Pak. J. Statist.
2021 Vol. 37(1), 1-18

THE TRANSMUTED EXPONENTIAL- EXPONENTIAL DISTRIBUTION WITH APPLICATION TO BREAST CANCER DATA

Aminu Suleiman Mohammed[1§] and **Fidelis Ifeanyi Ugwuowo**[2]
[1] Department of Statistics, Ahmadu Bello University, Zaria-Nigeria.
Email: mohammedas@abu.edu.ng
[2] Department of Statistics, University of Nigeria, Nsukka, Nigeria.
Email: fidelis.ugwuowo@unn.edu.ng
[§] Corresponding author Email: aminusmohammed@gmail.com

ABSTRACT

A new three-parameter lifetime model entitled Transmuted Exponential-Exponential Distribution has been introduced in this research. The new distribution is capable of modeling a skewed dataset. Numerous statistical properties like the reliability function, failure rate, moments, moment generating function, quantile function, odd function, Renyi entropy, and order statistic were derived and presented in explicit form. An approach to the method of maximum likelihood is employed in estimating the distribution parameters. Also, a simulation study is carried out to examine the structure of the different estimates by changing the sample sizes and initial values of the parameters. The results prove empirically by using real-life and simulated datasets that the new model gives sufficient fits in comparison with other competing distribution in modeling skewed datasets.

KEYWORDS

Exponential distribution, Reliability function, Maximum Likelihood, Order statistics.

1. INTRODUCTION

Statistical literature nowadays revealed that exponential distribution (ED) plays an important role in the areas of real-life testing (engineering, biomedicine, finance, agriculture, etc.), in contrast to that of normal distribution. There are several continuous distributions for modeling lifetime data like the Weibull, gamma, exponential, Lindley, and lognormal. The ED, Lindley, and the Weibull distributions are more common in practice than those of gamma and lognormal distributions. This is because of the required numerical integrations of the survival functions of both the gamma and lognormal distributions which are not easily expressed in closed forms.

These continuous distributions are very good examples used in modeling real-life datasets. For the detailed discussion and application of ED, one can refer to Davis (1952), Epstein, and Sobel (1953), Epstein (1958), Feigl, and Zelen (1965), and more. The gamma and lognormal distributions and the likes which are not expressed in closed forms have little applications whereas, ED suitably attracts the attention of researchers because of its tractability which have shown popularity in practice (Shanker, et al. 2015).

Efforts to generalize the one-parameter ED have given rise to the development of new flexible distributions such as the Exponentiated ED (Gupta 2001), Beta ED (Nadarajah and

Kotz 2006), a three-parameter Kumaraswamy ED (Cordeiro and de Castro 2011), the Generalized ED Gupta and Kundu (1999, 2007), the transmuted ED by Owoloko *et al.*, (2015), and the odd generalized exponential-ED by Maiti and Pramanik (2015). Others include the new Lindley-ED by Oguntunde et al., (2016), Mathematical Study on Kumaraswamy New Weighted ED by Mohammed and Abdullahi (2017) and Transmuted Kumaraswamy-inverse ED by Yahaya and Mohammed (2017), Abba and Singh (2018) proposed and study the New Odd Generalized Exponential- ED. These compound models have outperformed the ED when applied to real-life datasets.

Recently, numerous classes of continuous probability distributions have been proposed in the statistical literature. It has been proved that they are useful for adding skewness and flexibility to the existing models. Some of these classes include the transformed-transformer technique (T-X class) by Alzaatreh et al. (2013), the Exponentiated-G (EG) class by Cordeiro *et al*. (2013), the Lomax-G family by Cordeiro *et al.* (2014), the Weibull-G class by Bourguignon *et al*. (2014), On Generating a New Class of Distributions Using the Tangent Function by Al-Mofleh (2018), The Alpha Power Transformation Class: Properties and Applications by Mead et al. (2019), A New Family of Distributions For Generating Skewed Models: Properties and Applications by (Mohammed and Ugwuowo 2020). Mohammed (2019) studied theoretical analysis of the Exponentiated Transmuted Kumaraswamy Distribution, Mohammed, and Yahaya (2019) studied Exponentiated Transmuted Inverse Exponential Distribution. The quasixgamma-geometric distribution with application in medicine by (Sen 2019).

In this article, the Transmuted Exponential-Exponential Distribution was proposed. For the remaining sections of this research work, the following arrangement is considered. In section 2, we presented the cumulative distribution function (cdf) and the probability density function (pdf) of the proposed distribution together with some of its properties. In section 3, an approach to the maximum likelihood technique is employed in estimating the model parameters. The entropy and order statistics of the proposed distribution are presented in section 4. A Simulation study for the unknown parameters of the distribution are presented in section 5. In section 6, the results prove empirically by using real-life and simulated datasets that the new model gives sufficient fits in comparison with other competing distribution in modeling skewed datasets. Finally, the paper is concluded in section 7.

2. TRANSMUTED EXPONENTIAL- EXPONENTIAL (TE-E) DISTRIBUTION

The cdf and pdf TE- G family of distributions are given in equation (1) and (2) which defined for any continuous random variable X.

$$F(x;\lambda,\theta,\xi)=\left(1-\left(1-G(x,\xi)\right)^{\lambda}\right)\left(1+\theta\left(1-G(x,\xi)\right)^{\lambda}\right) \tag{1}$$

and

$$f(x;\lambda,\theta,\xi)=\frac{g(x,\xi)}{1-G(x,\xi)}\lambda\left(1-G(x,\xi)\right)^{\lambda}\left(1-\theta+2\theta\left(1-G(x,\xi)\right)^{\lambda}\right) \tag{2}$$

where, $G(x,\xi)$ and $g(x,\xi)$ are the baseline cdf and pdf respectively depending on a vector parameter ξ whereas, $\lambda > 0$, $-1 \le \theta \le 1$ are two additional parameters i.e. scale and transmuted (shape) parameter respectively.

