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THE TOPP-LEONE ODD EXPONENTIAL HALF LOGISTIC-G FAMILY OF DISTRIBUTIONS: MODEL, PROPERTIES AND APPLICATIONS

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ABSTRACT

We developed a new generalized distribution referred to as the Topp-Leone Odd Exponential Half Logistic-G (TL-OEHL-G) distribution. The proposed distribution is an infinite linear combination of the exponentiated-G distribution. Some special cases from the TL-OEHL-G distribution are presented. The special cases of the TL-OEHL-G distribution apply to high skewed data and different forms of the hazard rate. Simulation study results for a selected special case are presented. Real data examples to demonstrate flexibility of the new model compared to other models are also provided.

KEYWORDS

Topp-Leone Distribution, Odd Exponential Half Logistic-G, Maximum Likelihood Estimation.

1. INTRODUCTION

There is increased demand for extended distributions in reliability and lifetime data analysis. Great work has been done in the generalization of classical models. Generalized distributions are flexible in data analysis since they can model data that exhibit monotonic or non-monotonic hazard rates. Established generators available in the literature are Marshall-Olkin-G by Marshall and Olkin [21], Weibull-G by Bourguignon et al. [7], exponentiated-G by Gupta et al. [15], beta-G by Jones [17], Kumaraswamy-G by Cordeiro and de Castro [11], T-X by Alzaatreh et al. [4], Type I half-logistic-G Cordeiro et al. [8], gamma-G by Ristic and Balakrishnan [28], and Topp-Leone-G by Al-Shomrani et al. [3].

Topp and Leone [31] presented an extension of the triangular distribution defined in the domain (0,1). The Topp-Leone distribution has a bathtub hazard rate function. The distribution has a closed quantile function that makes it easy to generate data. The Topp-Leone distribution function is given by

$$F_{TL}(x) = [1-(1-x)^2]^b, \tag{1.1}$$

for $0 < x < 1$ and $b > 0$. The generalizations of the TL distribution include work by Nadarajah and Kotz [25], Ghitany et al. [14], Kotz and Nadarajah [18], Kotz and Seier [19], Vicari et al. [32], Genc [13] and Bayoud [6] and Hassan et al. [16].

Furthermore, Cordeiro et al. [8], developed the type 1 half-logistic distribution with cumulative distribution function (cdf)

$$\begin{aligned} F(x;\lambda,\psi) &= \int_0^{-ln(1-G(x;\psi))} \frac{2\lambda\exp\{-\lambda x\}}{(1+\exp\{-\lambda x\})^2}dx \\ &= \frac{1-[1-G(x;\psi)]^{\lambda}}{1+[1-G(x;\psi)]^{\lambda}}, \end{aligned} \tag{1.2}$$

where $G(x;\psi)$ is the cdf of the baseline distribution and $\lambda > 0$ is the shape parameter. We obtain a special case, namely, half-logistic-G (HL-G) model, with cdf and probability density function (pdf)

$$F_{HL-G}(x;\psi) = \frac{G(x;\psi)}{1+\overline{G}(x;\psi)}, \tag{1.3}$$

and

$$f_{HL-G}(x;\psi) = \frac{2g(x;\psi)}{(1+\overline{G}(x;\psi))^2} \tag{1.4}$$

respectively, by setting $\lambda = 1$.

Afify et al. [1] generalized the HL-G distribution to develop the Exponentiated Odd Exponential Half Logistic-G (EOEHL-G) family of distributions. Their generalization exhibits interesting shapes for both the density and hazard rate function, demonstrating its usefulness in lifetime data analysis. Other generalizations of the HL-G distribution include the exponentiated half-logistic generated family by Cordeiro et al. [9], Kumaraswamy type 1 half-logistic family of distributions with applications by El-Sayed [12], the type I generalized half-logistic distribution based on upper record values by Kumar et al. [20] and generalized half-logistic Poisson distributions by Muhammad et al. [22]. The cdf and pdf of the EOEHL-G distribution are

$$F_{EOEHL-G}(x;\alpha,\lambda,\psi) = \left[\frac{1-\exp\left(\frac{-\lambda G(x;\psi)}{\bar{G}(x;\psi)}\right)}{1+\exp\left(\frac{-\lambda G(x;\psi)}{\bar{G}(x;\psi)}\right)}\right]^{\alpha} \tag{1.5}$$

and

$$f_{EOEHL-G}(x;\alpha,\lambda,\psi) = \frac{2\alpha\lambda g(x;\psi)\exp\left(\frac{-\lambda G(x;\psi)}{\bar{G}(x;\psi)}\right)\left[1-\exp\left(\frac{-\lambda G(x;\psi)}{\bar{G}(x;\psi)}\right)\right]^{\alpha-1}}{\bar{G}^2(x;\psi)\left(1+\exp\left(\frac{-\lambda G(x;\psi)}{\bar{G}(x;\psi)}\right)\right)^{\alpha+1}}, \tag{1.6}$$

where $\alpha, \lambda > 0, g(x;\psi) = \frac{dG(x;\psi)}{dx}$ and $G(x;\psi)$ is the baseline distribution. When $\alpha = 1$, we have the Odd Exponential Half Logistic-G (OEHL-G) distribution.

We were motivated by the fact that the TL distribution has a domain that is limited to (0,1). We, therefore, propose a new generalization of the TL distribution with the following desirable properties:

- the new distribution is flexible since the domain is not restricted to (0,1);
- the pdf of the new distribution takes various shapes for selected parameters values including almost symmetric, reverse-J, right and left-skewed;

- the hazard rate function takes various shapes that include J-shape, reverse-J, and upside bathtub other than the bathtub shape.

We develop the Topp-Leone Odd Exponential Half Logistic-G (TL-OEHL-G) family of distributions. In Section 2, we present the TL-OEHL-G distribution. In Section 3 we present some special cases of the TL-OEHL-G distribution. Structural properties are presented in Section 4. Section 5 contain the maximum likelihood estimates of the model parameters. Simulation study results are presented in Section 6. Section 7 contain applications of the proposed model to real data examples. Section 8 contain some concluding remarks.

2. THE MODEL

We generalize the Topp-Leone distribution using the OEHL-G distribution, to derive the TL-OEHL-G family of distributions. Therefore, the cdf and pdf of the TL-OEHL-G family of distributions are given by

$$F_{TL-OEHL-G}(x;b,\lambda,\psi)=\left[1-\left[1-\frac{1-\exp\left(-\lambda\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right)}{1+\exp\left(-\lambda\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right)}\right]^2\right]^b \tag{2.1}$$

and

$$\begin{aligned}f_{TL-OEHL-G}(x;b,\lambda,\psi) &= \frac{8b\lambda g(x;\psi)\exp\left(-2\lambda\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right)}{\bar{G}^2(x;\psi)\left(1+\exp\left(-\lambda\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right)\right)^3}\\ &\times \left[1-\left[1-\frac{1-\exp\left(-\lambda\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right)}{1+\exp\left(-\lambda\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right)}\right]^2\right]^{b-1},\end{aligned} \tag{2.2}$$

respectively, for $b,\lambda>0$ and ψ is a vector of parameters.

2.1 Quantile Function

We invert the cdf of the TL-OEHL-G distribution to obtain the quantile function as follows:

$$\left[1-\left[1-\frac{1-\exp\left(-\lambda\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right)}{1+\exp\left(-\lambda\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right)}\right]^2\right]^b=u$$

can be written as

$$[1-u^{1/b}]^{1/2}=\left[1-\frac{1-\exp\left(-\lambda\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right)}{1+\exp\left(-\lambda\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right)}\right],$$

which reduces to

$$-\lambda\frac{G(x;\psi)}{\bar{G}(x;\psi)}=\ln\left[\frac{[1-u^{1/b}]^{1/2}}{2-[1-u^{1/b}]^{1/2}}\right],$$

so that

$$G(x;\psi)=\left[\frac{\ln\left[\frac{[1-u^{1/b}]^{1/2}}{2-[1-u^{1/b}]^{1/2}}\right]}{\ln\left[\frac{[1-u^{1/b}]^{1/2}}{2-[1-u^{1/b}]^{1/2}}\right]-\lambda}\right].$$

Therefore, we obtain quantile values for the TL-OEHL-G distribution by solving the non-linear equation

$$x(u)=G^{-1}\left[\frac{\ln\left[\frac{[1-u^{1/b}]^{1/2}}{2-[1-u^{1/b}]^{1/2}}\right]}{\ln\left[\frac{[1-u^{1/b}]^{1/2}}{2-[1-u^{1/b}]^{1/2}}\right]-\lambda}\right], \tag{2.3}$$

via iterative methods using MATLAB or R software.

