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# THE EXPONENTIATED ODD WEIBULL-TOPP-LEONE-G FAMILY OF DISTRIBUTIONS: MODEL, PROPERTIES AND APPLICATIONS

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## ABSTRACT

In this paper, a new generalized family of distributions called the exponentiated odd Weibull-Topp-Leone-G (EOW-TL-G) family of distributions is presented. A linear representation of the proposed model is also presented. A simulation study to examine the consistency of the maximum likelihood estimates is conducted. Usefulness of the new proposed model was assessed by means of applications to two real data examples.

## KEYWORDS

Exponentiated-G Distribution, Odd Weibull-G Distribution, Topp-Leone-G Distribution, Maximum Likelihood Estimation.

## 1. INTRODUCTION

In recent years, generalized distributions have been widely used in modeling different real life phenomena as they possess more flexibility compared to classical distributions. Generalized and extended families of distributions have been introduced and studied over past decades for modeling data in many applied areas such as economics, engineering, biological studies, environmental sciences, medical sciences and finance. Statistical distributions are important in modeling the real life of an item and therefore proper distributions that provide useful information for sound conclusions and decisions are needed. For that reason, generalized distributions have become appropriate for data that have both monotonic and non-monotonic hazard rate functions.

Many generators are proposed in the literature and these include Kumaraswamy-G (Kw-G) by Cordeiro et al. [16], Kumaraswamy odd Lindley-G (Kw-OLG) by Chipepa et al. [13], beta-G by Eugene et al. [17], beta odd Lindley-G (BOL-G) by Chipepa et al. [12], Marshall and Olkin-G (MO-G) by Marshall and Olkin [21], gamma-G by Zografos and Balakrishnan [25], Weibull-G (W-G) by Bourguignon et al. [8], T-X family by Alzaatreh et al. [5], Topp-Leone odd log-logistic-G (TLOLL-G) by Brito et al. [9], Topp-Leone-Marshall-Olkin-G (TL-MO-G) by Chipepa et al. [11], to mention a few.

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More recently, Chipepa et al. [10] developed the odd Weibull-Topp-LeoneG (OW-TL-G) family of distributions using the generalization by Gurvich et al. [18]. Gurvich et al. [18] developed the generalized family of the Weibull distribution with cumulative distribution function (cdf) given by

$$F(x;\ \alpha,\xi) = 1 - exp(-\alpha H(x;\xi)), \tag{1.1}$$

for $\alpha > 0$ and $\xi$ a vector of parameters from the baseline distribution. The cdf of the OW-TL-G family of distributions is given by

$$F(x;\ b,\beta,\xi) = 1 - exp\,\{-t\}, \tag{1.2}$$

where $t = \left[\frac{[1-\bar{G}^2(x;\xi)]^b}{\left[1-(1-\bar{G}^2(x;\xi))^b\right]}\right]^\beta$ is the exponentiated odds ratio of the Topp-Leone-G (TL-G) distribution by Al-Shomrani et al. [4], for $b$, $\beta$>0 and parameter vector $\xi$.

The primary motivations for considering this new family of distributions are the advantages it offers with respect to having hazard functions that exhibits increasing, decreasing, bathtub and upside bathtub shapes, as well as the versatility and flexibility in modeling lifetime data. We were motivated by the properties shown by the OW-TL-G family of distributions. We therefore propose an extension of the OW-TL-G family of distributions called the exponentiated odd Weibull-Topp-Leone-G (EOW-TL-G) family of distributions which

- can fit data sets with various shapes of hazard rate functions (J, reverse J, uni-modal, bathtub, upside bathtub followed by bathtub);
- Is different from the Topp-Leone distribution in the sense that there is no restriction in the domain (0, 1).

The rest of the paper is organized as follows: In Section 2, we present the new generalized family of distributions, some sub-families, quantile function and linear representation. Parameter estimation via the method of maximum likelihood is given in Section 3. Some special cases of the EOW-TL-G family of distributions are presented in Section 4. Monte Carlo simulation study results are presented in Section 5. Applications of the EOW-TL-log logistic (EOWTL-LLoG) distribution to real data sets are given in Section 6, followed by concluding remarks.

## 2. THE MODEL AND PROPERTIES

In this section, we introduce the proposed distribution, namely, EOW-TL-G family of distributions. We added an extra shape parameter to equation (1.2) to get the EOW-TL-G family of distributions with cdf and probability density function (pdf) given by

$$F(x) = [1 - \exp\{-t\}]^\alpha \tag{2.1}$$

and

$$f(x) = \frac{2b\alpha\beta g(x;\xi)\overline{G}(x;\xi)[1-\overline{G}^2(x;\xi)]^{b\beta-1}}{[1-(1-\overline{G}^2(x;\xi))^b]^{\beta+1}} \exp\{-t\}[1-\exp\{-t\}]^{\alpha-1}, \tag{2.2}$$

respectively, where $t = \left[\frac{[1-\bar{G}^2(x;\xi)]^b}{\left[1-(1-\bar{G}^2(x;\xi))^b\right]}\right]^\beta$, for $b\ \alpha, \beta > 0$ and parameter vector $\xi$.

### 2.1 Some Sub-Families

Some sub-families of the EOW-TL-G distribution are presented in this subsection.

