Pak. J. Statist.
2020 Vol. 36(4), 341-353

INVERSE RAYLEIGH MINIMAX DISTRIBUTION: PROPERTIES AND APPLICATIONS

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ABSTRACT

Here we have proposed a new probability distribution by compounding Rayleigh with Manimax distribution. Important mathematical and statistical properties of the distribution have been derived and discussed. Then, the parameter estimation is discussed using maximum likelihood method of estimation. Finally, real data set is analyzed to investigate the suitability of the proposed distribution in modeling count dataset.

KEYWORDS

Rayleigh distribution, Minimax distribution, compound distribution, Count data, Maximum likelihood estimation.

1. INTRODUCTION

Variety of probability models have been fitted to different types of real life phenomenon's depending on the nature of real life problems. There are some situations in real life like insurance claims in a locality, deaths due to COVID-19 in Europe where parameter of fitted model to such data sets will vary according to person's profession and age respectively and follow some other distribution either continuous or discrete, in these situations we use compounding technique to analyze the data. Data analysts have widely used compounding technique for analyzing data in real life as compounding technique not only brings extra elasticity in applying probability models to data but it also captures more variation from data. In the direction of using compounding probability models Sankaran (1970) formulated the Discrete Poisson-Lindley Distribution and studied its vital properties [1]. Hassan, Wani, Shafi (2020) introduced Poisson Pranav distribution and obtained its various mathematical properties along with obtaining applications of the proposed model [2]. Zeghdoudi and Grine (2017) introduced Poisson Quasi-Lindley distribution and its applications by combining Poisson and Quasi-Lindley distribution [3]. Merovci (2013) obtained transmuted Rayleigh distribution and studied its necessary properties [4]. Hassan, Dar & Para (2019) introduced a new generalization of Ishita distribution and obtained vital properties of the distribution along with applications of the proposed model [5]. Shanker (2017) obtained the discrete Poisson-Akash Distribution and studied its mathematical properties [6]. Hassan, Wani & Para (2018) obtained three parameter Quasi Lindley distribution by using weighting technique and obtained various properties of that model [7].

In this paper we propose a new compounding distribution by compounding Inverse Rayleigh distribution with Minimax distribution, as there is a need to find more flexible model for analyzing statistical data.

2. DEFINITION OF PROPOSED MODEL (INVERSE RAYLEIGH MINIMAX DISTRIBUTION)

If $Z|p \sim \mathrm{R}\left(p\right)$, where p is itself a random variable having Minimax distribution with parameter (μ,σ), then the resulting distribution obtained by marginalizing over p will be known as a compound of Inverse Rayleigh distribution with that of Minimax distribution, which is denoted by $IRMD\left(Z;\mu,\sigma\right)$. The proposed model will be discrete as parent distribution is discrete.

Theorem 2.1:

The probability mass function of a Inverse Rayleigh Minimax Distribution i.e., $IRMD\left(Z;\mu,\sigma\right)$ is given by

$$P(Z=z)=\sigma\left[B\left(\sigma,\frac{(z+1)^{-2}}{\mu}+1\right)-B\left(\sigma,\frac{z^{-2}}{\mu}+1\right)\right];z=0,1,2,3,...,;\mu>0,\sigma\geq 0\,.$$

Proof:

By the definition (2), the pmf of a IRMD $\left(Z;\mu,\sigma\right)$ can be obtained as

$$g(z\mid p)=p^{(z+1)^{-2}}-p^{(z)^{-2}} \quad , \; z=0,1,2,3,...,;p>0$$

When its parameter P follows Minimax distribution (MD) with pdf

$$h(p;\mu,\sigma)=\mu\sigma z^{\mu-1}(1-z^{\alpha})^{\sigma-1}, \qquad \mu>0,\sigma>0,0<z<1$$

We have

$$P(Z)=\int_0^1 g(z\mid p).h(p;\mu,\sigma)dp$$

$$P(Z)=\int_0^1\left(p^{(z+1)^{-2}}-p^{(z)^{-2}}\right)\mu\sigma z^{\mu-1}\left(1-z^{\alpha}\right)^{\sigma-1},z=0,1,2,3,...,;\mu>0,\sigma\geq 0 \quad (2.1)$$

Put $(1-z^{\mu})=u$

$$P(Z)=\sigma\left(\int_0^1 u^{\sigma-1}(1-u)^{\frac{(z+1)^{-2}}{\mu}}du-\int_0^1 u^{\sigma-1}(1-u)^{\frac{z^{-2}}{\mu}}du\right)$$

$$P(Z)=\sigma\left[B\left(\sigma,\frac{(z+1)^{-2}}{\mu}+1\right)-B\left(\sigma,\frac{z^{-2}}{\mu}+1\right)\right]$$

$$P(Z)=\sigma\left[\frac{\Gamma(\sigma)\Gamma\left(\frac{(z+1)^{-2}}{\mu}+1\right)}{\Gamma\left(\sigma+\frac{(z+1)^{-2}}{\mu}+1\right)}-\frac{\Gamma(\sigma)\Gamma\left(\frac{z^{-2}}{\mu}+1\right)}{\Gamma\left(\sigma+\frac{(z)^{-2}}{\mu}+1\right)}\right],\quad z=0,1,2....,\mu,\sigma>0$$

which is the p.m.f. of IRMD

The corresponding c.d.f of IRMD is obtained as:

$$F_Z(z)=\sigma B(z+1;\mu,\sigma), z=0,1,2...\text{and } \mu,\sigma>0 \tag{2.2}$$

where $\mathrm{B}(z+1;\mu,\sigma)=\frac{\Gamma\sigma\Gamma\left(\frac{(z+1)^{-2}}{\mu}+1\right)}{\Gamma\left(\sigma+\frac{(z+1)^{-2}}{\mu}+1\right)}$.