2.2 Linear Representation

A series representation of the TL-OEHL-G pdf is provided in this section. By considering the following series expansions

$$\left[1-\left[1-\frac{1-\exp\left(-\lambda\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right)}{1+\exp\left(-\lambda\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right)}\right]^2\right]^{b-1}$$
$$=\sum_{m=0}^{\infty}\frac{(-1)^m\Gamma(b)}{\Gamma(b-m)m!}\left[1-\frac{1-\exp\left(-\lambda\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right)}{1+\exp\left(-\lambda\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right)}\right]^{2m},$$

$$\left[1-\frac{1-\exp\left(-\lambda\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right)}{1+\exp\left(-\lambda\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right)}\right]^{2m}$$
$$=\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(2m+1)}{\Gamma(2m+1-n)n!}\left[\frac{1-\exp\left(-\lambda\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right)}{1+\exp\left(-\lambda\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right)}\right]^{n},$$

$$\left[1+\exp\left(-\lambda\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right)\right]^{-(3+n)}=\sum_{w=0}^{\infty}\binom{-(3+n)}{w}\exp\left(-w\lambda\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right),$$

$$\left[1-\exp\left(-\lambda\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right)\right]^{n}=\sum_{z=0}^{\infty}(-1)^z\binom{n}{z}\exp\left(-z\lambda\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right),$$

$$\exp\left(-\lambda(w+z+2)\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right)=\sum_{p=0}^{\infty}\frac{(-1)^p[w+z+2]^p\lambda^p}{p!}\left[\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right]^p$$

and

$$[\bar{G}(x;\psi)]^{-(p+2)}=[1-G(x;\psi)]^{-(p+2)}=\sum_{q=0}^{\infty}(-1)^q\binom{-(p+2)}{q}G^q(x;\psi),$$

the TL-OEHL-G pdf can be expressed as a linear combination of exponentiated-G (Exp-G) densities as follows:

$$\begin{aligned} f_{TL-OEHL-G}(x;b,\lambda,\psi) &= \sum_{m,n,w,z,p,q=0}^{\infty} \frac{(-1)^{m+n+z+p+q} b\lambda^{p+1}[w+z+2]^p}{m!n!p!(p+q+1)} \frac{\Gamma(b)}{\Gamma(b-m)} \\ &\times \frac{\Gamma(2m+1)}{\Gamma(2m+1-n)} \binom{-(3+n)}{w} \binom{n}{z} \binom{-(p+2)}{q} \\ &\times (p+q+1) g(x;\psi) G^{p+q}(x;\psi) \\ &= \sum_{p,q=0}^{\infty} v_{p,q} g_{p+q}(x;\psi), \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} v_{p,q} &= \sum_{m,n,w,z=0}^{\infty} \frac{(-1)^{m+n+z+p+q} b\lambda^{p+1}[w+z+2]^p}{m!n!p!(p+q+1)} \frac{\Gamma(b)}{\Gamma(b-m)} \\ &\times \frac{\Gamma(2m+1)}{\Gamma(2m+1-n)} \binom{-(3+n)}{w} \binom{n}{z} \binom{-(p+2)}{q}, \end{aligned} \tag{2.5}$$

and $g_{p+q}(x;\psi) = (p+q+1)\, g(x;\psi)[G(x;\psi)]^{p+q}$.

3. SOME SUB-MODELS

We present in this section, some sub-models of the TL-OEHL-G distribution with the Uniform, log-logistic and Weibull distributions as baseline distributions.

3.1 Topp-Leone Odd Exponential Half Logistic-Uniform (TL-OEHL-U) Distribution

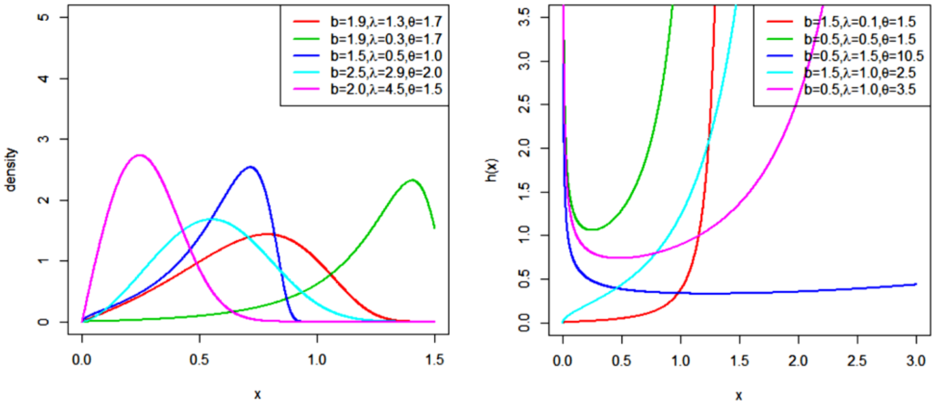
If we take the uniform distribution as the baseline distribution with pdf and cdf $g(x;\theta) = 1/\theta$ and $G(x;\theta) = x/\theta$, respectively, we obtain the TL-OEHL-U distribution with cdf and pdf

$$F_{TL-OEHL-U}(x;b,\lambda,\theta) = \left[1 - \left[1 - \frac{1-\exp\left(-\frac{\lambda x}{\theta - x}\right)}{1+\exp\left(-\frac{\lambda x}{\theta - x}\right)}\right]^2\right]^b$$

and

$$\begin{aligned} f_{TL-OEHL-U}(x;b,\lambda,\theta) &= \frac{8b\lambda \exp\left(-\frac{2\lambda x}{\theta - x}\right)}{\theta[1-x/\theta]^2\left(1+\exp\left(-\frac{\lambda x}{\theta - x}\right)\right)^3} \\ &\times \left[1 - \left[1 - \frac{1-\exp\left(-\frac{\lambda x}{\theta - 1}\right)}{1+\exp\left(-\frac{\lambda x}{\theta - x}\right)}\right]^2\right]^{b-1}, \end{aligned}$$

respectively, for $b, \delta > 0$ and $0 < x < \theta$.



(a) (b)

Figure 3.1: Pdf and hrf Plots for the TL-OEHL-U Distribution

The TL-OEHL-U pdf exhibits various shapes for the pdf and hazard rate function. The hazard rate function exhibits the bathtub, J-shaped, and reverse-J shapes for selected parameter values.

3.2 Topp-Leone Odd Exponential Half Logistic-Log-Logistic Distribution

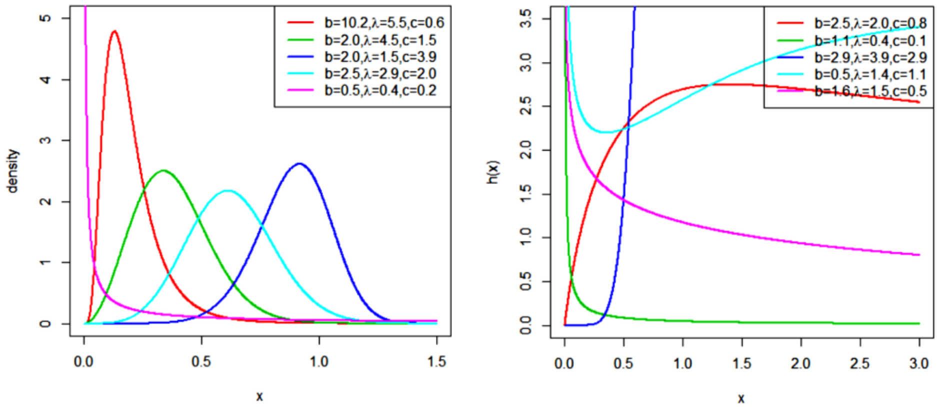
By taking the log-logistic distribution as the baseline distribution, with pdf and cdf $g(x) = cx^{c-1}(1+x^c)^{-2}$ and $G(x) = 1-(1+x^c)^{-1}$, respectively, we obtain the TL-OEHL-LLoG distribution with cdf and pdf

$$F_{TL-OEHL-LLoG}(x;b,\lambda,c) = \left[1-\left[1-\frac{1-\exp\left(-\lambda\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right)}{1+\exp\left(-\lambda\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right)}\right]^2\right]^b$$

and

$$\begin{aligned} f_{TL-OEHL-LLoG}(x;b,\lambda,c) &= \frac{8b\lambda c x^{c-1}(1+x^c)^{-2}\exp\left(-2\lambda\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right)}{(1+x^c)^{-2}\left(1+\exp\left(-\lambda\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right)\right)^3} \\ &\times \left[1-\left[1-\frac{1-\exp\left(-\lambda\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right)}{1+\exp\left(-\lambda\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right)}\right]^2\right]^{b-1}, \end{aligned}$$

respectively, for $b, \lambda, c > 0$.