- When $\beta = 1$, we obtain the exponentiated odd exponential-Topp-Leone-G (EOE-TL-G) family of distributions.
- When $\beta = 2$, we obtain the exponentiated odd Rayleigh-Topp-Leone-G (EOR-TL-G) family of distributions.
- When $\alpha = 1$, we obtain the Odd Weibull-Topp-Leone-G (OW-TL-G) family of distributions.
- When $b = \beta = 1$ we obtain a new family of distributions with the following cdf,

$$F(x;\alpha,\xi) = \left[1 - \exp\left\{-\left[\frac{[1-\overline{G}^2(x;\xi)]}{\overline{G}^2(x;\xi)}\right]\right\}\right]^{\alpha}.$$

- Furthermore, we obtain a new family of distributions by letting $b = 1$ and $\beta = 2$ with the following cdf,

$$F(x;\alpha,\xi) = \left[1 - \exp\left\{-\left[\frac{[1-\overline{G}^2(x;\xi)]}{\overline{G}^2(x;\xi)}\right]^2\right\}\right]^{\alpha}.$$

### 2.2 Quantile Function

We derive the quantile function by inverting the cdf given by equation (2.1). We invert the function

$$\left[1 - \exp\left\{-\left[\frac{[1-\overline{G}^2(x;\xi)]^b}{[1-(1-\overline{G}^2(x;\xi))^b]}\right]^{\beta}\right\}\right]^{\alpha} = u,$$

for $0 \leq u \leq 1$, which simplifies to

$$(-\ln(1-u^{\frac{1}{\alpha}}))^{\frac{1}{\beta}} = \frac{(1-\overline{G}^2(x;\xi)]^b)}{(1-(1-\overline{G}^2(x;\xi))^b)}.$$

This can be written as

$$[1-\overline{G}^2(x;\xi)]^b = \frac{[-\ln(1-u^{\frac{1}{\alpha}})]^{\frac{1}{\beta}}}{\left[1+\left(-\ln(1-u^{\frac{1}{\alpha}})\right)^{\frac{1}{\beta}}\right]}.$$

The equation simplifies to

$$G(x;\xi) = 1 - \left[1 - \left(\frac{(-\ln(1-u^{\frac{1}{\alpha}}))^{\frac{1}{\beta}}}{1+(-\ln(1-u^{\frac{1}{\alpha}}))^{\frac{1}{\beta}}}\right)^{\frac{1}{b}}\right]^{\frac{1}{2}}.$$

Therefore, the quantiles of the EOW-TL-G family of distributions is given by

$$x(u) = G^{-1}\left[1 - \left[1 - \left(\frac{(-\ln(1-u^{\frac{1}{\alpha}}))^{\frac{1}{\beta}}}{1+(-\ln(1-u^{\frac{1}{\alpha}}))^{\frac{1}{\beta}}}\right)^{\frac{1}{b}}\right]^{\frac{1}{2}}\right]. \tag{2.3}$$

**2.3 Linear Representation**

The linear representation of the EOW-TL-G family of distributions is presented in this subsection. Using the following series expansions

$$[1-\exp\{-t\}]^{\alpha-1} = \sum_{q=0}^{\infty}(-1)^q\binom{\alpha-1}{q}\exp\{-qt\},$$

$$\exp\{-(q+1)t\} = \sum_{z=0}^{\infty}\frac{(-1)^z(q+1)^z}{z!}t^z,$$

$$[1-(1-\overline{G}^2(x;\xi))^b]^{-(\beta(z+1)+1)} = \sum_{k=0}^{\infty}(-1)^k\binom{-(\beta(z+1)+1)}{k}(1-\overline{G}^2(x;\xi))^{bk},$$

$$[1-\overline{G}^2(x;\xi)]^{b\beta(z+1)+bk-1} = \sum_{m=0}^{\infty}(-1)^m\binom{b\beta(z+1)+bk-1}{m}\overline{G}^{2m}(x;\xi)$$

and

$$\overline{G}^{2m+1}(x;\xi) = \sum_{j=0}^{\infty}(-1)^j\binom{2m+1}{j}G^j(x;\xi),$$

we write the EOW-TL-G pdf as

$$\begin{aligned} f(x) &= \sum_{q,z,k,m,j=0}^{\infty}\frac{(-1)^{q+z+k+m+j}(q+1)^z 2b\alpha\beta}{z!}\binom{-(\beta(z+1)+1)}{k} \\ &\times \binom{\alpha-1}{q}\binom{b\beta(z+1)+bk-1}{m}\binom{2m+1}{j}g(x;\xi)G^j(x;\xi) \\ &= \sum_{j=0}^{\infty}v_j g_j(x;\xi), \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} v_j &= \sum_{q,z,k,m=0}^{\infty}\frac{(-1)^{q+z+k+m+j}(q+1)^z 2b\alpha\beta}{z!(j+1)}\binom{\alpha-1}{q}\binom{-(\beta(z+1)+1)}{k} \\ &\times \binom{b\beta(z+1)+bk-1}{m}\binom{2m+1}{j} \end{aligned} \tag{2.5}$$

and $g_j(x;\xi) = (j+1)g(x;\xi)[G_j(x;\xi)]$ is an exponentiated-G (Exp-G) with power parameter $j$. The EOW-TL-G distribution is a linear combination of Exp-G densities and

the mathematical properties of the EOW-TL-G family of distributions follows from those of the Exp-G family of distributions.

## 3. MAXIMUM LIKELIHOOD ESTIMATION

If $X_i \sim EOW-TL-G(b,\alpha,\beta;\xi)$ with the parameter vector $\Delta = (b,\alpha,\beta;\ \xi)^T$. The total log-likelihood $\ell = \ell(\Delta)$ from a random sample of size n is given by

$$\begin{aligned}\ell &= n\log(2b) + n\log\beta + n\log\alpha + \sum_{i=1}\log[g(x_i;\xi)] \\ &+ \sum_{i=1}^{n}\log[\overline{G}(x_i;\xi)] + (b\beta-1)\sum_{i=1}^{n}\log[1-\overline{G}^2(x_i;\xi)] \\ &- (\beta+1)\sum_{i=1}^{n}\log[1-(1-\overline{G}^2(x_i;\xi))^b] - \sum_{i=1}^{n}t \\ &+ (\alpha-1)\sum_{i=1}^{n}\log[1-\exp\{-t\}], \end{aligned} \tag{3.1}$$

where $t = \left[\frac{[1-\bar{G}^2(x;\xi)]^b}{[1-(1-\bar{G}^2(x;\xi))^b]}\right]^\beta$. The elements of the score vector $U = \left(\frac{\partial\ell}{\partial\alpha}, \frac{\partial\ell}{\partial b}, \frac{\partial\ell}{\partial\beta}, \frac{\partial\ell}{\partial\xi_k}\right)$ are obtained by taking the partial derivatives from equation (3.1).

