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EXPLICIT FORMS OF GENERALIZED ORDER STATISTICS OF BURR III DISTRIBUTION

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ABSTRACT

In this paper, the generalized order statistics (GOS) pertaining to Burr III distribution has been considered. The joint and the conditional pdf and cdf of the GOS for the Burr III distribution is studied within a limited scope. In addition, the current paper seeks to spot the light on some special forms derived through the mentioned joint and conditional pdf and cdf of the GOS for the Burr III.

KEYWORDS

Generalization of order statistics, joint distribution, conditional distribution, Burr type III.

1. INTRODUCTION

The Generalized Order Statistics (GOS) gripped the attention of numerous researchers during the past century. Kamps (1995) introduced the GOS as an inclusive frame for ordered random variable models. Moreover, various models of ascendingly ordered random variables were consolidated with different related topics such record values, also ordinary, Stigler and sequential order statistics by (Kamps, 1995). Ahsanullah (1995) has derived the marginal and joint moment generating functions from Gumbel and power distributions by recursive relations, respectively. Raqab (2003) has established the moment generating function of the generalized order statistics of the exponential distribution which derived as recurrence relations satisfied by the single and the product moments for order statistics from.

In this article, the researcher presents the GOS of the Burr III distribution and some of its properties. Generally, Burr's family of twelve distributions for modeling and fitting life time data was introduced by Burr (Para, Jabeen and Jan, 2015). The Burr type XII and III have many applications in physics, actuarial studies, reliability and applied statistics. The estimation of their parameters has a wide range of functions such as reliability, hazard rate and mode. A more detailed description of this topic can be referred to in Johnson, Kotz, and Balakrishnan (1995). Moreover, Shekh Ahmed (2013) applied the GOS for the exponential distribution and derived some special cases for these forms.

The contents of this article are presented in six sections. In the next section, the basic concepts and facts related to the GOS are presented. Section three will be confined to reviewing the Burr III density (pdf) and cumulative (cdf) functions, then joint distributions of the GOS for Burr III are driven and discussed with some special cases in section four. In section five, the conditional distribution for the GOS of Burr III distribution is highlighted. Finally, some conclusions and suggestions for further research are presented in section six.

2. BASIC CONCEPTS

The generalized order statistics are widely used in several statistical subjects. The order statistics, serial statistics and representations of marginal density and distribution functions are directly related with the concept of GOS. In the following, some basic definitions will be presented.

Definition 2.1

Let $F(x)$ refers to an absolutely continuous distribution function with density function $f(x)$, then the sequence of random variables $X_{1:n,m,k}$, $X_{2:n,m,k}$, ... $X_{n:n,m,k}$ is called 'n' Generalized Order Statistics (GOS), where ($k \geq 1$, m is a real number) (Kamps, 1995).

Definition 2.2

The random variables $X(1,n,m,k),\ldots,X(n,n,m,k)$are called GOS based on the (cdf), $F(x)$, if their joint (pdf) is given by

$$f(x_1,\ldots,x_n) = \begin{cases} k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}\left(1-F(x_i)\right)^{m_i} f(x_i)\right)\left(1-F(x_i)\right)^{k-1} f(x_n) \\ 0, \qquad \text{otherwise} \end{cases} \tag{1}$$

for $F^{-1}(0) = x_1 < \cdots < x_n = F^{-1}(1)$, with parameters $n \epsilon N, n \geq 2, k > 0,\ m = (m_1, m_2, \ldots, m_{n-1})$, $M_r = \sum_{j=r}^{n-1} m_j$, such that $\gamma_r = k + n - r + M_r > 0$, for all $r \in \{1,\ldots,n-1\}$, let $c_{r-1} = \prod_{j=1}^{r}\gamma_j, r = 1,2,\ldots,n-1$ and $\gamma_n = k$, (Shekh Ahmed, 2013).

The (cdf) of such generalized order statistics random variable $X(r,n,m,k)$ is given by,

$$F_{1,\ldots,n}(X) = 1 - c_{r-1}\sum_{j=1}^{r}\frac{a_j(r)}{\gamma_j}\left(1-F(X)\right)^{\gamma_j}, \tag{2}$$

where

$$a_j(r) = \prod_{i=1, i\neq j}^{r}\left(\frac{1}{\gamma_i - \gamma_j}\right), 1 \leq j < r \leq n.$$

The marginal density function of r^{th} GOS $X(1,n,\tilde{m},k), r = 1,\ldots,n$ based on F is formulated by [4] as,

$$f_{X(1,n,m,k)}(x) = \varphi_{r,n}\left(F(x)\right)f(x),$$

where,

$$\varphi_{r:n}(x) = \frac{c_{r-1}}{(r-1)!}(1-x)^{\gamma_r - 1} g_m^{r-1}(x), \tag{3}$$

for $x \epsilon [0,1)$

$$g_m(x) = \begin{cases} \frac{1}{m+1}(1-(1-x)^{m+1}), \text{if } m \neq -1 \\ -ln(1-x) \text{ if } m = -1 \end{cases}$$

$c_{r-1} = \prod_{i=1}^{r} \gamma_i$.