2.1 Definition of the TE-ED

Consider the $f(x) = \alpha e^{-\alpha x}$ and $F(x) = 1 - e^{-\alpha x}$ of the one parameter ED having a scale parameter $\alpha > 0$. Substituting the functions in (1) and (2), the density and distribution function of the Transmuted Exponential - Exponential Distribution (TE-ED) are as;

Definition 1:

A random variable X is said to follow a TE-ED if its density function has the form;

$$F(x;\lambda,\theta,\alpha) = \left(1 - e^{-\lambda\alpha x}\right)\left(1 + \theta e^{-\lambda\alpha x}\right) \tag{3}$$

and,

$$f(x;\lambda,\theta,\alpha) = \lambda\alpha\left(1-\theta\right)e^{-\alpha\lambda x} + 2\lambda\theta\alpha e^{-2\alpha\lambda x} \tag{4}$$

where,

$\lambda, \alpha > 0$ stand for scale parameters and $-1 \le \theta \le 1$ is transmuted parameter.

2.1.1 Model Validity Check

Proposition 1:

The TE-ED is a well valid density function

$$\int_0^\infty f(x;\lambda,\alpha,\theta)dx = 1$$

Proof:

$$\int_0^\infty \left(\lambda\alpha\left(1-\theta\right)e^{-\alpha\lambda x} + 2\lambda\theta\alpha e^{-2\alpha\lambda x}\right)dx$$

$$= \lambda\alpha\left(1-\theta\right)\int_0^\infty e^{-\alpha\lambda x}dx + 2\lambda\theta\alpha\int_0^\infty e^{-2\alpha\lambda x}dx$$

$$= \lambda\alpha\left(1-\theta\right)\left(\frac{-e^{-\alpha\lambda x}}{\lambda\alpha}\right)_0^\infty + 2\lambda\theta\alpha\left(\frac{-e^{-2\alpha\lambda x}}{2\lambda\alpha}\right)_0^\infty$$

$$= \left(1-\theta\right)\left(-e^{-\alpha\lambda x}\right)_0^\infty + \theta\left(-e^{-2\alpha\lambda x}\right)_0^\infty = \left(1-\theta\right)\left(0+1\right) + \theta\left(0+1\right) = 1$$

2.1.2 Graphical Illustration of the Density and Distribution Function of TE-ED

The plots of the cdf and pdf of the TE-ED are respectively displayed in Figure 1 for selected values $\lambda = a, \alpha = b$ *and* $\theta = c$.

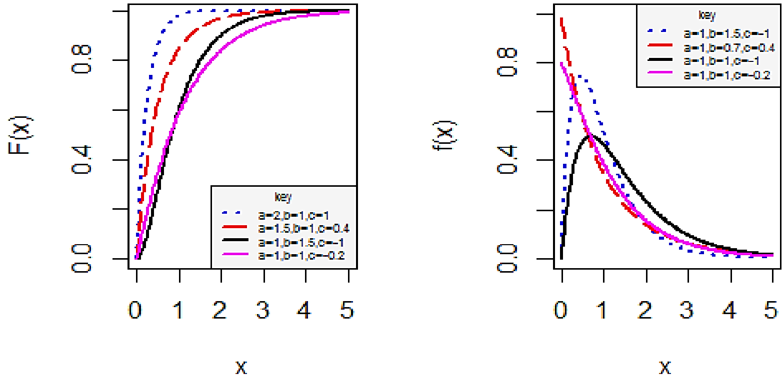

Figure 1: Plots of the cdf and pdf of TE-ED

2.2 Statistical Properties of TE-ED

Here, we present some properties of the TE-E distribution. They include the following:

2.2.1 Moments and Moment Generating Function

Definition 2:

Let X be a random variable with the TE-E density function (4).

The r[th] Moments of TE-ED is given by;

$$\mu_r' = \left(\frac{(1-\theta)}{(\alpha\lambda)^r} + \frac{\theta}{(2\alpha\lambda)^r}\right)\Gamma(r+1) \tag{5}$$

when $r = 1, 2, 3$ and 4

We have the results as;

$$\mu_1' = \left(\frac{(1-\theta)}{\alpha\lambda} + \frac{\theta}{2\alpha\lambda}\right)\Gamma(2) = \frac{(1-\theta)}{\alpha\lambda} + \frac{\theta}{2\alpha\lambda} = \frac{2-\theta}{2\alpha\lambda}$$

$$\mu_2' = \left(\frac{(1-\theta)}{(\alpha\lambda)^2} + \frac{\theta}{(2\alpha\lambda)^2}\right)\Gamma(3) = \frac{2(1-\theta)}{(\alpha\lambda)^2} + \frac{2\theta}{(2\alpha\lambda)^2}$$

$$= \frac{8(1-\theta)+2\theta}{4(\alpha\lambda)^2} = \frac{8-6\theta}{4(\alpha\lambda)^2} = \frac{4-3\theta}{2(\alpha\lambda)^2}$$

$$\mu_3' = \left(\frac{(1-\theta)}{(\alpha\lambda)^3} + \frac{\theta}{(2\alpha\lambda)^3}\right)\Gamma(4) = \frac{6(1-\theta)}{(\alpha\lambda)^3} + \frac{6\theta}{(2\alpha\lambda)^3}$$

$$= \frac{48(1-\theta)+6\theta}{8(\alpha\lambda)^3} = \frac{48-42\theta}{8(\alpha\lambda)^3} = \frac{24-21\theta}{4(\alpha\lambda)^3}$$

$$\mu_4' = \left(\frac{(1-\theta)}{(\alpha\lambda)^4} + \frac{\theta}{(2\alpha\lambda)^4} \right)\Gamma(5) = \frac{24(1-\theta)}{(\alpha\lambda)^4} + \frac{24\theta}{16(\alpha\lambda)^4}$$

$$= \frac{384(1-\theta)+24\theta}{16(\alpha\lambda)^4} = \frac{384-360\theta}{16(\alpha\lambda)^4} = \frac{48-45\theta}{8(\alpha\lambda)^4}$$