## 4. SOME SPECIAL CASES

We present two special cases of the EOW-TL-G family of distributions in this section. We take the log-logistic and Kumaraswamy distributions as the baseline distributions.

### 4.1 Exponentiated Odd Weibull-Topp-Leone-Log-Logistic (EOW-TL-LLoG) Distribution

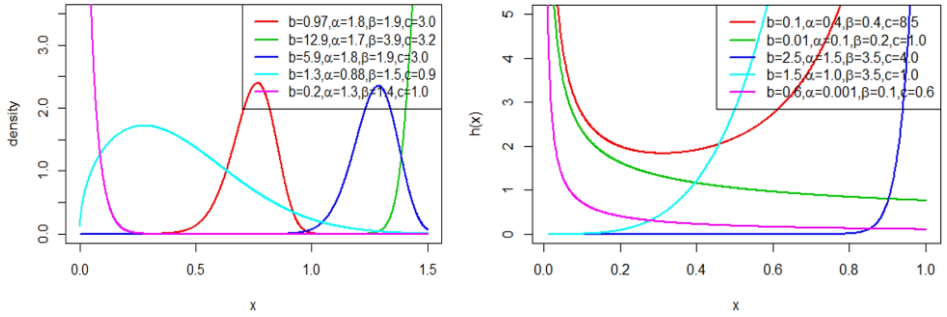
Consider the log-logistic distribution with pdf and cdf given by $g(x) = cx^{c-1}(1+x^c)^{-2}$ and $G(x) = 1-(1+x^c)^{-1}$, for $c>0$, respectively, as the baseline distribution. The EOW-TL-LLoG distribution has cdf and pdf given by

$$F_{EOW-TL-LLoG}(x;b,\alpha,\beta,c) = \left[1-\exp\left\{-\left[\frac{(1-(1+x^c)^{-2})^b}{(1-(1-(1+x^c)^{-2})^b)}\right]^\beta\right\}\right]^\alpha$$

and

$$\begin{aligned} f_{EOW-TL-LLoG}(x;b,\alpha,\beta,c) &= \frac{2b\alpha\beta c x^{c-1}(1+x^c)^{-3}[1-(1+x^c)^{-2}]^{b\beta-1}}{(1-[1-(1+x^c)^{-2}]^b)^{\beta+1}} \\ &\times \exp\left\{-\left[\frac{(1-(1+x^c)^{-2})^b}{(1-(1-(1+x^c)^{-2})^b)}\right]^\beta\right\} \\ &\times \left[1-\exp\left\{-\left[\frac{(1-(1+x^c)^{-2})^b}{(1-(1-(1+x^c)^{-2})^b)}\right]^\beta\right\}\right]^{\alpha-1}, \end{aligned}$$

respectively, for $b,\alpha,\beta,c>0$.




(a) (b)

**Figure 4.1: Plots of the pdf and hrf for the EOW-TL-LLoG Distribution**

Figure 4.1(a) shows the pdfs for the EOW-TL-LLoG distribution for selected parameter values. The pdf takes various shapes including J-shaped, reverse J-shaped, almost symmetric, right and left skewed. The hazard rate function (hrf) for the EOW-TL-LLoG distribution exhibit increasing, decreasing, reverse J, bathtub and J shapes as shown in Figure 4.1(b).

## 4.2 Exponentiated Odd Weibull-Topp-Leone-Kumaraswamy (EOW-TL-K) Distribution

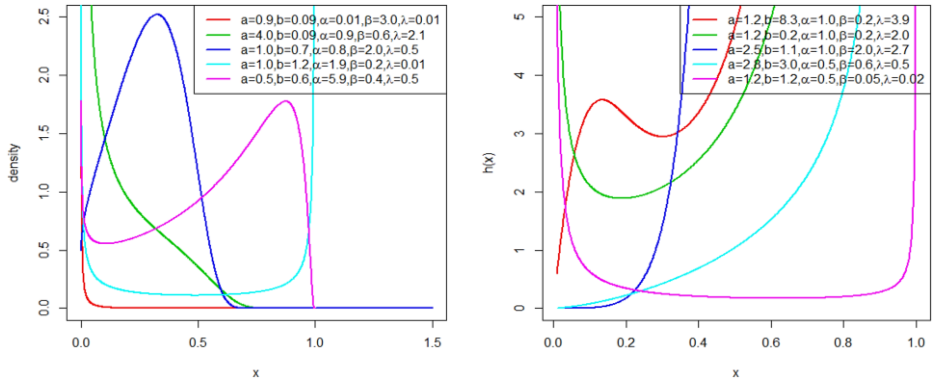
If the Kumaraswamy distribution is the baseline distribution with pdf and cdf given by $g(x;a,\lambda) = a\lambda x^{a-1}(1-x^a)^{\lambda-1}$ and $G(x;a,\lambda) = 1-(1-x^a)^{\lambda}$, for $x > 0$, $a$, and $c > 0$, respectively, then the EOW-TL-K distribution has cdf and pdf given by

$$F_{EOW-TL-K}(x;b,\alpha,\beta,a,\lambda) = \left[1-\exp\left\{-\left[\frac{(1-(1-x^a)^{2\lambda})^b}{(1-(1-(1-x^a)^{2\lambda})^b)}\right]^{\beta}\right\}\right]^{\alpha}$$

and

$$\begin{aligned} f_{EOW-TL-K}(x;b,\alpha,\beta,a,\lambda) &= \frac{2b\alpha\beta a\lambda x^{a-1}(1-x^a)^{2\lambda-1}(1-(1-x^a)^{2\lambda})^{b\beta-1}}{(1-(1-(1-x^a)^{2\lambda})^b)^{\beta+1}} \\ &\times \exp\left\{-\left[\frac{(1-(1-x^a)^{2\lambda})^b}{(1-(1-(1-x^a)^{2\lambda})^b)}\right]^{\beta}\right\} \\ &\times \left[1-\exp\left\{-\left[\frac{(1-(1-x^a)^{2\lambda})^b}{(1-(1-(1-x^a)^{2\lambda})^b)}\right]^{\beta}\right\}\right]^{\alpha-1}, \end{aligned}$$

respectively, for $b,\alpha,\beta,a > 0,\lambda$ and $x > 0$.