The corresponding marginal distribution of r^{th} GOS based on F is denoted by,

$$F_{X(r,n,m,k)}(x) = \varphi_{r,n}\big(F(x)\big),$$

where

$$\varphi_{r:n}(x) = 1 - \sum_{i=0}^{r-1} \frac{c_{r-1}}{r!}(1-u)^{\gamma_r+1}\big(g_m(u)\big)^i, u \in (0,1). \quad (4)$$

3. JOINT DISTRIBUTION OF THE GENERALIZED ORDER STATISTICS

In this section, we scrutinize the joint distribution of GOS for Burr III distribution.

3.1 The Burr Type III Distribution

A random variable X which has a Burr III distribution with location parameter a and shape parameter b, has a continuous probability density function (pdf) and formulated as,

$$f(x) = \left(\frac{\ell c}{b}\right)\left(\frac{b}{x}\right)^{c+1}\left[1+\left(\frac{b}{x}\right)^{c}\right]^{-\ell-1}, x > 0, (b, c, \ell > 0), \quad (5)$$

The distribution function (cdf) is:

$$F(x) = \left[1+\left(\frac{b}{x}\right)^{c}\right]^{-\ell}, x > 0, (b, c, \ell > 0), \quad (6)$$

Now, once we substitute equations (5) and (6) into equations (1), we get the formula of the joint pdf of the GOS of Bur III distribution will be obtained:

$$f(x_1, \dots, x_n) = \left[\prod_{j=1}^{n-1} \gamma_j\right]\left[\prod_{i=1}^{n-1}\left(1-\Delta_i{}^{-\ell}\right)^{m_i}.\left(\frac{\ell c}{b}\right)\left(\frac{b}{x_i}\right)^{c+1} \Delta_i{}^{-\ell-1}\right] \times \left(1-\Delta_n{}^{-\ell}\right)^{k-1}.\left(\frac{\ell c}{b}\right)\left(\frac{b}{x_n}\right)^{c+1} \quad (7)$$

where

$$\Delta_i = \left[1+\left(\frac{b}{x_i}\right)^{c}\right], i = 1,2, \dots, n.$$

In Equation (7), let $k = 1$ and $m = 0$ then it will be observed that

$$\prod_{j=1}^{n-1} \gamma_j = \prod_{j=1}^{n-1}(1+n-j) = n(n-1) \dots 3.2.1 = n!.$$

Therefore the joint pdf of all ordinary order statistics of Burr III distribution becomes

$$f(x_1,\dots,x_n)=n!\left[\prod_{i=1}^{n-1}\Delta_i{}^{-\ell-1}.\left(\frac{\ell c}{b}\right)\left(\frac{b}{x_i}\right)^{c+1}\right]\left(\frac{\ell c}{b}\right)\left(\frac{b}{x_n}\right)^{c+1}\Delta_n{}^{-\ell-1}$$
$$=n!\prod_{i=1}^{n}\left(\frac{\ell c}{b}\right)\left(\frac{b}{x_i}\right)^{c+1}\Delta_i{}^{-\ell-1}. \qquad (8)$$

3.2 Joint Distribution of Two Generalized Order Statistics of Generalized Burr III Distribution

In this section concerns with investigating the joint pdf of $\boldsymbol{X(i,n,m,k)}$ and $\boldsymbol{X(j,n,m,k)}$ for GOS of Burr III distribution. And in order to reach this, we introduce the following theorem from (Qiu, Wang, 2007).

Theorem 3.1

The joint pdf of i^{th} and j^{th} generalized order statistics $\boldsymbol{X(i,n,m,k)}$ and $\boldsymbol{X(j,n,m,k)}$ is represented via

$$f_{i,j,n,m,k}(x_i,x_j)=\frac{c_j}{(i-1)!\,(j-i-1)!}\left(1-F(x_i)\right)^m\left(1-F(x_j)\right)^{\gamma_j-1}\left(g_m(F(x_i))\right)^{i-1}D \qquad (9)$$

where

$$\boldsymbol{D}=\left(\boldsymbol{g_m}\left(\boldsymbol{F(x_j)}-\boldsymbol{g_m}\left(\boldsymbol{F(x_j)}\right)\right)\right)^{\boldsymbol{j-i-1}}\boldsymbol{f(x_i)f(x_j)},$$

for $\boldsymbol{0<x_i<x_j<\infty,c_r=\prod_{j=1}^{r}\gamma_j,\gamma_j=k+(n-j)(m+1)}$.

