Pak. J. Statist.
2021 Vol. 21(1), 39-56

ON WEIGHTED (SHIFT-DEPENDENT) GENERALIZED CUMULATIVE RESIDUAL INACCURACY AND ITS DYNAMIC VERSION

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ABSTRACT

In the present paper, we introduce weighted (length biased) generalized cumulative residual inaccuracy measure of order γ and dynamic (residual) version of it. The general expressions of the proposed generalized cumulative residual inaccuracy and the weighted version of it have been calculated for some well-known lifetime distributions by using the proportional hazard rate model. Based on this model, Rayleigh distribution has been characterized and also it is shown that the dynamic version of the proposed inaccuracy measure uniquely determines the survival function. Further, several important properties of the dynamic measure and their relationships with other reliability measures are studied.

KEYWORDS

Weighted generalized inaccuracy, Weighted mean residual life function, Proportional hazard rate model, Characterization results, Lifetime distributions.

1. INTRODUCTION

A very important and a famous concept in the area of information theory most commonly known as Shannon's entropy was principally originated by Claude Shannon in 1948. The measure has been extended and generalized by various researchers and one of the important generalizations is inaccuracy measure, also called as Kerridge's (1961) measure of inaccuracy. The measure plays an eminent role in estimation, statistical inference, coding theory etc. in addition, the measure of inaccuracy has been significantly used as an effective device for the assessment of error in empirical conclusions.

Let the two absolutely continuous random variables X and Y with density functions $f(x)$ and $g(x)$ respectively, represent the time to failure of two components of a system, then the Shannon's (1948) entropy and Kerridge's (1961) measure of inaccuracy are respectively defined as

$$SE_{(X)} = -\int_0^\infty f(x)\log f(x)\,dx \tag{1}$$

$$\xi_{(X,Y)} = -\int_0^\infty f(x)\log g(x)\,dx \tag{2}$$

A well-known fact about the information measures given in (1) and (2), is that they are shift-independent, and can be thought of a limitation of these measures, as they give equal preferences to the occurrence of every event. However, in real life situations such as a system flop to work, or a neuron discharge spikes in a given time interval occurs in different equally wide interval. So to overcome this limitation, Belis and Guiasu (1968) first introduced the notion of discrete weighted (length biased) entropy.

In agreement with Belis and Guiasu (1968), Dicrescenzo and Longobardi (2006) introduced the shift-dependent (length biased) differential entropy of an absolutely continuous random variable X as

$$SE_{(w,X)} = -\int_0^\infty x\, f(x) \log f(x)\, dx, \tag{3}$$

where, the coefficient x in the integral serve as the weight function $w(x)$ of the elementary events.

If $w(x) = x$, i.e. if $w(x)$ rely on the length (size) of unit of concern, then X^w is said to be length (size) biased random variable.

Kumar et al. (2010) made an effort to introduce the weighted (length biased) measure of inaccuracy as

$$\xi_{(w,X,Y)} = -\int_0^\infty x\, f(x) \log g(x)\, dx. \tag{4}$$

When, $g(x) = f(x)$, then (4) reduces to (3), the weighted entropy.

As an alternative measure to (1), Rao et al. (2004) introduced the concept of cumulative residual entropy which is defined as follows

$$CSE_{(X)} = -\int_0^\infty \overline{F}(x) \log \overline{F}(x)\, dx. \tag{5}$$

In the same way of cumulative residual entropy (5), Taneja and Kumar (2012) defined the cumulative residual inaccuracy as

$$C\xi_{(X,Y)} = -\int_0^\infty \overline{F}(x) \log \overline{G}(x)\, dx, \tag{6}$$

where, $\overline{F}(x) = 1 - F(x)$ and $\overline{G}(x) = 1 - G(x)$, indicate survival functions of the random variables X and Y respectively.

The measures given in (5) and (6), are valid both in discrete as well as in continuous domains and have more advantages than Shannon's entropy. For some general results and applications of these measures refer to Rao (2005), Drissi et al. (2008), Navarro et al. (2010), Wang and Vemuri (2007) and Wang et al. (2003a, 2003b).

Asadi and Zohrevand (2007), introduced the dynamic version of (5), known as dynamic cumulative residual entropy and is given by

$$CSE_{(X)}(t) = -\int_t^\infty \frac{\overline{F}(x)}{\overline{F}(t)} \log \frac{\overline{F}(x)}{\overline{F}(t)} dx . \tag{7}$$

In the similar way of (7), Taneja and Kumar (2012), defined dynamic cumulative residual inaccuracy as

$$C\xi_{(X,Y)}(t) = -\int_t^\infty \frac{\overline{F}(x)}{\overline{F}(t)} \log \frac{\overline{G}(x)}{\overline{G}(t)} dx . \tag{8}$$

Analogous to (3), Misagh et al. (2011) proposed the weighted version of cumulative residual entropy (CRE) as

$$CSE_{(w,X)} = -\int_0^\infty x\overline{F}(x) \log \overline{F}(x) dx . \tag{9}$$

Based on the concept (9), Daneshi et al. (2019), explained the weighted (length-biased) cumulative residual inaccuracy (WCRI) as

$$C\xi_{(w,X,Y)} = -\int_0^\infty x\overline{F}(x) \log \overline{G}(x) dx . \tag{10}$$

Mirali and Baratpour (2017) made an effort to develop the weighted version of dynamic cumulative residual entropy known as weighted dynamic cumulative residual entropy given as follows

$$CSE_{(w,X)}(t) = -\int_t^\infty x \frac{\overline{F}(x)}{\overline{F}(t)} \log \frac{\overline{F}(x)}{\overline{F}(t)} dx . \tag{11}$$

Based on this concept, the weighted dynamic cumulative residual inaccuracy can be defined as

$$C\xi_{(w,X,Y)}(t) = -\int_t^\infty x \frac{\overline{F}(x)}{\overline{F}(t)} \log \frac{\overline{G}(x)}{\overline{G}(t)} dx . \tag{12}$$

There has been a spacious interest among researchers to extend the generalizations of basic inaccuracy measures to make them more flexible. So, in this way Nath (1968) made an attempt to define the generalized inaccuracy measure as

$$\xi^{\gamma}_{(X,Y)} = \frac{1}{1-\gamma} \log\left[\int_0^\infty f(x)\big(g(x)\big)^{\gamma-1} dx\right]; \ \gamma \neq 1,\ \gamma > 0 . \tag{13}$$

where, $\lim_{\gamma\to 1} \xi^{\gamma}_{(X,Y)} = -\int_0^\infty f(x) \log g(x) dx$, which is the Kerridge's inaccuracy as given in (2).