(a) (b)

**Figure 4.2: Plots of the pdf and hrf for the EOW-TL-K Distribution**

Figures 4.2(a) and 4.2(b) shows the plots of the pdfs and hrfs of the EOW-TLK distribution. The pdf takes various shapes including U-shape, right skewed, left skewed and reverse-J. Graphs of the hrf exhibits increasing, upside bathtub followed by bathtub, U-shape and J-shape.

## 5. SIMULATION STUDY

We conduct a Monte Carlo simulation study to assess the finite sample behavior of the maximum likelihood estimates (MLEs) of the parameters: $b, \alpha, \beta$, and $c$. The exactness of the MLEs is examined by means of the Mean Estimate, RMSE and Average Bias. We generate $N = 1000$ samples of size $n = 25$, 50, 100, 200, 400, 800 and 1000 from simulations carried out using R software. The exact parameter values used in the data generating process are I. $b = 0.7$, $\alpha = 1.0$, $\beta = 1.0$, $c = 1.5$, II. $b = 0.5$, $\alpha = 1$, $\beta = 1.0$, $c = 1.5$ III. $b = 2$, $\alpha = 0.5$, $\beta = 0.5$, $c = 1.5$ and IV. $b = 1.5$, $\alpha = 1.5$, $\beta = 0.5$, $c = 1$. Simulation study results are shown in Table 5.1. The results shows that as the sample size increases, the mean estimates of the parameters tend to be closer to the true parameter values, since RMSEs and average bias decays toward zero for all the parameters. The bias and RMSE for the estimated parameter, say, $\hat{\Delta}$, are given by:

$$Bias(\hat{\Delta}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\Delta}_i - \Delta), \quad \text{and} \quad RMSE(\hat{\Delta}) = \sqrt{\frac{\sum_{i=1}^{N}(\hat{\Delta}_i - \Delta)^2}{N}},$$

respectively.

## 6. APPLICATIONS

Two real data examples are used to illustrate the usefulness of the EOWTL-LLoG distribution compared to other models in the literature. We used the following goodness-of-fit statistics: -2loglikelihood (-2 log L), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICC), Bayesian Information Criterion (BIC). Other goodness-of-fit statistics including Cramer von Mises ($W^*$) and Andersen-Darling ($A^*$) (described by Chen and Balakrishnan [15]) to assess model adequacy. The smaller the values of these statistics, the better the model. Kolmogorov-Smirnov (K-S) statistic, its

P-value and sum of squares (SS) from the probability plots were also used to assess goodness-of-fit. The model with the smaller SS, smaller KS value and the highest p-value for the KS statistic, is regarded as the best fitting model.

The nlm function in R was used to estimate the model parameters. We present the model parameters estimates (standard errors in parenthesis) and the goodness-of-fit-statistics in Tables 6.1 and 6.2. Plots of the fitted densities, the histogram of the data and probability plots (Chambers et al. [14]) are shown in Figures 6.1(a), 6.1(b), 6.2(a) and 6.2(b).

We compare the EOW-TL-LLoG distribution to several competing models with equal number of parameters. These are the exponential Lindley odd log-logistic Weibull (ELOLLW) by Korkmaz et al. [20], Topp-Leone-WeibullLomax (TL-WLx) by Jamal et al. [19], odd log-logistic exponentiated Weibull (OLLEW) by Afify et al. [1], transmuted Topp-Leone-Weibull (TTL-W) by Yousof et al. [24], beta odd Lindley-Uniform (BOL-U) and beta odd LindleyExponential (BOL-E) by Chipepa et al. [12] distributions. The pdfs of the non-nested models are as follows:

**Table 5.1**
**Monte Carlo Simulation Results for EOW-TL-LLoG Distribution: Mean, RMSE and Average Bias**

| Parameter | $n$ | I | | | II | | |
|---|---|---|---|---|---|---|---|
| | | $b = 0.7, \alpha = 1.0, \beta = 1.0, c = 1.5$ | | | $b = 0.5, \alpha = 1, \beta = 1.0, c = 1.5$ | | |
| | | **Mean** | **RMSE** | **Bias** | **Mean** | **RMSE** | **Bias** |
| $b$ | 25 | 1.026171 | 0.743755 | 0.326171 | 0.961294 | 0.805669 | 0.461294 |
| | 50 | 1.042169 | 0.755269 | 0.342169 | 0.963837 | 0.827576 | 0.463837 |
| | 100 | 0.968534 | 0.731997 | 0.268534 | 0.902000 | 0.794575 | 0.402000 |
| | 200 | 0.987959 | 0.718185 | 0.287959 | 0.914573 | 0.785340 | 0.414573 |
| | 400 | 0.947881 | 0.675301 | 0.247881 | 0.933307 | 0.788136 | 0.433307 |
| | 800 | 0.903689 | 0.619922 | 0.203689 | 0.873062 | 0.700903 | 0.373062 |
| | 1000 | 0.895861 | 0.596405 | 0.195861 | 0.857656 | 0.686260 | 0.357656 |
| $\alpha$ | 25 | 3.711141 | 7.416697 | 2.711141 | 2.159265 | 3.482272 | 1.159265 |
| | 50 | 2.813582 | 4.547072 | 1.813582 | 2.190897 | 3.432464 | 1.190897 |
| | 100 | 2.984122 | 4.742774 | 1.984122 | 2.274526 | 3.389202 | 1.274526 |
| | 200 | 2.305695 | 3.647250 | 1.305695 | 1.811647 | 2.592503 | 0.811647 |
| | 400 | 1.991520 | 2.861926 | 0.991520 | 1.564228 | 2.031483 | 0.564228 |
| | 800 | 1.673338 | 1.989741 | 0.673338 | 1.316782 | 1.495097 | 0.316782 |
| | 1000 | 1.517417 | 1.608903 | 0.517417 | 1.223410 | 1.213598 | 0.223410 |
| $\beta$ | 25 | 2.404789 | 3.997805 | 1.404789 | 2.566932 | 2.541629 | 1.566932 |
| | 50 | 1.961080 | 1.625191 | 0.961080 | 2.323544 | 2.133604 | 1.323544 |
| | 100 | 1.761839 | 1.407402 | 0.761839 | 2.141224 | 1.962895 | 1.141224 |
| | 200 | 1.731529 | 1.341714 | 0.731529 | 2.092653 | 1.872663 | 1.092653 |
| | 400 | 1.615978 | 1.200257 | 0.615978 | 2.092919 | 1.844199 | 1.092919 |
| | 800 | 1.511846 | 1.101519 | 0.511846 | 1.865381 | 1.583201 | 0.865381 |
| | 1000 | 1.488135 | 1.075183 | 0.488135 | 1.834162 | 1.548678 | 0.834162 |
| $c$ | 25 | 1.326043 | 1.193069 | 0.173957 | 1.313948 | 1.181470 | 0.186052 |
| | 50 | 1.250399 | 0.946489 | 0.249601 | 1.217784 | 1.005159 | 0.282216 |
| | 100 | 1.278580 | 0.776774 | 0.221420 | 1.223640 | 0.841001 | 0.276360 |
| | 200 | 1.275327 | 0.719317 | 0.224673 | 1.203471 | 0.774068 | 0.296529 |
| | 400 | 1.269246 | 0.632559 | 0.230754 | 1.149221 | 0.735506 | 0.350779 |
| | 800 | 1.302536 | 0.573604 | 0.197464 | 1.188717 | 0.653995 | 0.311283 |
| | 1000 | 1.301701 | 0.547147 | 0.198299 | 1.195440 | 0.636021 | 0.304560 |

