Pak. J. Statist.
2021 Vol. 37(3), 189-213

THE EXPONENTIATED GUMBEL WEIBULL DISTRIBUTION: PROPERTIES AND APPLICATIONS

Uchenna U. Uwadi[1§], **Elebe E. Nwezza**[1], **Chukwuemeka O. Omekara**[2] and **Chukwunenye I. Okonkwo**[2]

[1] Department of Mathematics/Computer Science/Statistics/Informatics Alex Ekwueme Federal University, Ndufu-Alike, Nigeria

[2] Department of Statistics, Michael Okpara University of Agriculture Umudike, Nigeria

[§] Corresponding author's Email: uchenna.uwadi@funai.edu.ng

ABSTRACT

In this research article, a new model called exponentiated Gumbel Weibull distribution from the exponentiated Gumbel–G family is introduced. The new model can be expressed as a linear combination of exponentiated Weibull and Weibull distribution. The expressions for, quantile function, ordinary, incomplete and Trimmed-L moments, mode, moment of residual life, moment of reversed residual life, inequality measures, entropy, and order statistics of the subject distribution is derived. Estimation of the unknown parameters is done using the maximum likelihood method. A simulation study results to assess the performance of the maximum likelihood estimates are presented. The importance of the new distribution is illustrated using two data sets.

KEYWORDS

Exponentiated Gumbel; Exponentiated Gumbel Family; Moments; Trimmed-L Moments; Order Statistics; Maximum Likelihood.

1. INTRODUCTION

The Weibull distribution is a very popular lifetime distribution that is widely used in different fields for modelling data sets. It is an easy distribution to work with, given that the cumulative distribution function (cdf) is in a closed-form; in addition, the parameters have physical meaning and interpretation.

Its wide applicability notwithstanding, a major challenge with the distribution is the limited shape of its hazard rate function (hrf) which can only be monotonically increasing or decreasing or constant (Corderio et al. 2018). Usually, some lifetime data (for instance machine life cycle, human mortality, and some data from medical studies) require non-monotone shaped hazard function. Over the years, researchers have proposed several generalizations of Weibull distribution to make it flexible enough to fit data sets whose hazard functions are non-monotonic.

Some generalizations of Weibull distribution existing in the literature include exponentiated Weibull by Mudholkar and Srivastava (1993), beta-Weibull by Lee et al. (2005), Marshall-Olkin extended Weibull by Ghitany et al. (2005), beta modified Weibull by Silva et al. (2010), Kumaraswamy Weibull by Cordeiro et al. (2010), transmuted Weibull by Aryal and Tsokos (2011), exponentiated generalized Weibull by Cordeiro et al. (2013), Gumbel Weibull distribution by Al-qtash et al. (2014), transmuted exponentiated generalized Weibull by Yousof et al. (2015), Marshall-Olkin additive Weibull by Afify et al. (2016), Kumaraswamy transmuted exponentiated additive Weibull by Nofal et al. (2016), exponentiated Kumaraswamy Weibull distribution by Eisse (2017), generalized transmuted Weibull by Nofal et al. (2017), Lindley Weibull distribution by Corderio et al. (2018) and exponentiated Weibull distribution by Hassan et al. (2019).

A family of distribution based on the logit of the cdf known as the exponentiated Gumbel generated (EGu-G) family was proposed by Uwadi et al. (2019). The cdf of EGu-G is given by

$$G(x) = 1 - \left[1 - \exp\left\{-B\left(\frac{F(x)}{1-F(x)}\right)^{-\frac{1}{\sigma}}\right\}\right]^{\alpha} \qquad x > 0, \alpha, \sigma > 0, -\infty < \mu < \infty \tag{1}$$

where $B = \exp\left(\frac{\mu}{\sigma}\right)$. Letting the baseline distribution $F(x)$ be a Weibull distribution. The cdf and probability density function (pdf) of the Weibull distribution is given by

$$F(x) = 1 - exp\left[-\left(x/b\right)^{a}\right] \quad x \geq 0, b > 0, a > 0 \tag{2}$$

and

$$f(x,a,b) = \frac{a}{b}\left(\frac{x}{b}\right)^{a-1} exp\left[-\left(x/b\right)^{a}\right] \tag{3}$$

respectively, where $a > 0$ is the shape parameter and $b > 0$ is the scale parameter.

An extension of Weibull distribution called exponentiated Gumbel Weibull (EGuW) distribution is proposed in this paper. The motivation for EGuW is to produce a flexible and heavy-tailed model that can provide a better fit to data sets when compared to other generalized distributions with the same baseline.

The rest of the paper is arranged as follows. The EGuW distribution is defined in Section 2. Some useful expansions are introduced in Section 3. Various properties of EGuW distributions such as ordinary, incomplete, and Trimmed-L moments, mode, and reliability and inequality measures are studied in Section 4. The entropy and order statistics of EGuW distribution are discussed in Section 5 and 6 respectively. Maximum likelihood estimates (MLEs) of parameters of EGuW distribution are derived in Section 7 while simulation studies to determine the performance and stability of the estimates of the proposed model are done in Section 8. In Section 9, an application showing that EGuW distribution provides a better fit than other distributions with the same baseline is presented. The concluding remarks are given in Section 10.

2. THE EGuW DISTRIBUTION

The cdf of EGuW distribution is defined in this section using the cdf of EGu-G family by substituting (2) in (1) we have

$$G(x) = 1 - \left[1 - exp\left\{-B\left(exp\left(\left(\frac{x}{b}\right)^a\right) - 1\right)^{-1/\sigma}\right\}\right]^\alpha \quad (4)$$

By differentiating (4), we obtain pdf of EGuW distribution as

$$g(x) = \frac{\alpha B a}{\sigma b}\left(\frac{x}{b}\right)^{a-1} exp\left[-\frac{1}{\sigma}\left(\frac{x}{b}\right)^a\right] exp\left\{-B\left(exp\left(\left(\frac{x}{b}\right)^a\right) - 1\right)^{-1/\sigma}\right\}$$

$$\times\left(1 - exp\left[-\left(\frac{x}{b}\right)^a\right]\right)^{-\left(1/\sigma+1\right)}\left[1 - exp\left\{-B\left(exp\left(\left(\frac{x}{b}\right)^a\right) - 1\right)^{-1/\sigma}\right\}\right]^{\alpha-1} \quad (5)$$

$$a > 0, b > 0, \alpha > 0, \sigma > 0, \quad -\infty < \mu < \infty$$

Theorem 1:

If Y is distributed as exponentiated Gumbel distribution with location, scale and shape parameters, μ, σ and α respectively, then $X = b\left[\log\left(e^Y + 1\right)^{\frac{1}{a}} - 1\right]$ is distributed as EGuW distribution with parameters, μ, σ, α, a and b.

Proof:

Given that Y is distributed as exponentiated Gumbel distribution due to Nadarajah (2006). The pdf of Y is given by

$$r(y) = \frac{\alpha}{\sigma} B \exp\left(-\frac{y}{\sigma}\right)\exp\left\{-B\exp\left(-\frac{y}{\sigma}\right)\right\}\left[1 - \exp\left\{-B\exp\left(-\frac{y}{\sigma}\right)\right\}\right]^{\alpha-1}$$

and

$$x = b\left[\log\left(e^Y + 1\right)^{\frac{1}{a}} - 1\right] \Rightarrow y = \ln\left[\exp\left(\frac{x}{b}\right)^a - 1\right]$$

The pdf of EGuW distribution is derived through transformation as

$$g(x) = r(x)\left|\frac{dy}{dx}\right| \quad (6)$$

But $\frac{dy}{dx} = \frac{a x^{a-1} \exp\left(\frac{x}{b}\right)^a}{b\left[\exp\left(\frac{x}{b}\right)^a - 1\right]}$

Substituting for $\left|\frac{dy}{dx}\right|$ and $r(x)$ in (6) gives

$$g(x) = \frac{\alpha B a}{\sigma b}\left(\frac{x}{b}\right)^{(a-1)} \exp\left(-\frac{1}{\sigma}\left(\frac{x}{b}\right)^a\right) \exp\left\{-B\left(\exp\left(\frac{x}{b}\right)^a - 1\right)^{-\frac{1}{\sigma}}\right\}$$

$$\times\left(1 - \exp\left(-\left(\frac{x}{b}\right)^a\right)\right)^{-\left(\frac{1}{\sigma}+1\right)}\left[1 - \exp\left\{-B\left(\left(1 + \frac{x}{b}\right)^a - 1\right)^{-\frac{1}{\sigma}}\right\}\right]^{(\alpha-1)}.$$

This pdf is the same as given in (5). Hence X is distributed as EGwW distribution.