To obtain the variance, Standard Deviation and Coefficient of Variation of TE-ED, we simply compute;

$$Var(x) = \mu_2' - (\mu_1')^2 = \frac{4-3\theta}{2(\alpha\lambda)^2} - \left(\frac{2-\theta}{2\alpha\lambda}\right)^2$$

$$= \frac{2(4-3\theta)-(4-4\theta+\theta^2)}{4(\alpha\lambda)^2} = \frac{4-2\theta-\theta^2}{4(\alpha\lambda)^2}$$

$$SD(x) = \sqrt{\mu_2' - (\mu_1')^2} = \sqrt{\frac{4-2\theta-\theta^2}{4(\alpha\lambda)^2}} = \frac{\sqrt{4-2\theta-\theta^2}}{2\alpha\lambda}$$

$$CV = \frac{\sigma}{\bar{X}} = \frac{\dfrac{\sqrt{4-2\theta-\theta^2}}{2\alpha\lambda}}{\dfrac{2-\theta}{2\alpha\lambda}} = \frac{\sqrt{4-2\theta-\theta^2}}{2-\theta}$$

The rth Moment about the Mean is given by;

$$E(x-\mu)^r = \sum_{k=0}^{r}(-1)^k \binom{r}{k} \mu^k \, \Gamma(r-k+1) \left((1-\theta)\left(\frac{1}{\lambda\alpha}\right)^{r-k} + \theta\left(\frac{1}{2\lambda\alpha}\right)^{r-k} \right) \qquad (6)$$

Corollary: if $\mu = 0$, we have the moment about the origin as;

$$E(x^r) = \left((1-\theta)\left(\frac{1}{\lambda\alpha}\right)^{r} + \theta\left(\frac{1}{2\lambda\alpha}\right)^{r} \right)\Gamma(r+1)$$

If $r=1$, the rth Moment about the Mean

$$E(x-\mu) = \sum_{k=0}^{1}(-1)^k \binom{1}{k} \mu^k \, \Gamma(2-k) \left((1-\theta)\left(\frac{1}{\lambda\alpha}\right)^{1-k} + \theta\left(\frac{1}{2\lambda\alpha}\right)^{1-k} \right)$$

If r=2, we have;

$$E(x-\mu)^2 = \sum_{k=0}^{2}(-1)^k \binom{2}{k} \mu^k \, \Gamma(3-k) \left((1-\theta)\left(\frac{1}{\lambda\alpha}\right)^{2-k} + \theta\left(\frac{1}{2\lambda\alpha}\right)^{2-k} \right)$$

If r=3

$$E(x-\mu)^3 = \sum_{k=0}^{3}(-1)^k\binom{3}{k}\mu^k\,\Gamma(4-k)\left((1-\theta)\left(\frac{1}{\lambda\alpha}\right)^{3-k}+\theta\left(\frac{1}{2\lambda\alpha}\right)^{3-k}\right)$$

If r=4

$$E(x-\mu)^4 = \sum_{k=0}^{4}(-1)^k\binom{4}{k}\mu^k\,\Gamma(5-k)\left((1-\theta)\left(\frac{1}{\lambda\alpha}\right)^{4-k}+\theta\left(\frac{1}{2\lambda\alpha}\right)^{4-k}\right)$$

The Coefficient of Variation is therefore given as;

$$CV = \frac{2\alpha\lambda\left(\sqrt{\sum_{k=0}^{2}(-1)^k\binom{2}{k}\mu^k\,\Gamma(3-k)\left((1-\theta)\left(\frac{1}{\lambda\alpha}\right)^{2-k}+\theta\left(\frac{1}{2\lambda\alpha}\right)^{2-k}\right)}\right)}{2-\theta}$$

The Coefficient of Skewness is given as;

$$CS = \frac{E(x-\mu)^3}{\left(E(x-\mu)^2\right)^{\frac{3}{2}}}$$

$$CS = \frac{\sum_{k=0}^{3}(-1)^k\binom{3}{k}\mu^k\,\Gamma(4-k)\left((1-\theta)\left(\frac{1}{\lambda\alpha}\right)^{3-k}+\theta\left(\frac{1}{2\lambda\alpha}\right)^{3-k}\right)}{\left(\sum_{k=0}^{2}(-1)^k\binom{2}{k}\mu^k\,\Gamma(3-k)\left((1-\theta)\left(\frac{1}{\lambda\alpha}\right)^{2-k}+\theta\left(\frac{1}{2\lambda\alpha}\right)^{2-k}\right)\right)^{\frac{3}{2}}} \quad (7)$$

The Coefficient of Kurtosis is given as;

$$CK = \frac{E(x-\mu)^4}{\left(E(x-\mu)^2\right)^2}$$

$$CK = \frac{\sum_{k=0}^{4}(-1)^k\binom{4}{k}\mu^k\,\Gamma(5-k)\left((1-\theta)\left(\frac{1}{\lambda\alpha}\right)^{4-k}+\theta\left(\frac{1}{2\lambda\alpha}\right)^{4-k}\right)}{\left(\sum_{k=0}^{2}(-1)^k\binom{2}{k}\mu^k\,\Gamma(3-k)\left((1-\theta)\left(\frac{1}{\lambda\alpha}\right)^{2-k}+\theta\left(\frac{1}{2\lambda\alpha}\right)^{2-k}\right)\right)^2} \quad (8)$$

The Moment Generating Function of TE-ED is given as;

$$M_X(t) = \frac{\lambda\alpha(1-\theta)}{\alpha\lambda - t} + \frac{2\lambda\theta\alpha}{2\alpha\lambda - t}. \tag{9}$$

2.2.2 Survival Function of TE-ED

$$S(x) = 1 - F(x) \tag{10}$$

By substituting the *cdf* of the TE-ED in (10), the survival function for the TE-ED is obtained as:

$$S(x) = \left(1-\theta+\theta e^{-\lambda\alpha x}\right)e^{-\lambda\alpha x}.$$

2.2.3 Hazard Function of TE-ED

$$h(x) = \frac{f(x;\lambda,\theta,\alpha)}{S(x)}$$

$$h(x) = \frac{\left(\lambda\alpha(1-\theta) + 2\lambda\theta\alpha e^{-\alpha\lambda x}\right)}{\left(1-\theta+\theta e^{-\lambda\alpha x}\right)}. \tag{11}$$