**Table 5.1 (contd….)**

| Parameter | $n$ | III | | | IV | | |
|---|---|---|---|---|---|---|---|
| | | $b$ = 2, $\alpha$ = 0.5, $\beta$ = 0.5, $c$ = 1.5 | | | $b$ = 1.5, $\alpha$ = 1.5, $\beta$ = 0.5, $c$ = 1 | | |
| | | **Mean** | **RMSE** | **Bias** | **Mean** | **RMSE** | **Bias** |
| $b$ | 25 | 2.394293 | 1.953255 | 0.394293 | 1.671549 | 1.181900 | 0.171549 |
| | 50 | 2.293820 | 1.508499 | 0.293820 | 1.625654 | 1.023952 | 0.125654 |
| | 100 | 2.359151 | 1.437413 | 0.359151 | 1.626163 | 1.046194 | 0.126163 |
| | 200 | 2.183945 | 1.080241 | 0.183945 | 1.635575 | 1.016390 | 0.135575 |
| | 400 | 2.180735 | 0.895075 | 0.180735 | 1.656262 | 0.880372 | 0.156262 |
| | 800 | 2.120294 | 0.756320 | 0.120294 | 1.578163 | 0.620511 | 0.078163 |
| | 1000 | 2.161777 | 0.707396 | 0.161777 | 1.557434 | 0.592760 | 0.057433 |
| $\alpha$ | 25 | 0.799755 | 0.950978 | 0.299755 | 2.813335 | 4.794793 | 1.313335 |
| | 50 | 0.772164 | 1.172585 | 0.272164 | 2.545840 | 3.704997 | 1.045840 |
| | 100 | 0.660956 | 0.638790 | 0.160956 | 2.273527 | 3.005821 | 0.773527 |
| | 200 | 0.630921 | 0.473820 | 0.130921 | 1.757650 | 1.279780 | 0.257650 |
| | 400 | 0.581591 | 0.347505 | 0.081591 | 1.690216 | 1.012882 | 0.190216 |
| | 800 | 0.563556 | 0.283081 | 0.063556 | 1.625341 | 0.785369 | 0.125341 |
| | 1000 | 0.534412 | 0.244724 | 0.034412 | 1.617311 | 0.627314 | 0.117311 |
| $\beta$ | 25 | 0.552763 | 0.556871 | 0.052763 | 0.883997 | 1.984103 | 0.383997 |
| | 50 | 0.573619 | 0.488145 | 0.073619 | 0.823807 | 1.419116 | 0.323807 |
| | 100 | 0.666005 | 0.803268 | 0.166005 | 0.682168 | 1.019452 | 0.182168 |
| | 200 | 0.628739 | 0.543570 | 0.128739 | 0.560805 | 0.726909 | 0.060805 |
| | 400 | 0.602503 | 0.446510 | 0.102503 | 0.536720 | 0.509966 | 0.036720 |
| | 800 | 0.588893 | 0.301658 | 0.088893 | 0.491598 | 0.214735 | 0.008403 |
| | 1000 | 0.565867 | 0.268874 | 0.065867 | 0.508146 | 0.295476 | 0.008146 |
| $c$ | 25 | 2.315733 | 1.609009 | 0.815733 | 2.004662 | 1.956339 | 1.004662 |
| | 50 | 2.039529 | 1.311810 | 0.539529 | 1.610771 | 1.458041 | 0.610771 |
| | 100 | 1.785716 | 1.023443 | 0.285716 | 1.499460 | 1.274868 | 0.499460 |
| | 200 | 1.667077 | 0.904958 | 0.167076 | 1.444355 | 1.014534 | 0.444355 |
| | 400 | 1.582006 | 0.718628 | 0.082006 | 1.283543 | 0.816208 | 0.283542 |
| | 800 | 1.500133 | 0.600907 | 0.000133 | 1.187551 | 0.612842 | 0.187551 |
| | 1000 | 1.552370 | 0.610614 | 0.052370 | 1.098263 | 0.415586 | 0.098263 |

$$
\begin{aligned}
& f_{ELOLLW}(x;\alpha,\beta,\gamma,\theta,\lambda) \\
&= \frac{\alpha\theta^2\gamma\lambda^{\gamma}x^{\gamma-1}e^{-(\lambda x)^{\gamma}}(e^{-(\lambda x)})^{\alpha\theta-1}(1-e^{-(\lambda x)^{\gamma}})^{\alpha-1}}{(\theta+\beta)((1-e^{-(\lambda x)^{\gamma}})^{\alpha}+e^{-\alpha(\lambda x)^{\gamma}})^{\theta-1}} \\
&\times \left(1-\beta\log\left[\frac{e^{-(\lambda x)^{\gamma}}}{(1-e^{-(\lambda x)^{\gamma}})^{\alpha}+e^{-\alpha(\lambda x)^{\gamma}}}\right]\right),
\end{aligned}
$$

