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A NEW BATHTUB SHAPED FAILURE RATE MODEL: PROPERTIES, AND APPLICATIONS TO ENGINEERING SECTOR

Muhammad Zeshan Arshad[1§], Muhammad Zafar Iqbal[1], Asad Anees[2]
Zahoor Ahmad[3] and **Oluwafemi Samson Balogun[4]**

[1] Department of Mathematics and Statistics, University of Agriculture, Faisalabad, Pakistan. Email: mzts2004@hotmail.com

[2] Institute of Mathematics, Clausthal University of Technology, Erzstraße 1, D-38678 Clausthal-Zellerfeld, Germany. Email: asad.anees@tu-clausthal.de

[3] Department of Mathematics and Statistics, University of Lahore, Sargodha Campus, Pakistan. Email: zahoor.ahmad@math.uol.edu.pk

[4] School of Computing, University of Eastern Finland, Kuopio Campus, FI-70211. Email: samson.balogun@uef.fi

[§] Corresponding author Email: profarshad@yahoo.com

ABSTRACT

In this paper, we proposed a new model, capable of modeling against the bathtub shaped failure rate phenomena, referred to as the Lehmann-II Muhammad (L-II-M) distribution. We derived some mathematical and reliability characteristics developed some explicit expressions for the moments, quantile function, and order statistics. We suggested a method of maximum likelihood estimator for the estimation of model parameters, and a simulation study was recommended to observe the asymptotic behavior of MLEs. The strength of L-II-M distribution was affirmed by modeling in two suitable lifetime engineering datasets.

KEYWORDS

Lehmann-II Distribution; Mustapha Type-I Distribution; Failure Rate Function; Moments; Oder Statistics; Simulation; Maximum Likelihood Estimation.

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1. INTRODUCTION

Over the past three decades, modeling against the bathtub shaped failure rate phenomena is considered a riddle for scientists and practitioners. To overcome these challenges scientists developed and discussed a range of lifetime models. For a comprehensive study, we refer the readers to see the credible work of some notable researchers including Wang (2000); Lai et al. (2003); Dimitrakopoulou *et al*. (2007); Pappas et al. (2012); Sarhan and Apaloo (2013); Sarhan et al. (2013); Ahsanullah et al. (2013); Saran and Pandey (2004); Doostmoradi et al. (2014); Ghnimi and Gasmi (2014); Tahir et al. (2014); Tahir et al. (2015); Fatima and Roohi (2015); Oluyede et al. (2015); Muhammad (2016); Pu et al. (2016); Okorie et al. (2017); Abdul-Moniem (2017); Ahmad and Hussain (2017); El-Morshedy et al. (2017); Mdlongwa et al. (2017); Al-Salafi and

Adham (2018); Al-Abbasi et al. (2019); Korkmaz et al. (2019); Okorie and Akpanta (2019); and Arshad et al. (2020).

The rest of the paper is organized as: the new model is proposed in Sec 2, mathematical properties are discussed in Sec 3, maximum likelihood estimation is explored in Sec 4, Sec 5 detailed the simulation study, real data application is discussed in Sec 6 and finally conclusions are given in Sec 7.

2. NEW MODEL

In this work, we prefer the model termed as Mustapha Type – I (2016) (MT-I). He proposed, asymmetric, upper bounded $0 < x \le \mathrm{g}, \alpha > 0$, a limited scoped model that was inappropriate fit for the bathtub shaped real-time data. The associated CDF and the corresponding PDF of MT-I are given, respectively, by

$$P(x;\alpha,\mathrm{g}) = \frac{\left(\frac{x}{\mathrm{g}}\right)(1+\alpha)}{\left(\frac{x}{\mathrm{g}}+\alpha\right)}, \tag{1}$$

and

$$p(x;\alpha,\mathrm{g}) = \frac{(1+\alpha)\alpha\mathrm{g}}{(\alpha+x\mathrm{g})^2}. \tag{2}$$

We introduce a two-parameter, competent lifetime model that exhibits the capacity of modeling in complex and sophisticated lifetime phenomena like bathtub shaped failure rate data, quite proficiently. The new model is obtained by following the P class known by Lehmann-II (1953). Mathematically the CDF of Lehmann-II is presented by

$$F(x) = 1 - \left(1 - P(x)\right)^{\beta}, \tag{3}$$

where $P(x)$ is a baseline model and $\beta > 0$ is a shape parameter.

In our new model, we induce a additional shape parameter $(\beta > 0)$ to the baseline model (present in equation (1)) and it starts providing a better fit to the bathtub shaped failure rate data. The additional shape parameter advances the tail weight and controls the skewness and kurtosis of the density function. The new model is obtained by placing the equation (1) in equation (3) and henceforth, it is referred to as the Lehmann-II Muhammad (L-II-M) distribution.

Formally, a random variable X is said to follow the Lehmann-II Muhammad (L-II-M) distribution for bounded interval (0, g), if the associated CDF and the corresponding PDF are given, respectively, by

$$F(x;\alpha,\beta,\mathrm{g}) = 1 - \left(1 - \frac{x}{\mathrm{g}}\right)^{\beta}\left(1 + \frac{x}{\alpha\mathrm{g}}\right)^{-\beta}, \tag{4}$$

and

$$f(x;\alpha,\beta,\mathrm{g}) = \frac{\beta(\alpha+1)}{\alpha\mathrm{g}}\left(1 - \frac{x}{\mathrm{g}}\right)^{\beta-1}\left(1 + \frac{x}{\alpha\mathrm{g}}\right)^{-\beta-1}, \tag{5}$$

where $\alpha > 0$ and $\beta > 0$ are the scale and additional shape parameter respectively, and g be the possible maximum value of $x(0 < x \le \mathrm{g})$. Further, at $\beta = 1$, the L-II-M distribution reduces to the baseline model.

3. MATHEMATICAL PROPERTIES

3.1 Linear Representation

Linear representation of CDF and PDF often makes the calculations much easier than the conventional integral computation corresponding to determining the mathematical properties of the subject function.

For power series expansion, if 'β' is real non-integer and -1 <z<1, then it can be written as:

$$(1-z)^{\beta-1}=\sum_{j=0}^{\infty} w_j z^j,$$

where $w_j=(-1)^j\binom{\beta-1}{j}=\frac{(-1)^j\Gamma(\beta)}{j!\Gamma(\beta-j)}$.

From equations (4) and (5), linear representation of CDF and PDF are given, respectively, by

$$F(x)=1-\sum_{i,j=0}^{\infty}\frac{(-1)^i}{\alpha^j \mathrm{g}^{i+j}}\binom{\beta}{i}\binom{-\beta}{j}x^{i+j}. \tag{6}$$

$$f(x)=\beta(\alpha+1)\sum_{i,j=0}^{\infty}\frac{(-1)^i}{\alpha^{j+1}\mathrm{g}^{i+j+1}}\binom{\beta-1}{i}\binom{-\beta-1}{j}x^{i+j}. \tag{7}$$

The expressions develop in equations (6) and (7) will be quite helpful in the forthcoming computations of various mathematical properties of the L-II-M distribution.

3.2 Reliability Characteristics

One of the key roles of probability distribution in reliability engineering is to analyze and predict the lifetime of a component. Noteworthy contribution of reliability measures including reliability function, hazard rate function, cumulative hazard rate function, reverse hazard rate function, Mills ratio, and Odd function are highly obliged.

