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A NEW VALUE FOR TU-GAMES: THE RANK-SHAPLEY VALUE

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ABSTRACT

The vast interpretation of weight in weighted Shapley value has offered opportunities for different scheme in sharing the dividend of a coalition in a transferable utility game. In this paper, a new allocation scheme (which makes use of linear ranks as a weight function of an individual player) as well as its axiomatic characterization is proposed. The use of rank, a linear function of player's stand-alone value as a weight of an individual player is introduced and the dividend accruable to any coalition is shared in proportion to the player's rank. Also, the dual equivalence of the new method is presented. This method is feasible in all class of cooperative game with transferable utility unlike the proportional Shapley value of Beal et al. (2018) that is restricted to only individually positive and individually negative games.

KEYWORDS

Rank; stand-alone value; dividend; Shapley value.

1. INTRODUCTION

A cooperative game with transferable utility is an aspect of game theory where players are permitted to cooperate or form coalitions by pulling their resources or strategies together so as to maximize their payoff. It captures a situation in which players can achieve certain payoffs from cooperation (Huettner, 2015). The application of cooperative game theory in solving real life problems has generally enlarged the scope of game theory. Thus, its application has found essence in many fields of study, especially where conflict of interest is involved (see, Shenoy, 1980; Zara et al., 2006; Naima and Odd, 2010). In a cooperative game, it is basically assumed that the coalition involving all the players in the game will form. When the coalition is formed, the question remains on how to allocate in a fair way, the gain of the coalition among the players. In the literature of cooperative game theory with transferable utility, different allocation schemes based on various notion of fairness have been proposed. However, the most popular is the solution concept that is based on average marginal contribution, attributed to Shapley (1953b). The Shapley allocation scheme popularly known as the Shapley value assumes equal weights for all the players in a particular coalition and allocates to each player the average of his marginal contribution in all the coalition in which the player is a member. The Shapley value is characterized by the following axioms: efficiency, symmetry, additivity and dummy axiom.

In many applications, however, the assumption that every player has equal weight in a game may not be appropriate (Levy & McLean, 1989) as players sometimes, represent constituencies of different sizes, or have different bargaining powers. This idea is reflected in the family of weighted Shapley value where every player in a game is associated with a positive weight according to which the dividend accruable to any coalition is shared. Harsanyi (1959) introduced the concept of dividend which was defined as what remains of a coalition say, θ after subtracting the worth of all nonempty sub-coalitions $T \subset \theta$.

Kalai and Samet (1987) popularized the weighted Shapley value by proposing a scheme expressed in terms of weighted dividend for all positive weights $\omega_i \in \mathbb{R}_+$. However, no particular weights were recommended, rather it was noted that the weights can be determined by considering such factors as bargaining ability, patience rates, past experience or size of the players (where players themselves are group of individuals). Haeringer (1998) argued that the sharing weight has to be inverted from maximization to minimization direction in any coalition for which the dividend is less than zero. This is to reflect the players' interest which is centered on minimizing losses and maximizing gains. Based on this contention, a framework for weighted Shapley value was developed in a context where weights are interpreted as measures of the bargaining power of the players. Hence, two different weight vectors: $\omega^+ = (\omega_1, \omega_2, \dots, \omega_n)$ and $\omega^- = \left(\frac{1}{\omega_1}, \frac{1}{\omega_2}, \dots, \frac{1}{\omega_n}\right)$ were recommended as the sharing weight of players for dividends greater than zero and dividends less than zero, respectively. Dehez (2017) proposed an allocation scheme by specifically assigning a unit weight to each player and sharing the dividend of every coalition in proportion to the unit weight. The weighting system in the scheme was explicit as it was specified to be a ratio based on the cardinality of the coalition containing player i.

In an attempt to adjust the allocation scheme of Kalai and Samet (1987), Beal et al., (2018) introduced a non-linear weighted Shapley value for TU (transferable utility) cooperative games in characteristics function form, in which the weights are endogenously given by the players' stand-alone value, $v(i)$. The allocation scheme is known as the proportional Shapley value since the share of the dividend was based on the proportion of the players' stand-alone value. In proportional Shapley value, player's stand-alone value is used as weight. Based on this, the proportional Shapley value has the possibility of allocating an undefined or irrational value to a player in a TU-cooperative game. In a general sense, the stand-alone value can take any sign, hence the allocation of an undefined (unfair) value to the players whenever $v(i) = 0$ for all $i \in N$. Consequent upon this, the application of the allocation scheme is limited to a certain class of games which are individually positive or individually negative. Due to these shortfalls, there is a need to relax the condition of individually positive and individually negative games and develop an allocation scheme based on the framework of weighted Shapley value whose weight system is a function of the stand-alone value instead of the stand-alone values themselves. In this note, a new allocation scheme that is based on the ranks of the stand-alone values of the players, as well as its dual equivalence is proposed in this work. The proposed scheme has practical essence and it is feasible in all kinds of TU-game.

2. BASIC DEFINITION AND NOTATIONS

In this work, a complete coalition structure in which every coalition in 2^n is feasible is adopted. Throughout this work, capital letters are used to denote coalition and the corresponding small case to denote their cardinality. Where Greek letters are used to denote coalition, $|.|$ is used to denote the cardinality. Let $\theta \in 2^n$ denote a coalition. For any game involving $n-$ players, there are 2^n possible coalitions that can form the coalition structure.