Note that the measure given in (13) has been extended and studied by several authors. For further on this direction, one may refer to Kayal and Sunoj (2017), Ghosh et al. (2018), Ghosh et al. (2019).

Analogous to (6), (8) and on the basis of (13), the generalized cumulative residual and dynamic cumulative residual inaccuracy of order γ are defined as

$$C\xi^{\gamma}_{(X,Y)} = \frac{1}{1-\gamma}\log\left[\int_0^{\infty}\overline{F}(x)\left(\overline{G}(x)\right)^{\gamma-1}dx\right];\ \gamma \neq 1,\ \gamma > 0\ , \tag{14}$$

and

$$C\xi^{\gamma}_{(X,Y)}(t) = \frac{1}{1-\gamma}\log\left[\int_t^{\infty}\left(\frac{\overline{F}(x)}{\overline{F}(t)}\right)\left(\frac{\overline{G}(x)}{\overline{G}(t)}\right)^{\gamma-1}dx\right];\ \gamma \neq 1,\ \gamma > 0 \tag{15}$$

respectively.

When $t = 0$, then (15) reduces to (14).

The objective of this paper is to develop weighted (length biased) generalized cumulative residual inaccuracy measure of order γ and dynamic (residual) version of it. The framework of this paper is organized as follows: In section 2, we discuss the weighted generalized cumulative residual inaccuracy (WGCRI). Also we calculate general expressions of (GCRI) and (WGCRI) for some well-known lifetime distributions by using the proportional hazard rate model (PHRM). In section 3, we discuss the weighted generalized dynamic cumulative residual inaccuracy (WGDCRI) and based on the proportional hazard rate model (PHRM), some characterization results of Rayleigh distribution and proposed (WGDCRI) are explored. Further, in section 4, we present several important properties of (WGDCRI) and their relationships with other reliability measures. Finally, in section 5, we illustrate some concluding remarks.

2. WEIGHTED GENERALIZED CUMULATIVE RESIDUAL INACCURACY (WGCRI)

Under this section, we discuss the length-biased version of GCRI (14), which is known as weighted generalized cumulative residual inaccuracy (WGCRI). We also explore the general expressions of GCRI and WGCRI corresponding to some well-known lifetime distributions.

In the analogy of (10) and on the basis of (13), the WGCI is given by

$$C\xi^{\gamma}_{(w,X,Y)} = \frac{1}{1-\gamma}\log\left[\int_0^{\infty}x\overline{F}(x)\left(\overline{G}(x)\right)^{\gamma-1}dx\right];\gamma \neq 1,\gamma > 0\,, \tag{16}$$

where, the coefficient x in the integral denotes the weight function assigning greater importance to larger values of the random variable X.

In order to verify some general results, we define the proportional hazard rate model (PHRM). The notion of this model was introduced by Cox (1959). The model has been widely used in the variety of fields such as survival analysis, reliability, economics etc. For the application of this model one may refer to Cox and Oakes (1984), Ebrahimi and Kirmani (1996) and Nair and Gupta (2007).

Definition 2.1

Two random variables X and Y are said to satisfy proportional hazard rate model (PHRM), if there exists (proportionality constant) $\theta > 0$ such that

$$\lambda_G(x) = \theta\lambda_F(x). \text{ Or, equivalently, } \overline{G}(x) = \left[\overline{F}(x)\right]^{\theta}, \text{ for some } \theta. \tag{17}$$

where, $\lambda_F(x)$ and $\lambda_G(x)$ represent the hazard rate functions of X and Y respectively.

Remark 2.1

If $\overline{F}(x) = \overline{G}(x)$, then (16) reduces to weighted cumulative residual Renyi's entropy of order γ, and if X and Y satisfy the proportional hazard rate model PHRM (17) with proportionality constant $\theta = 1$, then (16) again reduces to weighted cumulative residual Renyi's entropy of order γ.

The following example exhibits the difference between GCRI (14) and its weighted version (16).

Example 2.1

Let the two random variables X and Y satisfy the PHRM and let X be distributed as

$$f_1(x) = \begin{cases} \frac{1}{2}; 0 \le x \le 2 \\ 0,\ otherwise \end{cases} \quad \text{and} \quad f_2(x) = \begin{cases} \frac{1}{2}; 2 \le x \le 4 \\ 0,\ otherwise \end{cases}$$

then, by simple calculation, we obtain

$$C\xi^{\gamma}_{1(X,Y)} = C\xi^{\gamma}_{2(X,Y)} = \frac{1}{1-\gamma}\log\left(\frac{2}{\theta(\gamma-1)+1}\right).$$

and

$$C\xi^{\gamma}_{1(w,X,Y)} = \frac{1}{1-\gamma}\log\left(\frac{4}{\left(\theta(\gamma-1)+2\right)\left(\theta(\gamma-1)+3\right)}\right),$$

$$C\xi^{\gamma}_{2(w,X,Y)} = \frac{1}{1-\gamma}\log\left(\frac{4\left(\theta(\gamma-1)+4\right)}{\left(\theta(\gamma-1)+2\right)\left(\theta(\gamma-1)+3\right)}\right).$$

Thus from the above calculation, we observed that the generalized cumulative residual inaccuracy (GCRI) of both the density functions is same, but their weighted versions are different. i.e. $C\xi^{\gamma}_{1(X,Y)} = C\xi^{\gamma}_{2(X,Y)}$, but $C\xi^{\gamma}_{1(w,X,Y)} \neq C\xi^{\gamma}_{2(w,X,Y)}$.