One may explain the reliability function as the probability of a component that survives till the time *x*. Analytically it is written as: $R(x)=1-F(x)$. Reliability function of X is given by

$$R(x)=\left(1-\frac{x}{\mathrm{g}}\right)^{\beta}\left(1+\frac{x}{\alpha\mathrm{g}}\right)^{-\beta}. \tag{8}$$

In reliability theory, one of the significantly contributed function, considers as a failure rate function, or hazard rate function and sometimes it is called the force of mortality, is used to measure the failure rate of a component in a particular period of time *x*. Hazard rate function is mathematically expressed as $h(x)=f(x)/R(x)$. Hazard rate function of X is given by

$$h(x)=\frac{\beta(\alpha+1)}{\alpha\mathrm{g}\left(1-\frac{x}{\mathrm{g}}\right)\left(1+\frac{x}{\alpha\mathrm{g}}\right)}. \tag{9}$$

The conditional survivor function is the probability that a component whose life say x, survives in additional interval at z. It can be expressed as:

$$R(z/x) = P(X > z + x/X > t) = \frac{R(X > z + x)}{P(X > x)} = \frac{R(x+z)}{R(x)}.$$

Conditional survivor function of X is given by

$$R(^{z}/_{x}) = \frac{\left(1 - \frac{x+z}{\mathrm{g}}\right)^{\beta}\left(1 + \frac{x}{\alpha \mathrm{g}}\right)^{\beta}}{\left(1 - \frac{x}{\mathrm{g}}\right)^{\beta}\left(1 + \frac{x+z}{\alpha \mathrm{g}}\right)^{\beta}}, t > 0. \tag{10}$$

Further, the reverse residual life function is given by $\bar{R}(z/x) = \frac{R(x-z)}{R(x)}$. Reverse residual life function of X is given by

$$\bar{R}(^{z}/_{x}) = \frac{\left(1 - \frac{x-z}{\mathrm{g}}\right)^{\beta}\left(1 + \frac{x}{\alpha \mathrm{g}}\right)^{\beta}}{\left(1 - \frac{x}{\mathrm{g}}\right)^{\beta}\left(1 + \frac{x-z}{\alpha \mathrm{g}}\right)^{\beta}}, t > 0. \tag{11}$$

Most of the time, it is assumed that the mechanical components/parts of some systems follow the bathtub-shaped failure rate phenomena. For this, several well established and useful reliability measures are available in the literature to discuss the significance of L-II-M distribution. Cumulative hazard rate function is expressed by $h_c(x) = -\,log\big(R(x)\big)$. Cumulative hazard rate function of X is given by

$$h_c(x) = -\,log\left(\left(1 - \frac{x}{\mathrm{g}}\right)^{\beta}\left(1 + \frac{x}{\alpha \mathrm{g}}\right)^{-\beta}\right). \tag{12}$$

Reverse hazard rate function is expressed by $h_r(x) = f(x)/R(x)$. Reverse hazard rate function of X is given by

$$h_r(x) = \frac{\beta(\alpha+1)\left(1 - \frac{x}{\mathrm{g}}\right)^{\beta-1}\left(1 + \frac{x}{\alpha \mathrm{g}}\right)^{\beta}}{\alpha \mathrm{g}\left(1 - \frac{x}{\mathrm{g}}\right)^{\beta}\left(1 + \frac{x}{\alpha \mathrm{g}}\right)^{1+\beta}}. \tag{13}$$

Mills ratio is expressed by $M(x) = R(x)/f(x)$. Mills ratio of X is given by

$$M(x) = \frac{\alpha \mathrm{g}\left(1 - \frac{x}{\mathrm{g}}\right)^{\beta}\left(1 + \frac{x}{\alpha \mathrm{g}}\right)^{1+\beta}}{\beta(\alpha+1)\left(1 - \frac{x}{\mathrm{g}}\right)^{\beta-1}\left(1 + \frac{x}{\alpha \mathrm{g}}\right)^{\beta}}. \tag{14}$$

Odd function is expressed by $O(x) = F(x)/R(x)$. Odd function of X is given by

$$O(x) = \frac{\left(1+\frac{x}{\alpha g}\right)^{\beta}\left(\frac{x}{g}\right)(1+\alpha)}{\left(\frac{x}{g}+\alpha\right)\left(1-\frac{x}{g}\right)^{\beta}}. \tag{15}$$

We may obtain the reliability characteristics for linear expressions mention in section 3.2. The survival and hazard rate functions of X are given by

$$R^{*}(x) = \sum_{i,j=0}^{\infty} \frac{(-1)^{i}}{\alpha^{j} g^{i+j}} \binom{\beta}{i}\binom{-\beta}{j} x^{i+j}, \tag{16}$$

and

$$h^{*}(x) = \frac{\beta(\alpha+1)\sum_{i,j=0}^{\infty} \frac{(-1)^{i}}{\alpha^{j+1} g^{i+j+1}} \binom{\beta-1}{i}\binom{-\beta-1}{j} x^{i+j}}{\sum_{i,j=0}^{\infty} \frac{(-1)^{i}}{\alpha^{j} g^{i+j}} \binom{\beta}{i}\binom{-\beta}{j} x^{i+j}}. \tag{17}$$

3.3 Shapes

Possible shapes of probability density function (PDF), and failure rate function (FRF) of L-II-M distribution are presented in Figures 1 and 2. Figures 1 illustrate the flexible shapes of PDF including the increasing, decreasing, U and upside down bathtub shapes, and Figure 2 reveals the increasing and upside down bathtub shapes of FRF over the selected combinations of the parameters.

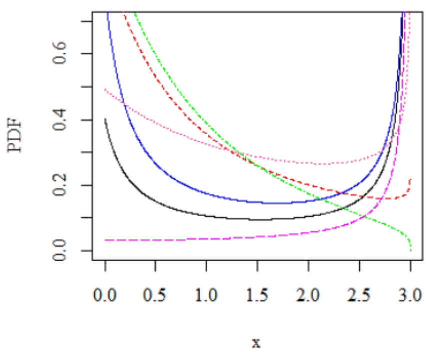

Figure 1: Density Function Plot for the fixed value of g = 3, and different values of α, β. For Black($\alpha = 0.09, \beta = 0.1$), Blue ($\alpha = 0.09, \beta = 0.5$), Red ($\alpha = 0.5$, $\beta = 0.9$), Hot-Pink ($\alpha = 0.9, \beta = 0.7$), Green ($\alpha = 0.8, \beta = 1.3$), and Magenta ($\alpha = 1.1, \beta = 0.05$)

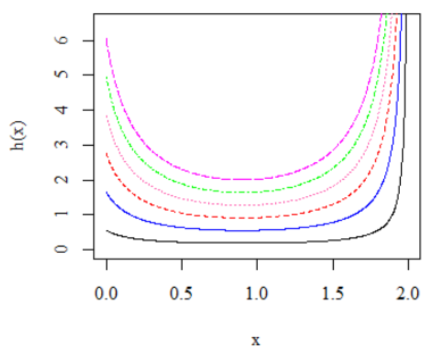

Figure 2: Failure Rate Function Plot for the fixed value of g = 2, $\alpha = 0.1$ and different values of β. For Black ($\beta = 0.1$), Blue($\beta = 0.3$), Red ($\beta = 0.5$), Hot-Pink ($\beta = 0.7$), Green ($\beta = 0.9$), and Magenta ($\beta = 1.1$)

3.4 Quantile, Skewness and Kurtosis

The p^{th} quantile function of $X\sim$ L-II-M $(x;\alpha,\beta,\mathrm{g})$ is obtained by inverting the CDF mentioned in the equation (4). Quantile function is defined by

$$p=F(x_p)=P(X\le x_p),0<p<1.$$

Quantile function of X is given by

$$x_p=\frac{\alpha \mathrm{g}\left(1-(1-p)^{1/\beta}\right)}{\left(\alpha+(1-p)^{1/\beta}\right)}. \tag{18}$$

To obtain the first quartile, median and third quartile of X, place q = 0.25, 0.50, and 0.75 respectively in equation (18). Henceforth, to generate random numbers, we assume that CDF in equation (4) follows to uniform distribution $u= U$ (0, 1).

Skewness (symmetry) and kurtosis (tailedness) of the L-II-M distribution can be calculated by the following useful measures

$$S_B=\frac{Q_{\left(\frac{3}{4}\right)}+Q_{\left(\frac{1}{4}\right)}-2Q_{\left(\frac{1}{2}\right)}}{Q_{\left(\frac{3}{4}\right)}-Q_{\left(\frac{1}{4}\right)}},\quad \text{and}\quad K_M=\frac{Q_{\left(\frac{3}{8}\right)}-Q_{\left(\frac{1}{8}\right)}-Q_{\left(\frac{5}{8}\right)}+Q_{\left(\frac{7}{8}\right)}}{Q_{\left(\frac{6}{8}\right)}-Q_{\left(\frac{2}{8}\right)}},$$

introduced by Bowley (1920) and Moors (1988) respectively. These descriptive measures, based on quartiles and octiles, provide more robust estimates than the traditional skewness and kurtosis measures. These measures are quite insensitive to outliers and work more effectively in moment deficient distributions.