2.2.4 Reserved Hazard Function of TE-ED

$$r(x) = \frac{f(x)}{F(x)} = \frac{\lambda\alpha(1-\theta)e^{-\alpha\lambda x} + 2\lambda\theta\alpha e^{-2\alpha\lambda x}}{\left(1-e^{-\lambda\alpha x}\right)\left(1+\theta e^{-\lambda\alpha x}\right)}$$

$$r(x) = \left(\lambda\alpha(1-\theta)e^{-\alpha\lambda x} + 2\lambda\theta\alpha e^{-2\alpha\lambda x}\right)\left(\left(1-e^{-\lambda\alpha x}\right)\left(1+\theta e^{-\lambda\alpha x}\right)\right)^{-1}. \tag{12}$$

2.2.5 Odd Function of TE-ED

The odd function of TE-ED is given by

$$O(x) = \frac{F(x;\lambda,\theta,\alpha)}{S(x)}$$

$$O(x) = \frac{\left(1-e^{-\lambda\alpha x}\right)\left(1+\theta e^{-\lambda\alpha x}\right)}{\left(1-\theta+\theta e^{-\lambda\alpha x}\right)e^{-\lambda\alpha x}}. \tag{13}$$

The plots of the survival and hazard function of the TE-ED are respectively displayed in Figure 2 for selected values $\lambda = a, \alpha = b$ and $\theta = c$.

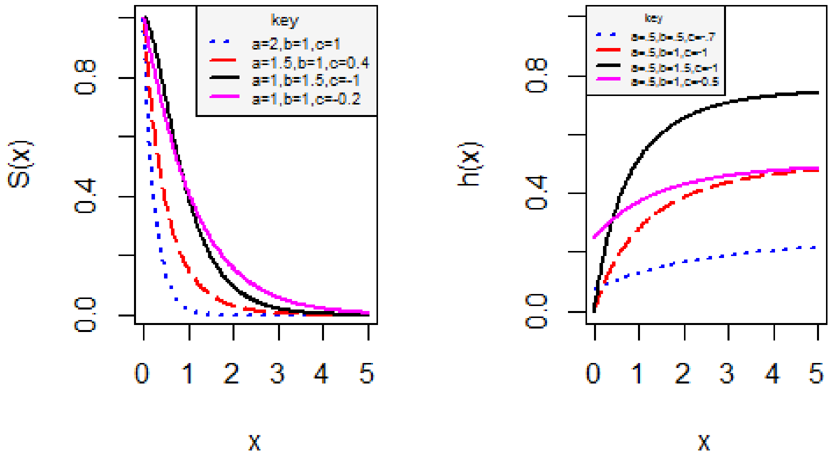

Figure 2: Plots of the Survival and Hazard Function of TE-ED

2.2.6 Quantile Function of TE-ED

Definition 3:

For a nonnegative continuous random variable X that follows the TE-ED, the quantile function is given by;

$$Q(u) = \frac{-1}{\lambda\alpha} \ln\left(\frac{(\theta-1) + \sqrt{(\theta-1)^2 + 4\theta(1-u)}}{2\theta} \right) \qquad \textit{for } \theta \neq 0 \qquad (14)$$

A simulation study was used to assess the behavior of skewness, kurtosis, mean and variance of the TE-E model. The results of these summary statistics are given in table 1 for some arbitrary values of the model parameters. The results has shown that the skewness, kurtosis, mean and variance decreases as the values of the parameter □ increase.

Table 1
Skewness, Kurtosis, Mean and Variance for Some Arbitrary Choices of the Parameter Values

PARAMETERS	$\boldsymbol{\lambda = 1, \alpha = 1}$		$\boldsymbol{\lambda = 1, \alpha = 1}$		
$\boldsymbol{\theta \downarrow}$	**Skewness**	**Kurtosis**	$\boldsymbol{\theta \downarrow}$	**Mean**	**Variance**
-0.8	0.7656	2.5878	**-0.8**	0.6392	0.3512
-0.6	0.5327	2.1627	**-0.6**	0.3500	0.1839
-0.4	0.3706	1.9634	**-0.4**	0.0192	0.0914
1.0	1.8342	7.5848	**1.0**	1.0059	1.0011

3. ESTIMATION OF PARAMETERS OF THE TRANSMUTED EXPONENTIAL – EXPONENTIAL DISTRIBUTION

The estimation of the parameters of the *TE-E* distribution is done by using the method of maximum likelihood estimation. Let $x_1, x_2, ..., x_n$ be a random sample from the *TE-E* Distribution with unknown vector of parameter $\psi = (\alpha, \lambda, \theta)^T$. For determining the MLE of ψ, we have the log-likelihood function:

$$L(\psi) = n\log\alpha + n\log\lambda - \alpha\lambda\sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\log\left(1-\theta+2\theta e^{-\alpha\lambda x_i}\right) \tag{15}$$

$$U_\alpha = \frac{\delta L(\psi)}{\delta\alpha} = \frac{n}{\alpha} - \lambda\sum_{i=1}^{n} x_i - 2\lambda\theta\sum_{i=1}^{n}\frac{x_i\, e^{-\alpha\lambda x_i}}{\left(1-\theta+2\theta e^{-\alpha\lambda x_i}\right)}$$

$$U_\lambda = \frac{\delta L(\psi)}{\delta\lambda} = \frac{n}{\lambda} - \alpha\sum_{i=1}^{n} x_i - 2\alpha\theta\sum_{i=1}^{n}\frac{x_i\, e^{-\alpha\lambda x_i}}{\left(1-\theta+2\theta e^{-\alpha\lambda x_i}\right)}$$

$$U_\theta = \frac{\delta L(\psi)}{\delta\theta} = \sum_{i=1}^{n}\frac{2e^{-\alpha\lambda x_i} - 1}{\left(1-\theta+2\theta e^{-\alpha\lambda x_i}\right)}$$

Setting and solving the nonlinear system of equations $U_\alpha = U_\lambda = U_\theta = 0$ at the same time yields the MLE $\hat{\psi} = (\hat{\alpha}, \hat{\lambda}, \hat{\theta})^T$. Generally it is more practical to use nonlinear methods of optimization such as the BFGS algorithm to maximize Log-likelihood function numerically.

