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PROPOSED BAYESIAN ESTIMATORS OF THE SCALE PARAMETER IN NAKAGAMI DISTRIBUTION

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ABSTRACT

In this paper, we proposed new Bayes estimators to estimate the scale parameter of the Nakagami distribution. The proposed estimators are based on a proposed loss function and two different prior forms were used. A simulation study showed that the proposed estimators are the best compared with other estimators.

KEYWORDS

Nakagami Distribution, Bayes Method, Jeffreys, Invers Gamma, square error loss function, Proposed loss function.

1. INTRODUCTION

Nakagami distribution is one of the flexible lifetime distributions used in the analysis of radio signals and the propagation of radio waves. It is also used in the study of medical imaging of ultrasound, especially in testing the efficiency of the heart. The Nakagami distribution is useful in engineering reliability and reliability theory. This distribution was proposed by (Nakagami, 1960). Many studies have dealt with the Nakagami distribution. (Strohbehn, Wang, & Speck, 1975) used Nakagami distribution in an optical signal on a line of sight path. Their study compared three different estimators for the Nakagami-m parameter and these estimators are Tolparev-Polyakov, Lorenz, and inverse normalized variance estimators. (Cheng & Beaulieu, 2001) estimated the Nakagami distribution parameters using the maximum likelihood method and two new estimators were also proposed. (Shankar, 2003) studied the relationship of this compound Gamma distribution to the Nakagami and Rayleigh distributions. This distribution is explored through signal analysis to-noise ratio of the envelopes and simulations of random numbers. (Kolár, Jirík, & Jan, 2004) studied Nakagami distribution that has been used in many engineering applications and also used in other applications in biomedical engineering such as in the ultrasound tissue characterization of the echocardiographic application. (Shittu, Adepoju, Shittu, & Kazeem, 2013) compared between a beta distribution and Nakagami distribution to get another distribution is called Beta-Nakagami distribution. (Adepoju, Chukwu, & Shittu, 2014) compared between an exponential distribution and Nakagami distribution to get another distribution is called exponential-Nakagami distribution. (Kumar & Vaish, 2015) developed Sequential Probability Ratio Test (SPRT) for the scale parameter of Nakagami distribution also the robustness of scale parameter. (Ahmad, Ahmad, & Ahmed, 2016) estimated scale parameters of Nakagami distribution by using maximum likelihood and Bayesian methods They also used three different loss functions also Jeffery and quasi prior distribution. (Kumar, Garg, & Krishna, 2016) derived the reliability function for the Nakagami distribution under progressive censoring data. In this paper we proposed a new Bayesian method to estimate the scale parameters of Nakagami Distribution. The paper is organized into several sections that include methods for estimating the parameters of Nakagami distribution, then simulation results, and finally recommendations.

The probability density function of Nakagami distribution is

$$f(x,k,\theta) = \frac{2\lambda^{\lambda}}{\Gamma(\lambda)\theta^{\lambda}} x^{2\lambda-1}\, e^{-\frac{\lambda x^2}{\theta}};\ x > 0, \lambda, \theta > 0 \tag{1}$$

where λ and θ represent the shape and the scale of parameters respectively.

The cumulative and reliability function of the Nakagami distribution are given by the following formulas respectively:

$$F(x) = \frac{\gamma\left(\lambda, \frac{\lambda}{\theta}\right)}{\Gamma(k)} \tag{2}$$

$$R(x) = 1 - \frac{\gamma\left(\lambda, \frac{\lambda}{\theta}\right)}{\Gamma(\lambda)} \tag{3}$$

where $\gamma(.)$ is incomplete gamma function which is computed by the following formula:-

$$\gamma(s,x) = \int_0^x t^{s-1}\, e^{-t} dt \tag{4}$$

The mean and variance of the Nakagami distribution are given by the following formulas respectively:

$$E(X) = \frac{\Gamma\left(\lambda + \frac{1}{2}\right)\left(\frac{\theta}{\lambda}\right)^{\frac{1}{2}}}{\Gamma(k)} \tag{5}$$

$$Var(X) = \theta\left\{1 - \frac{1}{\lambda}\left(\frac{\Gamma\left(\lambda + \frac{1}{2}\right)}{\Gamma(\lambda)}\right)^2\right\} \tag{6}$$

And the mode of Nakagami distribution is

$$\frac{\sqrt{2}}{2}\left(\frac{(2\lambda-1)\theta}{\lambda}\right)^{\frac{1}{2}}. \tag{7}$$