(a) (b)

Figure 3.2: Pdf and hrf Plots for the TL-OEHL-LLoG Distribution

The pdfs for TL-OEHL-LLoG distribution take various shapes including reverse-J, symmetric and right, or left-skewed. The hazard rate function exhibit reverse-J, J-Shape, bathtub, and upside bathtub shapes.

3.3 The Topp-Leone Odd Exponential Half Logistic-Weibull Distribution

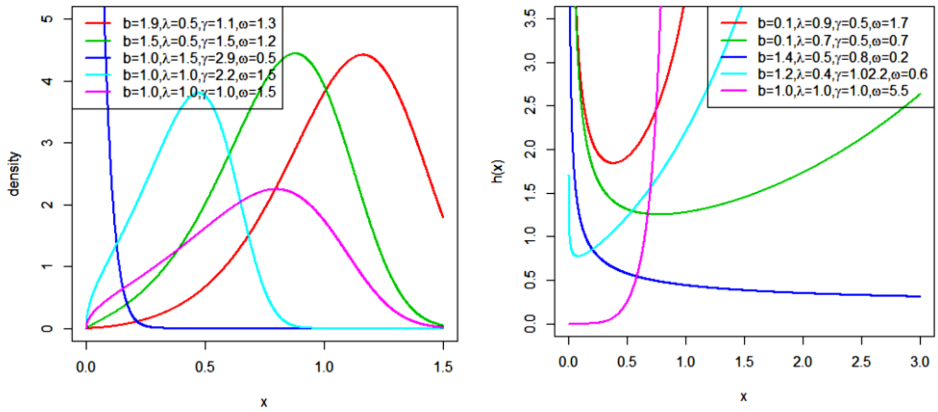
If we take the baseline distribution to be the Weibull distribution with pdf and cdf $g(x;\gamma,\omega) = \gamma\omega x^{\omega-1}\, exp(-\gamma x^{\omega})$ and $G(x;\gamma,\omega) = 1 - exp(-\gamma x^{\omega})$, respectively, we obtain the Topp-Leone Odd Exponentiated Half Logistic-Weibull (TL-OEHLW) distribution with cdf and pdf

$$F_{TL-OEHL-W}(x;b,\lambda,\gamma,\omega) = \left[1-\left[1-\frac{1-\exp\left(-\lambda\frac{1-e^{-\gamma x^{\omega}}}{e^{-\gamma x^{\omega}}}\right)}{1+\exp\left(-\lambda\frac{1-e^{-\gamma x^{\omega}}}{e^{-\gamma x^{\omega}}}\right)}\right]^{2}\right]^{b}$$

and

$$\begin{aligned} f_{TL-OEHL-W}(x;b,\lambda,\gamma,\omega) &= \frac{8b\lambda\gamma\omega x^{\omega-1}e^{-\gamma x^{\omega}}\exp\left(-2\lambda\frac{1-e^{-\gamma x^{\omega}}}{e^{-\gamma x^{\omega}}}\right)}{e^{-2\gamma x^{\omega}}\left(1+\exp\left(-\lambda\frac{1-e^{-\gamma x^{\omega}}}{e^{-\gamma x^{\omega}}}\right)\right)^{3}} \\ &\times \left[1-\left[1-\frac{1-\exp\left(-\lambda\frac{1-e^{-\gamma x^{\omega}}}{e^{-\gamma x^{\omega}}}\right)}{1+\exp\left(-\lambda\frac{1-e^{-\gamma x^{\omega}}}{e^{-\gamma x^{\omega}}}\right)}\right]^{2}\right]^{b-1}, \end{aligned}$$

respectively, for $b, \lambda, \gamma, \omega > 0$.



(a) (b)

Figure 3.3: Pdf and hrf Plots for the TL-OEHL-W Distribution

The TL-OEHL-W pdf applies to data sets of varying skewness and kurtosis. The hazard rate exhibits increasing, decreasing and bathtub shapes.

4. STRUCTURAL PROPERTIES

4.1 Distribution of Order Statistics

We derive the distribution of order statistics from the TL-OEHL-G distribution using equation (4.1),

$$f_{i:n}(x;b,\lambda,\psi)=\frac{f(x)}{B(i,n-i+1)}\sum_{j=0}^{n-j}\binom{n-i}{j}F(x)^{j+i-1}, \tag{4.1}$$

where $B(.,.)$ is the beta function. Using equations (2.1) and (2.2), we get

$$\begin{aligned} f(x)F(x)^{j+i-1} &= \frac{8b\lambda g(x;\psi)\exp\left(-2\lambda\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right)}{\bar{G}^2(x;\psi)\left(1+\exp\left(-\lambda\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right)\right)^3} \\ &\times \left[1-\left[1-\frac{1-\exp\left(-\lambda\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right)}{1+\exp\left(-\lambda\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right)}\right]^2\right]^{b(j+i)-1}. \end{aligned}$$

Applying the following series expansions

$$\begin{aligned} &\left[1-\left[1-\frac{1-\exp\left(-\lambda\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right)}{1+\exp\left(-\lambda\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right)}\right]^2\right]^{b(j+i)-1} \\ &=\sum_{m=0}^{\infty}\frac{(-1)^m\Gamma(b(j+i))}{\Gamma(b(j+i)-m)m!}\times\left[1-\frac{1-\exp\left(-\lambda\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right)}{1+\exp\left(-\lambda\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right)}\right]^{2m}, \end{aligned}$$

$$\left[1-\frac{1-\exp\left(-\lambda\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right)}{1+\exp\left(-\lambda\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right)}\right]^{2m}$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(2m+1)}{\Gamma(2m+1-n)n!}\left[\frac{1-\exp\left(-\lambda\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right)}{1+\exp\left(-\lambda\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right)}\right]^{n},$$

$$\left[1+\exp\left(-\lambda\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right)\right]^{-(3+n)}$$

$$=\sum_{w=0}^{\infty}\binom{-(3+n)}{w}\exp\left(-w\lambda\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right),$$

$$\left[1-\exp\left(-\lambda\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right)\right]^{n}$$

$$=\sum_{z=0}^{\infty}(-1)^z\binom{n}{z}\exp\left(-z\lambda\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right),$$

$$\exp\left(-\lambda(w+z+2)\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right)$$

$$=\sum_{p=0}^{\infty}\frac{(-1)^p[w+z+2]^p\lambda^p}{p!}\left[\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right]^{p}$$

and

$$[\bar{G}(x;\psi)]^{-(p+2)}=[1-G(x;\psi)]^{-(p+2)}$$

$$=\sum_{q=0}^{\infty}(-1)^q\binom{-(p+2)}{q}G^q(x;\psi),$$

yields

$$\begin{aligned} & f(x)F(x)^{j+i-1} \\ &= \sum_{m,n,w,z,p,q=0}^{\infty} \frac{(-1)^{m+n+z+p+q} b\lambda^{p+1}[w+z+2]^p}{m!n!p!(p+q+1)} \frac{\Gamma(b(j+i))}{\Gamma(b(j+i)-m)} \\ &\times \frac{\Gamma(2m+1)}{\Gamma(2m+1-n)} \binom{-(3+n)}{w} \binom{n}{z} \binom{-(p+2)}{q} \\ &\times (p+q+1) g(x;\psi) G^{p+q}(x;\psi). \end{aligned} \tag{4.2}$$

Therefore, the distribution of the i^{th} order statistic from the TL-OEHL-G is given by

$$\begin{aligned} & f_{i:n}(x;b,\lambda,\xi) \\ &= \sum_{m,n,w,z,p,q=0}^{\infty} \sum_{j=0}^{n-j} \frac{(-1)^{m+n+z+p+q} b\lambda^{p+1}[w+z+2]^p}{B(i,n-i+1)m!n!p!(p+q+1)} \frac{\Gamma(b(j+i))}{\Gamma(b(j+i)-m)} \\ &\times \frac{\Gamma(2m+1)}{\Gamma(2m+1-n)} \binom{n-i}{j} \binom{-(3+n)}{w} \binom{n}{z} \binom{-(p+2)}{q} \\ &\times (p+q+1) g(x;\psi) G^{p+q}(x;\psi) \\ &\times \sum_{p,q=0}^{\infty} v^{*}_{p,q} g_{p+q}(x;\psi), \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} v^{*}_{p,q} = & \sum_{m,n,w,z=0}^{\infty} \sum_{j=0}^{n-j} \frac{(-1)^{m+n+z+p+q} b\lambda^{p+1}[w+z+2]^p}{B(i,n-i+1)m!n!p!(p+q+1)} \frac{\Gamma(b(j+1))}{\Gamma(b(j+i)-m)} \\ &\times \frac{\Gamma(2m+1)}{\Gamma(2m+1-n)} \binom{n-i}{j} \binom{-(3+n)}{w} \binom{n}{z} \binom{-(p+2)}{q} \end{aligned} \tag{4.4}$$

and $g_{p+q}(x;\psi) = (p+q+1)\, g(x;\psi) G^{p+q}(x;\psi)$ is the Exp-G distribution with power parameter $(p+q)$.