4. ENTROPY AND ORDER STATISTIC OF THE TE-ED

Here, The Renyi entropy and order statistics of the TE-ED were presented.

4.1 Renyi Entropy of the TE-ED

The Renyi Entropy of a random variable *X* having Transmuted Exponential-Exponential Distribution is given as;

$$I_R(\rho) = \frac{\rho-1}{1-\rho}\log(\lambda) + \frac{\rho-1}{1-\rho}\log(\alpha) + \frac{\rho-1}{1-\rho}\log\left(\sum_{k,l=0}^{\infty} m_{k,l,\rho}\right) \tag{16}$$

where,

$$m_{k,l,\rho} = \sum_{k,l=0}^{\infty}\frac{(-1)^{k+l}\,\Gamma(\rho+1)\Gamma(k+1)}{k!\,l!\,\Gamma(\rho+1-k)\Gamma(k+1-l)}\theta^k 2^l .$$

4.2 Order Statistic of the TE-ED

The probability density function of the jth order statistics for a given random samples $X_{1,} X_2, ..., X_n$ from the cdf and of TE-ED is given as;

$$f_{j:n}(x)=\frac{n!\lambda\alpha}{(j-1)!(n-j)}\left(1-\theta+2\theta e^{-\alpha\lambda x}\right)w_{k,l,m,p,q} \tag{17}$$

where,

$$w_{k,l,m,p,q}=\sum_{k,l,m,p,q=0}^{\infty}\frac{\theta^{l+m}(\lambda\alpha)^{q}(-1)^{k+l+m+p+q}(1+k+l+p+n-j)^{q}\, x^{q}\,\Gamma j\,\Gamma j\,\Gamma\left(n-j+1\right)\Gamma(m+1)}{k!l!m!p!q!\Gamma(j-l)\Gamma(j-k)\Gamma(n-j+1-m)\Gamma(m+1-p)}.$$

5. SIMULATION STUDY

Here, we conducted a simulation study in order to examine the performance of the MLEs of the TE-ED parameters. 5000 samples was generated of varying sizes, $n=25$, 50, 100, 150 and 300 of the TE-ED for arbitrary choice of parameters for $\alpha=3,\lambda=2,$ $\theta=0.5$, $\alpha=2,\lambda=5,\theta=0.4$ and $\alpha=6,\lambda=5,\theta=0.5$. The assessment of estimates is based upon the average of the maximum likelihood estimates (MLEs) of the model parameters, bias and the mean squared error (MSE) of the maximum likelihood estimates. The empirical study was performed using the R programming language and the results are presented in Table 2, Table 3 and Table 4. The values in Table 2, Table 3 and Table 4 show that the estimates are quite steady and more significantly as the sample size increases the estimates tend to be closer to the true values of the parameters. Moreover, from Table 2, Table 3 and Table 4 show that the biases and MSEs decrease as the sample size increases. Furthermore, from this simulation study we conclude that the maximum likelihood technique is appropriate in estimating the parameters of the TE-ED.

Table 2
Average Values of the MLEs, Biases and MSEs of the TE-ED
for $\boldsymbol{\alpha=3,\lambda=2,\theta=0.5}$

n	Parameter	Estimate	Bias	MSE
n=25	α	3.1430	0.1430	0.1083
	λ	2.1688	0.1688	0.2209
	θ	0.3252	-0.1748	0.2914
n=50	α	3.1201	0.1201	0.0817
	λ	2.1269	0.1269	0.1662
	θ	0.3558	-0.1442	0.2026
n=100	α	3.0802	0.0802	0.0583
	λ	2.0912	0.0912	0.1172
	θ	0.3985	-0.1015	0.1375
n=150	α	3.0627	0.0627	0.0466
	λ	2.0777	0.0777	0.0960
	θ	0.4148	-0.0852	0.1116
n=300	α	3.0424	0.0424	0.0330
	λ	2.0544	0.0544	0.0669
	θ	0.4414	-0.0586	0.0761

Table 3
Average Values of the MLEs, Biases and MSEs of the TE-ED for $\alpha = 2, \lambda = 5, \theta = 0.4$

n	Parameter	Estimate	Bias	MSE
n=25	α	2.2533	0.2533	0.3731
	λ	5.1339	0.1339	0.0964
	θ	0.1989	-0.2011	0.3031
n=50	α	2.1830	0.1830	0.2663
	λ	5.1005	0.1005	0.0636
	θ	0.2479	-0.1521	0.2025
n=100	α	2.1185	0.1185	0.1759
	λ	5.0638	0.0638	0.0414
	θ	0.3018	-0.0982	0.1343
n=150	α	2.0844	0.0844	0.1385
	λ	5.0480	0.0480	0.0326
	θ	0.3296	-0.0704	0.1063
n=300	α	2.0417	0.0417	0.0911
	λ	5.0284	0.0284	0.0216
	θ	0.3657	-0.0343	0.0713

Table 4
Average Values of the MLEs, Biases and MSEs of the TE-ED for $\alpha = 6, \lambda = 4, \theta = 0.5$

n	Parameter	Estimate	Bias	MSE
n=25	$\boldsymbol{\alpha}$	6.2231	0.2231	0.3271
	$\boldsymbol{\lambda}$	4.2673	0.2673	0.7177
	$\boldsymbol{\theta}$	0.3601	-0.1399	0.2842
n=50	$\boldsymbol{\alpha}$	6.1854	0.1854	0.2782
	$\boldsymbol{\lambda}$	4.2085	0.2085	0.5890
	$\boldsymbol{\theta}$	0.3826	-0.1174	0.1985
n=100	$\boldsymbol{\alpha}$	6.1334	0.1334	0.2050
	$\boldsymbol{\lambda}$	4.1419	0.1419	0.4367
	$\boldsymbol{\theta}$	0.4185	-0.0815	0.1358
n=150	$\boldsymbol{\alpha}$	6.1038	0.1038	0.1701
	$\boldsymbol{\lambda}$	4.1208	0.1208	0.3612
	$\boldsymbol{\theta}$	0.4316	-0.0684	0.1111
n=300	$\boldsymbol{\alpha}$	6.0781	0.0781	0.1228
	$\boldsymbol{\lambda}$	4.0920	0.0920	0.2601
	$\boldsymbol{\theta}$	0.4486	-0.0514	0.0760

6. APPLICATIONS

Here, to demonstrate the potentiality of the TE-E model, we used both real-life and simulated datasets. The MLEs and the performance of the models are computed via R software.