for $\alpha,\beta,\gamma,\theta,\lambda>0$,

$$
\begin{aligned}
f_{TL-WLx}(x;a,b,\alpha,\theta) &= 2\theta\alpha ab(1+bx)^{a\alpha-1}(1-(1+bx)^{-a})^{\alpha-1} \\
&\times \exp\left(-2\left(\frac{1-(1+bx)^{-a}}{(1+bx)^{-a}}\right)\right) \\
&\times \left[1-\exp\left(-2\left(\frac{1-(1+bx)^{-a}}{(1+bx)^{-a}}\right)\right)\right]^{\theta-1},
\end{aligned}
$$

for $a,b,\alpha,\theta>0$,

$$f_{TTL-W}(x;a,b,\alpha,\lambda) = 2\alpha b a^b x^{b-1} e^{-2(ax)^b}(1-e^{-2(ax)^b})^{\alpha-1} \times (1+\lambda-2\lambda(1-e^{-2(ax)^b})^{\alpha}),$$

for $a,b,\alpha,\lambda>0$,

$$f_{BOL-U}(x;a,b,\lambda,\theta) = \frac{1}{B(a,b)}\left[1-\frac{\lambda+(1-x/\theta)}{(1+\lambda)(1-x/\theta)}\exp\left\{-\lambda\frac{x}{(\theta-x)}\right\}\right]^{a-1} \times \left[\frac{\lambda+(1-x/\theta)}{(1+\lambda)(1-x/\theta)}\exp\left\{-\lambda\frac{x}{(\theta-x)}\right\}\right]^{b-1} \times \frac{\lambda^2}{(1+\lambda)}\frac{\theta^2}{(\theta-x)^3}\exp\left\{-\lambda\frac{x}{(\theta-x)}\right\},$$

for $a,b,\lambda>0$ and $0<x<\theta$,

$$f_{BOL-E}(x;a,b,\lambda,\theta) = \frac{1}{B(a,b)}\left[1-\frac{\lambda+e^{-\theta x}}{(1+\lambda)e^{-\theta x}}\exp\left\{-\lambda\frac{(1-e^{-\theta x})}{e^{-\theta x}}\right\}\right]^{a-1} \times \left[\frac{\lambda+e^{-\theta x}}{(1+\lambda)e^{-\theta x}}\exp\left\{-\lambda\frac{(1-e^{-\theta x})}{e^{-\theta x}}\right\}\right]^{b-1} \times \frac{\lambda^2}{(1+\lambda)}\frac{(\theta e^{-\theta x})}{e^{-3\theta x}}\exp\left\{-\lambda\frac{1-e^{-\theta x}}{e^{-\theta x}}\right\},$$

for $a,b,\lambda,\theta>0$ and

$$f_{OLLEW}(x;\alpha,\beta,\gamma,\theta) = \frac{\theta\beta\gamma x^{\beta-1}e^{-(x/\alpha)^\beta}[1-e^{-(x/\alpha)^\beta}]^{\gamma\theta-1}(1-[1-e^{-(x/\alpha)^\beta}]^\gamma)^{\theta-1}}{\alpha\beta([1-e^{-(x/\alpha)^\beta}]^{\theta\gamma}+(1-[1-e^{-(x/\alpha)^\beta}]^\gamma)^\theta)^2},$$

for $\alpha,\beta,\lambda,\gamma,\theta>0$ and for the ELOLLW distribution, we considered the case when $\alpha=1$.

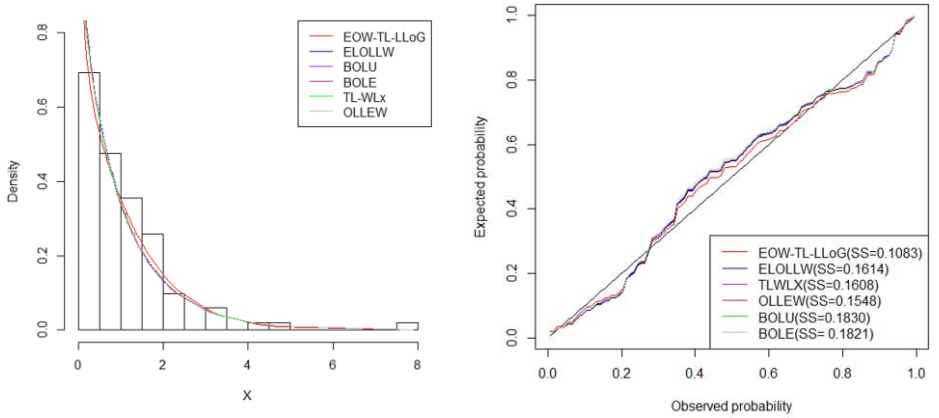
### 6.1 Kevlar 49/Epoxy Strands Failure at 90% Data

We fit the EOW-TL-LLoG distribution to the data set reported by Andrews and Herzberg [6] and by Barlow, Toland and Freeman [7]. The data set was also analyzed by Chipepa et al. [11]. The data represents failure times (in hours) of kevlar 49/epoxy strands subjected to constant sustained pressure at the 90% stress level. The observations are as follows:

0.047, 0.115, 0.121, 0.132, 0.164, 0.197, 0.203, 0.260, 0.282, 0.296, 0.334, 0.395, 0.458, 0.466, 0.501, 0.507, 0.529, 0.534, 0.540, 0.641, 0.644, 0.696, 0.841, 0.863, 1.099, 1.219, 1.271, 1.326, 1.447, 1.485, 1.553, 1.581, 1.589, 2.178, 2.343, 2.416, 2.444, 2.825, 2.830, 3.578, 3.658, 3.743, 3.978, 4.003, 4.033.