2. BAYES METHOD

The Bayes method depends on the amount of information available about the random parameter θ to be estimated that can be formulated as a probability distribution called a prior distribution and symbolized by the symbol $\pi(\theta)$.In the Bayes method, the prior distribution function is combined with the likelihood function of observation and by using the inverse Bayes rule, we get the posterior probability distribution as in the following formula.

$$h(\theta|x_1, x_2, \dots, x_n) = \frac{L(\theta|x_1, x_2, \dots, x_n)\pi(\theta)}{\int_{\forall\theta} L(\theta|x_1, x_2, \dots, x_n)\pi(\theta)\, d\theta} \quad (8)$$

By using square error loss function which is according to the following formula:-

$$\mathrm{L}(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2 \quad (9)$$

Then we get the Bayes estimator, which will be the mean for the posterior distribution according to the following formula:-

$$\hat{\theta}_{Bayes}\,(t) = \int_{\forall\theta} h(\theta|x_1, x_2, \dots, x_n)\, d\theta. \quad (10)$$

1.2 Bayes Method with Jeffreys' Prior

A prior distribution of the scale parameter θ of the Nakagami distribution which is based on Jeffrey method (Ahmad, Ahmad, & Ahmed, 2016) as following:

$$\pi_1(\theta) = \frac{1}{\theta^c}, c > 0 \quad (11)$$

Then the posterior distribution of scale parameter θ with prior distribution in (11) is obtained as in the following formula:-

$$h_1(\theta|x_1, x_2, \dots, x_n) = \frac{\frac{1}{\theta^{n\lambda+c}}\ exp\left(\frac{-\lambda}{\theta}\sum_{i=1}^{n} x_i^2\right)}{\int_0^\infty \frac{1}{\theta^{n\lambda+c}} exp\left(\frac{-\lambda}{\theta}\sum_{i=1}^{n} x_i^2\right) d\theta} \quad (12)$$

The integral of the numerator for equation (12) is equal to:-

$$\frac{\Gamma(n\lambda + c - 1)}{(\lambda \sum_{i=1}^{n} x_i^2)^{n\lambda+c-1}} \quad (13)$$

Then the posterior distribution of scale parameter θ is given by:-

$$h_1(\theta|x_1, x_2, \dots, x_n) = \frac{(\lambda \sum_{i=1}^{n} x_i^2)^{n\lambda+c-1} exp\left(\frac{-\lambda}{\theta}\sum_{i=1}^{n} x_i^2\right)}{\Gamma(nk + c - 1)\theta^{n\lambda+c}} \quad (14)$$

Then the Bayes estimator for the scale parameter θ with Jeffreys' prior is equal to mean of the posterior distribution as follow:-

$$\hat{\theta}_{B1} = \int_0^\infty \theta\ \frac{(\lambda \sum_{i=1}^{n} x_i^2)^{n\lambda+c-1} exp\left(\frac{-\lambda}{\theta}\sum_{i=1}^{n} x_i^2\right)}{\Gamma(n\lambda + c - 1)\theta^{n\lambda+c}}\ d\theta \quad (15)$$

$$= \frac{\lambda \sum_{i=1}^{n} x_i^2}{nk + c - 2}. \quad (16)$$

2.2 Bayes Method with Invers Gamma Prior

In the case of using a prior probability inverted gamma which is as in the following formula:-

$$\pi_2(\theta) = \frac{\theta^{-a-1}}{\Gamma(a)}\, e^{-\frac{1}{\theta}} \tag{17}$$

Then the posterior distribution of the scale parameter θ with prior distribution in (17) is as in the following formula:-

$$h_2(\theta|x_1, x_2, \dots, x_n) = \frac{\frac{1}{\theta^{n\lambda+a+1}}\, exp\left(-\frac{(1+\lambda x_i^2)}{\theta}\right)}{\int_0^\infty \frac{1}{\theta^{n\lambda+a+1}}\, exp\left(-\frac{(1+\lambda\sum_{i=1}^n x_i^2)}{\theta}\right)} \tag{18}$$

Therefore, the integral of the numerator for equation (18) is as in the following formula:-

$$\frac{\Gamma(n\lambda+a)}{(1+\lambda\sum_{i=1}^n x_i^2)^{n\lambda+a}} \tag{19}$$

Then we get

$$h_2(\theta|x_1, x_2, \dots, x_n) = \frac{(1+\lambda\sum_{i=1}^n x_i^2)^{n\lambda+a} exp\left(-\frac{(1+\lambda\sum_{i=1}^n x_i^2)}{\theta}\right)}{\Gamma(n\lambda+a)\theta^{n\lambda+a+1}} \tag{20}$$