6.1 Real Dataset

The dataset represents the survival times of 121 patients with breast cancer obtained from a large hospital in the period from 1929 to 1938 (Lee, 1992). This dataset has been used by Tahir et al. (2014) and more recently by (Al-kadim and Mahdi, 2018). We equally used this dataset to compare the TE-E model with Beta Exponential (BE), Kumaraswamy Exponential (KwE), and Exponentiated Generalized Exponential (EGE) distribution.

In order to determine the best out of the competing models, we will make use of some criteria including *AIC* (Akaike Information Criterion), *CAIC* (Consistent Akaike Information Criterion), *HQIC* (Hannan-Quinn Information Criteria) and *BIC* (Bayesian Information Criterion). These criteria are mathematically expressed as:

$$AIC = -2L + 2k, CAIC = -2L + 2kn/(n-k-1), HQIC = -2L + 2klog(log(n)) \text{ and } BIC = -2L + klog(n)$$

where L stand for log-likelihood function, k is the number of model parameters and n stand for the size of the sample. Furthermore, we equally compute other measures such as Anderson- Darling (A*), Cramer- Von Mises (W*), Kolmogorov- Smirnov Statistic and P- value.

Note: The model with the smallest value of these measures and highest p- value is consider to be the best among the competing models.

Table 5
Gives the Summary Statistics of the Breast Cancer Dataset

n	Min.	Median	Mean	Variance	Max.	Skewness	Kurtosis
121	0.30	40.00	46.33	1244.46	154.00	1.04318	3.4021

Table 6
Estimated Parameters and their Standard Errors (in Brackets) for the Breast Cancer Dataset

Model	$\hat{\lambda}$	$\hat{\theta}$	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$
TE-E	0.6581 (1.5256)	–0.7803 (0.1407)	-	-	0.0455 (0.1055)
KwE	-	-	1.4508 (0.1657)	2.0547 (1.5954)	1.0149 (0.0099)
BE	-	-	1.5065 (0.1819)	1.7926 (1.5077)	0.0163 (0.0126)
EGE	-	-	1.1056 (1.4624)	1.5155 (0.1926)	0.0251 (0.0331)

Table 5 provides the descriptive statistics of the real dataset while in Table 6, we provide the estimates of the parameters and their standard errors (in parentheses) for the TE-E and the competing distributions.

Table 7
Goodness-of-Fit Statistics for Breast Cancer Dataset

Model	-LL	AIC	CAIC	HQIC	BIC
TE-E	**578.9776**	**1163.955**	**1164.16**	**1167.362**	**1172.343**
KwE	579.6644	1165.329	1165.534	1168.735	1173.716
BE	579.9223	1165.845	1166.05	1169.251	1174.232
EGE	580.0936	1166.187	1166.392	1169.594	1174.575

Table 8
Goodness-of-Fit Statistics for Breast Cancer Dataset

Model	W*	A*	K-S	P-value
TE-E	**0.0532**	**0.3773**	**0.0676**	**0.6375**
KwE	0.0560	0.4013	0.0735	0.5301
BE	0.0593	0.4176	0.0593	0.4538
EGE	0.0615	0.4288	0.0615	0.4123

From Table 5, the skewness of the dataset is 1.0432 which shown clearly that the dataset is asymmetric and positively skewed. The results of Tables 7 and 8 show that the Transmuted Exponential-Exponential (TE-E) distribution has the smallest values of the goodness-of-fit statistics and with a high P-value. Therefore, the TE-E model is considered as the best among the competing distributions.

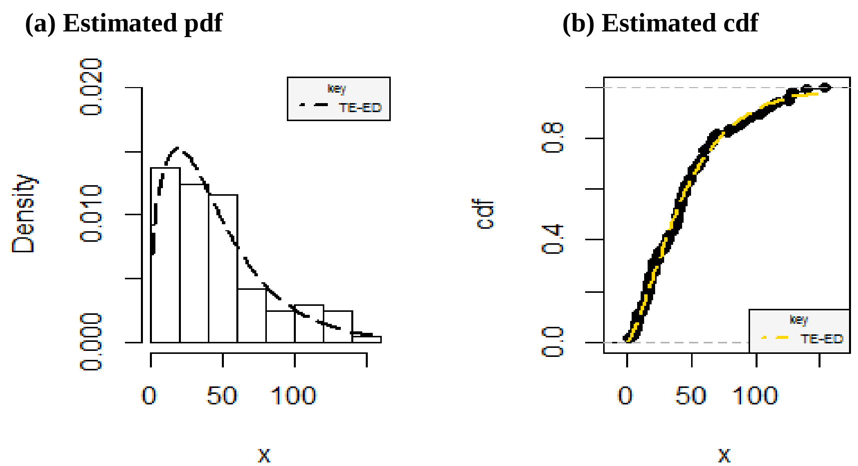

Figure 3: Plots of the Estimated pdf and cdf of TE-E Distribution

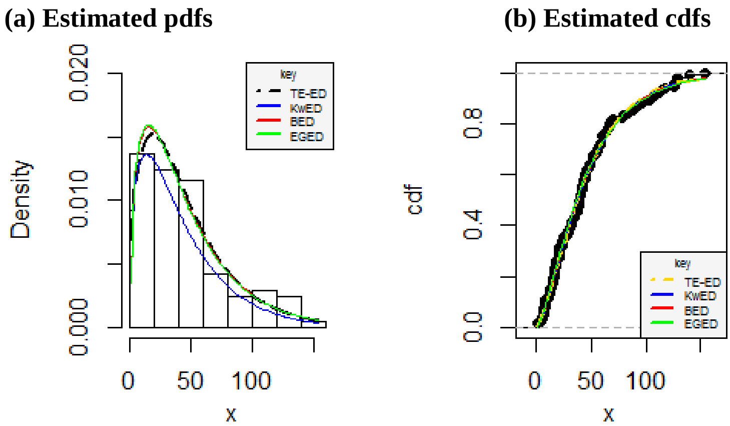

Figure 4: Plots of the Estimated pdfs and cdfs of TE-E, KwE, BE and EGE Distributions