For any coalition $\theta \in 2^n$, $v(\theta)$: $2^n \to \mathbb{R}$ is a function that assigns a real value to each coalition. By convention, $v(\emptyset) = 0$, hence, the coalition space is restricted to $2^n - 1$. Consider a particular player of interest, say player i. In a complete coalition structure, every player i participates in 2^{n-1} different coalitions.

Definition 1.

Let $v(1)$, $v(2)$, …, $v(n)$ be a random set of stand-alone values of n-players in a game (N, v). Let $v(1) < v(2) < \cdots < v(n)$ be the corresponding order of the values. If the stand-alone values are uniquely defined, $v(i)$ is weighted with a rank r_i among $v(1)$, $v(2)$, …, $v(n)$. By being uniquely defined, it implies non-existence of ties. That is $v(i) \neq v(j)$ for all $i \neq j$. However, if there is an existence of ties ($v(i) = v(j)$ for at least one $i \neq j$), the average rank of the tied stand-alone values is assigned to each of the affected players.

Assuming that all the players have played individually (non-cooperatively) and their stand-alone values $v(i)$ are known. Let the stand-alone values of the players be ranked as follows: $r_1, r_2, \dots, r_n$ where r_i is a positive integer (but sometimes fractions in a case of ties). It is assumed that the players enter into coalitions according to their ranks. Let ${r_i}^{max}$ be the rank of the player with highest stand-alone value. The player with ${r_i}^{max}$ will be more marketable during coalition formation than others since it is expected by a binding agreement that the player will go into any coalition with full capacity. Similarly, let ${r_i}^{min}$ be the rank of the player with smallest stand-alone value. In practical sense, the player with ${r_i}^{min}$ will be the least player to be considered during coalition formation. The marketability of the players in every coalition can therefore, be expressed in terms of probability as

$$P_\theta(i = r_i) = \frac{r_i}{\pi_\theta} \quad (1)$$

where $\pi_\theta = \sum_{i \in \theta} r_i$.

In the rest of this work, $\sum_{i \in \theta} r_i$ and π_θ will be used interchangeably for convenience. Specifically, the marketability of a player in the formation of the grand coalition is

$$P_N(i = r_i) = \frac{r_i}{\pi_N} = \frac{2r_i}{n(n+1)} \quad (2)$$

The marketability measure $P_\theta(i = r_i)$ is interpreted as the proportion in which player i share the dividend accruable to any coalition θ of which player i is a member.

3. THE PROPOSED METHOD

Let $\mathbb{N}$ be a universe of players. Let $N = \{1, 2, \ldots, n\}: n \geq 2, N \subset \mathbb{N}$ be a fixed and finite set of players, such that the players can freely cooperate through a binding agreement. Any subset of N is called a coalition and N itself is called a grand coalition. The space for all the feasible coalition is 2^n. For any coalition $\theta \in 2^n$, $v(\theta): 2^n \to \mathbb{R}$ is a function that assigns a real value to each coalition. By convention, $v(\emptyset) = 0$. Define Ω^{N} to be a collection of all games. For any two games $(G_i)_{i=1,2} = (N, v_i) \in \Omega^{\mathrm{N}}$ defined on the same set of players, the addition of the games is defined as $(v_1 + v_2)(\theta) = v_1(\theta) + v_2(\theta)$ and for any scalar α, the product is defined as $(\alpha v)(\theta) = \alpha v(\theta)$. Thus, Ω^{N} is a vector space. Let u_θ be the unanimity game for coalition θ. For any $T \subseteq N$,

$$u_\theta(T) = \begin{cases} 1 & if\ \theta \subseteq T \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

In other words, a coalition T has value 1 (is winning) if it contains all the players of θ and value 0 (is losing) if it does not contain all the players of θ. The unanimity game forms a basis for the vector space Ω^{N} since the vector space is endowed with the usual linear operation of addition and scalar product (Dragan, 1991). Thus, any game $v \in \Omega^{\mathrm{N}}$ can be decomposed as a linear combination of unanimity games.

$$v = \sum_{\theta \in 2^n} \beta_\theta . u_\theta \tag{4}$$

where β_θ, the Harsanyi dividend is the coordinate of the basis. $\beta_\theta = H_v(\theta)$ is defined as the dividend accruable to a coalition θ based on v (Harsanyi, 1959). In a functional form, the Harsanyi dividend is defined as

$$H_v(\theta) = v(\theta) - \sum_{T \subset \theta} H_v(T) = \sum_{T \subseteq \theta} (-1)^{|\theta| - t}\, v(T) \tag{5}$$

For any unanimity game u_θ, Shapley (1953a) assigned a unit weight to each of the players and shared a unit dividend among the members of the coalition $\theta \ni i$ in proportion to the unit weight. Thus the Weighted Shapley value is given as

$$\varphi_i(u_\theta) = \sum_{i \in \theta;\, \theta \in 2^n} \frac{1}{|\theta|} \mathrm{H}_v(\theta) = \frac{1}{|\theta|} \tag{6}$$

Recall that the essence of coalition formation in a cooperative game is to maximize player's payoff such that for every player i, the payoff $x_i \geq v(i)$. As defined earlier (definition 1), let r_i be the rank of player i. The ranking is done in descending order such that the player with highest stand-alone value takes the highest rank. We propose the proportion for sharing the dividend of any coalition to be $\frac{r_i}{\pi_\theta}$, where r_i is the rank of player i and $\pi_\theta = \sum_{i \in \theta} r_i$ is the sum of the ranks of the players in the coalition θ of which i is a member. Following the framework of Shapley (1953a), we propose a single value solution (allocation scheme) known as Rank-Shapley (Rsh) value which assigns an efficient payoff vector for any game v with transferable utility,

$$Rsh_i(N,v,r) = \sum_{i\in\theta;\ \theta\in 2^n} \frac{r_i}{\pi_\theta} \mathrm{H}_v(\theta) \tag{7}$$

The proposed method shares the dividend in each coalition in proportion to the player's rank and it is feasible in all class of TU-games. However, if the stand-alone values are equal, the Rank-Shapley value coincides with the Shapley value.