3. USEFUL EXPANSIONS

A useful mixture representation of the pdf and cdf of EGuW distribution is given in this section. The pdf in (5) can be rewritten as

$$g(x) = \frac{\alpha B a}{\sigma b}\left(\frac{x}{b}\right)^{a-1} \frac{\left(1 - exp\left[-\left(\frac{x}{b}\right)^a\right]\right)^{-\left(1/\sigma+1\right)}}{\left(1 - \left(1 - exp\left[-\left(\frac{x}{b}\right)^a\right]\right)\right)^{-\frac{1}{\sigma}+1}} exp\left\{-B\left(\frac{1 - exp\left[-\left(\frac{x}{b}\right)^a\right]}{1 - \left(1 - exp\left[-\left(\frac{x}{b}\right)^a\right]\right)}\right)^{-1/\sigma}\right\}$$

$$\times exp\left[-\left(\frac{x}{b}\right)^a\right]\left[1 - exp\left\{-B\left(\frac{1 - exp\left[-\left(\frac{x}{b}\right)^a\right]}{1 - \left(1 - exp\left[-\left(\frac{x}{b}\right)^a\right]\right)}\right)^{-1/\sigma}\right\}\right]^{\alpha-1} \quad (7)$$

The general binomial expression for $\delta > 0$ and $|d| < 1$ is given as:

$$(1-d)^{\delta-1} = \sum_{i=1}^{\infty}(-1)^i \binom{\delta-1}{i} d^i \quad (8)$$

while the series of $\exp(-d)$ is

$$\exp(-d) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j} d^j . \tag{9}$$

Using the general binomial expression in (8) and the expansion of exponential function given in (9), the pdf of EGuW distribution assumes the following form

$$g(x) = \sum_{i,j,k}^{\infty} W_{i,j,k} h_{\left(k-\frac{1}{\sigma}(j+1)\right)}(x) \tag{10}$$

where

$$W_{i,j,k} = (-1)^{i+j+k} \frac{\alpha B^{j+1}(i+1)^j}{\sigma\left(k-\frac{1}{\sigma}(j+1)\right) j!} \binom{\alpha-1}{i} \binom{\frac{1}{\sigma}(j+1)-1}{k}$$

and $h_p(x) = pf(x)F(x)^{p-1}$. Hence the EGuW distribution can be expressed as a linear combination of exponentiated Weibull distribution with parameters a, b and $k-\frac{1}{\sigma}(j+1)$.

Furthermore, the general binomial expansion applied to (10) gives

$$g(x) = \sum_{i,j,k,m}^{\infty} \Lambda_{i,j,k,m} \left(\frac{x}{b}\right)^{a-1} exp\left[-(m+1)\left(\frac{x}{b}\right)^a\right] \tag{11}$$

where

$$\Lambda_{i,j,k,m} = \frac{\alpha a}{\sigma b}(-1)^{i+j+k+m} \frac{B^{j+1}(i+1)^j}{j!} \binom{\alpha-1}{i} \binom{\frac{1}{\sigma}(j+1)-1}{k} \binom{k-\frac{1}{\sigma}(j+1)-1}{m}.$$

The result in (11) shows that the EGuW distribution can also be expressed as an infinite linear combination of Weibull distribution.

The expansion of the cdf $(G(x))^s$ is obtained using (8) in (12)

$$(G(x))^s = \left(1-\left[1-exp\left\{-B\left(\frac{1-exp\left[-\left(\frac{x}{b}\right)^a\right]}{1-\left(1-exp\left[-\left(\frac{x}{b}\right)^a\right]\right)}\right)^{-1/\sigma}\right\}\right]^{\alpha}\right)^s \tag{12}$$

where s is a positive integer we have

$$\left(G(x)\right)^{s}=\sum_{p=0}^{s}(-1)^{p}\binom{s}{p}\left[1-exp\left\{-B\left(\frac{1-exp\left[-\left(\frac{x}{b}\right)^{a}\right]}{1-\left(1-exp\left[-\left(\frac{x}{b}\right)^{a}\right]\right)}\right)^{-1/\sigma}\right\}\right]^{\alpha p} \tag{13}$$

Using (8) and (9) in (13), $\left(G(x)\right)^{s}$ can be expressed as

$$\left(G(x)\right)^{s}=\sum_{qm_1lu=0}^{\infty}\sum_{p=0}^{s}\Lambda_{pqlm_1u}exp\left[-u\left(\frac{x}{b}\right)^{a}\right] \tag{14}$$

where

$$\Lambda_{pqlm_1u}=\frac{(-1)^{p+q+m_1+l+u}}{m_1!}\binom{s}{p}\binom{\alpha p}{q}\binom{\frac{m_1}{\sigma}}{l}\binom{l-\frac{m_1}{\sigma}}{u}(Bq)^{m_1}.$$

The expressions in (11) and (14) are very useful in obtaining other properties of EGuW distribution.

4. SOME PROPERTIES OF EGuW

Some properties of EGuW distribution are discussed in this section.

4.1 Quantile Function of EGuW

Theorem 2:

The quantile function of EGuW distribution is given by

$$Q(u)=b\left[\log\left(\left[-\frac{1}{B}\log\left(1-(1-u)^{\frac{1}{\alpha}}\right)\right]^{-\sigma}+1\right)\right]^{\frac{1}{a}} \tag{15}$$

where u is uniformly distributed between the interval $(0,1)$.

Proof:

Theorem 2 can easily be proved by inverting (4). Theorem 2 is very important in the simulation of random samples from EGuW distribution. The lower and upper quartiles of EGuW distribution is obtained respectively by setting u to $\frac{1}{4}$ and $\frac{3}{4}$ in (15). In particular, the median of EGuW distribution is derived by letting $u = \frac{1}{2}$ in (15). Hence the median of the proposed distribution is

$$Q\left(\tfrac{1}{2}\right) = b\left[\log\left[\left[-\frac{1}{B}\log\left[1-\left(\tfrac{1}{2}\right)^{\frac{1}{\alpha}}\right]\right]^{-\sigma}+1\right]\right]^{\frac{1}{a}}.$$

Alternative measures of kurtosis based on octiles can be computed with the relationship of Moor (1988). Using the quantile function in (15), the Moors kurtosis of the EGuW distribution is obtained using (16). Plots of Kurtosis of EGuW and Weibull distribution in Figure 1 shows that EGuW distribution has a heavier tail than the Weibull distribution.

$$K = \frac{Q\left(\tfrac{7}{8}\right)-Q\left(\tfrac{5}{8}\right)+Q\left(\tfrac{3}{8}\right)-Q\left(\tfrac{1}{8}\right)}{Q\left(\tfrac{6}{8}\right)-Q\left(\tfrac{2}{8}\right)}. \tag{16}$$

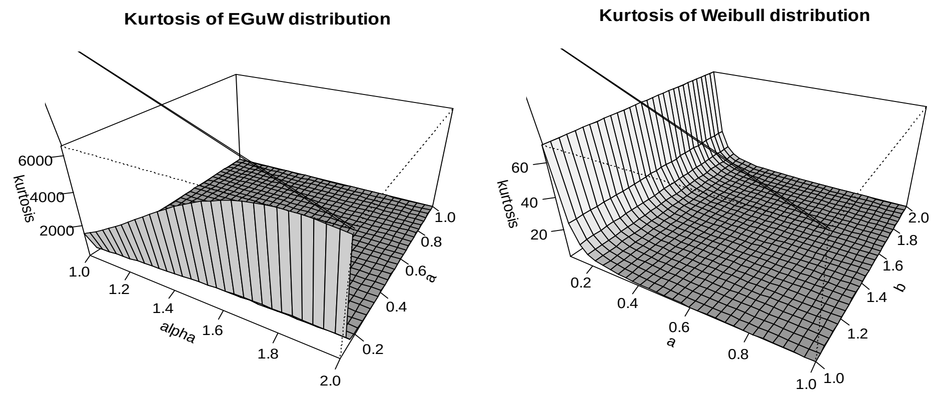

Figure 1: Plots of Kurtosis of EGuW and Weibull Distribution

4.2 Ordinary and Incomplete Moments

In this subsection we provide the explicit expressions for an ordinary moment, moment generating function, incomplete moment, mean and median deviation of EGuW distribution.