The Bowley skewness and Moors kurtosis plots present in Figures 3 and 4 respectively, exhibit that the L-II-M distribution might be positively skewed, as the two shape parameters (α,β) are accountable for the variability in measures.

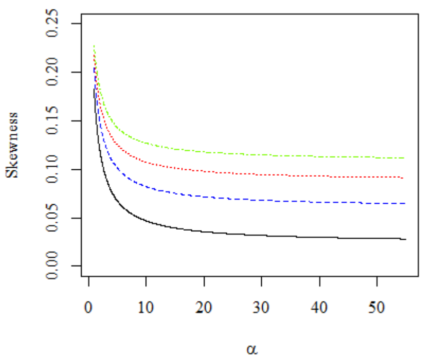

Figure 3: Skewness Plot for the different values of β. For Black ($\beta=0.77$), Blue($\beta=0.74$), Red ($\beta=0.75$), and Green ($\beta=0.76$)

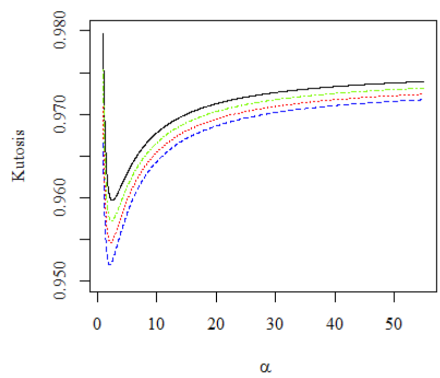

Figure 4: Kurtosis Plot for the different values of β. For Black($\beta=1.1$), Blue ($\beta=1.3$), Red ($\beta=1.5$), and Green ($\beta=1.7$)

3.5 Limiting Behavior

Here we study the limiting behavior of CDF, PDF, RF, and HRF of the L-II-M distribution present in equations (4), (5), (8), and (9) at $x \to 0$ and $x \to g$.

Proposition-1: *Limiting behavior of CDF, PDF, RF, and HRF of the L-II-M distribution at* $x \to 0$ *is followed by*

$$F(x) \sim 0,$$
$$f(x) \sim \frac{\beta(\alpha+1)}{\alpha \mathrm{g}},$$
$$R(x) \sim 1,$$
$$FHRF \sim \frac{\beta(\alpha+1)}{\alpha \mathrm{g}},$$

Proposition-2: *Limiting behavior of CDF, PDF, RF, and HRF of the L-II-M distribution at* $x \to \mathrm{g}$ *is followed by*

$$F(x) \sim 1,$$
$$f(x) \sim 0,$$
$$R(x) \sim 0,$$
$$HRF \sim 0,$$

The limiting behaviors of CDF, PDF, RF, and HRF are presented in the above expressions at $x \to 0$ and $x \to g$, illustrate the effect of parameters on the tail of the L-II-M distribution.

3.6 Moments and Its Associated Measures

Moments have a remarkable role in the discussion of distribution theory, to study the momentous characteristics of a probability distribution.

Theorem 1: If $X \sim$ L-II-M $(x; \alpha, \beta, \mathrm{g})$, with $\alpha, \beta > 0$ and $x \leq \mathrm{g}$, then the r-th ordinary moment (say $\mu_r^{/}$) of X is given by

$$\mu_r^{/} = \beta(\alpha+1) \sum_{i,j=0}^{\infty} \frac{(-1)^i}{\alpha^{j+1}} \binom{\beta-1}{i} \binom{-\beta-1}{j} \left(\frac{\mathrm{g}^r}{r+i+j+1}\right).$$

Proof: Equation (7) can be written as

$$\mu_r^{/} = \beta(\alpha+1) \sum_{i,j=0}^{\infty} \frac{(-1)^i}{\alpha^{j+1}} \binom{\beta-1}{i} \binom{-\beta-1}{j} \int_0^{\mathrm{g}} x^{r+i+j}\, dx,$$

straightforward computation of the above equation yields $\mu_r^{/}$, it is given by

$$\mu_r^{/} = \beta(\alpha+1) \sum_{i,j=0}^{\infty} \frac{(-1)^i}{\alpha^{j+1}} \binom{\beta-1}{i} \binom{-\beta-1}{j} \left(\frac{\mathrm{g}^r}{r+i+j+1}\right), \tag{19}$$

where, $\alpha > 0$ be a scale and $\beta > 0$ be a shape parameter of X with $x \leq \mathrm{g}$. One may derive the number of statistic by following the equation (18). For instance: mean of X is obtained by setting r =1 in equation (19). To deduce the 2nd, 3rd, and 4th ordinary moments of X, place r = 2, 3, and 4 in equation (19) respectively. Further, for fractional positive and fractional negative moments of X, substitute r by (m/n) and $(-m/n)$ in the equation (19) respectively. To discuss the variability in X, the Fisher index ($F.I = Var(x)/E(x)$) plays a supportive role, it is given by

$$F.I = \frac{\left(\begin{array}{c}\beta(\alpha+1)\sum_{i,j=0}^{\infty}\frac{(-1)^i}{\alpha^{j+1}}\binom{\beta-1}{i}\binom{-\beta-1}{j}\left(\frac{g^2}{3+i+j}\right)- \\ \left(\beta(\alpha+1)\sum_{i,j=0}^{\infty}\frac{(-1)^i}{\alpha^{j+1}}\binom{\beta-1}{i}\binom{-\beta-1}{j}\left(\frac{g}{2+i+j}\right)\right)^2\end{array}\right)}{\beta(\alpha+1)\sum_{i,j=0}^{\infty}\frac{(-1)^i}{\alpha^{j+1}}\binom{\beta-1}{i}\binom{-\beta-1}{j}\left(\frac{g}{2+i+j}\right)}. \quad (20)$$

For negative moments of X, substitute r by – w in equation (19), it is given by

$$\mu_{-w}^{/} = \beta(\alpha+1)\sum_{i,j=0}^{\infty}\frac{(-1)^i}{\alpha^{j+1}}\binom{\beta-1}{i}\binom{-\beta-1}{j}\left(\frac{1}{g^w(1+i+j-w)}\right). \quad (21)$$

Also, moment generating function $M_X(t)$ can be presented

$$M_X(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\mu_r^{/}.$$

Moment generating function of X is given by

$$M_X(t) = \beta(\alpha+1)\sum_{r=0}^{\infty}\frac{t^r}{r!}\sum_{i,j=0}^{\infty}\frac{(-1)^i}{\alpha^{j+1}}\binom{\beta-1}{i}\binom{-\beta-1}{j}\left(\frac{g^r}{r+i+j+1}\right). \quad (22)$$

Similarly, characteristic function of X is given by

$$\phi_X(t) = M_X(it) = \sum_{r=0}^{\infty}\frac{(it)^r}{r!}\sum_{i,j=0}^{\infty}\frac{(-1)^i}{\alpha^{j+1}}\binom{\beta-1}{i}\binom{-\beta-1}{j}\left(\frac{g^r}{r+i+j+1}\right). \quad (23)$$

A well-established relationship between the ordinary moments $(\mu_r^{/})$and central moments (μ_s) is given by $\mu_s = \sum_{k=0}^{s}\binom{s}{k}(-1)^k\left(\mu_1^{/}\right)^i\mu_{s-k}^{/}$. This relationship is quite helpful to obtain the cumulants (K_s). First four cumulants are: $K_1 = \mu_1^{/}, K_2 = \mu_2^{/} - \mu_1^{/2}$, $K_3 = \mu_3^{/} - 3\mu_2^{/}\mu_1^{/} + 2\mu_1^{/3}$, and $K_4 = \mu_4^{/} - 4\mu_3^{/}\mu_1^{/} - 3\mu_2^{/2} + 12\mu_2^{/}\mu_1^{/2} - 6\mu_1^{/4}$.