3.1 Motivating Example

For justification of the practical essence of the proposed method, consider an example given below.

Example 1

Three transportation companies (P, R, and G) are in the business of commuting passengers from location A to location B. The profit made in a day is a linear function of the number of passengers commuted. Based on past records, it is expected that the companies will have number of passengers as presented in Table 1 below. The profit function is given as

$$f(x) = \begin{cases} 0 \text{ if } x < 10 \\ 10x \text{ if } 10 \leq x \leq 20 \\ 10x + 20 \text{ if } 20 < x \leq 30 \\ 10x + 30 \text{ if } x > 30 \end{cases} \tag{8}$$

Table 1
No. of Passengers

Companies	***P***	***R***	***G***
No. of Passengers	**15**	**12**	**7**

The three companies can cooperate in commuting the passengers so as to maximize their profit. To share the profit, the problem can be formulated as a cooperative game and solved as follows:

Table 2
The Game

$\boldsymbol{\theta}$	$\{P\}$	$\{R\}$	$\{G\}$	$\{P,R\}$	$\{P,G\}$	$\{R,G\}$	$\{P,R,G\}$
$\boldsymbol{v(\theta)}$	150	120	0	290	240	190	370
$\boldsymbol{H_v(\theta)}$	150	120	0	20	90	70	-80
Rank $(\boldsymbol{\pi_\theta})$	3	2	1	5	4	3	6

Assuming the proportional Shapley value of Beal et al. (2018) is used in solving the problem, the following payoff will be allocated to the companies.

$$PSh_i(N,v) = \sum_{i\in\theta;\ \theta\in 2^n} \frac{v(i)}{\sum_{k\in\theta} v(k)} \mathrm{H}_v(\theta)$$
$$= (206.7, 163.3, 0)$$

This solution is not rational (fair) to company-G. It is obvious from the example that company-G has a non-zero contribution in all the coalitions formed with other companies, yet company-G is allocated a zero value by proportional Shapley value. It means that company-G is not recognized to have had any incentive by forming coalition with other companies.

However, if the problem is solved using Rank-Shapley value, an improvement in the allocation to company-G is made.

$$RSh_i(N,v) = \sum_{i\in\theta;\ \theta\in 2^n} \frac{r_i}{\pi_\theta} \mathrm{H}_v(\theta)$$
$$= (189.5, 148, 32.5)$$

The Rank-Shapley value is fair enough to allocate a non-zero value to company-G because it shares the dividend of every coalition in proportion to the rank of the player's stand-alone value instead of sharing the dividend in proportion to the player's stand-alone value. This is a reflection of the fact that company-G deserves to be rewarded for its contribution in all the coalitions formed with other companies.

4. CHARACTERIZATION OF RANK-SHAPLEY VALUE

Let Π be a specific and unique order (ranking). By being unique, it implies that Π does not involve ties. To generate a Π-dependent basis, we adjust the unanimity basis through elementary row operation to reflect the order Π. The elementary row operation is summarized as follows: add $\sum_{i\in T}(r_i)a\,; a \geq 2, i = 1, 2, \dots, n$ to $R_{\{T\}}$ (Row-$\{T\}$) in the unanimity basis $\mu_{\{\theta\}}$. The essence of the adjustment is to ensure a specific order Π and supper-additivity in the new basis. Thus,

$$\mu_{\{\theta\}}^{\Pi}(T) = \begin{cases} 1 + \sum_{i\in T}(r_i)a \ \text{ if } \theta \subseteq T \\ \sum_{i\in T}(r_i)a \quad \text{otherwise} \end{cases}$$

We call $\mu_{\{\theta\}}^{\Pi}$ a Π-dependent game (basis), interpreted as a collection of games in which every player enters coalitions with pre-determined weight ar_i. When all the players in θ cooperate, there is an additional reward of 1 which is to be shared among the cooperating players in accordance with their pre-determined weights. In a Π-dependent basis, the cooperation of all the players in θ yields an additional unit payoff with $\sum_{i\in T}(r_i)a$ reflecting the aggregate weights of players in T. Unlike the unanimity basis, the Π-dependent basis is not a simple game and $\mu^{\Pi}_{\{\theta\}}$ is an inessential game for all $|\theta| = 1$.