4.2 Entropy

We derive in this section Rényi entropy [27] of the TL-OEHL-G distribution. Rényi entropy encompasses other entropy measures, for example Shannon entropy by Shannon [29]. Rényi entropy is given by

$$I_R(\nu) = (1-\nu)^{-1} \log \left[\int_0^{\infty} f^{\nu}(x) dx \right], v \neq 1, v > 0. \tag{4.5}$$

Substituting Equation (2.2) for $f(x)$, we get

$$f^{\nu}(x;b,\lambda,\psi) = \frac{(8b)^{\nu}\lambda^{\nu}g^{\nu}(x;\psi)\exp\big(-2\nu\lambda\frac{G(x;\psi)}{\bar{G}(x;\psi)}\big)}{\bar{G}^{2\nu}(x;\psi)\big(1+\exp\big(-\lambda\frac{G(x;\psi)}{\bar{G}(x;\psi)}\big)\big)^{3\nu}} \times \left[1-\left[1-\frac{1-\exp\big(-\lambda\frac{G(x;\psi)}{\bar{G}(x;\psi)}\big)}{1+\exp\big(-\lambda\frac{G(x;\psi)}{\bar{G}(x;\psi)}\big)}\right]^2\right]^{(b-1)\nu}.$$

Applying the following expansions

$$\left[1-\left[1-\frac{1-\exp\big(-\lambda\frac{G(x;\psi)}{\bar{G}(x;\psi)}\big)}{1+\exp\big(-\lambda\frac{G(x;\psi)}{\bar{G}(x;\psi)}\big)}\right]^2\right]^{(b-1)\nu} = \sum_{m=0}^{\infty}\frac{(-1)^m\Gamma((b-1)\nu+1)}{\Gamma((b-1)\nu+1-m)m!} \times 1-\frac{1-\exp\big(-\lambda\frac{G(x;\psi)}{\bar{G}(x;\psi)}\big)}{1+\exp\big(-\lambda\frac{G(x;\psi)}{\bar{G}(x;\psi)}\big)}\Bigg]^{2m},$$

$$\left[1-\frac{1-\exp\big(-\lambda\frac{G(x;\psi)}{\bar{G}(x;\psi)}\big)}{1+\exp\big(-\lambda\frac{G(x;\psi)}{\bar{G}(x;\psi)}\big)}\right]^{2m} = \sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(2m+1)}{\Gamma(2m+1-n)n!}\left[\frac{1-\exp\big(-\lambda\frac{G(x;\psi)}{\bar{G}(x;\psi)}\big)}{1+\exp\big(-\lambda\frac{G(x;\psi)}{\bar{G}(x;\psi)}\big)}\right]^n,$$

$$\left[1+\exp\left(-\lambda\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right)\right]^{-(3\nu+n)} = \sum_{w=0}^{\infty}\binom{-(3\nu+n)}{w}\exp\left(-w\lambda\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right),$$

$$\left[1-\exp\left(-\lambda\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right)\right]^n = \sum_{z=0}^{\infty}(-1)^z\binom{n}{z}\exp\left(-z\lambda\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right),$$

$$\exp\left(-\lambda(w+z+2\nu)\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right) = \sum_{p=0}^{\infty}\frac{(-1)^p[w+z+2\nu]^p\lambda^p}{p!}\left[\frac{G(x;\psi)}{\bar{G}(x;\psi)}\right]^p$$

and

$$[\bar{G}(x;\psi)]^{-(p+2\nu)} = [1-G(x;\psi)]^{-(p+2\nu)} = \sum_{q=0}^{\infty}(-1)^q \binom{-(p+2\nu)}{q} G^q(x;\psi),$$

we get

$$\begin{aligned} I_R(\nu) &= (1-\nu)^{-1}\log \\ &\Bigg[\sum_{m,n,w,z,p,q=0}^{\infty} \frac{(-1)^{m+n+z+p+q} b\lambda^{p+1}[w+z+2\nu]^p}{m!n!p!} \\ &\times \frac{\Gamma((b-1)\nu+1)}{\Gamma((b-1)\nu+1-m)} \frac{\Gamma(2m+1)}{\Gamma(2m+1-n)} \binom{-(3\nu+n)}{w}\binom{n}{z}\binom{-(p+2\nu)}{q} \\ &\times \left(\frac{1}{\frac{p+q}{\nu}+1}\right)^{\nu} \int_0^{\infty}\left[\left(\frac{p+q}{\nu}+1\right) g(x;\psi) G^{(p+q)/\nu}(x;\psi)\right]^{\nu} dx\Bigg] \\ &= (1-\nu)^{-1}\log\left[\sum_{p,q=0}^{\infty} w^*_{p,q} e^{(1-\nu)I_{REG}}\right], \end{aligned} \tag{4.6}$$

where

$$\begin{aligned} w^*_{p,q} &= \sum_{m,n,w,z=0}^{\infty} \frac{(-1)^{m+n+z+p+q} b\lambda^{p+1}[w+z+2\nu]^p}{m!n!p!} \frac{\Gamma((b-1)\nu+1)}{\Gamma((b-1)\nu+1-m)} \\ &\times \frac{\Gamma(2m+1)}{\Gamma(2m+1-n)} \binom{-(3\nu+n)}{w}\binom{n}{z}\binom{-(p+2\nu)}{q}\left(\frac{1}{\frac{p+q}{\nu}+1}\right)^{\nu} \end{aligned} \tag{4.7}$$

and $I_{REG} = \int_0^{\infty}\left[\left(\frac{p+q}{v}+1\right) g(x;\psi) G^{(p+q)/v}(x;\psi)\right]^{v} dx$ is the Rényi entropy of Exp-G distribution with power parameter $\left(\frac{p+q}{v}\right)$.

4.3 Moments and Probability Weighted Moments

Ordinary moments, incomplete moments and moment generating function of the TL-OEHL-G distribution are derived in this section. The s^{th} ordinary moment is given by

$$\mu'_s = E(X^s) = \sum_{p,q=0}^{\infty} v_{p,q}\, E\left(Y^s_{p+q}\right) \tag{4.8}$$

where Y^s_{p+q} has an Exp-G distribution and $v_{p,q}$ is given by Equation (2.5).
The r^{th} central moment of X is

$$\mu_r = \sum_{s=0}^{r}\binom{r}{s}(-\mu'_1)^{r-s} = E(X^s) = \sum_{s=0}^{r}\sum_{p,q=0}^{\infty} v_{p,q}\binom{r}{s}(-\mu'_1)^{r-s}E\left(Y^r_{p+q}\right).$$

The r^{th} incomplete moment of X is given by

$$\phi_r(z) = \int_{-\infty}^{z} x^r f(x)dx = \sum_{p=0}^{\infty} v_{p,q} \int_{-\infty}^{z} x^r g_{p+q}(x;\xi)dx. \qquad (4.9)$$

The moment generating function (mgf) is given by

$$M_x(t) = E(e^{tX}) = \sum_{p,q=0}^{\infty} v_{p,q} M_{p+q}(t),$$

where $M_{p+q}(t)$ is the mgf of Exp-G distribution with power parameter $(p+q)$.
The $(j,i)^{th}$ probability weighted moment (PWM), say $\eta_{j,i}$ of X is derived as follows:

$$\eta_{j,i} = E(X^j F(X)^i) = \int_{-\infty}^{\infty} x^j f(x)F(x)^i dx.$$

Using equation (4.2),

$$\begin{aligned} f(x)F(x)^i &= \sum_{m,n,w,z,p,q=0}^{\infty} \frac{(-1)^{m+n+z+p+q} b\lambda^{p+1}[w+z+2]^p}{m!n!p!(p+q+1)} \frac{\Gamma(bi)}{\Gamma(bi-m)} \\ &\times \frac{\Gamma(2m+1)}{\Gamma(2m+1-n)} \binom{-(3+n)}{w} \binom{n}{z} \binom{-(p+2)}{q} \\ &\times (p+q+1) g(x;\psi) G^{p+q}(x;\psi) \end{aligned}$$

so that

$$f(x)F(x)^i = \sum_{p,q=0}^{\infty} z^*_{p,q} g_{p+q}(x;\psi),$$

where

$$\begin{aligned} z^*_q &= \sum_{m,n,w,z=0}^{\infty} \frac{(-1)^{m+n+z+p+q} b\lambda^{p+1}[w+z+2]^p}{m!n!p!(p+q+1)} \frac{\Gamma(bi)}{\Gamma(bi-m)} \\ &\times \frac{\Gamma(2m+1)}{\Gamma(2m+1-n)} \binom{-(3+n)}{w} \binom{n}{z} \binom{-(p+2)}{q} \end{aligned}$$

and $g_{p+q}(x;\psi)$ is an Exp-G density. Thus,

$$\begin{aligned} \eta_{j,i} &= \sum_{p,q=0}^{\infty} z^*_{p,q} \int_{-\infty}^{\infty} x^j g_{p+q}(x;\psi)dx \\ &= \sum_{p,q=0}^{\infty} z^*_{p,q} E(T^j_{p+q}), \end{aligned}$$

where T_{p+q}^{j} is j^{th} power of a random variable with an Exp-G distribution.