**Table 6.1**
**Parameter Estimates and Goodness-of-Fit Statistics for Various Models Fitted for kevlar Data Set**

| | Estimates | | | | Statistics | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Model | $\alpha$ | $b$ | $\beta$ | $c$ | $-2\log L$ | $AIC$ | $AICC$ | $BIC$ | $W^*$ | $A^*$ | $KS$ | $P-value$ |
| EOW-TL-LLoG | 2.30830 | 0.77107 | 0.35578 | 1.39721 | 204.0 | 212.0 | 212.4 | 222.5 | 0.1143 | 0.7241 | 0.0763 | 0.5992 |
| | (1.1203) | (0.3688) | (0.1214) | (0.6383) | | | | | | | | |
| | $\beta$ | $\lambda$ | $\theta$ | $\gamma$ | | | | | | | | |
| ELOLLW | 7.06921 | 0.15560 | 7.39364 | 0.83740 | 205.2 | 213.2 | 213.6 | 223.7 | 0.1666 | 0.9601 | 0.0816 | 0.5123 |
| | (4.1287) | (0.1115) | (3.9446) | (0.1094) | | | | | | | | |
| | $a$ | $b$ | $\alpha$ | $\theta$ | | | | | | | | |
| TLWLX | 0.6616 | 0.7096 | 1.3632 | 0.6079 | 205.4 | 213.4 | 213.8 | 223.9 | 0.1625 | 0.9447 | 0.0835 | 0.4824 |
| | (0.6596) | (0.9640) | (0.9470) | (0.4456) | | | | | | | | |
| | $\alpha$ | $\beta$ | $\gamma$ | $\theta$ | | | | | | | | |
| OLLEW | 1.6251 | 1.1121 | 0.6081 | 1.1687 | 205.5 | 213.5 | 214.0 | 224.0 | 0.1616 | 0.9432 | 0.0798 | 0.5408 |
| | (2.6163) | (0.4424) | (0.9012) | (0.9463) | | | | | | | | |
| | $a$ | $b$ | $\lambda$ | $\theta$ | | | | | | | | |
| BOL-U | $8.7181\times10^{1}$ | $1.0323\times10^{-1}$ | $2.3456\times10^{4}$ | $2.8294\times10^{5}$ | 205.7 | 213.7 | 214.1 | 224.1 | 0.1820 | 1.0333 | 0.0893 | 0.394 |
| | $(1.0675\times10^{-1})$ | (1.6271) | $(7.1192\times10^{-4})$ | $(5.9016\times10^{-5})$ | | | | | | | | |
| | $a$ | $b$ | $\lambda$ | $\gamma$ | | | | | | | | |
| BOL-E | $8.7078\times10^{-1}$ | $3.0492\times10^{1}$ | $2.0405\times10^{1}$ | $1.4327\times10^{-3}$ | 205.7 | 213.7 | 214.1 | 224.1 | 0.1814 | 1.0308 | 0.089 | 0.3973 |
| | $(1.0643\times10^{-1})$ | $(4.9112\times10^{-6})$ | $(7.8702\times10^{-6})$ | $(2.3001\times10^{-5})$ | | | | | | | | |
| | $a$ | $b$ | $\alpha$ | $\lambda$ | | | | | | | | |
| TTLW | 0.4271 | 0.8883 | 0.7930 | 0.00000 | 205.6 | 213.6 | 214.0 | 224.0 | 0.1770 | 1.0103 | 0.1189 | 0.115 |
| | (0.6710) | (0.761) | (0.6054) | (2.7437) | | | | | | | | |




(a) (b)

**Figure 6.1: Fitted Densities and Probability Plots kevlar Data**

The estimated variance-covariance matrix is

$$\begin{bmatrix} 1.25500 & -0.36743 & 0.08278 & -0.50153 \\ -0.36743 & 0.13598 & -0.01127 & 0.08490 \\ 0.08278 & -0.01127 & 0.01473 & -0.07577 \\ -0.50153 & 0.08490 & -0.07577 & 0.40739 \end{bmatrix}$$

and the 95% confidence intervals for the parameters are

$$\alpha \in [2.3083 \pm 2.1957], b \in [0.7711 \pm 0.7228], \beta \in [0.3558 \pm 0.2379]$$
and $c \in [1.3972 \pm 1.2510]$.

From results presented in Table 6.1, we observe that the EOW-TL-LLoG model has the lowest values for the goodness-of-fit statistics and the highest P-value for the K-S statistic compared to the other models considered in this paper. Therefore, we conclude that the EOW-TL-LLoG model fit the kevlar data set better than the other models ELLOW, OLLEW, BOL-E, BOL-U, TLWLX and TTLW distributions. Also, Figures 6.1(a) and 6.1(b) shows that our proposed model performs better than the competing other models that were considered on kevlar data set.

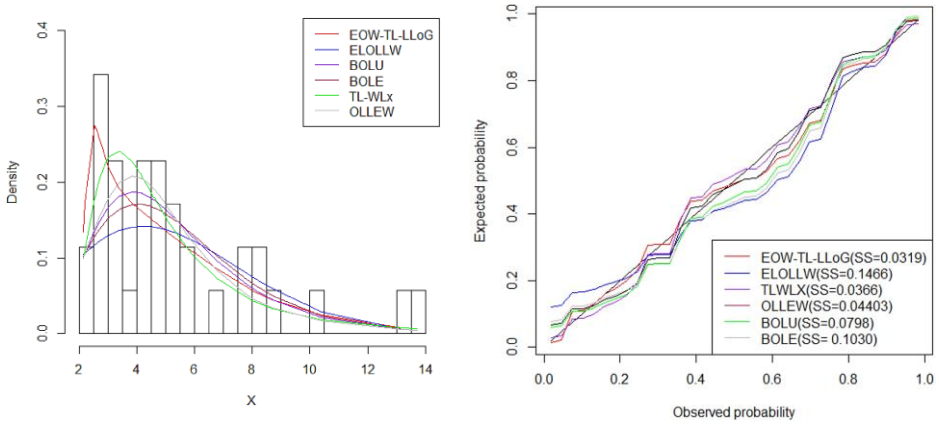
### 6.2 Growth Hormone Data

The second data set was used by Alizadeh et al. [3] to show the superiority of the exponentiated power Lindley power series (EPLPS) class of the distributions. The data are

2.15, 2.20, 2.55, 2.56, 2.63, 2.74, 2.81, 2.90, 3.05, 3.41, 3.43, 3.43, 3.84, 4.16, 4.18, 4.36, 4.42, 4.51, 4.60, 4.61, 4.75, 5.03, 5.10, 5.44,5.90, 5.96, 6.77, 7.82, 8.00, 8.16, 8.21, 8.72, 10.40, 13.20, 13.70.