Then the Bayes estimator for the scale parameter θ with Invers Gamma Prior is equal to mean of the posterior distribution as follow:-

$$\hat{\theta}_{B2} = \int_0^\infty \theta\, \frac{(1+\lambda\sum_{i=1}^n x_i^2)^{n\lambda+a} exp\left(-\frac{(1+\lambda\sum_{i=1}^n x_i^2)}{\theta}\right)}{\Gamma(nk+a)\theta^{n\lambda+a+1}}\, d\theta \tag{21}$$

$$= \frac{1+\lambda\sum_{i=1}^n x_i^2}{n\lambda+a-1}. \tag{22}$$

3. PROPOSED BAYES METHOD

In this article we proposed that the Bayes method is based on the following loss function:-

$$L(\hat{\theta}, \theta) = \left(\hat{\theta}^w - \theta^w\right)^2, w > 0 \tag{23}$$

By finding the expected error loss function according to the following formula:-

$$E\left[L(\hat{\theta}, \theta)\right] = \int\limits_{\forall\theta} \left(\hat{\theta}^w - \theta^w\right)^2 h(\theta|x_1, x_2, \dots, x_n) d\theta \tag{24}$$

By deriving the above equation with respect to $\hat{\theta}$ and then equalizing it to zero as follow:-

$$\frac{\partial}{\partial\hat{\theta}} E\left[L\left(\hat{\theta},\theta\right)\right] = \frac{\partial}{\partial\hat{\theta}} \int_{\forall\theta} \left(\hat{\theta}^{w} - \theta^{w}\right)^{2} h(\theta|x_1, x_2, \dots, x_n) d\theta$$

$$\Rightarrow \int_{\forall\theta} 2w\hat{\theta}^{w-1} \left(\hat{\theta}^{w} - \theta^{w}\right) h(\theta|x_1, x_2, \dots, x_n) d\theta = 0$$

$$\hat{\theta}^{w} \int_{\forall\theta} h(\theta|x_1, x_2, \dots, x_n) d\theta = \int_{\forall\theta} \theta^{w} h(\theta|x_1, x_2, \dots, x_n) d\theta$$

Then the proposed Byes estimator is illustrated according to the following formula:-

$$\hat{\theta}_{PBayes}\,(t) = \left[\int_0^\infty \theta^w\, h(\theta|x_1, x_2, \dots, x_n) d\theta\right]^{\frac{1}{w}}. \tag{25}$$

3.1 Proposed Bayes Method with Jeffreys' Prior

In the case that a proposed loss function is used in (23) and the prior distribution of the scale parameter θ in (11), then the proposed Bayes estimator for the scale parameter θ in Nakagami distribution is obtained as follow:-

$$\hat{\theta}_{B3} = \left[\int_0^\infty \theta^w\, h_1(\theta|x_1, x_2, \dots, x_n) d\theta\right]^{\frac{1}{w}} \tag{26}$$

$$= \left[\int_0^\infty \frac{\theta^w\, (k\sum_{i=1}^n x_i^2)^{nk+c-1} exp\left(\frac{-k}{\theta}\sum_{i=1}^n x_i^2\right)}{\Gamma(nk+c-1)\theta^{nk+c}}\right]^{1/w}$$

$$\hat{\theta}_{B3} = \left(\frac{(k\sum_{i=1}^n x_i^2)^w \Gamma(nk+c-w-1)}{\Gamma(nk+c-1)}\right)^{1/w}. \tag{27}$$

3.2 Proposed Bayes Method with Invers Gamma Prior

In the case that a proposed loss function is used in (23) and the prior distribution of the measurement parameter θ in (17), then Bayes estimator of the scale parameter θ is obtained as follow:-

$$\hat{\theta}_{B4} = \left[\int_0^\infty \theta^w\, h_2(\theta|x_1, x_2, \dots, x_n) d\theta\right]^{1/w} \tag{28}$$

$$= \left[\int_0^\infty \theta^w \frac{(1+\lambda\sum_{i=1}^n x_i^2)^{n\lambda+a}\, exp\left(-\frac{(1+\lambda\sum_{i=1}^n x_i^2)}{\theta}\right)}{\Gamma(n\lambda+a)\theta^{n\lambda+a+1}} d\theta\right]^{1/w}$$

$$\hat{\theta}_{B4} = \left[\frac{(1+\lambda\sum_{i=1}^n x_i^2)^w \Gamma(n\lambda+a-w)}{\Gamma(n\lambda+a)}\right]^{1/w}. \tag{29}$$

4. SIMULATION

To compare the estimator of the Nakagami distribution, the simulation was used by Mont Carlo and for different sample sizes and initial values for parameters based on mean squares errors. The simulation results are executed based on a program written in the language R and with iteration equal to (1000).