We derived the distribution of order statistics, Rényi entropy, moments and probability weighted moments in this section. The properties were obtained directly from the properties of the Exp-G distribution, since the TL-OEHL-G distribution is a linear combination of the Exp-G distribution.

5. MAXIMUM LIKELIHOOD ESTIMATION

If $X_i \sim TL - OEHL - G(b, \lambda; \psi)$, then the total log-likelihood $\ell = \ell(\Delta)$ from a random sample of size n is given by

$$\begin{aligned}
\ell &= n\log(8b) + n\log(\lambda) + \sum_{i=1}^{n}\log[g(x_i;\psi)] - 2\lambda\sum_{i=1}^{n}\frac{G(x_i;\psi)}{\bar{G}(x_i;\psi)} \\
&- 2\sum_{i=1}^{n}\log[\bar{G}(x_i;\psi)] - 3\sum_{i=1}^{n}\log\left[1+\exp\left(-\lambda\frac{G(x_i;\psi)}{\bar{G}(x_i;\psi)}\right)\right] \\
&+ (b-1)\sum_{i=1}^{n}\log\left[1-\left(1-\frac{1-\exp\left(-\lambda\frac{G(x_i;\psi)}{\bar{G}(x_i;\psi)}\right)}{1+\exp\left(-\lambda\frac{G(x_i;\psi)}{\bar{G}(x_i;\psi)}\right)}\right)^2\right].
\end{aligned}$$

The score vector $U = \left(\frac{\partial\ell}{\partial b}, \frac{\partial\ell}{\partial\lambda}, \frac{\partial\ell}{\partial\psi_k}\right)$ has elements given by:

$$\begin{aligned}
\frac{\partial\ell}{\partial b} &= \frac{n}{b} + \sum_{i=1}^{n}\log\left[1-\left(1-\frac{1-\exp\left(-\lambda\frac{G(x_i;\psi)}{\bar{G}(x_i;\psi)}\right)}{1+\exp\left(-\lambda\frac{G(x_i;\psi)}{\bar{G}(x_i;\psi)}\right)}\right)^2\right], \\
\frac{\partial\ell}{\partial\lambda} &= \frac{n}{\lambda} - 2\sum_{i=1}^{n}\frac{G(x_i;\psi)}{\bar{G}(x_i;\psi)} + 3\sum_{i=1}^{n}\frac{G(x_i;\psi)\exp\left(-\lambda\frac{G(x_i;\psi)}{\bar{G}(x_i;\psi)}\right)}{\bar{G}(x_i;\psi)\left(1+\exp\left(-\lambda\frac{G(x_i;\psi)}{\bar{G}(x_i;\psi)}\right)\right)} \\
&- (b-1)\sum_{i=1}^{n}\frac{8G(x_i;\psi)\exp\left(-2\lambda\frac{G(x_i;\psi)}{\bar{G}(x_i;\psi)}\right)\left(1+\exp\left(-\lambda\frac{G(x_i;\psi)}{\bar{G}(x_i;\psi)}\right)\right)^{-1}}{\bar{G}(x;\psi)\left(3\exp\left(-2\lambda\frac{G(x_i;\psi)}{\bar{G}(x_i;\psi)}\right) - 2\exp\left(-\lambda\frac{G(x_i;\psi)}{\bar{G}(x_i;\psi)}\right) - 1\right)}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial\ell}{\partial\psi_k} &= \sum_{i=1}^{n}\frac{1}{g(x_i;\psi)}\frac{\partial g(x_i;\psi)}{\partial\psi_k} - 2\lambda\sum_{i=1}^{n}\frac{G(x_i;\psi)\frac{\partial\bar{G}(x_i;\psi)}{\partial\psi_k} - \bar{G}(x_i;\psi)\frac{\partial G(x_i;\psi)}{\partial\psi_k}}{\bar{G}^2(x_i;\psi)} \\
&- 2\sum_{i=1}^{n}\frac{1}{\bar{G}(x_i;\psi)}\frac{\partial\bar{G}(x_i;\psi)}{\partial\psi_k} \\
&- 3\sum_{i=1}^{n}\frac{1}{\left[1+\exp\left(-\lambda\frac{G(x_i;\psi)}{\bar{G}(x_i;\psi)}\right)\right]}\frac{\partial\left[1+\exp\left(-\lambda\frac{G(x_i;\psi)}{\bar{G}(x_i;\psi)}\right)\right]}{\partial\psi_k}
\end{aligned}$$

$$+(b-1)\sum_{i=1}^{n}\frac{1}{\left[1-\left(1-\frac{1-\exp\left(-\lambda\frac{G(x_i;\psi)}{\bar{G}(x_i;\psi)}\right)}{1+\exp\left(-\lambda\frac{G(x_i;\psi)}{\bar{G}(x_i;\psi)}\right)}\right)^2\right]}\frac{\partial\left[1-\left(1-\frac{1-\exp\left(-\lambda\frac{G(x_i;\psi)}{\bar{G}(x_i;\psi)}\right)}{1+\exp\left(-\lambda\frac{G(x_i;\psi)}{\bar{G}(x_i;\psi)}\right)}\right)^2\right]}{\partial\psi_k},$$

respectively. These partial derivatives are not in closed form and can be solved using R, MATLAB and SAS software by use of iterative methods.

To obtain confidence intervals for model parameters (b,λ,ψ) and the hypotheses concerning these parameters, the observed information matrix is required and is given by

$$J(\Delta)=\begin{pmatrix} J_{bb}(\Delta) & J_{b\lambda}(\Delta) & J_{b\psi}(\Delta) \\ J_{\lambda b}(\Delta) & J_{\lambda\lambda}(\Delta) & J_{\lambda\psi}(\Delta) \\ J_{\psi b}(\Delta) & J_{\psi\lambda}(\Delta) & J_{\psi\psi}(\Delta) \end{pmatrix}, \tag{5.1}$$

where $J_{i,j}=\frac{-\partial^2\ell(\Delta)}{\partial_i\partial_j}$, for $i,j=b,\lambda,\psi$. Under the usual regularity conditions $\hat{\Delta}$ is asymptotically normal distributed, that is $\hat{\Delta}\sim N(0,I^{-1}(\Delta))$ as $n\rightarrow\infty$, where $I(\Delta)$ is the expected information matrix. The asymptotic behavior remains valid if $I(\Delta)$ is replaced by $J(\hat{\Delta})$, the information matrix evaluated at $\hat{\Delta}$.

6. SIMULATION STUDY

We conducted a simulation study to evaluate the consistency of the maximum likelihood estimators in this section. We simulated for N=1000 times with sample of sizes $n=$50, 100, 200, 400 and 800. The results of the simulation study are shown in Table 6.1. The mean values approximate the true parameter values as the sample size increases. The results also show that the RMSE and average bias decay to zero as the sample size increases. Consequently, the TL-OEHL-LLoG model produces consistent model parameter estimates.