In the following table 1, we present the general expressions of GCRI and WGCRI of some well-known lifetime distributions.

Table 1
The Expressions of GCRI and WGCRI for some well-known Lifetime Distributions

Distribution	$\overline{F}(x)$	x	$C\xi^{\gamma}_{(X,Y)}$	$C\xi^{\gamma}_{(w,X,Y)}$
Uniform	$\frac{b-x}{b-a}$	$a<x<b$	$P\log\left(\frac{b-a}{R+2}\right)$	$P\log\left(\frac{b-a}{R+2}\right)\left(a+\frac{b-a}{R+3}\right)$
Exponential	$e^{-\lambda x}$	$x\geq 0,\ \lambda>0$	$P\log\left(\frac{1}{\lambda(R+1)}\right)$	$P\log\left(\frac{1}{(\lambda(R+1))^2}\right)$
Pareto	$\left(1+\frac{x}{b}\right)^{-a}$	$x\geq 0, a>1,$ $b>0$	$P\log\left(\frac{b}{S-1}\right)$	$P\log\left(\left(b^2\right)B(2,\ S-2)\right)$
Finite Range	$(1-x)^a$	$0\leq x\leq 1,$ $a>1$	$P\log\left(\frac{1}{S+1}\right)$	$P\log\left(\frac{1}{(S+1)(S+2)}\right)$
Power	$1-x^a$	$0<x<1,$ $a>0$	$P\log\left(\left(\frac{1}{a}\right)B\left(\frac{1}{a},R+2\right)\right)$	$P\log\left(\left(\frac{1}{a}\right)B\left(\frac{2}{a},R+2\right)\right)$

where,

$P=\frac{1}{1-\gamma},\quad R=\theta(\gamma-1),\ S=a(R+1)$ and

$B(m,n)=\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ is a Beta function.

Definition 2.2

Let X be a non-negative random variable with survival function $\overline{F}(x)$, then the weighted mean residual lifetime (WMRL) is defined as follows

$$m_F^w(t)=\frac{\int_t^\infty x\overline{F}(x)\,dx}{\overline{F}(t)}. \tag{18}$$

In particular, $m_F^w(0)=\int_0^\infty x\overline{F}(x)\,dx$.

Theorem 2.1

Let X and Y be the two non-negative random variables satisfying the PHRM with proportionality constant $\theta(>0)$ and let $C\xi^{\gamma}_{(w,X,Y)} < \infty$, then

$$C\xi^{\gamma}_{(w,X,Y)} \geq \frac{1}{1-\gamma} \log m^{w}_{F}(0), \gamma > 0 .$$

Proof:

Since, X and Y satisfy the PHRM, so (16) can be written as

$$C\xi^{\gamma}_{(w,X,Y)} = \frac{1}{1-\gamma} \log\left[\int_{0}^{\infty} x\left(\overline{F}(x)\right)^{\theta(\gamma-1)+1} dx\right]; \gamma \neq 1, \gamma > 0 .$$

Put $\theta = 1$ and using $\left[\overline{F}(t)\right]^{\gamma} \leq (\geq) \overline{F}(t), t \geq 0$, when $\gamma \geq 1, (0 < \gamma \leq 1)$, the desired result is obtained.

3. WEIGHTED GENERALIZED DYNAMIC CUMULATIVE RESIDUAL INACCURACY (WGDCRI)

In this section, we discuss the weighted version of GDCRI (15) which leads to weighted generalized dynamic cumulative residual inaccuracy (WGDCRI). Based on PHRM (17), some characterization results of WGDCRI are also discussed.

Definition 3.1

Analogous to (12), the weighted form of (15) is defined as

$$C\xi^{\gamma}_{(w,X,Y)}(t) = \frac{1}{1-\gamma} \log\left[\int_{t}^{\infty} x \left(\frac{\overline{F}(x)}{\overline{F}(t)}\right)\left(\frac{\overline{G}(x)}{\overline{G}(t)}\right)^{\gamma-1} dx\right], \gamma \neq 1, \gamma > 0 . \tag{19}$$

General expressions of GDCRI and WGDCRI of some well-known lifetime distributions

i) If X has Uniform distribution over $(a,b), a < b,$ with $\overline{F}(x) = \frac{b-x}{b-a}$, and let X and Y satisfy the proportional hazard rate model with proportionality constant $\theta > 0$, then

$$C\xi^{\gamma}_{(X,Y)}(t) = \frac{1}{1-\gamma} \log\left(\frac{b-t}{r+1}\right)$$

and

$$C\xi^{\gamma}_{(w,X,Y)}(t) = \frac{1}{1-\gamma} \log\left(\frac{(b-t)\left(t(r+2)+(b-t)\right)}{(r+1)(r+2)}\right).$$

ii) If X is exponentially distributed with parameter $\mu > 0$, such that $\overline{F}(x) = e^{-\mu x}, \mu > 0$, then

$$C\xi^{\gamma}_{(X,Y)}(t) = \frac{1}{1-\gamma}\log\left(\frac{1}{\mu r}\right)$$

and

$$C\xi^{\gamma}_{(w,X,Y)}(t) = \frac{1}{1-\gamma}\left[\mu rt + \log\left(\frac{\Gamma(2,\mu rt)}{(\mu r)^2}\right)\right],$$

where,

$$r = \left[\theta(\gamma-1)+1\right].$$

Theorem 3.1

For the two non-negative random variables X and Y, if $C\xi^{\gamma}_{(w,X,Y)}(t) < \infty$, then under (17)

$$C\xi^{\gamma}_{(w,X,Y)}(t) \geq \frac{1}{1-\gamma}\log m^{w}_{F}(t), \gamma > 0.$$

Proof:

Since, we know that $\overline{G}(x) \leq \overline{G}(t)$, for all $x \geq t$. Using this fact in (19), we obtain the desired result.