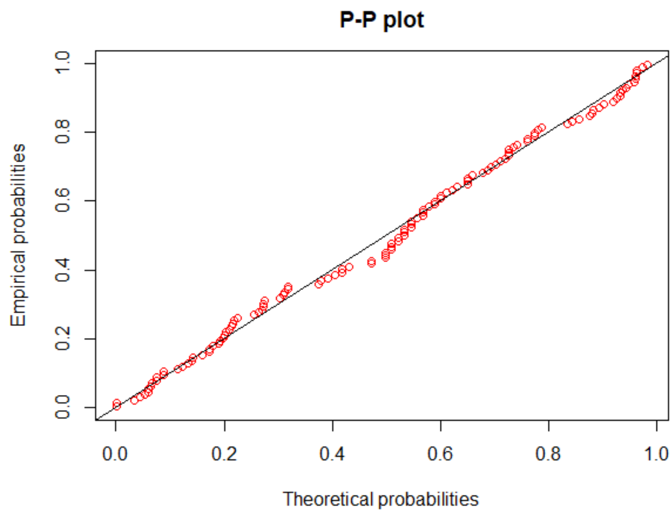

Figure 5: P-P Plot of TE-ED

6.2 Randomly Generated Datasets

We used an already known skewed distribution (Exponential distribution) with parameter alpha=1 to generate a data of size N=150. Method of simple random sampling (SRS) was adopted in drawing four different sizes from the whole dataset as follows; n = 30; 50; 100 and 140 via R package. Then, the three-parameter Transmuted Exponential Exponential (TE-E), Kumaraswamy Exponential (KwE), Beta Exponential (BE), and Exponentiated Generalized Exponential (EGE) distribution was fitted to the data sets. The idea is to identify at what degree of asymmetry does TE-E distribution outperformed the other competing models.

Table 9
Descriptive Statistics of the Simulated Data for Various Sizes

n	Min.	Median	Mean	Variance	Max.	Skewness	Kurtosis
140	0.0166	0.8070	1.1020	0.9852	5.2660	1.4223	5.1263
100	0.0166	0.7255	1.0350	0.8279	3.9200	1.2318	3.9912
50	0.0166	0.7128	0.9829	0.7469	3.9200	1.3086	4.5106
30	0.0166	0.7192	0.9756	0.8479	3.9200	1.5562	5.3195

From the summary statistics in Table 7, the skewness changes for different sample sizes. Table 10 and 11 give the goodness-of-fit statistics for the simulated datasets of TE-E, KwE, BE, and EGE model for different sample sizes. The values in the tables indicate that TE-E outperformed the competing models more especially when the skewness falls within the range of 1.0 and 1.6 as it can be seen from the summary statistics of both real and simulated datasets. The K-S test statistic of the TE-E model is lower than the competing models and has high P-values in all the datasets which shows that the TE-E model fits the datasets than the competing models.

Table 10
Goodness-of-Fit Statistics for Simulated Dataset of Various Sizes

	Model	-LL	AIC	CAIC	HQIC	BIC
n=140	TE-E	**153.0038**	**312.0076**	**312.1841**	**315.5938**	**320.8326**
	KwE	153.1208	312.2415	312.4180	315.8277	321.0664
	BE	153.2436	312.4871	312.6636	316.0733	321.3121
	EGE	153.2794	312.5587	312.7352	316.1449	321.3837
n=100	TE-E	**102.8156**	**211.6312**	**211.8812**	**214.7943**	**219.4467**
	KwE	102.8537	211.7075	211.9575	214.8705	219.523
	BE	102.9778	211.9557	212.2057	215.1188	219.7712
	EGE	103.0188	212.0376	212.2876	215.2007	219.8531
n=50	TE-E	**48.7236**	**103.4472**	**103.9689**	**105.6315**	**109.1833**
	KwE	48.8033	103.6065	104.1283	105.7909	109.3426
	BE	48.8645	103.7289	104.2506	105.9132	109.465
	EGE	48.8901	103.7801	104.3019	105.9644	109.5162
n=30	TE-E	**29.1894**	**64.3789**	**65.3020**	**65.7236**	**68.5825**
	KwE	29.2568	64.5136	65.4366	65.8583	68.7172
	BE	29.2569	64.5136	65.4368	65.8585	68.7174
	EGE	29.2555	64.5110	65.4340	65.8557	68.7145

Table 11
Goodness-of-Fit Statistics for Simulated Data of Various Sizes

n	Model	W*	A*	K-S	P-value
n=140	TE-E	**0.0227**	**0.2039**	**0.0334**	**0.9976**
	KwE	0.0278	0.2431	0.0344	0.9964
	BE	0.3118	0.2686	0.0373	0.9901
	EGE	0.3190	0.2741	0.0385	0.9857
n=100	TE-E	**0.0296**	**0.2102**	**0.0434**	**0.9917**
	KwE	0.3074	0.2257	0.0461	0.9836
	BE	0.0322	0.2425	0.0510	0.9572
	EGE	0.0327	0.2468	0.0524	0.9467
n=50	TE-E	**0.0176**	**0.1473**	**0.0481**	**0.9995**
	KwE	0.0191	0.1656	0.0492	0.9992
	BE	0.0201	0.1754	0.0542	0.9968
	EGE	0.0203	0.1785	0.0561	0.9950
n=30	TE-E	**0.0407**	**0.2637**	**0.0933**	**0.9345**
	KwE	0.0462	0.2949	0.1089	0.8313
	BE	0.0463	0.2957	0.1113	0.8118
	EGE	0.0463	0.2954	0.1122	0.8041

7. CONCLUSION

This paper proposed a new three-parameter exponential distribution called Transmuted Exponential- Exponential Distribution. Some statistical properties of the TE-ED such as reliability function, failure rate, moments, moment generating function, Renyi entropy, and order statistics were studied and presented. The maximum likelihood technique is employed to estimate the distribution parameters and a simulation study is carried out to examine the behavior of different estimates under varying sample sizes. Based on the selected parameter values, the shape of the distribution changes and it is a positive skewed model. The new model provides consistently better fits as compared to other competing models in modeling skewed datasets when applied real-life and simulated datasets.