Table 6.1
Simulation Study Results for TL-OEHL-LLoG Distribution

Parameter	n	$b=0.5,\lambda=0.5,c=1.5$			$b=0.5,\lambda=1.5,c=1.5$			$b=1.5,\lambda=0.5,c=0.5$		
		Mean	**RMSE**	**Bias**	**Mean**	**RMSE**	**Bias**	**Mean**	**RMSE**	**Bias**
b	50	0.6525	0.5344	0.1525	0.6401	0.5245	0.1401	2.5980	5.2701	1.0980
	100	0.5555	0.2799	0.0555	0.5597	0.2822	0.0597	1.8454	1.3166	0.3454
	200	0.5153	0.1691	0.0153	0.5187	0.1702	0.0187	1.6112	0.5932	0.1112
	400	0.5105	0.1087	0.0105	0.5123	0.1096	0.0123	1.5578	0.3644	0.0578
	800	0.5023	0.0763	0.0023	0.5057	0.0776	0.0057	1.5172	0.2526	0.0172
λ	50	0.5875	0.3684	0.0875	1.6053	0.4604	0.1053	0.6216	0.3980	0.1216
	100	0.5266	0.2652	0.0266	1.5455	0.3229	0.0455	0.5472	0.2570	0.0472
	200	0.4979	0.1867	-0.0020	1.5088	0.2240	0.0088	0.5115	0.1731	0.0115
	400	0.5027	0.1276	0.0027	1.5086	0.1507	0.0086	0.5086	0.1192	0.0086
	800	0.4991	0.0927	-0.0008	1.5052	0.1093	0.0052	0.5016	0.0869	0.0016
c	50	1.6609	0.8061	0.1609	1.7182	0.9102	0.2182	0.5197	0.2201	0.0197
	100	1.6520	0.6322	0.1520	1.6437	0.6254	0.1437	0.5183	0.1451	0.0183
	200	1.6064	0.4387	0.1064	1.5993	0.4239	0.0993	0.5158	0.1042	0.0158
	400	1.5382	0.2518	0.0382	1.5355	0.2525	0.0355	0.5051	0.0673	0.0051
	800	1.5245	0.1816	0.0245	1.5181	0.1805	0.0181	0.5036	0.0494	0.0036
		$b=1.5,\lambda=0.5,c=1.5$			$b=1.0,\lambda=1.5,c=1.5$			$b=1.5,\lambda=1.5,c=1.5$		
b	50	2.4637	5.0401	0.9637	2.4733	1.3730	1.6283	0.3730	4.9534	0.9249
	100	1.8237	1.2476	0.3237	1.8551	1.1392	0.6708	0.1392	1.2801	0.2909
	200	1.6091	0.5893	0.1091	1.6496	1.0376	0.3566	0.0376	0.5901	0.0886
	400	1.5525	0.3630	0.0525	1.6047	1.0221	0.2236	0.0221	0.3637	0.0484
	800	1.5152	0.2467	0.0152	1.5737	1.0047	0.1542	0.0047	0.2476	0.0132
λ	50	0.5986	0.3874	0.0986	1.6028	1.5636	0.4536	0.0636	0.4884	0.0779
	100	0.5435	0.2554	0.0435	1.5538	1.5232	0.3193	0.0232	0.3323	0.0290
	200	0.5112	0.1726	0.0112	1.5259	1.4942	0.2209	-0.0057	0.2259	-0.0012
	400	0.5065	0.1184	0.0065	1.5277	1.5013	0.1453	0.0013	0.1501	0.0040
	800	0.5008	0.0844	0.0008	1.5239	1.4990	0.1035	-0.0009	0.1071	-0.0004
c	50	1.6036	0.7061	0.1036	0.5542	1.7437	1.0202	0.2437	0.9079	0.1909
	100	1.5567	0.4419	0.0567	0.5227	1.6229	0.5628	0.1229	0.5046	0.0974
	200	1.5452	0.3101	0.0452	0.5113	1.5860	0.3789	0.0860	0.3378	0.0654
	400	1.5183	0.2008	0.0183	0.4984	1.5294	0.2177	0.0294	0.2039	0.0218
	800	1.5145	0.1446	0.0145	0.4956	1.5192	0.1540	0.0192	0.1459	0.0159

7. APPLICATIONS

We present in this section, three applications of the TL-OEHL-LLoG distribution so as to demonstrate the versatility of the model in data fitting. Model performance was assessed by means of goodness-of-fit statistics; Cramer-von-Mises (W^*) and Andersen-Darling (A^*), -2loglikelihood (-2 log L), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICC), Bayesian Information Criterion (BIC), Kolmogorov-Smirnov (K-S) statistic (and its p-value), and sum of squares (SS). The model with the smallest values of the goodness-of-fit statistics and a bigger p-value for the K-S statistic is regarded as the best model.

Model parameters were estimated using the maximum likelihood estimation technique with the aid of R software. Tables 7.1, 7.2 and 7.3 shows the parameter estimates (standard errors in parenthesis) and the goodness-of-fit-statistics for the various models considered in this paper. We present in Figures 7.1(a), 7.1(b), 7.2(a), 7.2(b), 7.3(a) and 7.3(b), plots of the fitted densities, the histogram of the data and probability plots.

The TL-OEHL-LLoG model was compared to a variety of equi-parameter models. The models considered in this paper are the Marshall-Olkin extended Fréchet (MOEFr) by Barreto-Souza et al. [5], type II generalized Topp-Leone-Rayleigh (TIGTLR), type II generalized Topp-Leone-exponential (TIGTLE) and type II generalized Topp-Leone-uniform (TIGTLU) by Hassan et al. [16], exponentiated-Fréchet (EFr) distribution by Nadarajah and Kotz [24], and the Marshall-Olkin extended inverse Weibull (IWMO) by Pakungwati et al. [26]. The pdfs of the non-nested models are given by:

$$f_{IWMO}(x;\alpha,\theta,\lambda)=\frac{\alpha\lambda\theta^{-\lambda}x^{-\lambda-1}e^{-(\theta x)^{-\lambda}}}{[\alpha-(\alpha-1)e^{-(\theta x)^{-\lambda}}]^2},$$

for $\alpha,\theta,\lambda>0$,

$$f_{EF_r}(x;a,\gamma,\theta)=a\gamma\theta^{\gamma}\left[1-e^{-(\theta/x)^{\gamma}}\right]^{a-1}x^{-(1+\gamma)}e^{-(\gamma+1)(\theta/x)^{\lambda}},$$

for $a,\gamma,\theta>0$,

$$f_{MOEF_r}(x;a,\gamma,\theta)=\frac{a\gamma\theta^{\gamma}x^{-(\gamma+1)}e^{-(\theta/x)^{\gamma}}}{[1-\bar{a}(1-e^{-(\theta/x)^{\gamma}})]^2},$$

for $a,\gamma,\theta>0$,

$$f_{TIIGTLU}(x;a,b,\gamma)=\frac{2ab}{\gamma}\left(\frac{x}{\gamma}\right)^{2b-1}\left(1-\frac{x}{\gamma}\right)^{a-1},$$

for $a,b,\gamma>0$,

$$f_{TIIGTLE}(x;a,b,\gamma)=2ab\gamma e^{\gamma x}[1-e^{-\gamma x}]^{2b-1}(1-(1-e^{-\gamma x})^{2b})^{a-1},$$

for $a,b,\gamma>0$ and

$$f_{TIIGTLR}(x;a,b,\gamma)=4ab\gamma xe^{\gamma x^2}[1-e^{-\gamma x^2}]^{2b-1}(1-(1-e^{-\gamma x^2})^{2b})^{a-1},$$

for $a,b,\gamma>0$.

7.1 1.5 cm Glass Fibres Data

The first data set represents strengths of 1.5 cm glass fibres. The data set was also analyzed by Bourguignon et al. [7], and Smith and Naylor [30]. See Bourguignon et al. [7] for further details on the data set.

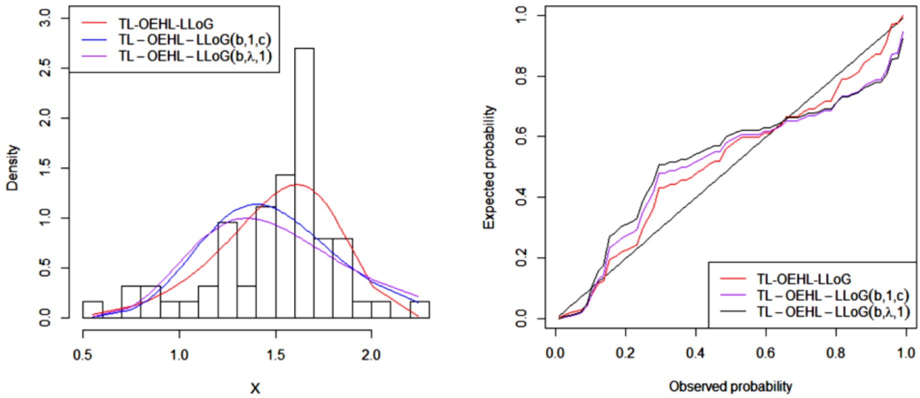
Table 7.1
Estimates of Parameters and Goodness-of-Fit Statistics