In the following theorem 3.2, we show that $C\xi^{\gamma}_{(w,X,Y)}(t)$ uniquely determines $\overline{F}(t)$.

Theorem 3.2

Let the two non-negative random variables X and Y with survival functions $\overline{F}(x)$ and $\overline{G}(x)$ respectively, satisfying the PHRM (17), with proportionality constant $\theta > 0$. Let $C\xi^{\gamma}_{(w,X,Y)}(t) < \infty, \forall t \geq 0$, be an increasing function of t, then $C\xi^{\gamma}_{(w,X,Y)}(t)$ uniquely determines the survival function $\overline{F}(x)$ of the random variable X.

Proof:

Rewriting (19) as

$$(1-\gamma)C\xi^{\gamma}_{(w,X,Y)}(t) = \log\int_{t}^{\infty} x\left(\frac{\overline{F}(x)}{\overline{F}(t)}\right)\left(\frac{\overline{G}(x)}{\overline{G}(t)}\right)^{\gamma-1} dx \;. \quad (20)$$

Differentiating (20) w.r.t t, we have

$$(1-\gamma)\frac{\partial}{\partial t}C\xi^{\gamma}_{(w,X,Y)}(t)$$

$$=\frac{1}{\int_t^{\infty} x\left(\frac{\overline{F}(x)}{\overline{F}(t)}\right)\left(\frac{\overline{G}(x)}{\overline{G}(t)}\right)^{\gamma-1}dx}\left[-t+\lambda_F(t)\left(\theta(\gamma-1)+1\right)\int_t^{\infty} x\left(\frac{\overline{F}(x)}{\overline{F}(t)}\right)\left(\frac{\overline{G}(x)}{\overline{G}(t)}\right)^{\gamma-1}dx\right].$$

Or,

$$(1-\gamma)\frac{\partial}{\partial t}C\xi^{\gamma}_{(w,X,Y)}(t)=\lambda_F(t)\left(\theta(\gamma-1)+1\right)-t\exp\left(-(1-\gamma)C\xi^{\gamma}_{(w,X,Y)}(t)\right). \quad (21)$$

Let F_1, G_1 and F_2, G_2 be two sets of the probability distribution functions satisfying PHRM, that is,

$\lambda_{G_1}(x)=\theta\lambda_{F_1}(x)$ and $\lambda_{G_2}(x)=\theta\lambda_{F_2}(x)$, and let

$$C\xi^{\gamma}_{(w,X_1,Y_1)}(t)=C\xi^{\gamma}_{(w,X_2,Y_2)}(t), \forall t\geq 0 \ . \quad (22)$$

Differentiating (22) on both sides w.r.t t, we have

$$\frac{\partial}{\partial t}C\xi^{\gamma}_{(w,X_1,Y_1)}(t)=\frac{\partial}{\partial t}C\xi^{\gamma}_{(w,X_2,Y_2)}(t), \forall t\geq 0 \ .$$

Or,

$$(1-\gamma)\frac{\partial}{\partial t}C\xi^{\gamma}_{(w,X_1,Y_1)}(t)=(1-\gamma)\frac{\partial}{\partial t}C\xi^{\gamma}_{(w,X_2,Y_2)}(t), \forall t\geq 0 \ . \quad (23)$$

Using (21) in (23), we obtain

$$\lambda_{F_1}(t)\left(\theta(\gamma-1)+1\right)-t\exp\left(-(1-\gamma)C\xi^{\gamma}_{(w,X_1,Y_1)}(t)\right)=\lambda_{F_2}(t)\left(\theta(\gamma-1)+1\right)$$
$$-t\exp\left(-(1-\gamma)C\xi^{\gamma}_{(w,X_2,Y_2)}(t)\right). \quad (24)$$

Using (22) in (24), we get $\lambda_{F_1}(t)=\lambda_{F_2}(t)$ or equivalently $\overline{F_1}(t)=\overline{F_2}(t)$. This concludes the proof.

Theorem 3.3

For all $t>0$ and under proportional hazard rate model with proportionality constant $\theta(>0)$, the following equality holds

$$C\xi^{\gamma}_{(w,X,Y)}(t)=\frac{1}{1-\gamma}\log\Bigg[t\exp\Big((1-\gamma)C\xi^{\gamma}_{(X,Y)}(t)\Big)+\int_{z=t}^{\infty}\left(\frac{\overline{F}(z)}{\overline{F}(t)}\right)^{\theta(\gamma-1)+1}\exp\Big((1-\gamma)C\xi^{\gamma}_{(X,Y)}(z)\Big)dz\Bigg].$$

Proof:

$$\int_{t}^{\infty}x\left(\frac{\overline{F}(x)}{\overline{F}(t)}\right)\left(\frac{\overline{G}(x)}{\overline{G}(t)}\right)^{\gamma-1}dx=\int_{t}^{\infty}\left(\int_{0}^{x}dz\right)\left(\frac{\overline{F}(x)}{\overline{F}(t)}\right)\left(\frac{\overline{G}(x)}{\overline{G}(t)}\right)^{\gamma-1}dx$$

$$=\int_{t}^{\infty}\left(\int_{0}^{t}dz+\int_{t}^{x}dz\right)\left(\frac{\overline{F}(x)}{\overline{F}(t)}\right)\left(\frac{\overline{G}(x)}{\overline{G}(t)}\right)^{\gamma-1}dx$$

$$=t\int_{t}^{\infty}\left(\frac{\overline{F}(x)}{\overline{F}(t)}\right)\left(\frac{\overline{G}(x)}{\overline{G}(t)}\right)^{\gamma-1}dx+\int_{z=t}^{\infty}\left(\int_{x=z}^{\infty}\left(\frac{\overline{F}(x)}{\overline{F}(t)}\right)\left(\frac{\overline{G}(x)}{\overline{G}(t)}\right)^{\gamma-1}dx\right)dz\,. \tag{25}$$