Table 1
Shows the Results of the First Simulation Experiment of the Scale Parameter Values when w=0.5 =, a = 1, c = 0.5

n	λ	Θ	$\hat{\theta}_{B1}$	$\hat{\theta}_{B2}$	$\hat{\theta}_{B3}$	$\hat{\theta}_{B4}$
20	0.7	1.3	1.56925	1.53818	1.47254	1.44648
	1.5	2	2.39908	2.37813	2.31246	2.29327
40	0.7	1.3	1.44596	1.43238	1.40421	1.39173
	1.5	2	2.17268	2.16342	2.13503	2.12615
100	0.7	1.3	1.61501	1.60913	1.59469	1.5890
	1.5	2	1.08850	1.98516	1.97528	1.97199
n	**λ**	**Θ**	$\mathrm{Mse}(\hat{\theta}_{B1})$	$\mathrm{Mse}(\hat{\theta}_{B2})$	$\mathrm{Mse}(\hat{\theta}_{B3})$	$\mathrm{Mse}(\hat{\theta}_{B4})$
20	0.7	1.3	0.00362	0.00284	0.00149	0.00107
	1.5	2	0.00796	0.00715	0.00488	0.00430
40	0.7	1.3	0.00053	0.00044	0.00027	0.00021
	1.5	2	0.00075	0.00067	0.00046	0.00040
100	0.7	1.3	0.00099	0.00096	0.00087	0.00084
	1.5	2	1.3e-06	2.2e-06	1.2e-06	1.1e-06

Table 2
Shows the Results of the Second Simulation Experiment of the Scale Parameter Values when w=0.2 =, a = 1.5, c = 1

n	λ	θ	$\hat{\theta}_{B1}$	$\hat{\theta}_{B2}$	$\hat{\theta}_{B3}$	$\hat{\theta}_{B4}$
20	0.7	1.3	1.50889	1.46352	1.42176	1.38335
	1.5	2	2.35772	2.32553	2.27455	2.24501
40	0.7	1.3	1.41918	1.39839	1.37957	1.36041
	1.5	2	2.15427	2.13974	2.11739	2.10346
100	1.5	2	1.60331	1.59405	1.58338	1.57443
	0.7	1.3	1.98183	1.97652	1.96872	1.96350
n	**λ**	**Θ**	$Mse(\hat{\theta}_{B1})$	$Mse(\hat{\theta}_{B2})$	$Mse(\hat{\theta}_{B3})$	$Mse(\hat{\theta}_{B4})$
20	0.7	1.3	0.00218	0.00134	0.00074	0.00035
	1.5	2	0.00640	0.00530	0.00377	0.00300
40	0.7	1.3	3.6e-04	2.4e-04	1.6e-04	9.1e-05
	1.5	2	0.00059	0.00049	0.00034	0.00027
100	1.5	2	0.00092	0.00086	0.00080	0.00075
	0.7	1.3	1.3e-06	2.2e-06	1.2e-06	1.1e-06

Table 3
Shows the Results of the Third Simulation Experiment of the Scale Parameter Values when w=0.8 =, a = 2, c = 1.5

n	λ	θ	$\hat{\theta}_{B1}$	$\hat{\theta}_{B2}$	$\hat{\theta}_{B3}$	$\hat{\theta}_{B4}$
20	0.7	1.3	1.45301	1.37437	1.43688	1.36064
	1.5	2	2.31776	2.23787	2.30597	2.22704
40	0.7	1.3	1.39338	1.35579	1.38578	1.34878
	1.5	2	2.13617	2.10003	2.13078	2.09487
100	1.5	2	1.59178	1.57223	1.58834	1.56891
	0.7	1.3	1.97520	1.96220	1.97322	1.96025
n	**λ**	**Θ**	$Mse(\hat{\theta}_{B1})$	$Mse(\hat{\theta}_{B2})$	$Mse(\hat{\theta}_{B3})$	$Mse(\hat{\theta}_{B4})$
20	0.7	1.3	0.00117	0.00028	0.00094	0.00013
	1.5	2	0.00505	0.00283	0.00468	0.00258
40	0.7	1.3	0.00021	0.00008	0.00018	0.00005
	1.5	2	0.00046	0.00025	0.00043	0.00022
100	1.5	2	0.00085	0.00074	0.00083	0.00072
	0.7	1.3	0.000006	0.000014	0.000007	0.000016

5. CONCLUSIONS

In this paper, we proposed a new Bayes estimators based on the proposed loss function. The simulation results showed Superiority of proposed Bayes estimators Bayes based on different prior distributions (non-informational and Inverse Gamma Priors) in all experiments. The optimal value of w in the proposed estimators is 0.5.

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