**Table 6.2**
**Parameter Estimates and Goodness-of-Fit Statistics for various Models Fitted for Growth Hormone Data Set**

| Model | Estimates | | | | Statistics | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $\alpha$ | $b$ | $\beta$ | $c$ | $-2\log L$ | $AIC$ | $AICC$ | $BIC$ | $W^*$ | $A^*$ | $KS$ | $P-value$ |
| EOW-TL-LLoG | 2.9282 | $1.2368\times10^{3}$ | $1.700\times10^{-1}$ | 3.2107 | 152.5 | 160.4 | 161.9 | 166.8 | 0.0357 | 0.2259 | 0.0808 | 0.9762 |
| | (1.1289) | $(5.5491\times10^{-4})$ | $(2.7594\times10^{-2})$ | $(3.7679\times10^{-1})$ | | | | | | | | |
| | $\beta$ | $\lambda$ | $\theta$ | $\gamma$ | | | | | | | | |
| ELOLLW | 0.0000 | 0.1137 | 2.1231 | 1.9932 | 165.0 | 173.0 | 174.3 | 179.2 | 0.1639 | 1.0262 | 0.1454 | 0.4500 |
| | (0.5800) | (0.0184) | (0.0005) | (0.2437) | | | | | | | | |
| | $a$ | $b$ | $\alpha$ | $\theta$ | | | | | | | | |
| TLWLX | $1.8270\times10^{-1}$ | $7.9361\times10^{2}$ | 1.1917 | $4.2730\times10^{3}$ | 155.1 | 163.1 | 164.4 | 169.3 | 0.0349 | 0.2470 | 0.0847 | 0.9634 |
| | $(2.1795\times10^{-2})$ | $(6.3586\times10^{-5})$ | $(2.2639\times10^{-1})$ | $(1.3787\times10^{-5})$ | | | | | | | | |
| | $\alpha$ | $\beta$ | $\gamma$ | $\theta$ | | | | | | | | |
| OLLEW | 0.0086 | 0.0363 | 2.0665 | 46.8716 | 158.6 | 166.6 | 168.0 | 172.9 | 0.0568 | 0.4102 | 0.0977 | 0.892 |
| | (0.0250) | (0.0054) | (0.2743) | (0.0013) | | | | | | | | |
| | $a$ | $b$ | $\lambda$ | $\theta$ | | | | | | | | |
| BOL-U | 6.3336 | $9.2737\times10^{-1}$ | $1.9053\times10^{6}$ | $3.8649\times10^{6}$ | 158.1 | 166.1 | 167.4 | 172.3 | 0.0696 | 0.4701 | 0.1073 | 0.8154 |
| | (1.6517) | $(1.9416\times10^{-1})$ | $(3.7663\times10^{-6})$ | $(1.8567\times10^{-6})$ | | | | | | | | |
| | $a$ | $b$ | $\lambda$ | $\gamma$ | | | | | | | | |
| BOL-E | 4.1652 | $3.8515\times10^{1}$ | $5.4038\times10^{1}$ | $3.6861\times10^{-4}$ | 160.2 | 168.2 | 169.6 | 174.4 | 0.1012 | 0.6566 | 0.1242 | 0.6531 |
| | $(2.5606\times10^{-9})$ | $(2.8036\times10^{-10})$ | $(2.1144\times10^{-10})$ | $(3.0504\times10^{-5})$ | | | | | | | | |
| | $a$ | $b$ | $\alpha$ | $\lambda$ | | | | | | | | |
| TTLW | $3.6610\times10^{2}$ | $2.2680\times10^{-1}$ | $3.0414\times10^{4}$ | 0.0000 | 155.1 | 163.1 | 164.4 | 169,3 | 0.0665 | 0.4503 | 0.9999 | $2.2\times10^{-16}$ |
| | $(6.1323\times10^{-4})$ | $(5.1260\times10^{-3})$ | $(8.4249\times10^{-6})$ | $(6.1456\times10^{-1})$ | | | | | | | | |




(a) (b)

**Figure 6.2: Fitted Densities and Probability Plots for Growth Hormone Data**

The estimated variance-covariance matrix is

$$\begin{bmatrix} 1.27446 & -0.00062 & -0.02222 & 0.38226 \\ -0.00062 & 0.00000 & 0.00001 & -0.00020 \\ -0.02222 & 0.00001 & 0.00076 & -0.00834 \\ 0.38226 & -0.00020 & -0.00834 & 0.14197 \end{bmatrix}$$

and the 95% confidence intervals for the parameters are

$$\alpha \in [2.9282 \pm 2.2127], b \in [(1.2368 \times 103) \pm 0.0011],$$
$$\beta \in [(1.700 \times 10 - 1) \pm 0.0541] \text{ and } c \in [3.2107 \pm 0.7385].$$

The second example further affirms the superiority of the EOW-TL-LLoG distribution compared to the selected models. The results are shown in Table 6.2. Figures 6.2(a) and 6.2(b) shows the flexibility enjoyed on fitting the growth hormone data set using the EOW-TL-LLoG distribution compared to the selected models.

## CONCLUDING REMARKS

We presented a new family of distributions, referred to as the Exponentiated Odd Weibull-Topp Leone-G distribution. The new family of distributions can handle heavy tailed data and also have non-monotonic hazard rate shapes. The new distribution has a desirable tractability property and can be expressed as a linear combination of the Exp-G distribution. The applications provided show that EOW-TL-LLoG distribution performs better than other several models in the literature.

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