Since, from (15), we have

$$\int_{t}^{\infty}\left(\frac{\overline{F}(x)}{\overline{F}(t)}\right)\left(\frac{\overline{G}(x)}{\overline{G}(t)}\right)^{\gamma-1}dx=\exp\Big((1-\gamma)C\xi^{\gamma}_{(X,Y)}(t)\Big). \tag{26}$$

and

$$\int_{t}^{\infty}\left(\overline{F}(x)\right)\left(\overline{G}(x)\right)^{\gamma-1}dx=\left(\overline{F}(t)\right)\left(\overline{G}(t)\right)^{\gamma-1}\exp\Big((1-\gamma)C\xi^{\gamma}_{(X,Y)}(t)\Big) \tag{27}$$

By using the PHRM (17), (27) reduces to

$$\int_{t}^{\infty}\left(\overline{F}(x)\right)\left(\overline{G}(x)\right)^{\gamma-1}dx=\left(\overline{F}(t)\right)^{\theta(\gamma-1)+1}\exp\Big((1-\gamma)C\xi^{\gamma}_{(X,Y)}(t)\Big). \tag{28}$$

Using (25), (26) and (28) in (19), the desired result is proved.

In the following lemma, a unique way of expressing the WGDCRI for symmetric random variables is obtained.

Lemma 3.1

Let X and Y be the random variables with finite support $[0,a]$ and symmetric with respect to $\frac{a}{2}$, i.e., $\overline{F}(x)=F(a-x)$ and $\overline{G}(x)=G(a-x)$ for $0\le x\le a$. Then,

$$C\xi^{\gamma}_{(w,X,Y)}(t)=\frac{1}{1-\gamma}\log\Big[a\exp\Big((1-\gamma)C\overline{\xi}^{\gamma}_{(X,Y)}(a-t)\Big)-\exp\Big((1-\gamma)C\overline{\xi}^{\gamma}_{(w,X,Y)}(a-t)\Big)\Big].$$

where, $C\overline{\xi}^{\gamma}_{(X,Y)}(t)=\frac{1}{1-\gamma}\log\left[\int_0^t\left(\frac{F(x)}{F(t)}\right)\left(\frac{G(x)}{G(t)}\right)^{\gamma-1}\right]$, is the dynamic cumulative past inaccuracy of order γ.

Proof:

Using the symmetry property $\overline{F}(x)=F(a-x)$ and $\overline{G}(x)=G(a-x)$ in (19), we have

$$C\xi^{\gamma}_{(w,X,Y)}(t)=\frac{1}{1-\gamma}\log\left[\int_t^a x\left(\frac{F(a-x)}{F(a-t)}\right)\left(\frac{G(a-x)}{G(a-t)}\right)^{\gamma-1}dx\right]$$

$$=\frac{1}{1-\gamma}\log\left[\int_0^{a-t}(a-z)\left(\frac{F(z)}{F(a-t)}\right)\left(\frac{G(z)}{G(a-t)}\right)^{\gamma-1}dz\right]$$

$$=\frac{1}{1-\gamma}\log\left[a\int_0^{a-t}\left(\frac{F(z)}{F(a-t)}\right)\left(\frac{G(z)}{G(a-t)}\right)^{\gamma-1}dz-\int_0^{a-t}z\left(\frac{F(z)}{F(a-t)}\right)\left(\frac{G(z)}{G(a-t)}\right)^{\gamma-1}dz\right].$$

By solving the above equation, the stated result can be verified.

The below given lemma provides the relationship between WGCRI and WGDCRI.

Lemma 3.2

If the two random variables X and Y satisfy the PHRM (17), then for all $t\geq 0$, the following equality holds

$$C\xi^{\gamma}_{(w,X,Y)}=\frac{1}{1-\gamma}\log\left[\int_0^t x\left(\overline{F}(x)\right)^{\theta(\gamma-1)+1}dx+\left(\overline{F}(t)\right)^{\theta(\gamma-1)+1}\exp\left((1-\gamma)C\xi^{\gamma}_{(w,X,Y)}(t)\right)\right].$$

Proof:

Rewriting (16) as

$$C\xi^{\gamma}_{(w,X,Y)}=\frac{1}{1-\gamma}\log\left[\int_0^t x\overline{F}(x)\left(\overline{G}(x)\right)^{\gamma-1}dx+\int_t^{\infty}x\overline{F}(x)\left(\overline{G}(x)\right)^{\gamma-1}dx\right]. \qquad (29)$$

Using PHRM and (17) in (29), the desired result is satisfied.

In the subsequent theorems, we prove characterization results of Rayleigh distribution in terms of WGDCRI by using the relationship between WGDCRI and WMRL.

Theorem 3.4

For the two random variables X and Y having WGDCRI $C\xi^{\gamma}_{(w,X,Y)}(t)$, the relationship $C\xi^{\gamma}_{(w,X,Y)}(t)=h$, where h is constant, is valid if and only if X has the Rayleigh distribution.

Proof:

Let X follows Rayleigh distribution and X and Y satisfy (17), then it is very easy to get $C\xi^{\gamma}_{(w,X,Y)}(t)=h,$ where, $h=\frac{1}{1-\gamma}\log\left(\frac{b^2}{\theta(\gamma-1)+1}\right).$

Conversely, let us suppose that $C\xi^{\gamma}_{(w,X,Y)}(t)=h,$ then

$$\frac{\partial}{\partial t}C\xi^{\gamma}_{(w,X,Y)}(t)=0.$$

Now, using (20), we have

$$\lambda_F(t)\big(\theta(\gamma-1)+1\big)-t\exp\Big(-(1-\gamma)C\xi^{\gamma}_{(w,X,Y)}(t)\Big)=0$$

After simplifying the above equation, we obtain

$$\lambda_F(t)=\frac{t}{b^2},$$

which is the hazard rate of Rayleigh distribution and this proves the result.