Let φ_i be the Rank-Shapley value of $\mu^{\Pi}_{\{\theta\}}$. As a value, it assigns to each player $i \in N$ a real number and it satisfies the conditions of the following axioms:

1. Efficiency: $\sum_{i\in N} \varphi_i\left(\mu^{\Pi}_{\{\theta\}}\right) = \mu^{\Pi}_{\{\theta\}}(N)$
2. Inessential game property: $\varphi_i\left(\mu^{\Pi}_{\{\theta\}}\right) = \mu^{\Pi}_{\{\theta\}}(i)$ for all $|\theta| = 1$.
3. Redundancy (dummy) axiom: $\varphi_i\left(\mu^{\Pi}_{\{\theta\}}\right) = \mu^{\Pi}_{\{\theta\}}(i)$ for all $i \notin \theta$. This axiom implies that any player that is not in θ is a redundant player since the player contributes nothing in cooperation. The value of a redundant player is unaffected by the value of the coalitions formed with other players.
4. Scaled balanced benefit axiom: For any $i, j \in \theta$, $\frac{\varphi_i\left(\mu^{\Pi}_{\{\theta\}}\right) - \mu^{\Pi}_{\{\theta\}}(i)}{ar_i} = \frac{\varphi_j\left(\mu^{\Pi}_{\{\theta\}}\right) - \mu^{\Pi}_{\{\theta\}}(j)}{ar_j}$
5. Π −Additivity: $\varphi_i\left(\mu^{\Pi}_{\{\theta\}} + \mu^{\Pi}_{\{\theta'\}}\right) = \varphi_i\left(\mu^{\Pi}_{\{\theta\}}\right) + \varphi_i\left(\mu^{\Pi}_{\{\theta'\}}\right)$ for all $\theta \subseteq N$.

Lemma 1

Let φ_i be a Rank-Shapley value,

$$\varphi_i\left(\mu^{\Pi}_{\{\theta\}}\right) = \begin{cases} \mu^{\Pi}_{\{\theta\}}(i) \; if \; |\theta| = 1 \\ \mu^{\Pi}_{\{\theta\}}(i) + \dfrac{r_i}{\pi_\theta} \text{ otherwise} \end{cases}$$

where r_i is the rank of player i stand-alone value, $\pi_\theta = \sum_{i\in\theta} r_i$ and $\mu^{\Pi}_{\{\theta\}}(i)$ is the stand-alone value of player i in $\mu^{\Pi}_{\{\theta\}}$.

Proof:

From axiom 4, $\left[\varphi_i\left(\mu^{\Pi}_{\{\theta\}}\right) - \mu^{\Pi}_{\{\theta\}}(i)\right]ar_j = ar_i\left[\varphi_j\left(\mu^{\Pi}_{\{\theta\}}\right) - \mu^{\Pi}_{\{\theta\}}(j)\right]$

Take the sum of both sides over $j \in \theta$

$$\left[\varphi_i\left(\mu^{\Pi}_{\{\theta\}}\right) - \mu^{\Pi}_{\{\theta\}}(i)\right]a\sum_{j\in\theta} r_j = ar_i \sum_{j\in\theta}\left[\varphi_j\left(\mu^{\Pi}_{\{\theta\}}\right) - \mu^{\Pi}_{\{\theta\}}(j)\right]$$

$$\varphi_i\left(\mu^{\Pi}_{\{\theta\}}\right) = \mu^{\Pi}_{\{\theta\}}(i) + \frac{r_i}{\sum_{j\in\theta} r_j}\sum_{j\in\theta}\left[\varphi_j\left(\mu^{\Pi}_{\{\theta\}}\right) - \mu^{\Pi}_{\{\theta\}}(j)\right]$$

For $|\theta| = 1$, $\sum_{j\in\theta}\left[\varphi_j\left(\mu^{\Pi}_{\{\theta\}}\right) - \mu^{\Pi}_{\{\theta\}}(j)\right] = 0$. This implies that $\varphi_i\left(\mu^{\Pi}_{\{\theta\}}\right) = \mu^{\Pi}_{\{\theta\}}(i)$.

However for $|\theta| \geq 2$, $\sum_{j\in\theta}\left[\varphi_j\left(\mu^{\Pi}_{\{\theta\}}\right) - \mu^{\Pi}_{\{\theta\}}(j)\right] = 1$ implying that

$$\varphi_i\left(\mu^{\Pi}_{\{\theta\}}\right) = \mu^{\Pi}_{\{\theta\}}(i) + \frac{r_i}{\sum_{j\in\theta} r_j} = \mu^{\Pi}_{\{\theta\}}(i) + \frac{r_i}{\pi_\theta}$$

By construction, $\mu^{\Pi}_{\{\theta\}}$ is an inessential (quasi-additive) game for all $|\theta| = 1$. As an inessential game, there is no incentive for cooperation, thus, $\varphi_i\left(\mu^{\Pi}_{\{\theta\}}\right) = \mu^{\Pi}_{\{\theta\}}(i)$. For $|\theta| \geq 2$, there is a unit incentive (dividend) only when all the members of θ cooperate. Recall that all the players in θ are not necessarily the same in rank. So in addition to the player's stand-alone value, the unit incentive (dividend) accrued to θ is shared according to the player's rank. Thus, $\varphi_i\left(\mu_{\{\theta\}}^{\Pi}\right) = \mu_{\{\theta\}}^{\Pi}(i) + \frac{r_i}{\pi_\theta}$.