The expression of s-th central moment of X is given by

$$\mu_s = \left(\begin{array}{c}\sum_{k=0}^{s}\binom{s}{k}(-1)^k\left(\beta(\alpha+1)\sum_{i,j=0}^{\infty}\frac{(-1)^i}{\alpha^{j+1}}\binom{\beta-1}{i}\binom{-\beta-1}{j}\left(\frac{g}{2+i+j}\right)\right)^i \\ \beta(\alpha+1)\sum_{i,j=0}^{\infty}\frac{(-1)^i}{\alpha^{j+1}}\binom{\beta-1}{i}\binom{-\beta-1}{j}\left(\frac{g^{s-k}}{s-k+i+j+1}\right)\end{array}\right). \quad (24)$$

One may perhaps further determine the statistic like skewness ($\gamma_1 = \mu_3^2/\mu_2^3$), kurtosis ($\gamma_2 = \mu_4/\mu_2^2$), and mode = $\left(\sqrt{\gamma_1}(\gamma_2+3)\mathrm{SD}/\left(2(5\gamma_2-6\gamma_1-9)\right)\right)$ (introduced by Karl Pearson) of X.

3.6.1 Incomplete Moments

Incomplete moments are classified into lower incomplete moments and upper incomplete moments.

Lower incomplete moments are defined as $M_r(l) = E_{X\le l}(x^r) = \int_0^l x^r f(x)dx.$ For X it given by

$$M_r(l) = \beta(\alpha+1)\sum_{i,j=0}^{\infty}\frac{(-1)^i}{\alpha^{j+1}}\binom{\beta-1}{i}\binom{-\beta-1}{j}\int_o^l x^{r+i+j}\,dx,$$

$$M_r(l) = \beta(\alpha+1)\sum_{i,j=0}^{\infty}\frac{(-1)^i}{\alpha^{j+1}}\binom{\beta-1}{i}\binom{-\beta-1}{j}\left(\frac{l^{r+i+j+1}}{r+i+j+1}\right). \tag{25}$$

Upper incomplete moments are defined as $M_r(u) = E_{X>u}(x^r) = \int_u^g x^r f(x)dx$ or more convenient, it can be written as $M_r(u) = \int_0^g x^r f(x)dx - \int_0^l x^r f(x)dx$. For X it given by

$$M_r(u) = \begin{pmatrix}\beta(\alpha+1)\sum_{i,j=0}^{\infty}\frac{(-1)^i}{\alpha^{j+1}}\binom{\beta-1}{i}\binom{-\beta-1}{j}\\ \left(\frac{1}{r+i+j+1}\right)(\mathrm{g}^{\,r} - l^{r+i+j+1})\end{pmatrix} \tag{26}$$

Let be the residual life function $m_n(w) = E[(X-w)^n/X>w] = \frac{1}{S(w)}\int_w^{\mathrm{g}}(x-w)^s f(x)dx$ of X has the n-th moment $m_n(w) = \frac{1}{S(w)}\sum_{s=0}^{n}\binom{n}{s}(-w)^{n-s}\left(\int_0^{\mathrm{g}} x^s f(x)dx - \int_0^w x^s f(x)dx\right)$.

Residual life function of X is given by

$$m_n(w) = \frac{1}{S(w)}\sum_{s=0}^{n}\binom{n}{s}(-w)^{n-s}\begin{pmatrix}\beta(\alpha+1)\sum_{i,j=0}^{\infty}\frac{(-1)^i}{\alpha^{j+1}}\binom{\beta-1}{i}\binom{-\beta-1}{j}\\ \left(\frac{1}{s+i+j+1}\right)(\mathrm{g}^s - w^{s+i+j+1})\end{pmatrix} \tag{27}$$

For mean residual life (MRL) function or life expectancy say $m_1(w)$ of X, put $n=1$ in equation (27).

Let be the reverse residual life (RRL) function $R_n(w) = E\left[\frac{(w-X)^n}{X}\le w\right] = \frac{1}{F(w)}\int_0^{\mathrm{g}}(w-x)^n f(x)dx$ of X has the n-th moment. $R_n(w) = \frac{1}{F(w)}\sum_{t=0}^{n}\binom{n}{t}(-1)^t w^{n-t}\int_0^{\mathrm{g}} x^t f(x)dx$.

$$R_n(w) = \frac{1}{F(w)}\sum_{t=0}^{n}\binom{n}{t}(-1)^t w^{n-t}\beta(\alpha+1) \sum_{i,j=0}^{\infty}\frac{(-1)^i}{\alpha^{j+1}}\binom{\beta-1}{i}\binom{-\beta-1}{j}\int_0^{\mathrm{g}} x^{t+i+j}\,dx,$$

Reverse residual life function of X is given by

$$R_n(w) = \left(\begin{array}{c}\frac{1}{F(w)}\sum_{t=0}^{n}\binom{n}{t}(-1)^t w^{n-t}\beta(\alpha+1) \\ \sum_{i,j=0}^{\infty}\frac{(-1)^i}{\alpha^{j+1}}\binom{\beta-1}{i}\binom{-\beta-1}{j}\left(\frac{\mathrm{g}^{\,t+i+j+1}}{t+i+j+1}\right)\end{array}\right) \tag{28}$$

For mean inactivity time, mean reversed residual life function or mean waiting time of X, put n = 1 in equation (28).

Kayid and Izadkhah (2014) defined, strong mean inactivity time (SMIT). For X, it is given by, $M(t) = t^2 - \frac{1}{f(t)}\int_0^t x^2 f(x)dx$ for g$, t > 0$

$$M(t) = \left(\begin{array}{c} t^2 - \left(1-\frac{t}{\mathrm{g}}\right)^{1-\beta}\left(1+\frac{t}{\alpha\mathrm{g}}\right)^{1+\beta} \\ \beta(1+\alpha)\sum_{i,j=0}^{\infty}\frac{(-1)^i\binom{\beta-1}{i}\binom{-\beta-1}{j}t^{i+j+3}}{\alpha^j \mathrm{g}^{i+j}(i+j+3)}\end{array}\right). \tag{29}$$

Mean past life time (MPL) for the conditional random variable $(x - X/X \le x)$ is given by $k(x) = E(x - X/X \le x)$. It can be presented by

$$k(x) = x - \frac{\int_0^t xf(x)dx}{F(x)}.$$

Mean past life time of X is given by

$$k(x) = x - \frac{\left(\begin{array}{c}\beta(\alpha+1) \\ \sum_{i,j=0}^{\infty}\frac{(-1)^i}{\alpha^{j+1}}\binom{\beta-1}{i}\binom{-\beta-1}{j}\left(\frac{t^{\,i+j+2}}{i+j+2}\right)\end{array}\right)}{1-\left(1-\frac{x}{\mathrm{g}}\right)^{\beta}\left(1+\frac{x}{\alpha\mathrm{g}}\right)^{-\beta}}. \tag{30}$$

3.7 Order Statistics

In reliability analysis and life testing of a component, order statistics (OS) and its moments are considered as noteworthy measures. Let $X_{1,}X_{2,}X_{3,}..., X_n$ be a random sample of size n follow to L-II-M distribution and $\{X_{(1)} < X_{(2)} < X_{(3)} < ... < X_{(n)}\}$ be the corresponding order statistics. The r.v's $X_{(i),}X_{(1),}X_{(n)}$ be the *i*-th, minimum, and maximum order statistics of X.

i-th OS PDF is given by

$$f_{(i:n)}(x)=\frac{1}{B(i,n-i+1)!}\left(F(x)\right)^{i-1}\left(1-F(x)\right)^{n-i}f(x), i=1,2,3,\ldots,n.$$

By equations (4) and (5), *i*-th OS PDF of X is given by

$$f_{(i:n)}(x)=\frac{\beta(\alpha+1)}{B(i,n-i+1)!}\begin{pmatrix}\left(1-\left(1-\frac{x}{\mathrm{g}}\right)^{\beta}\left(1+\frac{x}{\alpha\mathrm{g}}\right)^{-\beta}\right)^{i-1}\left(\left(1-\frac{x}{\mathrm{g}}\right)^{\beta}\left(1+\frac{x}{\alpha\mathrm{g}}\right)^{-\beta}\right)^{n-i}\\ \left(\frac{\beta(\alpha+1)}{\alpha\mathrm{g}}\left(1-\frac{x}{\mathrm{g}}\right)^{\beta-1}\left(1+\frac{x}{\alpha\mathrm{g}}\right)^{-\beta-1}\right)\end{pmatrix}. \quad (31)$$