First, using (11), the *rth* ordinary moment of EGuW distribution is obtained as

$$E\left(X^r\right) = \mu^r = \int_{-\infty}^{\infty} x^r g(x)\,dx$$

$$\mu^r = \sum_{i,j,k,m}^{\infty} \Lambda_{i,j,k,m} \int_0^{\infty} x^r \left(\frac{x}{b}\right)^{a-1} exp\left[-(m+1)\left(\frac{x}{b}\right)^a\right] dx$$

Let $w = (m+1)\left(\frac{x}{b}\right)^{a}$, then

$$\mu^{r} = \sum_{i,j,k,m}^{\infty} \Lambda_{i,j,k,m} \frac{b^{r+1}}{a(m+1)^{\frac{r}{a}+1}} \Gamma\left(\frac{r}{a}+1\right) \tag{17}$$

Second, the moment generating function of EGuW distribution is given by

$$M_X(t) = E\left(e^{tX}\right) = E\left[\sum_{r=0}^{\infty} \frac{t^r X^r}{r!}\right] = \sum_{r=1}^{\infty} \frac{t^r}{r!} E\left(X^r\right) \tag{18}$$

Substituting (17) into (18) we have

$$M_X(t) = \sum_{r,i,j,k,m=0}^{\infty} \Lambda_{i,j,k,m} \frac{t^r}{r!} \frac{b^{r+1}\Gamma\left(\frac{r}{a}+1\right)}{a(m+1)^{\frac{r}{a}+1}}$$

Third, rth incomplete moment $M_r(z)$ of EGuW distribution is derived as follows

$$M_r(z) = \int_0^z x^r g(x)dx \tag{19}$$

substituting (11) in (19) we have

$$M_r(z) = \sum_{i,j,k,m}^{\infty} \Lambda_{i,j,k,m} \int_0^z x^r \left(\frac{x}{b}\right)^{a-1} exp\left[-(m+1)\left(\frac{x}{b}\right)^{a}\right]dx$$

Letting $w = (m+1)\left(\frac{x}{b}\right)^{a}$, hence

$$M_r(z) = \sum_{i,j,k,m}^{\infty} \Lambda_{i,j,k,m_1} \frac{b^{r+1}}{a(m+1)^{\left(\frac{r}{a}+1\right)}} \nu\left(\left(\frac{r}{a}+1\right),(m+1)\left(\frac{z}{b}\right)^{a}\right) \tag{20}$$

where $\nu(r,z) = \int_0^z x^{r-1}e^{-x}dx$ is the lower incomplete gamma function.

Fourth, the mean deviation and the median deviation are very important measures of spread in statistics. The mean deviation about the mean is given by

$$D(\mu) = 2\mu G(\mu) - 2\int_0^{\mu} xg(x)dx$$

Using the result from (20), the mean deviation from the mean of EGuW distribution is given as

$$D(\mu) = 2\mu G(\mu) - 2\sum_{i,j,k,m}^{\infty} \Lambda_{i,j,k,m} \frac{b^2}{a(m+1)^{\left(\frac{1}{a}+1\right)}} \nu\left(\left(\frac{1}{a}+1\right),(m+1)\left(\frac{\mu}{b}\right)^a\right)$$

For the mean deviation from the median of EGuW distribution we have

$$D(m) = \mu - 2\int_0^m xg(x)dx$$

Using the result from (20) again we have

$$D(M) = \mu - 2\sum_{i,j,k,m}^{\infty} \Lambda_{i,j,k,m} \frac{b^2}{a(m+1)^{\left(\frac{1}{a}+1\right)}} \nu\left(\left(\frac{1}{a}+1\right),(m+1)\left(\frac{M}{b}\right)^a\right)$$

4.3 Trimmed-L Moments

Elamir and Seheult (2003) introduced Trimmed-L-moments based on the order statistics. The TL-moments are more robust than the traditional moments and exist even when the moment of the distribution does not exist. (Abdul-Moniem and Seham, 2015). The rth TL-moment with t_1 smallest and t_2 largest trimming denoted as $\lambda_r^{t_1+t_2}$ is given as

$$\lambda_r^{(t_1+t_2)} = \frac{1}{r}\sum_{k_1=0}^{r-1}(-1)^{k_1}\binom{r-1}{k_1}E\left(Y_{r+t_1-k_1,r+t_1-k_2}\right) \quad r=1,2 \tag{21}$$

For a symmetric case $t_1 = t_2 = t$ (21) reduces to

$$\lambda_r^{(t)} = \frac{1}{r}\sum_{k_1=0}^{r-1}(-1)^{k_1}\binom{r-1}{k_1}E\left(Y_{r+t-k_1,r+2t}\right) \quad r=1,2$$

But

$$E\left(Y_{r+t-k_1,r+2t}\right) = \frac{(r+2t)!}{(r+t-k_1-1)!(t+k_1)!}\int_0^\infty xg(x)G(x)^{r+t+k_1-1}\left(1-G(x)\right)^{(t+k_1)}dx \tag{22}$$

Applying the general binomial expansion to (23) we have

$$E\left(Y_{r+t-k_1,r+2t}\right) = \frac{(r+2t)!}{(r+t-k_1-1)!(t+k_1)!}\sum_{\ell=0}^{t+k_1}(-1)^\ell\binom{t+k_1}{\ell}\int_0^\infty xg(x)G(x)^{r+t+k_1+\ell-1}dx \tag{23}$$

and substituting (11) and (14) in (23) and replacing s in (14) with $r+t+k_1+\ell-1$ we obtain

$$E\left(Y_{r+t-k_1,r+2t}\right)=\frac{(r+2t)!}{(r+t-k_1-1)!(t+k_1)!}\sum_{\ell=0}^{(t+k_1)}\sum_{ijkm=0}^{\infty}\sum_{qm_1lu}^{\infty}\sum_{p=0}^{r+t+k_1+\ell-1}(-1)^{\ell}\binom{t+k_1}{\ell}\Lambda_{i,j,k,m}\Lambda_{p,q,l,m_1u}\times\int_0^{\infty}x\left(\frac{x}{b}\right)^{a-1}\exp\left[-(m+u+1)\left(\frac{x}{b}\right)^{a}\right]dx$$

Hence the TL-moment of EGuW distribution takes the following form

$$\lambda_r^{(t)}=\frac{b^2}{ar}\sum_{k_1=0}^{r-1}\sum_{\ell=0}^{(t+k_1)}\sum_{ijkm=0}^{\infty}\sum_{qm_1lu}^{\infty}\sum_{p=0}^{r+t+k_1+\ell-1}(-1)^{k_1+\ell}\frac{(r+2t)!}{(r+t-k_1-1)!(t+k_1)!}\binom{r-1}{k_1}\binom{t+k_1}{\ell}\times\Lambda_{i,j,k,m}\Lambda_{p,q,l,m_1u}\Gamma\left(\frac{1}{a}+1\right)(m+u+1)^{-\left(\frac{1}{a}+1\right)}.$$

4.4 Mode

The mode of the EGuW distribution can be described analytically by taking the log of (5), differentiating with respect to x and equating to zero. The mode of the EGuW distribution is determined by

$$\frac{(a-1)}{x}-\frac{ax^{a-1}}{b^a}\left[exp\left[\left(\frac{x}{b}\right)^a\right]-1\right]^{-1}-\frac{ax^{a-1}}{\sigma b^a}\left(1+\left[exp\left[\left(\frac{x}{b}\right)^a\right]-1\right]^{-1}\right)$$
$$+\frac{Bax^{a-1}\left(1-exp\left[-\left(\frac{x}{b}\right)^a\right]\right)^{-1/\sigma-1}}{\sigma b^a\left[exp\left[\frac{1}{\sigma}\left(\frac{x}{b}\right)^a\right]\right]}\left[1+\frac{(\alpha-1)exp\left\{-B\left(exp\left[\left(\frac{x}{b}\right)^a\right]-1\right)^{-1/\sigma}\right\}}{\left[1-exp\left\{-B\left(exp\left[\left(\frac{x}{b}\right)^a\right]-1\right)^{-1/\sigma}\right\}\right]}\right]=0 \quad (24)$$

There may be more than one root to (24). If $x=x_0$ is a root of (24), then it corresponds to a local maximum, local minimum or point of inflexion depending on whether $\varpi(x_0)<0$, $\varpi(x_0)>0$ or $\varpi(x_0)=0$ where $\varpi(x_0)=g''(x)$, The plots of the pdf of EGuW distribution for selected parameter values are given in Figure 2. Figure 2 reveals that the density, EGuW can be right-skewed; reverse J-shaped, and unimodal.