Lemma 2

Any game v defined on a fixed set of players N is a linear combination of $\mu_{\{\theta\}}^{\Pi}$.

$$v = \sum_{\theta \subseteq N} \delta_v(\theta)\, \mu_{\{\theta\}}^{\Pi}$$

where $\delta_v(\theta)$ is the coefficient of $\mu_{\{\theta\}}^{\Pi}$. The coefficient is a function of the Harsanyi dividend, $H_v(\theta)$ and it is dependent on N.

$$\delta_v(\theta) = \begin{cases} H_v(\theta) - \dfrac{a r_i v(N)}{1 + \sum_{i \in N}(r_i)a} & \text{for } |\theta| = 1 \\ H_v(\theta) & otherwise \end{cases} \tag{9}$$

Proof:

Let $w(T) = \sum_{i \in T}(r_i)a$ be an inessential game and let $\mu_{\{\theta\}}(T) = \begin{cases} 1 & \text{if } \theta \subseteq T \\ 0 & \text{otherwise} \end{cases}$ be the unanimity basis (Shapley, 1953b).

By definition, $\mu_{\{\theta\}}^{\Pi}(T) = \mu_{\{\theta\}}(T) + w(T)$.

$$v = \sum_{\theta \subseteq N} \delta_v(\theta)\, \mu_{\{\theta\}}^{\Pi} = \sum_{\theta \subseteq N} \delta_v(\theta) \left(\mu_{\{\theta\}} + w\right)$$

By decomposition, $w = \sum_{i \in N}(r_i)a\, \mu_{\{i\}}$ since $H_w(\theta) = 0\ \forall\ |\theta| \geq 2$ (inessential game).

$$\begin{aligned} v &= \sum_{\theta \subseteq N} \delta_v(\theta)\, \mu_{\{\theta\}} + \sum_{\theta \subseteq N} \delta_v(\theta) \sum_{i \in N} (r_i)a\, \mu_{\{i\}} \\ &= \sum_{i \in N} \delta_v(i)\, \mu_{\{i\}} + \sum_{\theta : |\theta| \geq 2} \delta_v(\theta)\, \mu_{\{\theta\}} + \sum_{\theta \subseteq N} \delta_v(\theta) \sum_{i \in N} (r_i)a\, \mu_{\{i\}} \\ &= \sum_{i \in N} \left[\delta_v(i) + \sum_{\theta \subseteq N} \delta_v(\theta)(r_i)a\right] \mu_{\{i\}} + \sum_{\theta : |\theta| \geq 2} \delta_v(\theta)\, \mu_{\{\theta\}} \end{aligned}$$

Recall that

$$v(N) = \sum_{\theta \subseteq N} \delta_v(\theta)\, \mu_{\{\theta\}}^{\Pi}(N) = \sum_{\theta \subseteq N} \delta_v(\theta) \left[1 + \sum_{i \in N} (r_i)a\right]$$

$$\sum_{\theta \subseteq N} \delta_v(\theta) = \frac{v(N)}{1 + \sum_{i \in N}(r_i)a}$$

Therefore,

$$v = \sum_{i \in N} \left[\delta_v(i) + \frac{a r_i v(N)}{1 + \sum_{i \in N}(r_i) a} \right] \mu_{\{i\}} + \sum_{\theta : |\theta| \geq 2} \delta_v(\theta)\, \mu_{\{\theta\}}$$

From (9), $H_v(i) = \delta_v(i) + \dfrac{a r_i v(N)}{1 + \sum_{i \in T}(r_i) a}$ and $H_v(\theta) = \delta_v(\theta)$ for $|\theta| \geq 2$.

This completes the proof.

Therefore,

$$v = \sum_{i \in N} H_v(i)\, \mu_{\{i\}} + \sum_{\theta : |\theta| \geq 2} H_v(\theta)\, \mu_{\{\theta\}} = \sum_{\theta \subseteq N} H_v(\theta)\, \mu_{\{\theta\}}.$$

Theorem 1

The Rank-Shapley value is the unique value that satisfies inessential game property, efficiency, redundancy (dummy), scaled balance benefit and Π −additivity axiom.

Proof:

Assuming that all these properties are satisfied, we want to derive the Rank-Shapley value of any game v.

Consider $\theta \sqsubseteq$ N and a Π-dependent basis $\mu^{\Pi}_{\{\theta\}}$ defined on θ. By efficiency,

$$\sum_{i \in N} \varphi_i\left(\mu^{\Pi}_{\{\theta\}}\right) = \sum_{i \in N} \mu^{\Pi}_{\{\theta\}}(i) = \left(1 + \sum_{i \in N} r_i a\right) = \mu^{\Pi}_{\{\theta\}}(N) \text{ for } |\theta| = 1.$$

For $|\theta| \geq 2$, $\displaystyle\sum_{i \in N} \varphi_i\left(\mu^{\Pi}_{\{\theta\}}\right) = \sum_{i \in N} \mu^{\Pi}_{\{\theta\}}(i) + 1 = \left(1 + \sum_{i \in N} r_i a\right) = \mu^{\Pi}_{\{\theta\}}(N)$