Since $\boldsymbol{\lim_{x\to-1}\left(\frac{1-(1-x)^{m+1}}{m+1}\right)=-ln(1-x)}$, we write $\boldsymbol{g_m=\frac{1-(1-x)^{m+1}}{m+1}}$ for all $\boldsymbol{x\in(0,1)}$ and for all $\boldsymbol{m}$ with $\boldsymbol{g_{-1}(x)=\lim_{x\to-1}g_m(x)}$. See (Shekh Ahmed, 2013) for further details.

Corollary 3.1

The joint pdf of $X(i,n,m,k)$, and $X(j,n,m,k)$ of Burr III distribution is approached through:

$$f_{i,j,n,m,k}(x_i,\ x_j)=\frac{c_j}{(i-1)!\,(j-i-1)!}([1-\Delta_i]^{-l})^m\left([1-\Delta_i{}^{-l}]\right)^{\gamma_{i-1}}\left[\frac{1-\left(1-\Delta_i{}^{-l}\right)}{m+1}\right]^{i-1}D, \qquad (10)$$

where,

$$D=\frac{(lc)^2b^{2c}}{(x_ix_j)^{c+1}}\left[\Delta_i\Delta_j\right]^{j-i-1}\left[\frac{(1-\Delta_i)^{-l}-\left(1-\Delta_i{}^{-l}\right)}{m+1}\right]$$

for, $0<x_i<x_j<\infty$, where $c_r=\prod_{j=1}^{r}\gamma_j,\gamma_j=1+(n-j)$.

Proof:

The distribution can be proved by substituting equations (5) and (6) by (9), resulting in equation (10) from which there are two special cases that can be driven.

3.3 The Joint Density Function of Two Ordinary Order Statistics of Burr III Distribution

Let $k = 1$ and $m = 0$, then Equation (10) reduces joint density of two ordinary order statistics $X(i, n, 0,1)$, $X(j, n, 0,1)$ in Burr III distribution as,

$$f_{i,j,n,0,1}(x_i,\ x_j) = \frac{c_j}{(i-1)!\,(j-i-1)!}\left(\left[1-\Delta_i{}^{-l}\right]\right)^{\gamma_{i-1}}\left[1 - \left(1-\Delta_i{}^{-l}\right)\right]^{i-1} D, \tag{11}$$

where,

$$D = \frac{(lc)^2 b^{2c}}{\left(x_i x_j\right)^{c+1}}\left[\Delta_i \Delta_j\right]^{j-i-1}\left[\left(1-\Delta_i{}^{-l}\right) - \left(1-\Delta_i{}^{-l}\right)\right].$$

4. DISTRIBUTION OF SINGLE GENERALIZED ORDER STATISTICS

In the following, the concept single generalized order statistics is presented and then the equivalent formulas of the Burr III distribution will be highlighted.

Theorem 4.1

Further integrating out $x_1, \dots, x_{r-1}, \dots, x_{r+1}, \dots, x_n$ from Definition 2.2, we get the pdf $f_{r,n,m,k}(x)$ of $X(r, n, m, k)$ as

$$f_{\ r,n,m,k}(x) = \frac{c_r}{(r-1)!}\left(1-F(x_r)\right)^{\gamma_r - 1} g_m^{r-1}\left(F(x_r)\right) f(x_r), \tag{12}$$

where,

$$c_r = \prod_{j=1}^{r} \gamma_j, \gamma_j = k + (n-r)(m+1), \text{ and } g_m = \frac{1-(1-x)^{m+1}}{m+1}$$

for all $x \in (0,1)$ for all m, with $g_{-1}(x) = \lim_{x \to -1} g_m(x)$, see (Garg, 2009).

From Theorem 4.1, we get the pdf for both the minimum and maximum GOS of the Bur III distribution.

Corollary 4.1

The pdf of the minimum generalized order statistic is

$$f_{1,n,m,k}(x) = \left(k + (n-1)(m+1)\right)\left(1-F(x_1)\right)^{k+(n-1)(m+1)-1} f(x_1) \tag{13}$$

Proof:

Using Equation (12), let $r = 1$, then $c_1 = \gamma_1 = k + (n-1)(m+1)$, we get Equation (13) and that completes the proof.