The equation (31) in the form of linear expression is given by

$$f_{(i:n)}(x)=\frac{\beta(\alpha+1)}{\alpha\mathrm{g}B(i,n-i+1)!}\left(\sum_{j,l,m=0}^{\infty}\begin{matrix}(-1)^{i+l}\mathrm{g}^{-l-m}\alpha^{-m}\\ \binom{n-i}{j}\binom{\beta(i+j)-1}{l}\binom{-\beta(i+j)-1}{m}\\ x^{l+m}\end{matrix}\right), \quad (32)$$

and the straight forward computation of equation (32), leads to the *w-th* moment OS of *X* and it is given by

$$\mu_{OS}^{w}=\frac{\beta(\alpha+1)\mathrm{g}^{r}}{\alpha B(i,n-i+1)!}\left(\sum_{j,l,m=0}^{\infty}\begin{matrix}(-1)^{i+l}\mathrm{g}^{-l-m}\alpha^{-m}\\ \binom{n-i}{j}\binom{\beta(i+j)-1}{l}\binom{-\beta(i+j)-1}{m}\\ (r+l+m+1)^{-1}\end{matrix}\right). \quad (33)$$

Further, the *minimum* and *maximum* order statistics of *X* follows directly from the equation (31) with *i* = 1 and *i* = *n*, respectively.

3.8 Stress – Strength Reliability

Let X_1 and X_2 be the strength and stress of a component respectively, followed by the same univariate family of distributions. A component will work improper and in-proper order, on the following conditions, if $X_2 > X_1$ and $X_2 < X_1$ respectively. To discuss the reliability (say *R*) of *X*, it is given by

$$R=P(X_2<X_1).$$

Let $X_1\sim$ L-II-M $(x;\alpha,\beta_1)$ and $X_2\sim$ L-II-M $(x;\alpha,\beta_2)$ be independent and follow to the L-II-M distribution; then the *R* is defined as

$$R=\int f_1(x)F_2(x)dx.$$

$$R = \int \left(\frac{\left(\frac{\beta_1(\alpha+1)}{\alpha \mathrm{g}} \left(1-\frac{x}{\mathrm{g}}\right)^{\beta_1-1} \left(1+\frac{x}{\alpha \mathrm{g}}\right)^{-\beta_1-1} \right)}{\left(1-\left(1-\frac{x}{\mathrm{g}}\right)^{\beta_2}\left(1+\frac{x}{\alpha \mathrm{g}}\right)^{-\beta_2}\right)} \right) dx. \quad (34)$$

Let's suppose

$$\left(1-\frac{x}{\mathrm{g}}\right)^{\beta_2}\left(1+\frac{x}{\alpha\mathrm{g}}\right)^{-\beta_2} = t$$

$$\Rightarrow \frac{\beta_1(\alpha+1)}{\alpha \mathrm{g}} \left(1-\frac{x}{\mathrm{g}}\right)^{\beta_2-1} \left(1+\frac{x}{\alpha\mathrm{g}}\right)^{-\beta_2-1} dx = -dt,$$

as $x \to 0$ and $t \to 1$; $x \to \mathrm{g}$ and $t \to 0$.

Substitute the above information in equation (34), simplify the stress-strength reliability (R) in terms of β_1 and β_2 and it is given by

$$R = \frac{\beta_1}{\beta_1+\beta_2}. \quad (35)$$

The expression in the equation (35) illustrates that the R of L-II-M distribution has increasing trend, as we presume that the R is the function of β_1.

3.9 Entropy

When a system is quantified by the disorderedness, randomness, diversity, or uncertainty, in general, it is known as entropy.

Rényi (1961) entropy of X is described by

$$H_\delta(X) = \frac{1}{1-\delta} log \int_0^{\mathrm{g}} f^\delta(x) dx, \delta > 0 \text{ and } \delta \neq 1.$$

First, we simplify $f(x)$ in terms of $f^\delta(x)$ by considering the equation (5)

$$f^\delta(x) = \left(\frac{\beta(\alpha+1)}{\alpha\mathrm{g}}\right)^\delta \left(1-\frac{x}{\mathrm{g}}\right)^{\delta(\beta-1)} \left(1+\frac{x}{\alpha\mathrm{g}}\right)^{-\delta(\beta+1)}, \quad (36)$$

by applying the binomial expansion on equation (36), we get

$$f^\delta(x) = \left(\frac{\beta(\alpha+1)}{\alpha\mathrm{g}}\right)^\delta \sum_{i,j=0}^{\infty} (-1)^i \alpha^{-j} \mathrm{g}^{-i-j} \binom{\delta(\beta-1)}{i} \binom{-\delta(\beta+1)}{j} x^{i+j}, \quad (37)$$

hence, to integrate the equation (37) yields the most simplified form of Rényi entropy of X and it is given by

$$H_\delta(X) = \frac{1}{1-\delta} log\left(\left(\frac{\beta(\alpha+1)}{\alpha g}\right)^{\delta} \sum_{i,j=0}^{\infty} \frac{(-1)^i \alpha^{-j} \binom{\delta(\beta-1)}{i}}{\binom{-\delta(\beta+1)}{j} \frac{g}{i+j+1}}\right). \tag{38}$$

The quadratic entropy is a special case of Rényi entropy, called quadratic Rényi entropy. It has wide range of application in economics, signal processing, and physics. It is followed the equation (38) with $\delta = 2$.

A generalization of the Boltzmann-Gibbs entropy is the η – entropy. Although in physics, it is referred as the Tsallis entropy. Tsallis (1988) / η – entropy is described by

$$H_\eta(X) = \frac{1}{\eta-1}\left(1 - \int_0^g f^{\eta-1}(x)dx\right), \eta > 0 \text{ and } \eta \neq 1\,.$$

η – entropy of X is given by

$$H_\eta(X) = \frac{1}{\eta-1}\left(1 - \left(\frac{\beta(\alpha+1)}{\alpha g}\right)^{\eta-1} \sum_{i,j=0}^{\infty} \frac{(-1)^i \alpha^{-j}}{\binom{(\eta-1)(\beta-1)}{i}\binom{-(\eta-1)(\beta+1)}{j}} \frac{g}{i+j+1}\right). \tag{39}$$

Another generalized version of the Shannon entropy is the φ - entropy. It is presented by

$$H_{\bar{\varphi}}(X) = \frac{1}{\bar{\varphi}-1}\left(1 - \int_0^g f^{\bar{\varphi}}(x)dx\right), \bar{\varphi} \neq 1.$$

$\bar{\varphi}$ - entropy of X is given by

$$H_{\bar{\varphi}}(X) = \frac{1}{\bar{\varphi}-1}\left(1 - \left(\frac{\beta(\alpha+1)}{\alpha g}\right)^{\bar{\varphi}} \sum_{i,j=0}^{\infty} \frac{(-1)^i \alpha^{-j}}{\binom{\bar{\varphi}(\beta-1)}{i}\binom{-\bar{\varphi}(\beta+1)}{j}\frac{g}{i+j+1}}\right). \tag{41}$$

Havrda and Charvat (1967) introduced the $\omega -$ entropy measure. It is presented by

$$H_\omega(X) = \frac{1}{2^{1-\omega}-1}\left(\int_0^g f^{\omega}(x)dx - 1\right), \omega > 0 \text{ and } \omega \neq 1\,.$$

$\omega -$ entropy of X is given by

$$H_\omega(X) = \frac{1}{2^{1-\omega}-1}\left(\left(\left(\frac{\beta(\alpha+1)}{\alpha g}\right)^{\omega} \sum_{i,j=0}^{\infty} \frac{(-1)^i \alpha^{-j} \binom{\omega(\beta-1)}{i}}{\binom{-\omega(\beta+1)}{j}\frac{g}{i+j+1}}\right) - 1\right). \tag{42}$$