The redundancy axiom follows from the above as the value (stand-alone value) of the complement of θ to achieve efficiency. For scaled balance benefit axiom, we concentrate on any two players say i and j, and consider their individual benefit due to cooperation. For $|\theta| = 1$, the individual benefit of any player due to cooperation is zero (trivial) since $\mu^{\Pi}_{\{\theta\}}\ \forall\ |\theta| = 1$ is an inessential game. However, for $|\theta| \geq 2$, the scaled balance benefit axiom follows from the fact that each player in θ gets in addition to his stand-alone value, a share of the unit incentive (dividend) in proportion to the player's pre-determined weight. For Π −additivity, consider the following game decomposition,

$$v = \sum_{\theta \subseteq N} \delta_v(\theta)\, \mu^{\Pi}_{\{\theta\}}$$

$$\varphi_i(v) = \varphi_i\left(\sum_{\theta \subseteq N} \delta_v(\theta)\, \mu^{\Pi}_{\{\theta\}}\right) = \sum_{\theta \subseteq N} \delta_v(\theta)\, \varphi_i\left(\mu^{\Pi}_{\{\theta\}}\right)$$
$$= \sum_{\theta \subseteq N} \delta_v(\theta) \left[\mu^{\Pi}_{\{\theta\}}(i) + \mu^{\Pi}_{\{\theta\}}(i) + \frac{r_i}{\pi_\theta}\right]$$

$$= \sum_{|\theta|=1} \delta_v(\theta)\, \mu^{\Pi}_{\{\theta\}}(i) + \sum_{|\theta|\geq 2} \delta_v(\theta)\, \mu^{\Pi}_{\{\theta\}}(i) + \sum_{|\theta|\geq 2} \delta_v(\theta) \frac{r_i}{\pi_\theta}$$

For $|\theta| = 1$, $\mu^{\Pi}_{\{\theta\}}(i) = (1 + ar_i)$ for one i and ar_i for others. Similarly, for $|\theta| \geq 2$, $\mu^{\Pi}_{\{\theta\}}(i) = ar_i$.

$$\varphi_i(v) = \delta_v(i)(1 + ar_i) + \sum_{i=1}^{n-1} \delta_v(i) ar_i + \sum_{|\theta|\geq 2} \delta_v(\theta) ar_i + \sum_{|\theta|\geq 2} \delta_v(\theta) \frac{r_i}{\pi_\theta}$$
$$= \delta_v(i)(1 + ar_i) + \sum_{\substack{\theta \in 2^n \\ \theta \neq i}} \delta_v(\theta) ar_i + \sum_{|\theta|\geq 2} \delta_v(\theta) \frac{r_i}{\pi_\theta}$$

Recall that

$$\sum_{\theta \subseteq N} \delta_v(\theta) = \frac{v(N)}{1 + \sum_{i\in N}(r_i)a}$$

It follows that

$$\sum_{\substack{\theta \in 2^n \\ \theta \neq i}} \delta_v(\theta) = \frac{v(N)}{1 + \sum_{i\in N}(r_i)a} - \delta_v(i).$$

$$\varphi_i(v) = \delta_v(i)(1 + ar_i) + \frac{v(N) ar_i}{1 + \sum_{i\in N}(r_i)a} - \delta_v(i) ar_i + \sum_{|\theta|\geq 2} \delta_v(\theta) \frac{r_i}{\pi_\theta}$$
$$= \delta_v(i) + \frac{v(N) ar_i}{1 + \sum_{i\in N}(r_i)a} + \sum_{|\theta|\geq 2} \delta_v(\theta) \frac{r_i}{\pi_\theta}$$

But $\delta_v(i) + \dfrac{v(N) ar_i}{1 + \sum_{i\in N}(r_i)a} = H_v(i)$ and $\delta_v(\theta) = H_v(\theta)$ for $|\theta| \geq 2$.

Also, $r_i = 0\ \forall\ i \notin \theta$. So, we restrict the summation to only $i \in \theta$. Therefore,

$$\varphi_i(v) = H_v(i) + \sum_{|\theta|\geq 2} \frac{r_i}{\pi_\theta} H_v(\theta) = \sum_{\substack{\theta \in 2^n \\ i \in \theta}} \frac{r_i}{\pi_\theta} H_v(\theta)$$

By the proof of theorem 1, the payoff of $\mu^{\Pi}_{\{\theta\}}$ can uniquely be determined by the above five axioms. These axioms are the general rules of division which justify the proposed value and validate the theory behind its formation.

Theorem 2

Let F_0 be a class of games for which $H_v(\theta) = c\ \forall\ |\theta| > 1, c > 0$. For any given value $\varphi = [\varphi_1, \varphi_2, \dots, \varphi_n]$, there is a unique game $v \in F_0$.

Proof:

We assume that φ is a Rank-Shapley value. Our interest is to generate a unique game $v \in F_0$. By efficiency property of Rank-Shapley value, $\sum_{i\in N} \varphi_i = v(N)$. From

$\varphi = [\varphi_1, \varphi_2, \dots, \varphi_n]$, the ranks $[r_1, r_2, \dots, r_n]$ of the stand-alone values can be deduced since there is ordinal equivalence between the rank of the stand-alone values $v(i)$ and RSh_i.