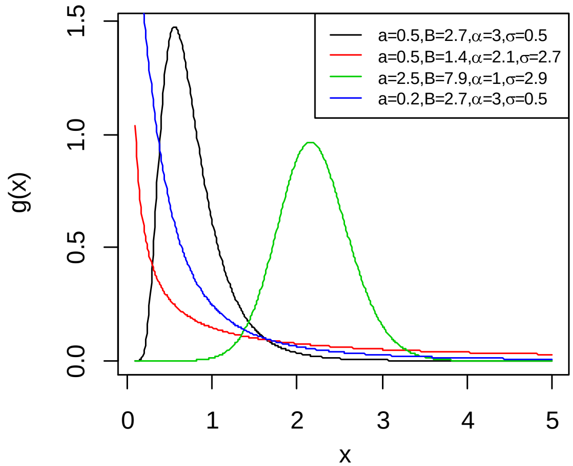

Figure 2: Plots of pdf of EGuW Distribution for Selected Parameter Values when $b = 1$

4.5 Reliability Analysis

In this section, some important reliability characteristics such as survival function $S(x)$, hazard rate function $h(x)$, reversed hazard rate function $R(x)$, the moment of residual life $m_n(z)$, and reversed moment of residual life $M_n(z)$ of EGuW distribution are introduced.

4.5.1 Survival Function

Survival (reliability) function $S(x)$ is the probability that a system will not fail in a given interval $(0,x)$. Mathematically, $S(x)=1-F(x)=P(X>x)$, hence $S(x)$ is the complementary distribution function. Using (4) the survival function of EGuW distribution is

$$G(x)=\left[1-exp\left\{-B\left(exp\left(\left(\frac{x}{b}\right)^{a}\right)-1\right)^{-1/\sigma}\right\}\right]^{\alpha}.$$

4.5.2 Hazard Rate Function

The hrf is a very important function in reliability studies. The hrf of the distribution of X is the instantaneous conditional probability of failure of a system given that the system has survived up to x. It is defined as $h(x)=f(x)/1-F(x)$, thus the $h(x)$ of EGuW distribution is given as

$$h(x)=\frac{\frac{\alpha B}{\sigma}\left(\frac{a}{b}\right)\left(\frac{x}{b}\right)^{a-1}\left[1-exp\left(-\left(\frac{x}{b}\right)^{a}\right)\right]exp\left\{-\left(\frac{1}{\sigma}\left(\frac{x}{b}\right)^{a}+B\left(exp\left(\left(\frac{x}{b}\right)^{a}\right)-1\right)^{-1/\sigma}\right)\right\}}{\left[1-exp\left\{-B\left(exp\left(\left(\frac{x}{b}\right)^{a}\right)-1\right)^{-1/\sigma}\right\}\right]}. \tag{25}$$

Figure 3 shows the hrf plots of EGuW distribution for selected parameter values. From Figure 3 it can be deduced that hrf of EGuW distribution can be decreasing, increasing and inverted bathtub shaped.

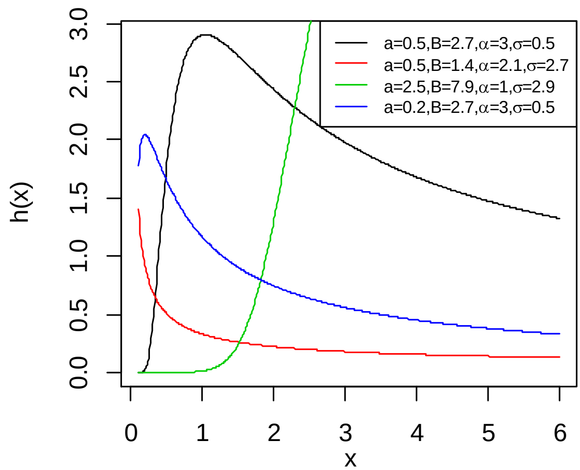

Figure 3: Plots of Hazard Rate Function of EGuW Distribution for Selected Parameter Values when $b = 1$

4.5.3 Reversed Hazard Rate Function

The reversed hazard rate function $R(x)$ extends the concept of hrf in a reverse time direction. The $R(x)$ for EGuW can be written as

$$R(x)=\frac{\frac{\alpha Ba}{\sigma b}\left(\frac{x}{b}\right)^{a-1}exp\left\{-\left(\frac{1}{\sigma}\left(\frac{x}{b}\right)^{a}+B\left(exp\left(\left(\frac{x}{b}\right)^{a}\right)-1\right)^{-1/\sigma}\right)\right\}\left[1-exp\left\{-B\left(exp\left(\left(\frac{x}{b}\right)^{a}\right)-1\right)^{-1/\sigma}\right\}\right]^{\alpha}}{\left(1-exp\left[-\left(\frac{x}{b}\right)^{a}\right]\right)^{\left(1/\sigma+1\right)}\left(1-\left[1-exp\left\{-B\left(exp\left(\left(\frac{x}{b}\right)^{a}\right)-1\right)^{-1/\sigma}\right\}\right]^{\alpha}\right)} \tag{26}$$

4.5.4 Moment of Residual Life Function

The nth moment of residual life function of a random variable X, denoted by $m_n(z)=E\left[(X-z)^n \mid X>z\right]; n=1,2\cdots$ and $z>0$; uniquely determines $G(x)$ and is given as

$$m_n(z)=\frac{1}{1-G(z)}\int_z^\infty (x-z)^n\, dG(x) \tag{27}$$

Applying the general binomial expansion to (27) and using (11), the nth moment of residual life function of EGuW distribution is given by

$$m_n(z)=\frac{\sum_{i,j,k,m}^{\infty}\sum_{r=0}^{n}(-1)^r z^r \Lambda_{i,j,k,m} b^{n-r+1}(m+1)^{-\left(\frac{n-r}{a}+1\right)}}{a\left[1-exp\left\{\left(exp\left[\left(\frac{z}{b}\right)^a\right]-1\right)^{-1/\sigma}\right\}\right]^{\alpha}}\Gamma\left(\left(\frac{n-r}{a}+1\right),\left((m+1)\left(\frac{z}{b}\right)^a\right)\right)$$

where $\Gamma(r,z)=\int_z^\infty x^{r-1}\,\mathrm{e}^{-x}\,dx$ is the upper incomplete gamma function. If $n=1$ we have the mean residual life or life expectation at age x, which is another interesting function in lifetime modelling. It represents the expected additional life length for a unit that is alive at age z.

4.5.5 Moment of the Reversed Residual Life

The nth moment of reversed residual life function, say $M_n(z)=E\left[(z-X)^n \mid X\le z\right]$ for $z>0,\ n=1,2\cdots,$ and $z>0$; uniquely defines $G(x)$ (Corderio et al. 2019). It is defined as

$$M_n(z)=\frac{1}{G(z)}\int_0^z (z-x)^n\, dG(x) \tag{28}$$

Using (11) and general binomial expansion in (28) we have

$$M_n(z)=\frac{\sum_{i,j,k,m}^{\infty}\sum_{r=0}^{n}(-1)^{n-r} z^r \Lambda_{i,j,k,m} b^{n-r+1}(m+1)^{-\left(\frac{n-r}{a}+1\right)}}{a\left\{1-\left[1-exp\left\{-B\left(exp\left(\left(\frac{z}{b}\right)^a\right)-1\right)^{-1/\sigma}\right\}\right]\right\}^{\alpha}}\nu\left(\left(\frac{n-r}{a}+1\right),\left((m+1)\left(\frac{z}{b}\right)^a\right)\right)$$

where $\nu(r,z)=\int_0^z x^{r-1}\,\mathrm{e}^{-x}\,dx$ is the lower incomplete gamma function. In particular, for $n=1$ $M_1(z)$ represents the mean inactivity time or mean reversed residual function which indicates the expected inactive life length for a unit which first observed down at age z .

4.6 Inequality Measures

The Lorenz and Bonferroni curves are very important inequality measures in income and wealth distribution. The Lorenz curve is given by $L_F(z)=\frac{\int_0^z xg(x)dx}{E(X)}$. Using the results in (11) and (17) for $r=1$ then Lorenz curve for EGuW distribution takes the form

$$L_F(z)=\frac{\sum\limits_{i,j,k,m}^{\infty}\Lambda_{i,j,k,m}\,\nu\left(\left(\frac{1}{a}+1\right),(m+1)\left(\frac{z}{b}\right)^{a}\right)(m+1)^{-\left(\frac{1}{a}+1\right)}}{\sum\limits_{i,j,k,m}^{\infty}\Lambda_{i,j,k,m}\,\Gamma\left(\frac{1}{a}+1\right)(m+1)^{-\left(\frac{1}{a}+1\right)}}$$

For the Bonferroni Curves, we have $B_F(z)=\frac{L_F(z)}{G(z)}$, hence the Bonferroni Curve for EGuW distribution can be written as

$$B_F(z)=\frac{\sum\limits_{i,j,k,m}^{\infty}\Lambda_{i,j,k,m}\,\nu\left(\left(\frac{1}{a}+1\right),(m+1)\left(\frac{z}{b}\right)^{a}\right)(m+1)^{-\left(\frac{1}{a}+1\right)}}{\sum\limits_{i,j,k,m}^{\infty}\Lambda_{i,j,k,m}\,\Gamma\left(\frac{1}{a}+1\right)(m+1)^{-\left(\frac{1}{a}+1\right)}\left(1-\left[1-exp\left\{-B\left(exp\left(\left(\frac{z}{b}\right)^{a}\right)-1\right)^{-1/\sigma}\right\}\right]^{\alpha}\right)}$$