Arimoto (1971) generalized the work of Havrda and Charvat by introducing $\varepsilon -$ entropy measure. It is presented by

$$H_{\varepsilon}(X)=\frac{1}{2^{1-\varepsilon}-1}\left(\left(\int_{0}^{g} f^{\frac{1}{\varepsilon}-1}(x)dx\right)^{\varepsilon}-1\right), \varepsilon>0 \text{ and } \omega\neq 1\,.$$

$\varepsilon -$ entropy of X is given by

$$H_{\varepsilon}(X)=\frac{1}{2^{1-\varepsilon}-1}\left(\left(\left(\frac{\beta(\alpha+1)}{\alpha g}\right)^{\left(\frac{1}{\varepsilon}-1\right)}\sum_{i,j=0}^{\infty}\frac{(-1)^{i}\alpha^{-j}\binom{\left(\frac{1}{\varepsilon}-1\right)(\beta-1)}{i}\binom{-\left(\frac{1}{\varepsilon}-1\right)(\beta+1)}{j}g}{i+j+1}\right)^{\varepsilon}-1\right). \quad (43)$$

Boekee and Lubba (1980) developed the $\tau -$ entropy measure. It is presented by

$$H_{\tau}(X)=\frac{\tau}{\tau-1}\left(1-\left(\int_{0}^{g} f^{\tau-1}(x)dx\right)^{\frac{1}{\tau}}\right), \tau>0 \text{ and } \tau\neq 1\,.$$

$\tau -$ entropy of X is given by

$$H_{\tau}(X)=\frac{\tau}{\tau-1}\left(1-\left(\left(\frac{\beta(\alpha+1)}{\alpha g}\right)^{\tau-1}\sum_{i,j=0}^{\infty}(-1)^{i}\alpha^{-j}\binom{(\tau-1)(\beta-1)}{i}\binom{-(\tau-1)(\beta+1)}{j}\frac{g}{i+j+1}\right)^{\frac{1}{\tau}}\right). \quad (44)$$

Mathai and Haubold (2013) generalized the classical Shannon entropy known by $\zeta -$ entropy. It is presented by

$$H_{\zeta}(X)=\frac{1}{\zeta-1}\left(\int_{0}^{g} f^{2-\zeta}(x)dx-1\right), \zeta>0 \; and \; \zeta\neq 1\,.$$

$\zeta -$ entropy of X is given by

$$H_{\zeta}(X)=\frac{1}{\zeta-1}\left(\left(1-\left(\frac{\beta(\alpha+1)}{\alpha g}\right)^{2-\zeta}\sum_{i,j=0}^{\infty}\frac{(-1)^{i}\alpha^{-j}\binom{(2-\zeta)(\beta-1)}{i}}{}\binom{-(2-\zeta)(\beta+1)}{j}\frac{g}{i+j+1}\right)-1\right). \quad (45)$$

4. ESTIMATION

In this section, we suggest the method of maximum likelihood estimation which provides the maximum information about the unknown model parameters.

From equation (4), likelihood function $L(\vartheta) = \prod_{j=1}^{n} f\ (x; \alpha, \beta)$ of X is given by

$$L\ (\vartheta) = \left(\frac{\beta(\alpha+1)}{\alpha \mathrm{g}}\right)^{n} \prod_{j=1}^{n} \left(1 - \frac{x}{\mathrm{g}}\right)^{\beta-1} \left(1 + \frac{x}{\alpha \mathrm{g}}\right)^{-\beta-1}.$$

The log likelihood function $LL(\vartheta) = l(\vartheta)$ of X is

$$l(\vartheta) = \left(\begin{array}{c} n(ln\beta + ln(1+\alpha) - ln\alpha - ln\mathrm{g}) + (\beta - 1) \sum_{j=1}^{n} \log\left(1 - \frac{x_j}{\mathrm{g}}\right) - \\ (\beta + 1) \sum_{j=1}^{n} \log\left(1 + \frac{x_j}{\alpha \mathrm{g}}\right) \end{array} \right). \quad (46)$$

Now we are concerned to obtain the ML estimates of L-II-M distribution. For this, first; we have to maximize equation (46), second; we calculate the partial derivatives with respect to unknown parameters (α, β) and equate to zero respectively.

The score vector components are given by

$$U(\vartheta) = \frac{\partial l}{\partial \vartheta} = \left(\frac{\partial l}{\partial \alpha}, \frac{\partial l}{\partial \beta}\right)^{T},$$

$$\frac{\partial l}{\partial \alpha} = \frac{n}{1+\alpha} + (\beta + 1) \sum_{j=1}^{n} \left(\frac{x_j}{\alpha(x_i + \alpha \mathrm{g})}\right), \quad (47)$$

$$\frac{\partial l}{\partial \beta} = \frac{n}{\beta} + \sum_{j=1}^{n} \log\left(1 - \frac{x_j}{\mathrm{g}}\right) - \sum_{j=1}^{n} \log\left(1 + \frac{x_j}{\alpha \mathrm{g}}\right). \quad (48)$$

The two non-linear equations (47) and (48) do not provide analytical solution for the MLEs and the optimum value of α, and β. The Newton-Raphson algorithm can be considered an appropriate technique in such kind of MLEs. For numerical solution, the R (statistical software) is used to estimate the parameters of the L-II-M distribution.

5. SIMULATION STUDY

There are several ways to conduct a simulation study, however the most demanding techniques are: (i) identity simulation; (ii) quasi-identity simulation; (iii) laboratory simulation; and (iv) computer simulation.

In this sub-section, the performance of MLE’s, we discuss by the following algorithm.

Step-1: A random sample $x_1, x_2, x_3, ..., x_n$ of sizes n = 50, 100, and 200 are generated from equation (18).

Step-2: Each sample is simulated 1000 times.

Step-3: The required results are obtained based on the different combinations of the parameters place in S-I (g = 1.1, $\alpha = 0.02, \beta = 0.30$), S-II (g = 1.1, $\alpha = 0.09, \beta = 0.20$), and S-III (g = 1.01, $\alpha = 1.9, \beta = 1.2$).

Step-4: Average maximum likelihood estimates (short MLEs) and their corresponding standard errors (short SEs) (present in parenthesis) are presented in Table 1.

Step-5: Gradual decrease in SEs and pretty close ML estimates to the true parameters are observed with increases in the sample size.

Step-6: Finally, the estimates present in Table 1 help us to specify that the method of maximum likelihood works quite well for the L-II-M distribution.

Table 1
Average MLEs and Standard Errors (in present in parenthesis)

n	S-I Estimates (Standard Errors)		S-II Estimates (Standard Errors)		S-III Estimates (Standard Errors)	
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$
50	0.0272 (0.0135)	0.3647 (0.0698)	0.1402 (0.1064)	0.2274 (0.0427)	1.1055 (1.2139)	1.0815 (0.3304)
100	0.0250 (0.0112)	0.2781 (0.0384)	0.1091 (0.0557)	0.1637 (0.0195)	1.2643 (0.9702)	1.1051 (0.2255)
200	0.0262 (0.2972)	0.0068 (0.0272)	0.1321 (0.0437)	0.1934 (0.0166)	1.4508 (0.8764)	1.1504 (0.1721)

6. REAL DATA APPLICATION

This section reports the application of the L-II-M distribution. For this, we explore the multidisciplinary areas of science and engage two suitable lifetime datasets related to the engineering sector.

The first data set relates to the study of failure times of fifty devices put on life test at time zero, discussed by Aarset (1987), and the second dataset relates to the study of the failure times of thirty devices, discussed by Meeker and Escobar (1998).

The L-II-M distribution is compared with its competing models (present in Table 2), based on some criteria: -Log-Likelihood (-Log*L*), Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), and Kolmogorov Smirnov (KS) (with P-value) test statistics. Numerous facts and figures are displayed in Tables 3-5. Some choices of the descriptive statistics are illustrated in Table 3. Tables 4 and 5, describe the estimates of the parameters with standard errors (in parenthesis), and various selection criteria and goodness-of-fit statistics as well.