Recall that the Rank-Shapley value

$$\varphi_i(N, v) = \sum_{\substack{i\in\theta \\ \theta\in 2^n}} \frac{r_i}{\pi_\theta} H_v(\theta) = v(i) + \sum_{\substack{i\in\theta;\, \theta\in 2^n \\ \theta\neq\{i\}}} \frac{r_i}{\pi_\theta} H_v(\theta)$$

Since $v \in \mathrm{F}_0$, $H_v(\theta) = c \;\forall\; |\theta| > 1$. Denote $\delta_i = \sum_{\substack{i\in\theta;\, \theta\in 2^n \\ \theta\neq\{i\}}} \frac{r_i}{\pi_\theta}$

Therefore,

$$\varphi_i = v(i) + c\delta_i \tag{10}$$

For any coalition θ,

$$\sum_{i\in\theta} \varphi_i = \sum_{i\in\theta} v(i) + c \sum_{i\in\theta} \delta_i \tag{11}$$

Recall also that

$$H_v(\theta) = v(\theta) - \sum_{T\subsetneq\theta} H_v(T) = c \;\text{ and }\; H_v(T) = c \;\forall\; |T| > 1.$$

In every coalition θ, there are $\left(2^{|\theta|} - 2 - |\theta|\right)$ of such coalitions $T \subsetneq \theta, |T| > 1$. Therefore,

$$H_v(\theta) = v(\theta) - \sum_{i\in\theta} v(i) - \left(2^{|\theta|} - 2 - |\theta|\right)c = c. \tag{12}$$

From (12),

$$\sum_{i\in\theta} v(i) = v(\theta) - c - \left(2^{|\theta|} - 2 - |\theta|\right)c \tag{13}$$

Substitute (13) into (11) and solve for $v(\theta)$. Thus,

$$v(\theta) = \sum_{i\in\theta} \varphi_i + \left(2^{|\theta|} - |\theta| - 1 - \sum_{i\in\theta} \delta_i\right)c \tag{14}$$

For $\theta = N, v(N) = \sum_{i\in N} \varphi_i$ since $\sum_{i\in N} \delta_i = 2^n - n - 1$.

(14) is a unique game $v \in \mathrm{F}_0$ for which $H_v(\theta) = c \;\forall\; |\theta| > 1, c > 0$. To determine c, we consider the grand dividend, $H_v(N)$.

$$H_v(N) = v(N) - \sum_{i\in N} v(i) - (2^n - 2 - n)c = c \tag{15}$$

Recall that for efficient value functions, $v(N) = \sum_{i\in N} \varphi_i$.

Substituting $\sum_{i\in N}\varphi_i$ for $v(N)$ in (15) and solving for c gives

$$c = \frac{\sum_{i\in N}\varphi_i - \sum_{i\in N}v(i)}{2^n - n - 1}$$

If c is desired to be strictly greater than zero, $\sum_{i\in N}v(i)$ is chosen such that $\sum_{i\in N}v(i) < \sum_{i\in N}\varphi_i$. This will yield a positive (essential) game. If $\sum_{i\in N}v(i) > \sum_{i\in N}\varphi_i$, c is negative and it will yield a negative (non-additive) game. Finally if $\sum_{i\in N}v(i) = \sum_{i\in N}\varphi_i$, $c = 0$ and it will yield an inessential game. Assuming that $\varphi_i = \varphi_j \ \forall\ i \neq j$. c is uniquely determined as $c = \frac{\varphi_i}{1+\delta_i}$ where $\delta_i = \delta_j \ \forall\ i \neq j$ and $H_v(\theta) = c \ \forall\ \theta \in 2^n$.

5. DUALITY

For any game (N, v), there exists an induced game known as the dual. Let (N, v) be any arbitrary game, we define a dual game (N, Λ) on the same set of players, where Λ is the dual characteristics function that "dualizes" every coalition $\theta \in 2^n$ in a specific way. Just like v, Λ maps from 2^n to a real line. Tadenuma (1990) introduced a simple notion of duality by defining the dual of every coalition to be a negative value of the coalition in the primal game. According to Tadenuma (1990),

$$\Lambda(\theta) = -v(\theta) \tag{16}$$

(16) is feasible in value but not feasible in a certain class of games restricted to non-negative values (e.g. the airport game). Also, the duality operator is not closed in a class of monotonic games (Funaki 1998). However, a popular notion of duality defines the dual of a game as

$$\Lambda(\theta) = v(\mathrm{N}) - v(\mathrm{N}\backslash\theta) \tag{17}$$

This is a game in which a coalition gets the rest of the grand coalition value after the complement of the coalition gets its value of the original (primal) game (Funaki, 1998). For any super-additive game (N, v), $\Lambda(\theta) = v(\mathrm{N}) - v(\mathrm{N}\backslash\theta) \geq v(\theta)$. However, in a quasi-additive (perfectly inessential) game, $\Lambda(\theta) = v(\theta) \ \forall\ \theta \in 2^n$.

The dual game is the antithesis of the primal game (depending on the interpretation given to the original game). Oishi et al. (2016) gives an interpretation to the dual of a given coalition $\theta \in 2^n$: If $v(\theta)$ represents what coalition θ can achieve on its own, the worth $\Lambda(\theta)$ represents the amount that the compliment $\mathrm{N}\backslash\theta$ cannot prevent θ from obtaining in v.

Let Ω_Λ^N be a collection of dual games established on a fixed set of players N.

$$\Omega_\Lambda^N = \left\{(N, \Lambda) | (N, v) \in \Omega^{\mathrm{N}}\right\}$$

Consider a peculiar basis for the dual of unanimity game introduced by Kalai and Samet (1987) as

$$d_\theta(T) = \begin{cases} 1 \ if \ T \cap \theta \neq \emptyset, T \neq \emptyset \\ \ 0 \quad \text{if} \quad \text{otherwise} \end{cases} \tag{18}$$

The game $d_\theta(T)$ has a natural interpretation as a cost game which posits that the presence of any number of members of θ in T incurs a unit cost (Kalai and Samet 1987).