Corollary 4.2

The pdf of the maximum generalized order statistic is

$$f_{n,n,m,k}(x) = \frac{\big(k+(n-1)(m+1)\big)\big(k+(n-2)(m+1)\ldots(k)\big)}{(n-1)!} \big(1 - F(x_n)\big)^{k-1}\left(\frac{1-\big(1-F(x_n)\big)^{m+1}}{m+1}\right). \quad (14)$$

Proof:

Using Theorem 4.1, let $r = n$, then $g_m = \frac{1-(1-F(x_n))^{m+1}}{m+1}$, $\gamma_n = k$, $c_n = \prod_{j=1}^{n} c + (n-j)(m+1) = \big(k+(n-1)(m+1)\big)\big(k+(n-2)(m+1)\big)\ldots(k)$, we get Equation (14) and that completes the proof.

4.1 The pdf of the r^{th} Generalized Order Statistic of Burr III Distribution

Using the pdf and cdf given in Equations (5) and (6) in Equation (12), and collecting terms we get the pdf of the r^{th} GOS for Burr III distribution

$$f_{r,n,m,k}(x) = \frac{c_r}{(r-1)!}([1-\Delta_r]^{-l})^{\gamma_{r-1}}\left[\frac{1-((1-\Delta_i)^{-l})^{m+1}}{m+1}\right]^{r-1} \left(\frac{lc}{b}\right)\left(\frac{b}{x_r}\right)^{c+1} {\Delta_r}^{-l-1}. \quad (15)$$

4.2 The pdf of the Minimum Generalized Order Statistic of Burr III Distribution

In Equation (15), let $r = 1$, $[c_1 = \gamma_1 = k + (n-1)(m+1)]]$, then the pdf of the minimum generalized order statistic for Burr III Distribution is

$$f_{1,n,m,k}(x) = [k+(n-1)(m+1)] ([1-\Delta_1]^{-l})^{k+(n-1)(m+1)-1}\left(\frac{lc}{b}\right)\left(\frac{b}{x_1}\right)^{c+1} {\Delta_1}^{-l-1}. \quad (16)$$

4.3 The pdf of the Minimum Ordinary Statistics of Burr III Distribution

In Equation (16), let $k = 1$ and $m = 0$ then,

$$f_{1,n,0,1}(x) = n([1-\Delta_1]^{-l})^{n-1}\left(\frac{lc}{b}\right)\left(\frac{b}{x_1}\right)^{c+1} {\Delta_1}^{-l-1}.$$

4.4 The pdf of the Minimum Generalized Order Statistic of Burr III Distribution

Using the pdf and cdf given in Equations (5) and (6) in Equation (14), and collecting terms we get the pdf of the maximum generalized order statistic for Burr III distribution,

$$f_{n,n,m,k}(x) = \frac{[k+(n-1)(m+1)][k+(n-2)(m+1)]\ldots[k]}{(n-1)!} \left[\frac{1-((1-\Delta_n)^{-l})^{m+1}}{m+1}\right]^{n-1}\left(\frac{lc}{b}\right)\left(\frac{b}{x_n}\right)^{c+1} {\Delta_n}^{-l-1}. \quad (17)$$

The pdf of the Maximum Ordinary Order Statistic of Burr III Distribution

In Equation (17), let $k = 1$ and $m = 0$, then the pdf of the maximum ordinary order statistic for Burr III distribution

$$f_{n,n,0,1}(x) = n\,([\Delta_n]^{-l})^{n-1}\left(\frac{lc}{b}\right)\left(\frac{b}{x_n}\right)^{c+1}([\Delta_n]^{-l})^{-l-1} \quad (18)$$

where,

$$\frac{[k+(n-1)(m+1)][k+(n-2)(m+1)]\ldots[k]}{(n-1)!}$$
$$= \frac{n(n-1)\ldots.3.2.1}{(n-1)!} = \frac{n!}{(n-1)!} = n.$$

5. CONDITIONAL DISTRIBUTION OF GENERALIZED ORDER STATISTICS OF BURR III DISTRIBUTION

In this section some previous literature of the conditional distribution of generalized order statistics is presented and then derived these results for Burr III distribution.