The performance of the L-II-M distribution eventually fulfills the criteria of a better fit among all competing models on both the datasets.

Moreover, the empirically fitted plots comprising PDFs, CDFs, Kaplan-Meier Survival, Total Time on Test transform (TTT), failure rate and Probability-Probability (PP), confirm the close fit to the datasets, are presented in Figures 5-16 respectively. All the numerical results in the following tables are calculated by the assist of statistical software R and the datasets are given in the appendix.

Table 2
CDF of Competing Models with Parameters, Variable Ranges alongside their References

Abbr.	Model	Parameters / Variable Range	Reference
GPF	$G_I(x) = 1 - \left(\frac{g-x}{g-m}\right)^{\alpha}$	$\alpha > 0$ $m \le x \le g$	Saran and Pandey (2004)
PF-I	$G_{II}(x) = \left(\frac{x}{g}\right)^{\alpha}$	$\alpha > 0$ $0 < x \le g$	Ahsanullah et al. (2013)
WPF	$G_{III}(x) = 1 - e^{-\alpha\left(\frac{x^{\beta}}{g^{\beta}-x^{\beta}}\right)^{\gamma}}$	$\alpha, \beta, \gamma > 0$ $0 < x \le g$	Tahir et al. (2014)
OGEPF	$G_{IV}(x) = \left(1 - e^{-\alpha\left(\frac{x^{\beta}}{g^{\beta}-x^{\beta}}\right)}\right)^{\gamma}$	$\alpha, \beta, \gamma > 0$ $0 < x \le g$	Tahir et al. (2015)
MT-I	$G_V(x) = \frac{\left(\frac{x}{g}\right)(1+\alpha)}{\left(\frac{x}{g}+\alpha\right)}$	$\alpha > 0$ $0 < x \le g$	Muhammad (2016)
MOPF	$G_{VI}(x) = 1 - \frac{\alpha\left(1-\left(\frac{x}{g}\right)^{\beta}\right)}{\left(\frac{x}{g}\right)^{\beta} + \alpha\left(1-\left(\frac{x}{g}\right)^{\beta}\right)}$	$\alpha, \beta > 0$ $0 < x \le g$	Okorie et al. (2017)
KPF	$G_{VII}(x) = 1 - \left(1 - \left(\frac{x}{g}\right)^{\alpha\beta}\right)^{\gamma}$	$\alpha, \beta, \gamma > 0$ $0 < x \le g$	Abdul-Moniem (2017)
Pareto	$G_{VIII}(x) = 1 - \left(\frac{m}{x}\right)^{\alpha}$	$\alpha > 0$ $m \le x < \infty$	Pareto
EPF	$G_{IX}(x) = \left(1 - \left(\frac{g-x}{g-m}\right)^{\alpha}\right)^{\beta}$	$\alpha, \beta > 0$ $m \le x \le g$	Arshad et al. (2020)

GPF=Generalized Power Function; PF-I=Power Function; WPF=Weibull Power Function; OGEPF=Odd Generalized Exponentiated Power Function; MT-I= Mustapha Type – I; MOPF=Marshall-Olkin Power Function; KPF=Kumaraswamy Power Function; EPF=Exponentiated Power Function.

Table 3
Descriptive Statistics

Data	Minimum	1st Quartile	Median	Mean	3rd Quartile	Maximum
Meeker and Escobar	2.00	68.75	196.50	177.03	298.25	300.00
Aarset	0.10	13.50	48.50	45.67	81.25	86.00

Table 4
Parameter Estimates and Standard Errors (in parenthesis)
for Aarset (1987) Data for $m \leq x \leq g$

Model	Estimates (Standard Errors)			Information Criterion			
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	-Log$L$	AIC	BIC	K-S (P-value)
L-II-M	0.07 (0.05)	0.29 (0.06)	-	203.66	411.32	415.15	0.11 (0.53)
KPF	1.06 (66.87)	0.37 (23.38)	0.42 (0.07)	201.58	409.16	414.89	0.08 (0.89)
WPF	0.74 (0.21)	1.49 (0.49)	0.34 (0.06)	205.18	416.36	422.09	0.92 (0.92)
OGEPF	2.64 (1.07)	0.18 (0.05)	0.06 (0.02)	205.34	416.68	422.42	0.08 (0.92)
MOPF	7.66 (5.71)	0.25 (0.15)	-	212.55	429.11	432.93	0.17 (0.10)
GPF	0.58 (0.08)	-	-	213.55	429.11	431.03	0.23 (0.01)
PF-II	0.58 (0.08)	-	-	213.54	429.12	431.03	0.01 (0.01)
PF-I	0.73 (0.10)	-	-	219.88	441.76	443.67	0.24 (0.01)
MT-I	20.47 (34.57)	-	-	222.87	447.75	449.63	0.29 (0.001)
Pareto	0.18 (0.03)	-	-	288.55	579.09	581.01	0.36 (0.00)
EPF	0.33 (0.07)	0.45 (0.07)	-	199.17	402.34	406.16	0.08 (0.08)

The Fitted PDF, CDF, Kaplan-Meier Survival Function, TTT, Failure Rate Function, and PP Plots of the L-II-M distribution for Aarset (1987) data.

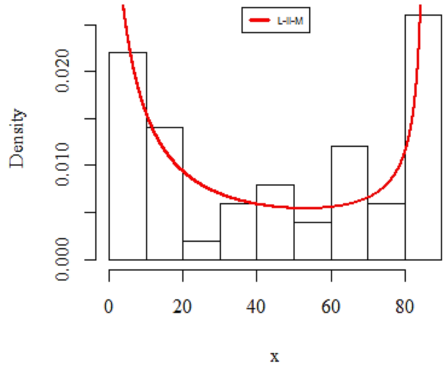

Figure 5: Empirically Fitted Density Function Plot

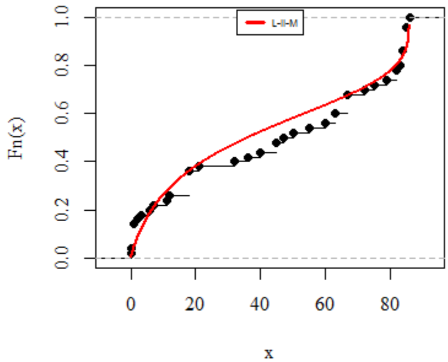

Figure 6: Empirically Fitted Distribution Function Plot

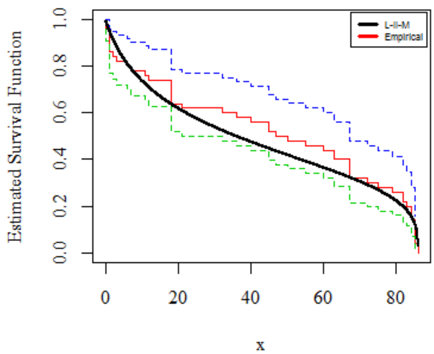

Figure 7: Kaplan-Meier Survival Function Plot

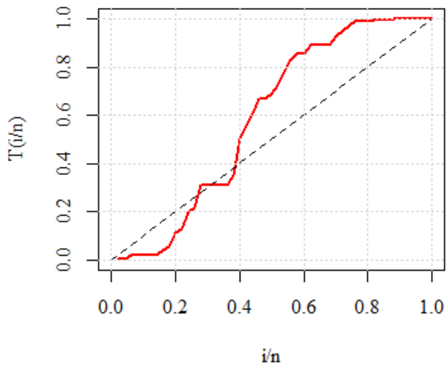

Figure 8: Total Time on Test Transform (TTT) Plot

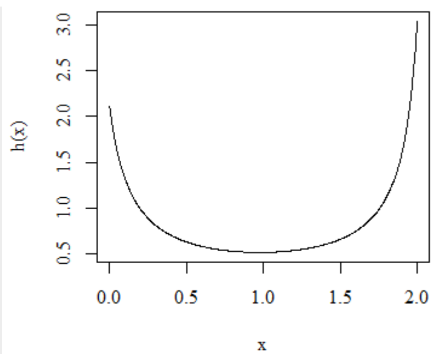

Figure 9: Empirically Fitted Failure Rate Function Plot

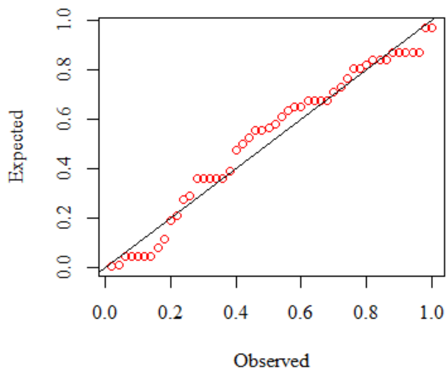

Figure 10: Probability-Probability (PP) Plot

Table 5
Parameter Estimates and Standard Errors (in parenthesis)
for Meeker and Escobar (1998) Data for $m \leq x \leq$ g