Theorem 3.5

Let X and Y be two non-negative random variables satisfying the PHRM (17) with proportionality constant $\theta(>0)$, and If $m_F^w(t)$ is the weighted mean residual life of X, then the relation

$$(1-\gamma)C\xi^{\gamma}_{(w,X,Y)}(t)=\log h+\log m_F^w(t),\ \text{where } h \text{ is constant,} \tag{30}$$

holds if and only if X has the survival function $\overline{F}(t)=\exp\left(\frac{-t^2}{2b^2}\right).$

Proof:

Let the random variable X has Rayleigh distribution with survival function $\overline{F}(t)=\exp\left(\frac{-t^2}{2b^2}\right)$, then $m_F^w(t)=\frac{\int_t^{\infty}x\overline{F}(x)\,dx}{\overline{F}(t)}=b^2.$

Using the above values in (19) and the relation given in (30) is satisfied.

To prove the converse part, let us assume that (30) holds, then under the PHRM, from (18) and (19), we obtain

$$\big(\theta(\gamma-1)+1\big)\overline{F}(t)\int_t^{\infty}x\left[\overline{F}(x)\right]^{\theta(\gamma-1)+1}dx=\left[\overline{F}(t)\right]^{\theta(\gamma-1)+1}\int_t^{\infty}x\overline{F}(x)\,dx.$$

Differentiating both sides w.r.t t, we get

$$-\left(\theta(\gamma-1)+1\right) f(t)\int_t^\infty x\left[\overline{F}(x)\right]^{\theta(\gamma-1)+1} dx$$

$$+\left(\theta(\gamma-1)+1\right)\left[\overline{F}(t)\right]^{\theta(\gamma-1)} f(t)\int_t^\infty x\overline{F}(x)dx = t\theta(\gamma-1)\left[\overline{F}(t)\right]^{\theta(\gamma-1)+2}. \quad (31)$$

After some simplification, the equation (31), reduces to

$$\lambda_F(t)m_F^w(t) = t. \quad (32)$$

On the other hand, differentiating (18) w.r.t t, we get

$$\frac{\partial}{\partial t}m_F^w(t) = \lambda_F(t)m_F^w(t) - t \quad (33)$$

Using (32) in (33), we obtain

$$\frac{\partial}{\partial t}m_F^w(t) = 0,$$

or precisely $m_F^w(t) = h$, where h is constant. Using this fact again in (32), we get

$$\lambda_F(t) = \frac{t}{h}$$

or analogously $\overline{F}(t) = \exp\left(\frac{-t^2}{h}\right)$, which is the survival function of Rayleigh distribution and hence the result is proved.

4. SOME PROPERTIES AND INEQUALITIES OF $C\xi_{(w,X,Y)}^{\gamma}(t)$

In this section, we present some significant properties and inequalities of Weighted Generalized Dynamic Cumulative Residual Inaccuracy (WGDCRI).

Definition 4.1

The survival function $\overline{F}$ is said to have increasing (decreasing) WGDCRI of order γ represented by IWGDCRI or DWGDCRI, if $C\xi_{(w,X,Y)}^{\gamma}(t)$ is increasing (decreasing) in $t, t > 0$.

It means $\overline{F}$ has IWGDCRI or DWGDCRI if $\frac{\partial}{\partial t}C\xi_{(w,X,Y)}^{\gamma}(t) \geq (\leq) 0$.

Theorem 4.1

Let the random variables X and Y have IWGDCRI, then under PHRM (17), $C\xi^{\gamma}_{(w,X,Y)}(t)$ obtains a lower bound as follows

$$C\xi^{\gamma}_{(w,X,Y)}(t) \geq \frac{1}{\gamma-1}\log\left[\left(\frac{\theta(\gamma-1)+1}{t}\right)\left(\frac{1+\frac{\partial}{\partial t}m_F(t)}{m_F(t)}\right)\right].$$

Proof:

From (21), we have

$$(1-\gamma)\frac{\partial}{\partial t}C\xi^{\gamma}_{(w,X,Y)}(t) = \lambda_F(t)\left(\theta(\gamma-1)+1\right) - t\exp\left(-(1-\gamma)C\xi^{\gamma}_{(w,X,Y)}(t)\right).$$

Using $\lambda_F(t) = \dfrac{1+\frac{\partial}{\partial t}m_F(t)}{m_F(t)}$, where $m_F(t) = \dfrac{\int_t^{\infty}\overline{F}(x)dx}{\overline{F}(t)}$ is the mean residual life function of X, we have

$$(1-\gamma)\frac{\partial}{\partial t}C\xi^{\gamma}_{(w,X,Y)}(t) = \left(\frac{1+\frac{\partial}{\partial t}m_F(t)}{m_F(t)}\right)\left(\theta(\gamma-1)+1\right) - t\exp\left(-(1-\gamma)C\xi^{\gamma}_{(w,X,Y)}(t)\right).$$

Since, $C\xi^{\gamma}_{(w,X,Y)}(t)$ is increasing w.r.t t. Therefore

$$C\xi^{\gamma}_{(w,X,Y)}(t) \geq \frac{1}{\gamma-1}\log\left[\left(\frac{\theta(\gamma-1)+1}{t}\right)\left(\frac{1+\frac{\partial}{\partial t}m_F(t)}{m_F(t)}\right)\right].$$

Theorem 4.2

Let the random variables X and Y be lifetimes of two components of a system with probability density functions $f(x)$ and $g(x)$ and with survival functions $\overline{F}(t)$ and $\overline{G}(t)$ respectively, $t>0$, then for $0<\gamma<1$, $C\xi^{\gamma}_{(w,X,Y)}(t)$ attains a lower bound as follows

$$C\xi^{\gamma}_{(w,X,Y)}(t) \geq \frac{1}{1-\gamma}\left[\log m_F(t) + \frac{1}{m_F(t)}\varphi(X;t)\right]. \tag{34}$$

where, $\varphi(X;t) = \left(\dfrac{1}{\overline{F}(t)}\int_t^{\infty}\overline{F}(x)\log x\,dx + (1-\gamma)C\xi_{(X,Y)}(t)\right)$.