Theorem 5.1

Let $X_1, \ldots., X_n$ be a random sample from a continuous population with cdf $F(x)$ and pdf $f(x)$. Let $X(1,n,m,k),\ X(2,n,m,k), \ldots,\ X(n,n,m,k)$ denote the generalized order statistics obtained from this sample. Then the conditional distribution of $X(s,n,m,k)|X(r,n,m,k) = x$, for $r < s$ is

$$h(X(s,n,m,k)|X(r,n,m,k)) = \frac{c_s\left(1-F(x)\right)^m\left(1-F(y)\right)^{\gamma_s-1}\left(g_m(F(y))-g_m(F(x))\right)^{s-r-1} f(y)}{c_r(s-r-1)!\left(1-F(x)\right)^{\gamma_r-1}} \quad (19)$$

$$F^{-1}(0) < x \le y < F^{-1}(1),$$

where,

$$c_r = \prod_{j=1}^{r}\gamma_j, \gamma_j = k+(n-j)(m+1), and\ g_m = \frac{1-(1-x)^{m+1}}{m+1}$$

for all $x \in (0,1)$ and for all m with $g_{-1}(x) = \lim_{x\to-1} g_m(x)$.

See (Samuel, 2008) for more details.

Corollary 5.1

The conditional pdf of $\boldsymbol{X(s,n,m,k)|X(r,n,m,k) = x}$ of Burr III distribution is given by

$$\boldsymbol{h(X(s,n,m,k)|X(r,n,m,k))} = \frac{\boldsymbol{n!\,c_s\left(1-\Delta_i^{-l}\right)^m\left(1-\Delta_j^{-l}\right)^{\gamma_s-1}\left(\frac{2-\Delta_i^{-l}-\Delta_j^{-l}}{m+1}\right)^{s-r-1}\prod_{i=1}^{n}\left(\frac{\ell c}{b}\right)\left(\frac{b}{x_i}\right)^{c+1}\Delta_i^{-\ell-1}}}{\boldsymbol{c_r(s-r-1)!\left(1-\Delta_i^{-l}\right)^{\gamma_r-1}}}, \quad (20)$$

Proof:

Substituting Equation (5) and (6) in Equation (19), we get Equation (20).

For more details, see (Shekh Ahmed, 2013).

Corollary 5.2

(Conditional distribution of an ordinary order statistics)

Using Theorem (5.1), if $k = 1$ and $m = 0$, then

$$h(X(s,n,0,1)|X(r,n,0,1)) = \frac{\left(1-F(x)\right)^{0}\left(1-F(y)\right)^{\gamma_s-1}\left(g_0\left(F(y)\right)-g_0\left(F(x)\right)\right)^{s-r-1} f(y)c_s}{(s-r-1)!\,c_r\left(1-F(x)\right)^{\gamma_r-1}}, \quad (21)$$

$X(r,n,0,1) \le Y(s,n,0,1) < \infty$,

$$\gamma_r = 1+(n-r), \gamma_s = 1+(n-s), \gamma_{r+1} = n-r, g_0\left(F(y)\right) = F(x), g_0\left(F(x)\right) = F(y), c_s = n(n-1)(n-2)\ldots(1+n-s) = \frac{n!}{(n-s)!}$$

and

$$c_r = n(n-1)(n-2)\ldots(1+n-r) = \frac{n!}{(n-r)!}.$$

Then**,**

$$h(X(s,n,0,1)|X(r,n,0,1)) = \frac{(n-r)!\left(1-F(y)\right)^{n-s}\left(F(y)-F(x)\right)^{s-r-1} f(y)}{(n-s)!\,(s-r-1)!\left(1-F(x)\right)^{n-r}}. \quad (22)$$

Corollary 5.3

The conditional terms of two ordinary statistics, $\boldsymbol{X(s,n,0,1)|X(r,n,0,1) = x}$, for Burr III distribution is

$$h(X(s,n,0,1)|X(r,n,0,1)) = \frac{\boldsymbol{n!\,(n-r)!\left(1-\Delta_j^{-l}\right)^{n-s}\left(2-\Delta_i^{-l}-\Delta_j^{-l}\right)^{s-r-1}\prod_{i=1}^{n}\left(\frac{\ell c}{b}\right)\left(\frac{b}{x_i}\right)^{c+1}\Delta_i^{-\ell-1}}}{\boldsymbol{(n-s)!\,(s-r-1)!\left(1-\Delta_i^{-l}\right)^{n-r}}}, \quad (23)$$

$\boldsymbol{X(s,n,0,1) \le Y(s,n,0,1) < \infty}$**.**

Proof:

In Equation (20), let $\boldsymbol{k = 1}$ and $\boldsymbol{m = 0}$, and collecting terms, we obtain eq. (23).

6. CONCLUSIONS AND SUGGESTIONS

In this paper, the Joint Distribution of the Generalized Order Statistics of Burr III distribution has been studied and established, while the joint density function of two ordinary Order Statistics was driven from them. In addition, some special cases of the conditional distribution of the generalized order statistic of Burr III has been concluded.

We suggest using the theoretical results for this paper in estimating the location and scale parameters. In addition, we suggest studying moments estimation of GOS, with applied practitioners.

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