3. SPECIAL CASES

Case 1:

If we put $\sigma,\mu=1$ the IRMD reduces to Discrete Rayleigh Uniform distribution with pmf as $P_1(Z=z)=\sigma[B(1,(z+1)^{-2}+1)-B(1,z^{-2}+1)]$

$$P_1(Z=z)=\left[\frac{z^{-2}-(z+1)^{-2}}{(z^{-2}+1)[(z+1)^{-2}+1]}\right],\qquad z=0,1,2...,$$

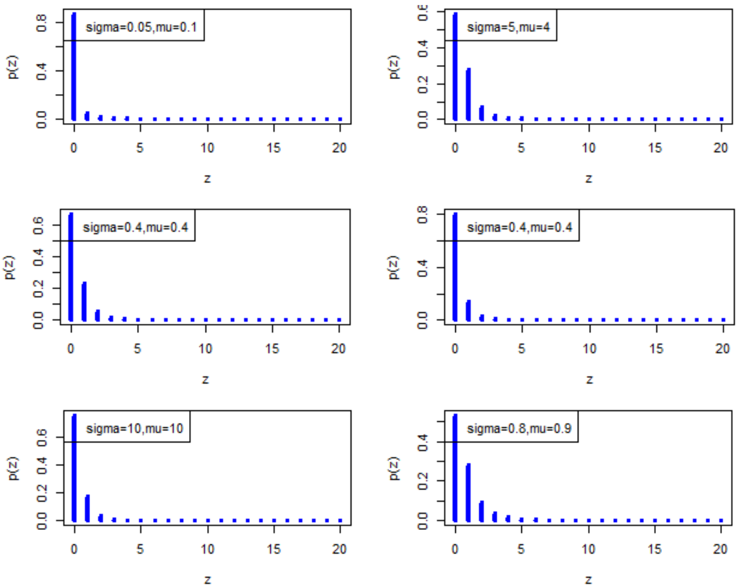

Figure 1: pmf Plot for different Values of μ and σ

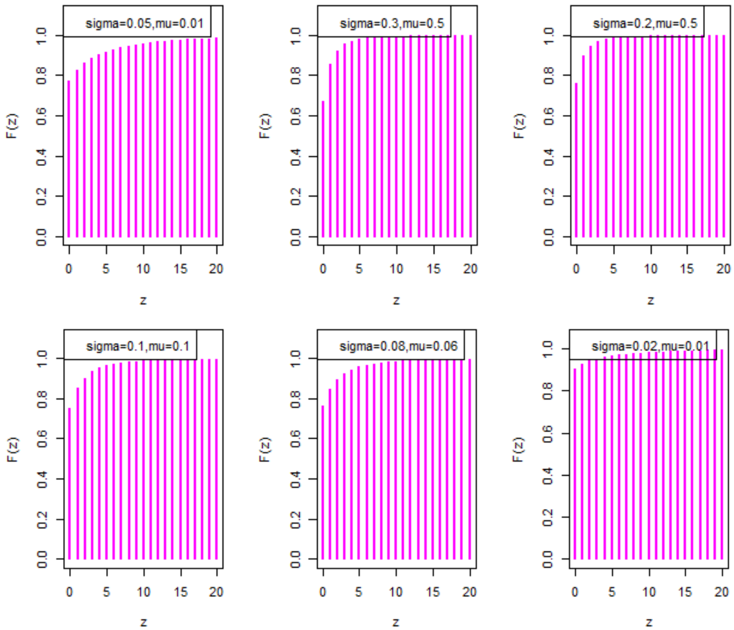

Figure 2: cdf Plot for different Values of μ and σ

4. RELIABILITY ANALYSIS

4.1 Reliability Function *R*(*z*):

The reliability function is defined as the probability that beyond a certain time period system will function. The reliability function or the survival function of IRMD is given as

$$R(z,\mu,\sigma)=1-\sigma B\left(\sigma,\frac{z^{-2}}{\mu}+1\right), z=0,1,2..,\mu,\alpha>0$$

where $B\left(\sigma,\frac{z^{-2}}{\mu}+1\right)=\dfrac{\Gamma(\sigma)\Gamma\left(\frac{z^{-2}}{\mu}+1\right)}{\Gamma\left(\sigma+\frac{(z)^{-2}}{\mu}+1\right)}$.