5. ENTROPY

The variation of uncertainty in a random variable X with pdf $f(x)$ is known as the entropy of X . It has various applications in science and engineering. A high value of entropy indicates greater uncertainty. One of the widely used measures of entropy is the Renyi entropy by Renyi (1961) and is defined as

$$I_{R(\gamma)}=\frac{1}{1-\gamma}\log\int_{-\infty}^{\infty} g^{\gamma}(x)dx,\ \ \gamma>0,\ \gamma\neq 1. \tag{29}$$

Theorem 3:

The Renyi entropy of a random variable X from EGuW distribution with pdf $g(x)$ is given by

$$I_R(\gamma)=\frac{1}{1-\gamma}\log\left\{\sum_{i,j,k,l}^{\infty}V_{ijkl}\frac{b}{a}\Gamma\left(\frac{1}{a}\left(\gamma\left(a-1\right)+1\right)\right)\left(\gamma+l\right)^{-\frac{1}{a}\left[\gamma(a-1)+1\right]}\right\}$$

for $\gamma>0$ and $\gamma\neq 1$ (30)

where

$$V_{ijkl}=\left(\frac{\alpha Ba}{\sigma b}\right)^{\gamma}\sum_{i,j,k,l=0}^{\infty}\frac{(-1)^{i+j+k+l}}{j!}B^{j}\left(\gamma+i\right)^{j}\binom{\gamma(\alpha-1)}{i}\binom{\frac{1}{\sigma}(\gamma+j)-\gamma}{k}\binom{k-\gamma-\frac{1}{\sigma}(\gamma+j)}{l}$$

Proof:

Applying the general binomial expansion and power series expansion of exponential function we have

$$g^{\gamma}(x)=\left(\frac{\alpha Ba}{\sigma b}\right)^{\gamma}\sum_{i,j,k,l=0}^{\infty}\frac{(-1)^{i+j+k+l}}{j!}B^{j}\left(\gamma+i\right)^{j}\binom{\gamma(\alpha-1)}{i}\binom{\frac{1}{\sigma}(\gamma+j)-\gamma}{k}$$
$$\binom{k-\gamma-\frac{1}{\sigma}(\gamma+j)}{l}\left(\frac{x}{b}\right)^{\gamma(a-1)}exp\left[-\left(\gamma+l\right)\left(\frac{x}{b}\right)^{a}\right]$$

Letting

$$V_{ijkl}=\left(\frac{\alpha Ba}{\sigma b}\right)^{\gamma}\sum_{i,j,k,l=0}^{\infty}\frac{(-1)^{i+j+k+l}}{j!}B^{j}\left(\gamma+i\right)^{j}\binom{\gamma(\alpha-1)}{i}\binom{\frac{1}{\sigma}(\gamma+j)-\gamma}{k}\binom{k-\gamma-\frac{1}{\sigma}(\gamma+j)}{l}$$

Then

$$I_{R(\gamma)}=\frac{1}{1-\gamma}\log\sum_{i,j,k,l}^{\infty}V_{ijkl}\int_0^{\infty}\left(\frac{x}{b}\right)^{\gamma(a-1)}exp\left[-\left(\gamma+l\right)\left(\frac{x}{b}\right)^{a}\right]dx$$

Therefore the Renyi entropy of EGuW is given by

$$I_R(\gamma)=\frac{1}{1-\gamma}\log\left\{\sum_{i,j,k,l}^{\infty}V_{ijkl}\frac{b}{a}\Gamma\left(\frac{1}{a}\left(\gamma\left(a-1\right)+1\right)\right)\left(\gamma+l\right)^{-\frac{1}{a}\left[\gamma(a-1)+1\right]}\right\}.$$

6. ORDER STATISTICS

Order statistics have many applications in statistical theory and practice. Let $X_1, X_2\cdots X_n$ be a random sample of size n from EGuW distribution. Suppose that $X_{r,n}$ denotes the rth order statistics. The pdf of the rth order statistics can be expressed as

$$g_{r;n}(x)=\frac{\sum_{l_1=0}^{n-r}(-1)^{l_1}\binom{n-r}{l_1}g(x)G(x)^{l_1+r-1}}{B(r,n-r+1)}, \quad (31)$$

where $B(.,.)$ is a beta function. Substituting (11) and (14) in (32) and replacing s with $l_1 + r - 1$ we obtain

$$g_{r;n}(x) = \frac{1}{B(r, n-r+1)} \sum_{l_1=0}^{n-r} \sum_{i,j,k,m_1,q,m,l,u}^{\infty} \sum_{p=0}^{l_1+r-1} \Lambda^* \left(\frac{x}{b}\right)^{a-1} exp\left[-(m+u+1)\left(\frac{x}{b}\right)^{a}\right] \tag{32}$$

where $\Lambda^* = (-1)^{l_1} \binom{n-r}{l_1} \Lambda_{i,j,k,m} \Lambda_{pqlm_1u}$.

The kth moment of rth order statistics for EGuW distribution is obtained as follows:

$$E\left(X_{r,n}^{k}\right) = \int_0^{\infty} x^k g_{r;n}(x)\,dx$$

$$E\left(X_{r,n}^{k}\right) = \frac{1}{B(r, n-r+1)} \sum_{l_1=0}^{n-r} \sum_{i,j,k,m,q,m_1,l,u}^{\infty} \sum_{p=0}^{l_1+r-1} \Lambda^* \int_0^{\infty} x^k \left(\frac{x}{b}\right)^{a-1} exp\left[-(m+u+1)\left(\frac{x}{b}\right)^{a}\right] dx$$

Let $z^* = (m+u+1)\left(\frac{x}{b}\right)^{a}$, then

$$E\left(X_{r,n}^{k}\right) = \frac{1}{B(r, n-r+1)} \sum_{l_1=0}^{n-r} \sum_{i,j,k,m,q,m_1,l,u}^{\infty} \sum_{p=0}^{l_1+r-1} \Lambda^* \frac{b^{k+1}}{a(m+u+1)^{1-\frac{k}{a}}} \Gamma\left(\frac{k}{a}+1\right).$$

7. MLE ESTIMATES OF EGuW PARAMETERS

Given that $x_1, x_2, \cdots x_n$ is a random sample from EGuW distribution and $\Psi = (a, b, B, \sigma, \alpha)$ be the unknown parameter vector for EGuW distribution. Using (5), the log-likelihood function of EGuW distribution is given by $l = \log\left(\prod_{i=1}^{n} g(x_i)\right)$

$$l = n\log\left(\frac{\alpha B a}{\sigma b}\right) + (a-1)\sum_{i=1}^{n}\log\left(\frac{x_i}{b}\right) - \left(\frac{1}{\sigma}+1\right)\sum_{i=1}^{n}\log\left(1-\exp\left(-\left(\frac{x_i}{b}\right)^{a}\right)\right)$$
$$+\sum_{i=1}^{n}\left(-\frac{1}{\sigma}\left(\frac{x_i}{b}\right)^{a}\right) - B\sum_{i=1}^{n} Y_i + (\alpha-1)\sum_{i=1}^{n}\log\left(1-\exp\left(-BY_i\right)\right) \tag{33}$$

where $Y_i = \left(exp\left(\left(\frac{x_i}{b} \right)^a \right) - 1 \right)^{-1/\sigma}$

The associated score function $V(\Psi) = \left(\frac{\partial l}{\partial \alpha}, \frac{\partial l}{\partial B}, \frac{\partial l}{\partial \sigma}, \frac{\partial l}{\partial a}, \frac{\partial l}{\partial b} \right)^T$ where $\frac{\partial l}{\partial \alpha}, \frac{\partial l}{\partial B}, \frac{\partial l}{\partial \sigma}, \frac{\partial l}{\partial a}$ and $\frac{\partial l}{\partial b}$ are partial derivatives of l given by