	Estimates			Statistics							
Model	b	λ	c	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	K-S	p-value
TL-OEHL-LLoG	0.9253	0.0571	5.3420	28.7	34.7	35.1	41.1	0.1907	1.0587	0.1461	0.1360
	(0.3349)	(0.0512)	(1.2489)								
TL-OEHL-LLoG$(b,1,c)$	7.8171	1	1.4957	48.9	52.9	53.1	57.2	0.5812	3.1945	0.2317	0.0023
	(1.0385)	-	(0.0911)								
TL-OEHL-LLoG$(b,\lambda,1)$	16.9817	1.4923	1	58.2	62.2	62.4	66.5	0.7125	3.9013	0.2215	0.0041
	(4.5125)	(0.1145)	-								
TL-OEHL-LLoG$(b,1,1)$	5.8803	1	1	77.8	79.8	79.9	81.9	0.5898	3.2372	0.3048	1.6530×10^{-5}
	(0.7409)	-	-								
TL-OEHL-LLoG$(1,\lambda,1)$	1	0.5821	1	160.1	162.1	162.1	164.2	0.5184	2.8447	0.4295	1.6150×10^{-10}
	-	(0.0602)	-								
TL-OEHL-LLoG$(1,1,c)$	1	1	1.2273	191.2	193.2	193.3	195.4	0.4654	2.5562	0.6743	$< 2.200 \times 10^{-16}$
	-	-	(0.1012)								
	a	γ	θ								
EFr	0.04621	0.4993	20.1145	189.1	195.1	195.5	201.6	1.1986	6.3098	0.4279	1.9210×10^{-10}
	(0.0153)	(0.0147)	(6.1482)								
MOEFr	54074	0.3858	7.9253	45.6	51.6	51.9	58.0	19.2509	122.7666	0.9997	$< 2.200 \times 10^{-16}$
	(3.8277×10^{-8})	(6.0532×10^{-2})	(0.8731)								
	α	λ	θ								
IWMO	52636	7.9256	2.5828	45.6	51.6	51.9	58.0	0.4974	2.7509	0.1536	0.1020
	(9.7035)	(0.1041)	(54.9189)								
	a	b	γ								
TIIGTLU	48.4266	2.8465	3.2268	30.4	36.4	36.8	42.8	19.4750	124.3383	0.9985	$< 2.200 \times 10^{-16}$
	(336.0839)	(0.4190)	(3.6401)								
TIIGTLE	530890	3.0526	0.0754	30.8	36.8	37.2	43.2	0.2464	1.3526	0.1549	0.0975
	(1.5076×10^{-8})	(0.3499)	(0.0195)								
TIIGTLR	52242	1.4595	0.0092	30.5	36.5	36.9	42.9	0.2410	1.3238	0.1485	0.1241
	(1.5128×10^{-8})	(0.1519)	(3.5319×10^{-5})								

The estimated variance-covariance matrix is

$$\begin{bmatrix} 0.112218 & 0.015979 & -0.377731 \\ 0.015979 & 0.002626 & -0.063259 \\ -0.377731 & -0.063259 & 1.559899 \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by $b \in [0.9253 \pm 0.6566]$, $\lambda \in [0.0571 \pm 0.1004]$ and $c \in [5.3420 \pm 2.4479]$.



(a) (b)

Figure 7.1: Fitted Densities and Probability Plots for Glass Fibre Data

Results from Table 7.1 shows that the TL-OEHL-LLoG model performs better than the non-nested models since it has the lowest values for the goodness-of-fit statistics. Also, Figures 7.1(a) and 7.1(b) demonstrate the flexibility gained by adding some extra parameters to the baseline distribution.

7.2 Silicon Nitride Data

The second data set represents fracture toughness of silicon nitride measured in MPa $m^{1/2}$ (See Nadarajah and Kotz [23], and Ali et al. [2] for details). The data are

5.50, 5.00, 4.90, 6.40, 5.10, 5.20, 5.20, 5.00, 4.70, 4.00, 4.50, 4.20, 4.10, 4.56, 5.01, 4.70, 3.13, 3.12, 2.68, 2.77, 2.70, 2.36, 4.38, 5.73, 4.35, 6.81, 1.91, 2.66, 2.61, 1.68, 2.04, 2.08, 2.13, 3.80, 3.73, 3.71, 3.28, 3.90, 4.00, 3.80, 4.10, 3.90, 4.05, 4.00, 3.95, 4.00, 4.50, 4.50, 4.20, 4.55, 4.65, 4.10, 4.25, 4.30, 4.50, 4.70, 5.15, 4.30, 4.50, 4.90, 5.00, 5.35, 5.15, 5.25, 5.80, 5.85, 5.90, 5.75, 6.25, 6.05, 5.90, 3.60, 4.10, 4.50, 5.30, 4.85, 5.30, 5.45, 5.10, 5.30, 5.20, 5.30, 5.25, 4.75, 4.50, 4.20, 4.00, 4.15, 4.25, 4.30, 3.75,3.95, 3.51, 4.13, 5.40, 5.00, 2.10, 4.60, 3.20, 2.50, 4.10, 3.50, 3.20, 3.30, 4.60, 4.30,4.30, 4.50, 5.50, 4.60, 4.90, 4.30, 3.00, 3.40, 3.70, 4.40, 4.90, 4.90, 5.00.

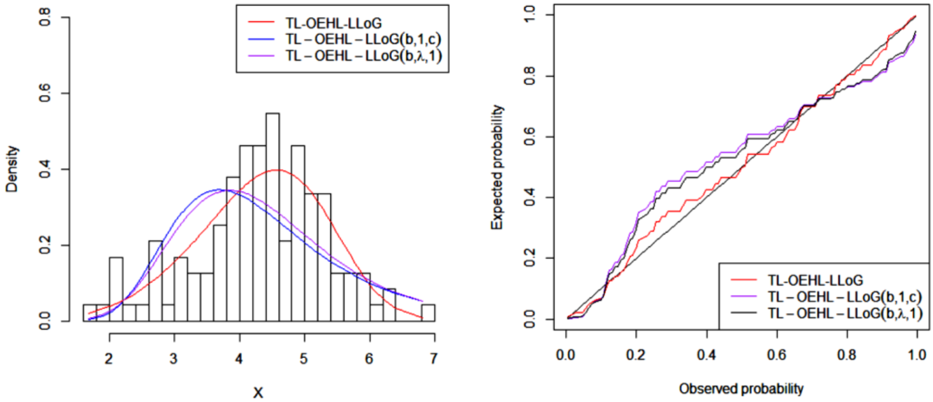
The estimated variance-covariance matrix is

$$\begin{bmatrix} 0.152359 & 1.035309 \times 10^{-3} & -0.311354 \\ 0.001035 & 7.800495 \times 10^{-6} & -0.002369 \\ -0.311354 & -2.369520 \times 10^{-3} & 0.722353 \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by $b \in [1.1509 \pm 0.7651]$, $\lambda \in [0.0018 \pm 0.0055]$ and $c \in [4.0107 \pm 1.6658]$.

Table 7.2
Estimates of Parameters and Goodness-of-Fit Statistics

Model	Estimates			Statistics							
	b	λ	c	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	K-S	p-value
TL-OEHL-LLoG	1.1509	0.0018	4.0107	336.8	342.8	343.0	351.1	0.0793	0.4803	0.0686	0.6297
	(0.3903)	(0.0028)	(0.8499)								
TL-OEHL-LLoG$(b,1,c)$	51.2826	1	0.7216	380.1	384.1	384.2	389.7	0.7841	4.5819	0.1679	0.0024
	(7.5382)	-	(0.0225)								
TL-OEHL-LLoG$(b,\lambda,1)$	16.2375	0.5170	1	368.4	372.4	372.5	377.9	0.5825	3.4679	0.1462	0.0123
	(3.2154)	(0.0299)	-								
TL-OEHL-LLoG$(b,1,1)$	190.4000	1	1	567.6	569.6	569.7	572.4	1.2396	7.0019	0.4958	$<2.2000\times10^{-16}$
	(17.4540)	-	-								
TL-OEHL-LLoG$(1,\lambda,1)$	1	0.2023	1	553.9	555.9	555.9	558.6	0.3256	1.9839	0.4001	$<2.2000\times10^{-16}$
	-	(0.0152)	-								
TL-OEHL-LLoG$(1,1,c)$	1	1	0.3878	888.5	890.5	890.5	893.3	0.4545	2.7321	0.8049	$<2.2000\times10^{-16}$
	-	-	(0.0235)								
	a	γ	θ								
EFr	0.0555	1.5719	18.4091	586.9	592.9	593.2	601.3	1.3837	7.6851	0.3861	7.772×10^{-16}
	(0.0244)	(0.0359)	(7.7522)								
MOEFr	2407.7	1.4344	7.0579	356.6	362.6	362.9	370.9	38.2495	235.4782	0.9989	$<2.2000\times10^{-16}$
	(7.7867×10^{-6})	(0.1296)	(0.5495)								
	α	λ	θ								
IWMO	2407.7	7.0579	0.6972	356.6	362.6	362.9	370.9	0.3596	2.2543	0.0804	0.4241
	(8.9422×10^{-7})	(0.5495)	(0.0630)								
	a	b	γ								
TIIGTLU	118.56	2.4689	12.4128	337.4	343.4	343.6	351.7	38.5743	237.3690	0.9981	$<2.200\times10^{-16}$
	(0.0167)	(0.1767)	(0.8163)								
TIIGTLE	52242	2.6498	0.0292	337.7	343.7	343.9	351.9	0.0978	0.6019	0.0727	0.5547
	(1.2314×10^{-7})	(0.2218)	(0.0053)								
TIIGTLR	4895.3	1.2617	0.0016	337.5	343.5	343.7	351.8	0.0938	0.5769	0.0730	0.5500
	(2.0859×10^{-7})	(0.0964)	(4.0089×10^{-4})								



(a) (b)

Figure 7.2: Fitted Densities and Probability Plots for Silicon Nitride Data

Furthermore, results shown in Table 7.2 affirms that the TL-OEHL-LLoG model performs better than the non-nested models considered in this paper. Also, the generalized model fit the silicon nitride data set better that the reduced models as shown in Figures 7.2(a) and 7.2(b).

