on the identification problem in other cases.

LIST OF ABBREVIATIONS

CDF	:	The cumulative distribution function
PDF	:	The probability density function
TL	:	The Topp-Leone distribution
ITL	:	The inverted Topp-Leone distribution
IPTL	:	The inverted power Topp-Leone
MGF	:	The moment generating function
MLE	:	The maximum likelihood estimation method

Availability of Data and Material

The real data set can be available at: http://www.npl.co.uk/

Competing Interests

The author declare that he has no competing interests

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Authors' Contributions

This manuscript has only one author. The author contributed 100% in drafting, giving the main proofs, reading and approving the final manuscript.

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APPENDICES

Appendix (I)

The Mean Deviation about Mean and about Median

They can be given by, respectively,

$$\delta_1(x)=\int_y |x-\mu|\ f(x)dx \text{ and } S_2(x)=\int_x |x-M|\, f(x)dx,$$

easily, it can be given by

$$S_1(x)=2\mu F(\mu)-2t(\mu) \text{ and } S_2(x)=\mu-2t(M),$$

where $T(q)=\int_{-\infty}^{q} x\, f(x)dx$ is the linear incomplete moment.

The proof:

First: mean deviation about mean:

Since

$$\delta_1(x)=\int_{-\infty}^{\infty} |x-\mu|\ f(x)\,dx,$$

then, $\delta_1(x)=\int_{\mu}^{\infty} (x-\mu)\ f(x)dx+\int_{-\infty}^{\mu} (\mu-x)\ f(x)dx,$

hence, $\delta_1(x)=\int_{\mu}^{\infty} x\ f(x)\ dx-\mu\int_{\mu}^{\infty} f(x)\ dx+\int_{-\infty}^{\mu} \mu\ f(x)\ dx-\int_{-\infty}^{\mu} x\ f(x)\ dx,$

so $\delta_1(x)=\int_{\mu}^{\infty} x\ f(x)\ dx-\mu+\mu\, F(\mu)+\mu\, F(\mu)-\int_{-\infty}^{\mu} x\ f(x)\ dx,$

adding and subtracting to $\int_{-\infty}^{\mu} x\ f(x)\ dx$ gives

$$\delta_1(x)=\int_{\mu}^{\infty} x\ f(x)\ dx-\mu+2\mu\, F(\mu)-\int_{-\infty}^{\mu} x\ f(x)\ dx$$
$$+\int_{-\infty}^{\mu} x\ f(x)\ dx-\int_{-\infty}^{\mu} x\ f(x)\ dx,$$

then, $\delta_1(x)=\int_{-\infty}^{\infty} x\, f(x)dx-\mu+2\mu F(x)-2\int_{-\infty}^{\mu} x\, f(x)\ dx,$

hence, $\delta_1(x)=2\mu F(\mu)-2\ T(\mu);\ T(\mu)=\int_{-\infty}^{\mu} x\ f(x)\ dx.$

Similarly, the mean deviation about median can be given.

Appendix (II)

Set(1): ($\alpha=0.5$, $\beta=0.5$)								
Sample Size	Parameters	Mean of Estimators	Biases	Total Bias	RMSE	Total RMSE	Pearson System Coefficients	Pearson Type
10	$\alpha=0.5$	39.192	38.692	38.712	100.254	100.415	-2.349	I
	$\beta=0.5$	1.752	1.252		5.688		0.316	IV
20	$\alpha=0.5$	11.264	10.764	10.796	41.81	42.043	0.314	IV
	$\beta=0.5$	1.327	0.827		4.415		0.344	IV
30	$\alpha=0.5$	4.048	3.548	3.571	19.769	19.957	0.364	IV
	$\beta=0.5$	0.897	0.397		2.734		0.441	IV
50	$\alpha=0.5$	1.74	1.24	1.283	10.374	10.682	0.623	IV
	$\beta=0.5$	0.83	0.33		2.546		0.511	IV
100	$\alpha=0.5$	0.842	0.342	0.378	1.356	2.003	0.344	IV
	$\beta=0.5$	0.662	0.162		1.475		0.752	IV
300	$\alpha=0.5$	0.542	0.042	0.043	0.710	1.012	0.113	IV
	$\beta=0.5$	0.510	0.010		0.722		0.902	IV