Model	Estimates (Standard Errors)			Information Criterion			
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	-Log*L*	AIC	BIC	K-S (P-value)
L-II-M	0.07 (0.05)	0.18 (0.04)	-	121.90	247.80	250.60	0.18 (0.28)
KPF	0.67 (94.55)	0.50 (70.79)	0.22 (0.04)	125.21	256.42	260.62	0.20 (0.17)
GPF	0.28 (0.05)	-	-	131.47	264.94	266.34	0.17 (0.02)
PF-II	0.28 (0.05)	-	-	131.47	264.93	266.34	0.27 (0.02)
WPF	0.81 (0.25)	3.39 (1.31)	0.21 (0.05)	148.72	303.45	307.66	0.30 (0.30)
OGEPF	6.29 (0.003)	0.11 (0.02)	0.01 (0.003)	150.31	306.62	310.83	0.19 (0.22)
MOPF	11.81 (13.33)	0.29 (0.27)	-	165.53	335.06	337.87	0.26 (0.03)
PF-I	0.99 (0.18)	-	-	171.11	344.22	345.63	0.28 (0.02)
MT-I	18.22 (23.59)	-	-	171.61	346.82	346.22	0.27 (0.02)
Pareto	0.25 (0.05)	-	-	212.67	427.34	428.74	0.36 (0.001)
EPF	0.15 (0.04)	0.41 (0.08)	-	119.63	243.26	246.06	0.18 (0.28)

The Fitted PDF, CDF, Kaplan-Meier Survival Function, TTT, Failure Rate Function, and PP Plots of the L-II-M distribution for Meeker and Escobar (1998) data.

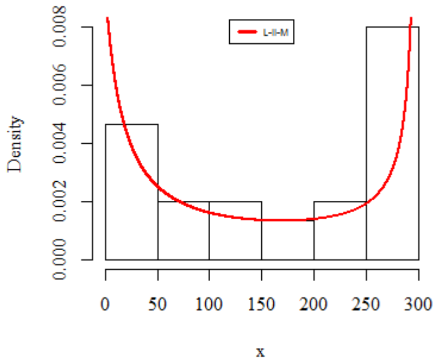

Figure 11: Empirically Fitted Density Function Plot

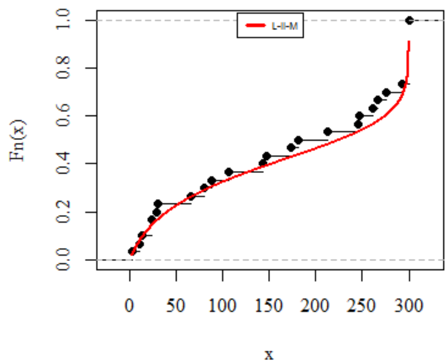

Figure 12:Empirically Fitted Distribution Function Plot

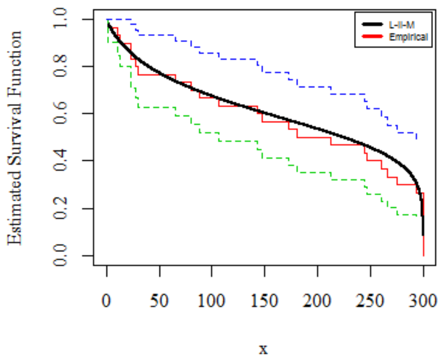

Figure 13: Kaplan-Meier Survival Function Plot

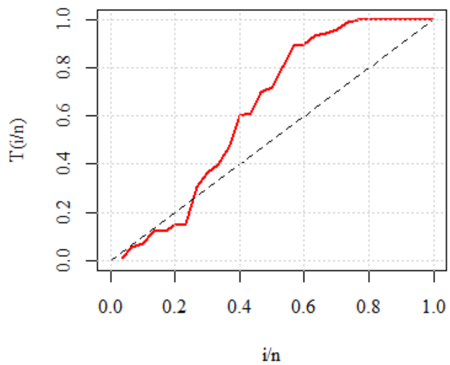

Figure 14: Total Time on Test Transform (TTT) Plot

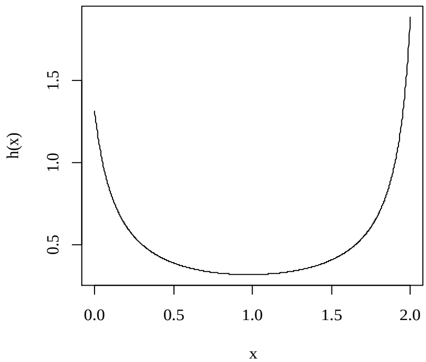

Figure 15: Empirically Failure Rate Function Plot

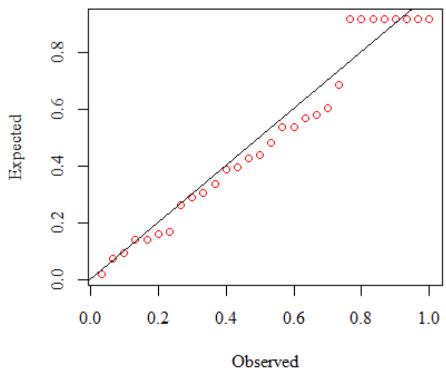

Figure 16: Probability-Probability (PP) Plot

7. CONCLUSION

In this article, we developed a flexible lifetime model that demonstrated the bathtub-shaped failure rate phenomena, so well. The proposed distribution was the Lehmann-II version of the Mustapha Type – I model and we referred to as the Lehmann-II Muhammad (L-II-M) distribution. Some mathematical and reliability measures are derived and discussed. Further, we developed some explicit expressions for the moments, quantile function, and order statistics. For model parameters, we followed the method of the maximum likelihood estimation, and the Monte Carole simulation was carried out to investigate the performance of the MLEs. The most efficient and consistent results explored the dominance of the L-II-M distribution by modeling in two well know real-time datasets belonged to the engineering sector, named after the Aarset (1987) and Meeker and Escobar (1998).

Furthermore, the CDF, PDF, and likelihood function of this model are simple to interpret; and it is computationally very simple. Its density and failure rate shapes follow the asymmetric and bathtub shaped characteristics. It offers more realistic and rationalized results explicitly on the bathtub shaped failure rate phenomena; and it provides a consistently better fit over its competing models. We hope in future the L-II-M distribution will be considered as a choice against the baseline model.

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APPENDIX

The real data sets analyze in the application section

Dataset-1: Aarset (1987)

0.1, 0.2, 1.0, 1.0, 1.0, 1.0, 1.0, 2.0, 3.0, 6.0, 7.0, 11.0, 12.0, 18.0, 18.0, 18.0, 18.0, 18.0, 21.0, 32.0, 36.0, 40.0, 45.0, 45.0, 47.0, 50.0, 55.0, 60.0, 63.0, 63.0, 67.0, 67.0, 67.0, 67.0, 72.0, 75.0, 79.0, 82.0, 82.0, 83.0, 84.0, 84.0, 84.0, 85.0, 85.0, 85.0, 85.0, 85.0, 86.0, 86.0.

Dataset-2: Meeker and Escobar (1998)

275, 13, 147, 23, 181, 30, 65, 10, 300, 173, 106, 300, 300, 212, 300, 300, 300, 2, 261, 293, 88, 247, 28, 143, 300, 23, 300, 80, 245, 266.