7.3 Breaking Stress of Carbon Fibres of 50 mm Length (GPa) Data

The third data set is on breaking stress of carbon fibres of 50 mm length (GPa). Cordeiro and Lemonte [10] also analyzed the same data set. The data are as follows:

0.39, 0.85, 1.08 ,1.25, 1.47, 1.57, 1.61, 1.61, 1.69, 1.80, 1.84, 1.87, 1.89, 2.03, 2.03, 2.05, 2.12, 2.35, 2.41, 2.43, 2.48, 2.50, 2.53, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.79, 2.81,2.82, 2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 3.09 ,3.11, 3.11, 3.15, 3.15,3.19, 3.22, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33, 3.39, 3.39, 3.56, 3.60, 3.65, 3.68, 3.70,3.75, 4.20, 4.38, 4.42, 4.70, 4.90.

Table 7.3
Estimates of Parameters and Goodness-of-Fit Statistics

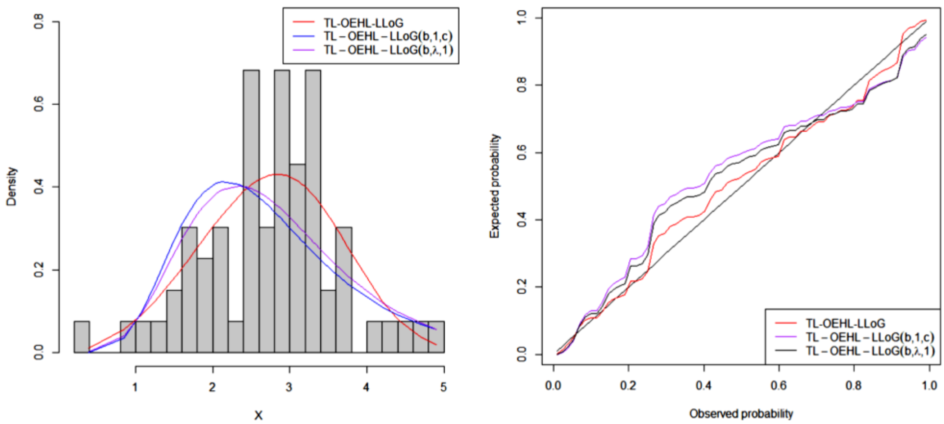
	Estimates			Statistics							
Model	b	λ	c	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	K-S	p-value
TL-OEHL-LLoG	1.1246	0.0381	2.8133	171.7	177.7	178.1	184.3	0.0816	0.4843	0.0797	0.7956
	(0.4785)	(0.0434)	(0.7468)								
TL-OEHL-LLoG$(b, 1, c)$	12.7788	1	0.7585	191.8	195.8	196.0	200.2	0.3692	2.0314	0.1675	0.0493
	(1.8128)	-	(0.0399)								
TL-OEHL-LLoG$(b, \lambda, 1)$	6.2597	0.6250	1	184.7	188.7	188.8	193.0	0.2727	1.4781	0.1402	0.1491
	(1.3160)	(0.0520)	-								
TL-OEHL-LLoG$(b, 1, 1)$	18.0600	1	1	231.2	233.2	233.2	235.4	0.4224	2.3252	0.3432	3.5330×10^{-7}
	(2.2230)	-	-								
TL-OEHL-LLoG$(1, \lambda, 1)$	1	0.3133	1	249.3	251.3	251.4	253.5	0.1982	1.0639	0.3483	2.2260×10^{-7}
	-	(0.0317)	-								
TL-OEHL-LLoG$(1, 1, c)$	1	1	0.5435	383.2	385.2	385.3	387.4	0.2614	1.4179	0.7357	$< 2.0000 \times 10^{-16}$
	-	-	(0.0439)								
	a	γ	θ								
EFr	0.3007	0.6302	2.3305	341.4	347.4	347.8	354.0	0.9810	5.5289	0.4060	7.0870×10^{-10}
	(0.0784)	(0.0913)	(0.3720)								
MOEFr	2.8873×10^{5}	0.2078	4.8956	183.3	189.3	189.7	195.9	20.9524	130.1468	0.9999	$< 2.0000 \times 10^{-16}$
	(6.6594×10^{-9})	(0.0574)	(0.5133)								
	α	λ	θ								
IWMO	3.1698×10^{5}	4.8954	4.9045	183.3	189.2	189.7	195.9	0.2593	1.3935	0.0939	0.6062
	(4.3098×10^{-6})	(0.5135)	(1.3648)								
	a	b	γ								
TIIGTLU	59.8163	1.7022	10.2218	172.1	178.1	178.5	184.7	20.9549	131.2354	0.9991	$< 2.0000 \times 10^{-16}$
	(0.0527)	(0.1630)	(1.1141)								
TIIGTLE	4.3593×10^{4}	1.7590	0.0161	172.2	178.2	178.6	184.8	0.0950	0.5340	0.0820	0.7669
	(1.2827×10^{-9})	(0.1791)	(0.0050)								
TIIGTLR	1.1821×10^{3}	0.8669	0.0018	172.2	178.2	178.6	184.7	0.0937	0.5285	0.0822	0.7644
	(6.9287×10^{-8})	(0.0857)	(7.1227×10^{-4})								

The estimated variance-covariance matrix is

$$\begin{bmatrix} 0.228940 & 0.019714 & -0.332124 \\ 0.019714 & 0.001884 & -0.032218 \\ -0.332124 & -0.032218 & 0.557642 \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by $b \in [1.1246 \pm 0.9378]$, $\lambda \in [0.0381 \pm 0.0851]$ and $c \in [2.8133 \pm 1.4636]$.

In addition, from the third example results shown in Table 7.3, we conclude that the TL-OEHL-LLoG model indeed performs better than the non-nested models considered in this paper. Also, the generalized model fit the carbon fibres data set better that the reduced models as shown in Figures 7.3(a) and 7.3(b).



(a) (b)

Figure 7.3: Fitted Densities and Probability Plots for Carbon Fibres Data

7.3.1 Likelihood Ratio Test

We present likelihood ratio test results in Table 7.4.

Table 7.4
Likelihood Ratio Test Results

Distribution	Data Set 1 χ^2 (p-value)	Data Set 2 χ^2 (p-value)	Data Set 3 χ^2 (p-value)
TL-OEHL-LLoG$(b, 1, c)$	20.2 (<0.00001)	43.3 (<0.00001)	20.1 (<0.00001)
TL-OEHL-LLoG$(b, \lambda, 1)$	29.5 (<0.00001)	31.6 (<0.00001)	13.0 (0.00031)
TL-OEHL-LLoG$(b, 1, 1)$	49.1 (<0.00001)	230.8 (<0.00001)	59.5 (<0.00001)
TL-OEHL-LLoG$(1, \lambda, 1)$	131.4 (<0.00001)	217.1 (<0.00001)	77.6 (<0.00001)
TL-OEHL-LLoG$(1, 1, c)$	162.5 (<0.00001)	551.7 (<0.00001)	211.5 (<0.00001)

Based on the results shown in Table 7.4, we conclude that the TL-OEHL-LLoG model performs better than its nested models.

8. CONCLUDING REMARKS

A new generalized distribution called the Topp-Leone-odd exponential half logistic-G (TL-OEHL-G) was developed. The new distribution applies to data sets with heavy tails and different shapes of hazard rate function. The statistical properties of the proposed distribution can be derived directly from those of the Exp-G distribution. The TL-OEHL-G distribution is a flexible and versatile distribution.

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