Proof:

From log-sum inequality, we have

$$\int_t^\infty \left(\frac{\bar{F}(x)}{\bar{F}(t)}\right) \log \left(\frac{\left(\frac{\bar{F}(x)}{\bar{F}(t)}\right)}{x\left(\frac{\bar{F}(x)}{\bar{F}(t)}\right)\left(\frac{\bar{G}(x)}{\bar{G}(t)}\right)^{\gamma-1}} \right) dx$$

$$\geq \int_t^\infty \frac{\bar{F}(x)}{\bar{F}(t)} dx \log \left(\frac{\int_t^\infty \frac{\bar{F}(x)}{\bar{F}(t)} dx}{\int_t^\infty x\left(\frac{\bar{F}(x)}{\bar{F}(t)}\right)\left(\frac{\bar{G}(x)}{\bar{G}(t)}\right)^{\gamma-1} dx} \right).$$

$$= m_F(t)\left[\log m_F(t) - (1-\gamma) C\xi^{\gamma}_{(w,X,Y)}(t)\right]. \tag{35}$$

where, (35) is obtained from (19).

The L.H.S of (34) leads to

$$-\frac{1}{\bar{F}(t)}\int_t^\infty \bar{F}(x)\log x\, dx - (\gamma-1)\int_t^\infty \left(\frac{\bar{F}(x)}{\bar{F}(t)}\right)\log\left(\frac{\bar{G}(x)}{\bar{G}(t)}\right) dx . \tag{36}$$

Using definition of $C\xi_{(X,Y)}(t)$ and (36) in (35), we obtain (34).

Theorem 4.3

Let $\bar{F}$ be an IWGDCRI (DWGDCRI) and $\gamma < 1$, then under PHRM with proportionality constant $\theta > 0$

$$\lambda_F(t) \geq (\leq) \frac{t\exp\left(-(1-\gamma) C\xi^{\gamma}_{(w,X,Y)}(t)\right)}{\theta(\gamma-1)+1} .$$

Proof:

From (21), we obtain

$$\frac{\partial}{\partial t} C\xi^{\gamma}_{(w,X,Y)}(t) = \frac{1}{1-\gamma}\left[\lambda_F(t)\left(\theta(\gamma-1)+1\right) - t\exp\left(-(1-\gamma) C\xi^{\gamma}_{(w,X,Y)}(t)\right)\right].$$

Since $\bar{F}$ is IWGDCRI (DWGDCRI) and $\gamma < 1$, therefore, we have

$$\frac{1}{1-\gamma}\left[\lambda_F(t)\left(\theta(\gamma-1)+1\right) - t\exp\left(-(1-\gamma) C\xi^{\gamma}_{(w,X,Y)}(t)\right)\right] \geq (\leq) 0$$

which leads to

$$\lambda_F(t) \geq (\leq) \frac{t \exp\left(-(1-\gamma) C\xi^{\gamma}_{(w,X,Y)}(t)\right)}{\theta(\gamma-1)+1}.$$

Theorem 4.4

For the random variables X and Y having support $(0,b]$, probability density functions $f(x)$ and $g(x)$ and survival functions $\overline{F}(t)$ and $\overline{G}(t)$ respectively, $t>0$, then for $\gamma<1$, the following upper bound of $C\xi^{\gamma}_{(w,X,Y)}(t)$ holds.

$$C\xi^{\gamma}_{(w,X,Y)}(t) \leq \frac{1}{1-\gamma}\left[\frac{\int_t^b x\left(\frac{\overline{F}(x)}{\overline{F}(t)}\right)\left(\frac{\overline{G}(x)}{\overline{G}(t)}\right)^{\gamma-1} \log x\left(\frac{\overline{F}(x)}{\overline{F}(t)}\right)\left(\frac{\overline{G}(x)}{\overline{G}(t)}\right)^{\gamma-1} dx}{\int_t^b x\left(\frac{\overline{F}(x)}{\overline{F}(t)}\right)\left(\frac{\overline{G}(x)}{\overline{G}(t)}\right)^{\gamma-1} dx} + \log(b-t)\right].$$

Proof:

From log-sum inequality and (19), we have

$$\int_t^b x\left(\frac{\overline{F}(x)}{\overline{F}(t)}\right)\left(\frac{\overline{G}(x)}{\overline{G}(t)}\right)^{\gamma-1} \log x\left(\frac{\overline{F}(x)}{\overline{F}(t)}\right)\left(\frac{\overline{G}(x)}{\overline{G}(t)}\right)^{\gamma-1} dx$$

$$\geq \int_t^b x\left(\frac{\overline{F}(x)}{\overline{F}(t)}\right)\left(\frac{\overline{G}(x)}{\overline{G}(t)}\right)^{\gamma-1} dx \log \frac{\int_t^b x\left(\overline{F}(x)\right)\left(\overline{G}(x)\right)^{\gamma-1} dx}{\int_t^b \left(\overline{F}(t)\right)\left(\overline{G}(t)\right)^{\gamma-1} dx}.$$

$$= \int_t^b x\left(\frac{\overline{F}(x)}{\overline{F}(t)}\right)\left(\frac{\overline{G}(x)}{\overline{G}(t)}\right)^{\gamma-1} dx \left[(1-\gamma) C\xi^{\gamma}_{(w,X,Y)}(t) - \log(b-t)\right].$$

After simplification, the proof is obvious.