Set(2): ($\alpha=0.5$, $\beta=1.5$)								
Sample Size	Parameters	Mean of Estimators	Biases	Total Bias	RMSE	Total RMSE	Pearson System Coefficients	Pearson Type
10	$\alpha=0.5$	62.749	62.249	62.36	149.86	150.767	-0.838	I
	$\beta=1.5$	5.221	3.721		16.521		0.309	IV
20	$\alpha=0.5$	15.023	14.523	14.788	61.444	63.044	0.314	IV
	$\beta=1.5$	4.287	2.787		14.114		0.327	IV
30	$\alpha=0.5$	4.759	4.259	4.839	26.752	29.875	0.443	IV
	$\beta=1.5$	3.797	2.297		13.298		0.342	IV
50	$\alpha=0.5$	1.259	0.759	1.248	3.674	7.69	0.455	IV
	$\beta=1.5$	2.491	0.991		6.756		0.442	IV
100	$\alpha=0.5$	0.91	0.41	0.584	2.294	4.496	0.713	IV
	$\beta=1.5$	1.915	0.415		3.867		0.704	IV
300	$\alpha=0.5$	0.570	0.070	0.094	1.271	2.214	0.913	IV
	$\beta=1.5$	1.564	0.064		1.814		0.904	IV

Set(3): (α=0.5, β=2.5)								
Sample Size	**Parameters**	**Mean of Estimators**	**Biases**	**Total Bias**	**RMSE**	**Total RMSE**	**Pearson System Coefficients**	**Pearson Type**
10	$\alpha=0.5$	66.59	66.09	66.322	157.106	158.949	-0.702	I
	$\beta=2.5$	8.038	5.538		24.138		0.299	IV
20	$\alpha=0.5$	16.682	16.182	17.441	72.838	79.026	0.314	IV
	$\beta=2.5$	9.007	6.507		30.655		0.318	IV
30	$\alpha=0.5$	4.284	3.784	5.216	27.948	34.348	0.545	IV
	$\beta=2.5$	6.089	3.589		19.967		0.36	IV
50	$\alpha=0.5$	1.472	0.972	1.843	9.381	13.08	0.839	IV
	$\beta=2.5$	4.066	1.566		9.114		0.362	IV
100	$\alpha=0.5$	0.679	0.179	0.69	0.619	7.822	0.21	IV
	$\beta=2.5$	3.166	0.666		7.797		0.908	IV
300	$\alpha=0.5$	0.549	0.049	0.1	0.221	0.846	0.843	IV
	$\beta=2.5$	2.588	0.088		0.816		0.993	IV

Set(4): ($\alpha=1.5$, $\beta=0.5$)								
Sample Size	**Parameters**	**Mean of Estimators**	**Biases**	**Total Bias**	**RMSE**	**Total RMSE**	**Pearson System Coefficients**	**Pearson Type**
10	$\alpha=1.5$	199.406	197.906	197.909	418.325	418.358	-0.303	I
	$\beta=0.5$	1.591	1.091		5.227		0.314	IV
20	$\alpha=1.5$	83.229	81.729	81.732	230.058	230.102	0.608	IV
	$\beta=0.5$	1.201	0.701		4.493		0.347	IV
30	$\alpha=1.5$	37.832	36.332	36.333	142.567	142.577	0.335	IV
	$\beta=0.5$	0.767	0.267		1.686		0.417	IV
50	$\alpha=1.5$	12.406	10.906	10.908	67.532	67.554	0.397	IV
	$\beta=0.5$	0.693	0.193		1.748		0.514	IV
100	$\alpha=1.5$	3.398	1.898	1.898	16.746	16.749	0.678	IV
	$\beta=0.5$	0.528	0.028		0.295		0.433	IV
300	$\alpha=1.5$	1.531	0.031	0.039	4.147	4.148	0.978	IV
	$\beta=0.5$	0.524	0.024		0.102		0.633	IV

Set(5): ($\alpha=2.5, \beta=0.5$)								
Sample Size	Parameters	Mean of Estimators	Biases	Total Bias	RMSE	Total RMSE	Pearson System Coefficients	Pearson Type
10	$\alpha=2.5$	365.9	363.4	363.402	687.582	687.601	-0.454	I
	$\beta=0.5$	1.658	1.158		5.163		0.327	IV
20	$\alpha=2.5$	178.436	175.936	175.937	452.336	452.348	2.647	VI
	$\beta=0.5$	0.949	0.449		3.204		0.364	IV
30	$\alpha=2.5$	97.837	95.337	95.337	308.219	308.223	0.409	IV
	$\beta=0.5$	0.688	0.188		1.54		0.465	IV
50	$\alpha=2.5$	38.834	36.334	36.334	163.886	163.89	0.32	IV
	$\beta=0.5$	0.577	0.077		1.046		0.709	IV
100	$\alpha=2.5$	8.088	5.588	5.588	31.391	31.393	0.464	IV
	$\beta=0.5$	0.517	0.017		0.315		0.368	IV
300	$\alpha=2.5$	2.584	0.084	0.103	5.539	5.540	0.164	IV
	$\beta=0.5$	0.561	0.061		0.108		0.268	IV