Any dual game $(N,\Lambda) \in \Omega_\Lambda^N$ can be decomposed into a linear combination of $d_\theta(T)$ with a unique coordinate h_θ. Thus,

$$\Lambda(\theta) = \sum_{T\subset N; T\neq\emptyset} d_\theta(T)h_\theta = \sum_{T:T\cap\theta\neq\emptyset} h_\theta \tag{19}$$

h_θ is analogous to the Harsanyi dividend in the maximization game and it is defined as

$$h_\theta = D_\Lambda(\theta) = \sum_{T:T\cup\theta=N} (-1)^{|\theta|+t+1-n}\Lambda(T) \tag{20}$$

(Dehez, 2011).

Following the (asymmetric) framework of weighted Shapley value, $D_\Lambda(\theta)$ is shared by coalition members in proportion to their ranks in the primal game. For the dual game (N,Λ), the Rank-Shapley value is given as

$$RSh_i(N,\Lambda) = \sum_{\substack{i\in\theta \\ \theta\in 2^n}} \frac{r_i}{\pi_\theta} D_\Lambda(\theta).$$

Theorem 3

For any given game $(N,v) \in \Omega^{\mathrm{N}}$ and its corresponding dual game $(N,\Lambda) \in \Omega_\Lambda^N$, $RSh_i(N,v) = RSh_i(N,\Lambda)\ \forall\ i \in N$.

Proof:

To prove this theorem, we simply show that $D_\Lambda(\theta) = H_v(\theta)$.

Recall that

$$D_\Lambda(\theta) = \sum_{T:T\cup\theta=N} (-1)^{|\theta|+t+1-n}\Lambda(T)$$

Let T be partitioned into two regions: $T = N$ and $T \neq N$.

$$D_\Lambda(\theta) = \sum_{\substack{T:T\cup\theta=N \\ T=N}} (-1)^{|\theta|+n+1-n}\Lambda(N) + \sum_{\substack{T:T\cup\theta=N \\ T\neq N}} (-1)^{|\theta|+t+1-n}\Lambda(T)$$

$$(-1)^{|\theta|+1}v(N) + \sum_{\substack{T:T\cup\theta=N \\ T\neq N}} (-1)^{|\theta|+t+1-n}\Lambda(T)$$

$$(-1)^{|\theta|+1}v(N) - \sum_{\substack{T:T\cup\theta=N \\ T\neq N}} (-1)^{|\theta|+t-n}\Lambda(T)$$

But $T: T \cup \theta = N\ \forall\ T \neq N$ is equivalent to $N\backslash R\ \forall\ R \subseteq \theta$. Based on this equivalence,

$$D_\Lambda(\theta) = (-1)^{|\theta|+1}v(N) - \sum_{R\subseteq\theta} (-1)^{|\theta|+n-r-n}\Lambda(N\backslash R)$$

$$(-1)^{|\theta|+1}v(N) - \sum_{R\subseteq\theta} (-1)^{|\theta|-r}[v(N) - v(R)]$$

$$v(\mathrm{N})\left[(-1)^{|\theta|+1} - \sum_{R\subseteq\theta}(-1)^{|\theta|-r}\right] + \sum_{R\subseteq\theta}(-1)^{|\theta|-r}v(\mathrm{R})$$

For any $\theta \in 2^n$, $\left[(-1)^{|\theta|+1} - \sum_{R\subseteq\theta}(-1)^{|\theta|-r}\right] = 0$. Therefore,

$$D_\Lambda(\theta) = \sum_{R\subseteq\theta}(-1)^{|\theta|-r}v(\mathrm{R}) = H_v(\theta)$$

Therefore, it follows that $RSh_i(N, v) = RSh_i(N, \Lambda)\ \forall\ i \in N$. It also holds as a primal-dual relationship that $RSh_i(N, \Lambda) = \sum_{i\in N} RSh_i(N, v) - \sum_{j\in N\backslash i} RSh_j(N, v)$.

6. CONCLUSION

In this article, a new allocation rule in the family of weighted Shapley value that makes use of linear ranks as the weight function of the individual players, as well as its dual equivalence has been proposed. The idea of Rank-Shapley value is a possibility offered by the vast interpretation of weight in the family of weighted Shapley value. The rank as a weight function in a TU-game is interpreted as the function that portrays the capacity of a player in a game and it is restricted only to natural numbers except in a case of ties. Rank as a weight function is coherent and definite. In other words, once a game is given, the ranks of the players can naturally be determined from the stand-alone values. The Rank-Shapley value therefore, has operational essence as it is feasible (defined) for any kind of cooperative game and can easily be manipulated. The feasibility and robustness of Rank-Shapley value is shown in example 1 where it allocates a fairer and rational payoff than the proportional Shapley value. In a cooperative game where at least one of the players has a stand-alone worth of zero, the Rank-Shapley value suffixes to be a powerful scheme for sharing the benefit of cooperation. The characterization of Rank-Shapley value offered in theorem 1 justifies it as the unique value that can be determined by the five axioms. While theorem 2 shows that for any given payoff vector, a unique game $v \in \mathrm{F}_0$ corresponding to the payoff vector can be generated, theorem 3 proves the equivalence of primal game solution and its dual solution. Finally, the Rank-Shapley value is feasible in all class of cooperative game unlike the proportional Shapley value of Beal et al. (2018) that is restricted to only individually positive and individually negative games.

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