Proposition 4.1

For the random variables X and Y having WGDCRI $C\xi^{\gamma}_{(w,X,Y)}(t)$ and $\gamma>1$, we have

$$C\xi^{\gamma}_{(w,X,Y)}(t) \geq \frac{1}{\gamma-1}\left(1-\int_t^{\infty} x\left(\frac{\overline{F}(x)}{\overline{F}(t)}\right)\left(\frac{\overline{G}(x)}{\overline{G}(t)}\right)^{\gamma-1} dx\right).$$

Proof:

Since, $-\log x \geq 1-x$, we have

$$C\xi^{\gamma}_{(w,X,Y)}(t)=\frac{1}{1-\gamma}\log\int_t^{\infty} x\left(\frac{\overline{F}(x)}{\overline{F}(t)}\right)\left(\frac{\overline{G}(x)}{\overline{G}(t)}\right)^{\gamma-1}dx$$

$$=-\frac{1}{\gamma-1}\log\int_t^{\infty} x\left(\frac{\overline{F}(x)}{\overline{F}(t)}\right)\left(\frac{\overline{G}(x)}{\overline{G}(t)}\right)^{\gamma-1}dx$$

$$\geq\frac{1}{\gamma-1}\left(1-\int_t^{\infty} x\left(\frac{\overline{F}(x)}{\overline{F}(t)}\right)\left(\frac{\overline{G}(x)}{\overline{G}(t)}\right)^{\gamma-1}dx\right).$$

5. CONCLUSION

In this paper, we developed and studied a weighted generalized cumulative residual inaccuracy measure of order γ with its dynamic (residual) version. We have characterized the Rayleigh distribution and presented several characterization results of the proposed weighted generalized dynamic cumulative residual inaccuracy (WGDCRI) measure. Finally, some significant properties and inequalities of this measure have been explored.

REFERENCES

1. Asadi, M. and Zohrevand, Y. (2007). On the dynamic cumulative residual entropy. *Journal of Stat. Planning and Inference*, 137, 1931-1941.
2. Belis, M. and Guiasu, S. (1968). A quantitative-qualitative measure of information in cybernetic systems. *IEEE Transactions on Information Theory*, *IT*, 4, 593-594.
3. Cox, D.R. and Oakes D. (1984). *Analysis of Survival Data*. CRC Press, Boca Raton, Florida, USA.
4. Cox, D.R. (1959). The analysis of exponentially distributed life-times with two types of failure. *J. R. Stat. Soc.*, Ser. B, 21, 411-421.
5. Daneshi, S., Nezakati, A. and Tahmasebi, S. (2019). On weighted cumulative past (residual) inaccuracy for record values. *Journal of Inequalities and Applications*, 2019(1), 1-22.
6. Di Crescenzo, A. and Longobardi, M. (2006). On weighted residual and past entropies, *Scientiae Mathematicae Japonicae*, 64, 255-266.
7. Drissi, N., Chonavel, T. and Boucher, J.M. (2008). Generalized cumulative residual entropy for distributions with unrestricted supports. *Research Letters in Signal Processing*, Volume 2008, Article ID 790607, 5 pages doi:10.1155/2008/790607.
8. Ebrahimi, N. and Kirmani, S.N.U.A. (1996). A characterization of the proportional hazards model through a measure of discrimination between two residual life distributions. *Biometrika*, 83(1), 233-235.
9. Ghosh, A. and Kundu, C. (2018). On generalized conditional cumulative past inaccuracy measure. *Applications of Mathematics*, 63, 167-193.
10. Ghosh, A. and Kundu, C. (2019). On generalized conditional cumulative residual inaccuracy measure. *Communications in Statistics-Theory and Methods*, 49, 1402-1421.

11. Kayal, S. and Sunoj, S.M. (2017). Generalized Kerridge's inaccuracy for conditionally specified models. *Communications in Statistics-Theory and Methods*, 46, 8257-8268.
12. Kerridge, D.F. (1961). Inaccuracy and inference. *J. Roy. Statist. Soc.*, Ser. B, 23, 184-194.
13. Kumar, V., Taneja, H.C. and Srivastava, R. (2010). Length biased weighted residual inaccuracy measure. *Metron*, LXVIII, 153-160.
14. Mirali, M. and Baratpour, S. (2017). Dynamic version of weighted cumulative residual entropy. *Communications in Statistics-Theory and Methods*, 46, 11047-11059.
15. Misagh, F., Panahi, Y., Yari, G.H. and Shahi, R. (2011). Weighted cumulative entropy and its estimation. In *2011 IEEE International Conference on Quality and Reliability* (pp. 477-480).
16. Nath, P. (1968). Inaccuracy and coding theory. *Metrika*, 13, 123-135.
17. Nair, N.U. and Gupta, R.P. (2007). Characterization of proportional hazard models by properties of information measures. *International Journal of Statistical Sciences*, 6, Special Issue, 223-231.
18. Navarro, J., del Aguila, Y. and Asadi, M. (2010). Some new results on the cumulative residual entropy. *Journal of Statistical Planning and Inference*, 140(1), 310-322.
19. Rao, M., Chen, Y., Vemuri, B.C. and Wang, F. (2004). Cumulative residual entropy: a new measure of information. *IEEE Transactions on Information Theory*, 50, 1220-1228.
20. Shannon, C.E. (1948). A mathematical theory of communications. *Bell System Technical Journal*, 27, 379-423.
21. Taneja, H.C. and Kumar, V. (2012). On dynamic cumulative residual inaccuracy measure. In *Proceedings of the World Congress on Engineering*, 1, 153-156.
22. Wang, F. and Vemuri, B.C. (2007). Non-rigid multi-modal image registration using cross cumulative residual entropy. *Internat. J. Compute. Vision*, 74, 201-215.
23. Wang, F., Vemuri, B.C., Rao, M. and Chen, Y. (2003a). A new and robust information theoretic measure and its application to image alignment. In: *Information Processing in Medical Imaging. Lecture Notes in Computer Science*, 2732, Springer, Heidelberg, 388-400.
24. Wang, F., Vemuri, B.C., Rao, M. and Chen, Y. (2003b). Cumulative residual entropy, a new measure of information and its application to image alignment. In: *Proceedings on the Ninth IEEE International Conference on computer Vision* (ICCV'03), Vol. 1, IEEE Computer Society Press, Silver Spring, MD, 548-553.