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \log\left(1 - \exp\left(-BY_i\right)\right)$$

$$\frac{\partial l}{\partial B} = \frac{n}{\sigma} + \sum_{i=1}^{n} Y_i + (\alpha - 1) \sum_{i=1}^{n} \left(\frac{Y_i \exp(-BY_i)}{1 - \exp(-BY_i)} \right)$$

$$\frac{\partial l}{\partial \sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^2} \sum_{i=1}^{n} \log\left(1 - \exp\left(-\left(\frac{x_i}{b} \right)^a \right) \right) + \frac{1}{\sigma^2} \sum_{i=1}^{n} \left(\frac{x_i}{b} \right)^a$$

$$-\frac{B}{\sigma^2} \sum_{i=1}^{n} \left[Y_i \left\{ \ln\left(\exp\left(\frac{x_i}{b} \right)^a - 1 \right) \right\} \right] + \frac{B(\alpha - 1)}{\sigma^2} \sum_{i=1}^{n} \left\{ \frac{Y_i \exp(-BY_i)}{\left(1 - \exp(-BY_i)\right)} \right\}$$

$$\frac{\partial l}{\partial a} = \frac{n}{a} + \sum_{i=1}^{n} \log\left(\frac{x_i}{b} \right) - \left(\frac{1}{\sigma} + 1 \right) \sum_{i=1}^{n} \left(\frac{\left(\frac{x_i}{b} \right)^a \ln\left(\frac{x_i}{b} \right) \exp\left(-\left(\frac{x_i}{b} \right)^a \right)}{\left(1 - \exp\left(-\left(\frac{x_i}{b} \right)^a \right) \right)} \right)$$

$$-\frac{1}{\sigma} \sum_{i=1}^{n} \ln\left(\frac{x_i}{b} \right) \exp\left(\frac{x_i}{b} \right)^a - B \sum_{i=1}^{n} \left(\exp\left(\left(\frac{x_i}{b} \right)^a \right) \left(\frac{x_i}{b} \right)^a \ln\left(\frac{x_i}{b} \right) \right)$$

$$+\frac{(\alpha - 1)B}{b^a} \sum_{i=1}^{n} \left(\frac{x_i^a \exp\left(\left(\frac{x_i}{b} \right)^a \right) \ln\left(\frac{x_i}{b} \right) \exp(-BY_i)}{\left(1 - \exp\left(-\left(\frac{x}{b} \right)^a \right) \right)} \right)$$

and

$$\frac{\partial l}{\partial b}=\frac{-n}{b}-\frac{n\left(a-1\right)}{b}+\frac{a\left(\frac{1}{\sigma}+1\right)}{b^{a+1}}\sum_{i=1}^{n}\left(\frac{x_i^{\,a}\exp\left(-\left(\frac{x_i}{b}\right)^{a}\right)}{\left(1-\exp\left(-\left(\frac{x}{b}\right)^{a}\right)\right)}\right)+\frac{a}{\sigma b^{a+1}}\sum_{i=1}^{n}x_i^{a}$$

$$-\frac{Ba}{\sigma b^{a+1}}\sum_{i=1}^{n}\left(x_i^{\,a}\exp\left(\left(\frac{x_i}{b}\right)^{a}\right)\left(\exp\left(\left(\frac{x_i}{b}\right)^{a}\right)-1\right)^{-\frac{1}{\sigma}-1}\right)$$

$$+\frac{a\left(\alpha-1\right)B}{\sigma b^{a+1}}\sum_{i=1}^{n}\left(\frac{x_i^{\,a}\left(\exp\left(\left(\frac{x_i}{b}\right)^{a}\right)-1\right)^{-\frac{1}{\sigma}-1}}{\left(1-\exp\left(-BY_i\right)\right)}\exp\left(-BY_i+\left(\frac{x_i}{b}\right)^{a}\right)\right)$$

Setting each element of the score function to zero, and solving the resulting nonlinear equations yield the estimates of the parameters of EGuW distribution. Since the elements of the score function are not in close form, their solutions can be found using some numerical optimization methods such as Quasi-Newton methods. An iterative method such as BFGS implemented in R software is an example of an iterative scheme based on the quasi-Newton method.

For interval estimation, a Fisher Information Matrix whose elements are the negative expected values of the second-order partial derivative of l in (33) is obtained as

$$J_{ij}\left(\Psi\right)=-E\left[\frac{\partial^2 l}{\partial\Psi_i\partial\Psi_j}\right]\; i,j=1,2,3,4,5$$

However, under some regularity conditions $\sqrt{n}\left(\widehat{\Psi}-\Psi\right)\sim N_5\left(0,\Phi\right)$. The Φ is the variance-covariance matrix obtained by taking the inverse of the Fisher Information Matrix $J_{ij}^{-1}\left(\Psi\right)$. Hence $\left(1-\alpha\right)100\%$ confidence interval for the estimates can be given as $\widehat{\Psi}\pm Z_{\frac{\alpha}{2}}\sqrt{\Phi_{ii}}$.

8. SIMULATION STUDY

This section is devoted to Monte Carlo simulation study aimed at ascertaining the performance of the maximum likelihood estimators of the parameters of EGuW distribution. Random samples are simulated using the inverse cdf method. The performance of the maximum likelihood estimators is examined using various simulations for different sample sizes and different parameter values. The simulation is repeated for $N = 500$ times each with sample size $n = 25{,}100{,}200{,}400$ and 800 and parameter values $a = 2.5, \alpha = 5, b = 3, B = 0.5, \sigma = 2$. Five quantities are computed in the simulations and these include:

Mean estimates (M) of the maximum likelihood estimator of the parameter $\Psi = (a, b, B, \sigma, \alpha)$ where

$$\mathrm{M} = \frac{1}{N}\sum_{i=1}^{N} \widehat{\Psi}\ ;$$

Average bias (AVB) of the maximum likelihood estimator of the parameter

$$\Psi = (a, b, B, \sigma, \alpha)$$

where

$$\mathrm{AVB} = \frac{1}{N}\sum_{i=1}^{N} \left(\widehat{\Psi} - \Psi\right);$$

Root mean squared error (RMSE) of the maximum likelihood estimator of the parameter

$$\Psi = (a, b, B, \sigma, \alpha)$$

where

$$\mathrm{RMSE} = \sqrt{\frac{1}{N}\sum_{i=1}^{N} \left(\widehat{\Psi} - \Psi\right)^2}\ ;$$

Coverage probability (CP) of 95% confidence intervals of the parameters $\Psi = (a, b, B, \sigma, \alpha)$. Average width (AW) of 95% confidence intervals of the parameter $\Psi = (a, b, B, \sigma, \alpha)$.

Table 1 contains the result for the M, AVB, RMSE, AW and CP values of the parameters a, α, b, B and σ for different sample sizes. Results from Table 1 show that the average biases for the parameter b are all negative. The average biases of the parameter b decrease as the sample size increases. It can also be observed that as the sample size increases, the root mean square errors of all the parameters decrease to zero underlining the consistency of the maximum likelihood estimators of the parameters. Also, the results in Table 1 show that the coverage probabilities of the confidence intervals are quite close to the nominal level of 95% and that the average confidence widths decrease as the sample size increases.

Table 1
Results of Monte Carlo Simulations for EGuW Parameters

Parameter	Sample size	M	AVB	RMSE	AW	CP
a	$n=25$	2.5061	-2.3422	5.5065	52.0324	0.92
	$n=100$	2.4213	3.3523	5.5014	48.3059	0.95
	$n=200$	2.5742	3.1506	5.4352	42.1781	0.97
	$n=400$	2.5197	-2.3165	5.3643	36.6572	0.99
	$n=800$	2.6592	2.4146	5.2116	29.8491	0.99
α	$n=25$	5.8352	-0.7345	2.1457	215.5985	0.87
	$n=100$	5.7893	0.7310	2.7055	169.8279	0.89
	$n=200$	5.6364	-0.5798	1.1612	143.9767	0.95
	$n=400$	5.9056	-0.1737	1.1095	104.8812	0.95
	$n=800$	5.8003	0.9967	1.0462	87.6121	0.97
b	$n=25$	4.3662	5.8004	9.0642	217.9356	0.99
	$n=100$	4.2965	5.6388	8.0083	105.2126	1
	$n=200$	3.9816	4.6599	7.1264	95.7382	1
	$n=400$	3.5643	4.1349	5.7750	87.3289	0.98
	$n=800$	3.3141	2.8513	4.2033	32.2321	1
B	$n=25$	0.8326	-1.8216	1.8604	27.2733	0.89
	$n=100$	0.9514	2.4452	1.7181	20.2095	0.92
	$n=200$	0.9030	4.2184	1.6636	17.2890	0.98
	$n=400$	0.5466	4.3237	1.5136	13.3731	0.99
	$n=800$	0.5110	-1.4986	1.5073	9.4237	1
σ	$n=25$	4.2490	-6.5072	15.1338	111.2375	0.91
	$n=100$	3.2558	-6.5192	13.6862	62.2417	0.95
	$n=200$	2.5316	-7.5709	9.5481	49.2168	0.89
	$n=400$	2.7840	8.9459	6.4391	29.3161	0.86
	$n=800$	2.7698	-0.7258	3.1707	14.7440	0.89