ACKNOWLEDGEMENTS

The authors wish to thank the Editor-in-Chief and the two anonymous referees for their valuable comments and suggestions which have greatly improved our paper.

REFERENCES

1. Abba, B. and Singh, V.V. (2018). New odd generalized exponential-exponential distribution: its properties and application. *Biostatistics and Biometrics Open Access Journal*, 6(3), 68-73.
2. Al-Kadim, K.A. and Mahdi, A.A. (2018). Exponentiated transmuted exponential distribution. *Journal of University of Babylon for Pure and Applied Sciences*, 26(2), 78-90.
3. Al-Mofleh, H. (2018). On Generating a New Family of Distributions Using the Tangent Function. *Pakistan Journal of Statistics and Operation Research*, 14(3), 471-499.
4. Alzaatreh, A., Lee, C., and Famoye, F. (2013). A new method for generating families of continuous distributions. *Metron*, 71, 63-79.
5. Bourguignon, M., Silva, R.B. and Cordeiro, G.M. (2014). The Weibull-G Family of Probability Distributions. *Journal of Data Science*, 12, 53-68.
6. Cordeiro, G.M. amd de Castro, M. (2011). A new family of generalized distributions. *Journal of Statistical Computation and Simulation*, *81*(7), 883-898.
7. Cordeiro, G.M., Ortega, E.M.M. and Da Cunha, D.C.C. (2013). The exponentiated generalized class of distribution. *Journal of Data Science*, 11, 1-27.
8. Cordeiro, G.M., Ortega, E.M.M., Popovic, B.V. and Pescim, R.R. (2014). The Lomax generator of distributions: Properties, minification process and regression model. *Applied Mathematics and Computation*, 247, 465-486.
9. Davis, D.J.F. (1952). An analysis of some failure data. *Journal of the American Statistical Association*, 47(258), 113-150.
10. Epstein, B. (1958). The exponential distribution and its role in Life –testing. *Ind. Qual. Control*, 15, 2-7.
11. Epstein, B. and Sobel, M. (1953). Life testing. *Journal of the American Statistical Association*, 48(263), 486-502.
12. Feigl, P. and Zelen, M. (1965). Estimation of exponential survival probabilities with concomitant information. *Biometrics*, 21(4), 826-838.
13. Gupta, R.D. and Kundu, D. (2001). Exponentiated exponential family: an alternative to gamma and Weibull distributions. *Biometrical Journal: Journal of Mathematical Methods in Biosciences*, 43(1), 117-130.
14. Gupta, R.D. and Kundu, D. (2007). Generalized exponential distribution: Existing results and some recent developments. *Journal of Statistical Planning and Inference*, 137(11), 3537-3547.
15. Gupta, R.D. and Kundu, D. (1999). Generalized exponential distributions, *Australian & New Zealand Journal of Statistics*, 41(2), 173-188.
16. Gupta, R.D. and Kundu, D. (2001). Letter to Editor. *Australian & New Zealand Journal of Statistics*, 43(4), 493-493.
17. Lee, E.T. (1992). *Statistical Methods for Survival Data Analysis*. New York: John Wiley.
18. Maiti, S.S. and Pramanik, S. (2015). Odds Generalized Exponential-Exponential Distribution. *Journal of Data Science*, 13, 733-754.
19. Mohammed, A.S. and Abdullahi, J. (2017). Mathematical Study on Kumaraswamy New Weighted Exponential Distribution. Journal of the Nigerian Association of Mathematical Physics, 43, 273-278.

20. Mead, M.E., Cordeiro, G.M., Afify, A. and Al-Mofleh, H. (2019). The Alpha Power Transformation Family: Properties and Applications. *Pakistan Journal of Statistics and Operation Research*, 15(3), 525-545.
21. Mohammed, A.S. (2019). Theoretical Analysis of the Exponentiated Transmuted Kumaraswamy Distribution with Application. *Annals of Statistical Theory and Applications (ASTA)*, 1, 51-60.
22. Mohammed, A.S. and Ugwuowo, F.I. (2020). A New Family of Distributions for Generating Skewed Models: Properties and Applications. *Pakistan Journal of Statistics*, 36(2), 149-168.
23. Mohammed, A.S. and Yahaya, A. (2019). Exponentiated Transmuted Inverse Exponential Distribution with Application. *Annals of Statistical Theory and Applications* (ASTA), 2, 71-80.
24. Nadarajah, S. and Kotz, S. (2006). The beta exponential distribution. *Reliability Engineering & System Safety*, 91(6), 689-697.
25. Oguntunde, P.E., Adejumo, A.O., Okagbue, H. and IRastogi, M.K. (2016). Statistical Properties and Applications of a New Lindley Exponential Distribution. *Gazi University Journal of Science,* 29(4), 831-838.
26. Owoloko, E.A., Oguntunde, P.E. and Adejumo, A.O. (2015). Performance rating of the transmuted exponential distribution: an analytical approach. *Springerplus*, 4, 818-829.
27. Sen, S., Afify, A., Al-Mofleh, H. and Ahsanullah, M. (2019). The quasixgamma-geometric distribution with application in medicine. *FILOMAT*, 33(16), 5293-5332.
28. Shanker, R., Fesshaye, H. and Selvaraj, S. (2015). On Modeling of Lifetimes Data Using Exponential and Lindley Distributions. *Biom. Biostat. Int. J.*, 2(5), 00042. DOI: 10.15406/bbij.2015.02.00042
29. Tahir, M.H., Mansoor, M., Zubair, M. and Hamedani, G.G. (2014). McDonald log-logistic distribution with an application to breast cancer data. *Journal of Statistical Theory and Applications*, 13(1), 65. https://doi.org/10.2991/jsta.2014.13.1.6
30. Yahaya, A. and Mohammed, A.S. (2017). Transmuted Kumaraswamy-Inverse Exponential Distribution and Its Properties. *Proceedings of 1st International Conference of Nigeria Statistical Society*, 1, 25-28.