9. APPLICATIONS OF EGuW TO REAL DATA SETS

In this section, two real data sets are used to illustrate the flexibility and superiority of EGuW distribution in modeling real data. EGuW distribution is compared with baseline distribution (4) and other generalized distribution with the same baseline distribution. Generalized distributions compared with EGuW include exponentiated Weibull (EW) Mudholkar and Srivastava (1993), Exponentiated Generalized Weibull (EGW) Cordeiro et al. (2013), Kumaraswamy Weibull (KW) by Cordeiro et al. (2010), Exponentiated Kumaraswamy Weibull (EKW) Eisse (2017) and Beta Weibull (BW) Famoye et al. (2005). The pdfs of EW, EGW, KW, EKW and BW distributions are respectively given by

$$EW:\ f(x)=\frac{a\theta}{b^{a}}x^{a-1}\exp\left(-\left(\frac{x}{b}\right)^{a}\right)\left(1-\exp\left(-\left(\frac{x}{b}\right)^{a}\right)\right)^{\theta}$$

$$EGW: \ f(x) = \alpha\theta\left(\frac{a}{b}\right)\left(\frac{x}{b}\right)^{a-1}\exp\left(-\left(\frac{x}{b}\right)^{a}\right)\left(\exp\left(-\left(\frac{x}{b}\right)^{a}\right)\right)^{\alpha-1}\left\{1-\exp\left(-\alpha\left(\frac{x}{b}\right)^{a}\right)\right\}^{\theta-1}$$

$$KW: \ f(x) = \alpha\theta\left(\frac{a}{b}\right)\left(\frac{x}{b}\right)^{a-1}\exp\left(-\left(\frac{x}{b}\right)^{a}\right)$$

$$\left\{1-\exp\left(-\left(\frac{x}{b}\right)^{a}\right)\right\}^{\alpha-1}\left(1-\left\{1-\exp\left(-\left(\frac{x}{b}\right)^{a}\right)\right\}^{\alpha}\right)^{\theta-1}$$

$$EKW: \ f(x) = \alpha\theta\beta\left(\frac{a}{b}\right)\left(\frac{x}{b}\right)^{a-1}\exp\left(-\left(\frac{x}{b}\right)^{a}\right)\left\{1-\exp\left(-\left(\frac{x}{b}\right)^{a}\right)\right\}^{\alpha-1}$$

$$\times\left(1-\left\{1-\exp\left(-\left(\frac{x}{b}\right)^{a}\right)\right\}^{\alpha}\right)^{\theta-1}\left(1-\left(1-\left\{1-\exp\left(-\left(\frac{x}{b}\right)^{a}\right)\right\}^{\alpha}\right)^{\theta}\right)^{\beta-1}$$

$$BW: \ f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\left(\frac{a}{b}\right)\left(\frac{x}{b}\right)^{a-1}\left(1-\exp\left(-\left(\frac{x}{b}\right)^{a}\right)\right)^{\alpha-1}\left(\exp\left(-\left(\frac{x}{b}\right)^{a}\right)\right)^{\beta}$$

In each case, the parameters in the model are estimated by the maximum likelihood (ML) method using *R* statistical software. The Anderson-Darling (A^*) and Cramer-von Mises (W^*) statistics are used in comparison EGuW with other competing models. These two statistics are widely used in the comparison of non-nested models and to determine how closely a specific cdf fits the empirical distribution (Cordeiro et al. 2018). In general, the smaller the values of these statistics, the better the fit of the distribution to the data.

The first data set (Data Set 1) is on the lifetime of 50 industrial devices see Mudholkar and Srivastava (1993). The data is as follows:

Dataset 1:

0.1, 0.2, 1.0, 1.0, 1.0, 1.0, 1.0, 2.0, 3.0, 6.0, 7.0, 11.0, 12.0, 18.0, 18.0, 18.0, 18.0, 18.0, 21.0, 32.0, 36.0, 40.0, 45.0, 45.0, 47.0, 50.0, 55.0, 60.0, 63.0, 63.0, 67.0, 67.0, 67.0, 67.0, 72.0, 75.0, 79.0, 82.0, 82.0, 83.0, 84.0, 84.0, 84.0, 85.0, 85.0, 85.0, 85.0, 85.0, 86.0, 86.0

Table 2
MLEs (Standard Errors in Parenthesis) and the Statistics W^* and A^* for Dataset 1

Distribution	Estimates					W^*	A^*
$EGuW(a,b,B,\alpha,\sigma)$	5.4500 (0.0093)	44.4737 (3.3431)	1.1964 (0.9780)	1.6040 (1.9168)	22.1067 (10.027)	0.0923	0.7969
$W(a,b)$	0.9488 (0.1195)	44.8762 (6.9396)				0.5281	3.4819
$EW(a,b,\theta)$	4.9359 (0.0113)	86.6696 (1.4912)	0.1458 (0.0206)			0.6499	4.1773
$EGW(a,b,\theta,\alpha)$	4.6913 (0.0229)	27.1445 (0.0231)	0.0034 (0.0011)	0.1459 (0.0217)		0.5503	3.3497
$KW(a,b,\theta,\alpha)$	1.6451 (0.0177)	10.9953 (0.0142)	0.1248 (0.0151)	0.0650 (0.0094)		0.3399	2.1972
$EKW(a,b,\theta,\alpha,\beta)$	5.7506 (0.0032)	50.7275 (0.0240)	0.1727 (0.0874)	0.0716 (0.0210)	0.3877 (0.1239)	0.1531	1.1771
$BW(a,b,\alpha,\beta)$	1.2549 (0.0011)	4.9088 (0.0011)	0.3357 (0.0981)	0.0508 (0.0075)		0.4225	2.4378

The maximum likelihood estimates and standard errors of the estimates of EGuW and other competing models are shown in Table 2. Also in Table 2 are the W^* and A^* statistics. We observe that the five-parameter EGuW distribution provides a better fit to the data set than the other distributions with the same baseline given that it has the lowest value in the goodness of fit criteria considered.

The plot of the histogram and estimated pdfs of $EGuW$, W, EW, EGW, KW, EKW and BW distributions are displayed in Figure 4 while the empirical and estimated cdfs are shown in Figure 5. Both plots affirm the results of the goodness of fit criteria that EGuW distribution provides a better fit to the data set than the other competing models.

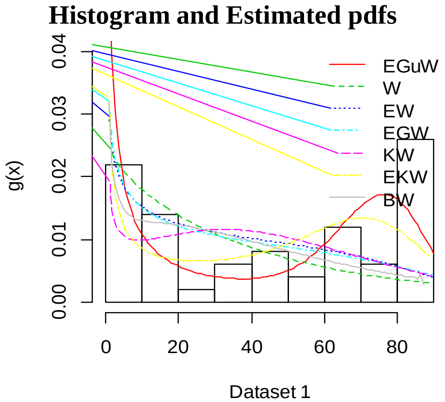

Figure 4: Plots of Estimated pdf of EGuW Distribution and other Competing Models based on Dataset 1

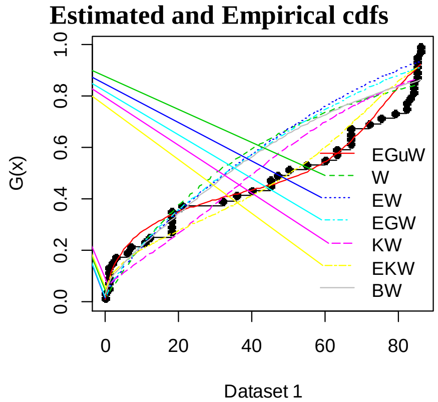

Figure 5: Plots of Estimated cdf of EGuW Distribution and other Competing Models based on Dataset 1

Second data (dataset 2) is on the strengths of 1.5 cm glass fibers initially reported by Smith and Naylor (1987) and reused by Alzaatreh et al. (2014). The data are:

Dataset 2:
0.55, 0.74, 0.77, 0.81, 0.84, 0.93, 1.04, 1.11, 1.13, 1.24, 1.25, 1.27,1.28, 1.29, 1.30, 1.36, 1.39, 1.42, 1.48, 1.48, 1.49, 1.49, 1.50, 1.50,1.51, 1.52, 1.53, 1.54, 1.55, 1.55, 1.58, 1.59, 1.60, 1.61, 1.61, 1.61,1.61, 1.62, 1.62, 1.63, 1.64, 1.66, 1.66, 1.66, 1.67, 1.68, 1.68, 1.69,2.00, 2.01, 2.24

Estimates of parameters of fitted distributions alongside their goodness of fit statistics for dataset 2 are given in Table 3. The EGuW provides a better fit to dataset 2 when compared to other competing models in this application because the values of W^* and A^* are the least among all the fitted models. This result is confirmed by the visual comparison provided in Figures 6 and 7.

Table 3

MLEs (Standard Errors in Parenthesis) and the Statistics W^* and A^* for Dataset 2

Distribution	Estimates					W^*	A^*
$EGuW(a,b,B,\alpha,\sigma)$	4.8649 (1.3036)	0.9887 (0.1381)	2.2515 (1.3267)	0.6686 (0.6180)	4.4792 (1.8535)	0.194	1.191
$W(a,b)$	5.2348 (0.5627)	1.5643 (0.0438)				0.302	1.681
$EW(a,b,\theta)$	5.5560 (1.4241)	1.5923 (0.1167)	0.8957 (0.3868)			0.299	1.660
$EGW(a,b,\theta,\alpha)$	5.5560 (1.4241)	2.0220 (52.2808)	3.7707 (541.69)	0.8957 (0.3868)		0.299	1.660
$KW(a,b,\theta,\alpha)$	5.0144 (8.1168)	2.2830 (8.0863)	1.0538 (1.9670)	7.4769 (145.35)		0.301	1.684
$EKW(a,b,\theta,\alpha,\beta)$	3.7223 (4.6716)	2.0410 (2.0678)	2.2058 (3.2514)	8.0476 (51.461)	0.6099 (0.7079)	0.295	1.640
$BW(a,b,\alpha,\beta)$	5.7530 (1.7688)	2.0461 (2.5360)	0.8502 (0.4205)	3.8818 (27.1293)		0.298	1.652

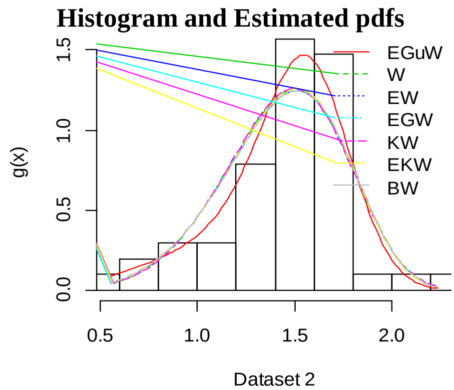

Figure 6: Plots of Estimated cdf of EGuW Distribution and other Competing Models based on Dataset 2

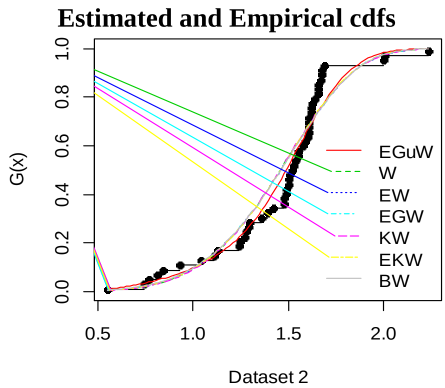

Figure 7: Plots of Estimated cdf of EGuW Distribution and other Competing Models based on Dataset 2

10. CONCLUSION

In this paper, we proposed a new five parameter distribution called exponentiated Gumbel Weibull (EGuW) distribution by using the Weibull distribution as the baseline distribution in the exponentiated Gumbel-G family of distributions. The density of the proposed family was expressed as a linear combination of the exponentiated Weibull and Weibull distribution. We derived the properties of the new model such as quantile function, moments, entropy, order statistics, and inequality measures. The estimation of the parameters of the new model was done using the method of maximum likelihood. Monte Carlo Simulation is used to examine the performance of the maximum likelihood estimates. Two datasets were used to authenticate the flexibility, potentiality, and usefulness of the proposed distribution.

REFERENCES

1. Abdul-Moniem, I.B. and Seham, M. (2015). Transmuted Gompertz Distribution. *Computational and Applied Mathematics Journal*, 1(3), 88-96.
2. Afify, A.Z., Cordeiro, G.M., Yousof, H.M., Abdus, S. and Ortega, E.M.M. (2016). The Marshall-Olkin additive Weibull distribution with variable shapes for the hazard rate. *Hacettepe Journal of Mathematics and Statistics*, doi:10.15672/hjms.201612618532.
3. Al-Aqtash, R., Lee, C. and Famoye, F. (2014). Gumbel-Weibull distribution: Properties and Applications. *Journal of Modern Applied Statistical Methods*, 13(2), 201-225.
4. Alzaatreh, A., Famoye, F. and Lee, C. (2014.). The gamma-normal distribution: Properties and applications. *Computational Statistics and Data Analysis*, 69, 67-80.
5. Aryal, G.R. and Tsokos, C.P. (2011). Transmuted Weibull distribution: a generalization of the Weibull probability distribution. *European Journal of Pure and Applied Mathematics*, 4, 89-102.
6. Cordeiro, G.M., Ortega, E.M. and Da Cunha, D.C.C. (2013). The exponentiated generalized class of distributions. *Journal of Data Sci.*, 11, 1-27.

7. Cordeiro, G.M., Ortega, E.M. and Nadarajah, S. (2010). The Kumaraswamy Weibull distribution with application to failure data. *Journal of the Franklin Institute*, 347, 1399-1429.
8. Cordeiro, G.M., Ortega, E.M. and Silva, G.O. (2014). The Kumaraswamy modified Weibull distribution: theory and applications. *Journal of Statistical Computation and Simulation*, 84, 1387-1411.
9. Corderio, G.M., Afify, A.Z., Yousof, H., Cakmakyapan, S. and Ozel, G. (2018). The Lindley Weibull Distribution: Properties and Applications. *Anais da Academia Brasileira de Ciencias*, 90(3), 2579-2598.
10. Eissa, F.H (2017). The exponentiated Kumaraswamy-Weibull Distribution with Application to Real Data. International. *J. Statist. Prob.*, 6(6), 167-182.
11. Elamir, E.A.H. and Scheult, A.H. (2003). Trimmed L-moments. *Computational Statistics & Data Analysis,* 43, 299-314.
12. Ghitany, M.E., Al-Hussaini, E.K. and Al-Jarallah, R.A. (2005). Marshall-Olkin extended Weibull distribution and its application to censored data. *Journal of Applied Statistics*, 32, 1025-1034.
13. Hassan, A. and Elgarhy, M. (2019). Exponentiated Weibull-Weibull distribution: Statistical Properties and Applications. *Gazi University Journal of Science*, 32(2), 616-635.
14. Lee, C., Famoye, F. and Olumolade, O. (2007). Beta-Weibull distribution: some properties and applications to censored data. *Journal of Modern Applied Statistical Methods*, 6(1), 173-186.
15. Moor, J.J. (1988). A quantile alternative for kurtosis. *The Statistician,* 37, 25-32.
16. Mudholkar, G.S. and Srivastava, D.K. (1993). Exponentiated Weibull family for analyzing bathtub failure-real data. *IEEE Transactions on Reliability*, 42, 299-302.
17. Nadarajah, S. (2006). The exponentiated Gumbel distribution with climate application: *Environmetrics*, 17, 13-23.
18. Nofal, Z.M., Afify, A.Z., Yousof, H.M. and Cordeiro, G.M. (2017). The generalized transmuted-G family of distributions. *Communications in Statistics-Theory and Methods*, 46, 4119-4136.
19. Nofal, Z.M., Afify, A Z., Yousof, H.M, Granzotto, D.C.T. and Louzada, F. (2016). Kumaraswamy transmuted exponentiated additive Weibull distribution. *International Journal of Statistics and Probability*, 5, 78-99.
20. Renyi, A. (1961). On Measures of entropy and information. *In Proceedings of the fourth Beckley Symposium on* Mathematical *Statistics and Probability*, 1(1), 547-561.
21. Silva, G.O., Ortega, E.M.M. and Cordeiro, G.M. (2010). The beta modified Weibull distribution. *Lifetime Data Analysis*, 6, 409-430.
22. Smith, R.L. and Naylor, J.C. (1987). A comparison of maximum likelihood and Bayesian estimators for the three-parameter Weibull distribution. *Journal of Royal Statistical Society,* Series C, 36, 358-369.
23. Uwadi, U.U., Okereke, E.W. and Omekara, C.O. (2019). Exponentiated Gumbel Family of Distributions: Properties and Applications. *International Journal of Basic Science and Technology*, 5(2), 100-117.
24. Yousof, H.M., Afify, A.Z., Alizadeh, M., Butt, N.S., Hamedani, G.G. and Ali, M.M. (2015). The transmuted exponentiated generalized-G family of distributions. *Pakistan Journal of Statistics and Operations Research*, 11, 441-464.