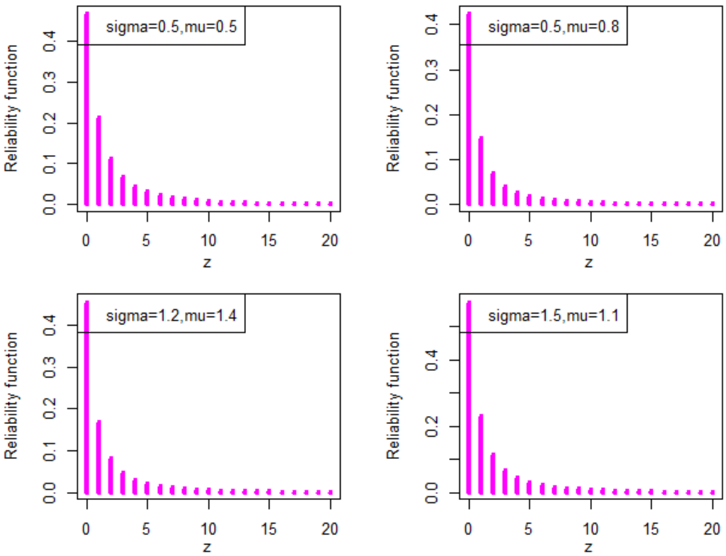

Figure 3: Reliability Function Plot for different Values of μ and σ

4.2 Hazard Function:

The hazard function is also known as hazard rate is given as:

$$\text{H.R} \quad = \text{h}(z,\mu,\sigma) = \frac{f(z;\mu,\sigma)}{R(z;\mu,\sigma)}$$

$$=\frac{\sigma\left[B\left(\sigma,\frac{(z+1)^{-2}}{\mu}+1\right)-B\left(\sigma,\frac{z^{-2}}{\mu}+1\right)\right]}{1-\sigma B\left(\sigma,\frac{z^{-2}}{\mu}+1\right)}, z=0,1,2..,\mu,\sigma>0.$$

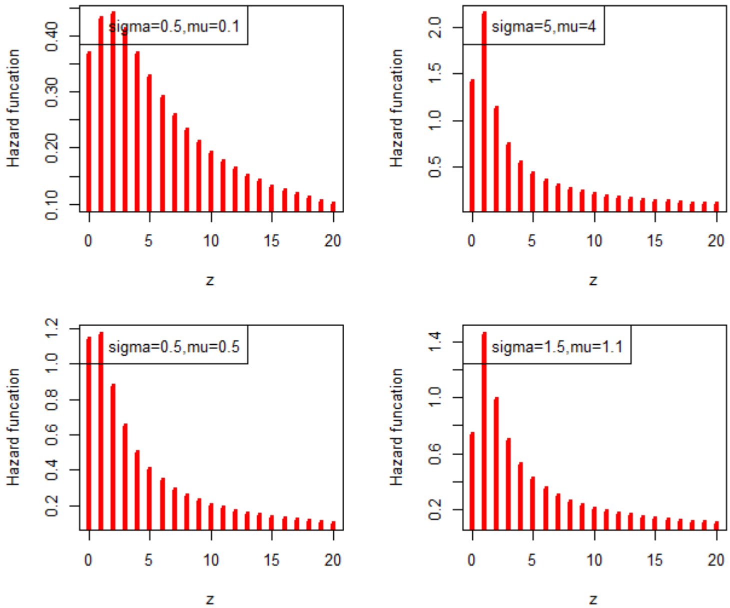

Figure 4: Hazard Rate Plot for different Values of μ and σ

4.3 Reverse Hazard Rate and Mills Ratio

The reverse hazard rate and the mills ratio of IRMD are respectively given as:

$$\text{R.H.R} = h_r(z,\mu,\sigma) = \frac{\sigma\left[B\left(\sigma,\frac{(z+1)^{-2}}{\mu}+1\right)-B\left(\sigma,\frac{z^{-2}}{\mu}+1\right)\right]}{1-\sigma B\left(\sigma,\frac{z^{-2}}{\mu}+1\right)}$$

$$\text{Mills Ratio} = \frac{1-\sigma B\left(\sigma,\frac{z^{-2}}{\mu}+1\right)}{\sigma\left[B\left(\sigma,\frac{(z+1)^{-2}}{\mu}+1\right)-B\left(\sigma,\frac{z^{-2}}{\mu}+1\right)\right]}.$$

4.4 Second Rate of Failure Rate

$$h_s(z)=\log\left(\frac{R(z)}{R(z+1)}\right)=\log\left(\frac{1-\sigma B\left(\sigma,\frac{z^{-2}}{\mu}+1\right)}{1-\sigma B\left(\sigma,\frac{(z+1)^{-2}}{\mu}+1\right)}\right), z=0,1,2...,\mu,\sigma>0.$$

5. MOMENT GENERATING FUNCTION (MGF) AND PROBABILITY GENERATING FUNCTION (PGF) OF INVERSE RAYLEIGH MANIMAX DISTRIBUTION

(a) MGF of Inverse Rayleigh Manimax Distribution is given by

$$M_z(t) = \sum_{z=0}^{\infty} e^{tz} p(z)$$

$$M_z(t) = \sum_{z=0}^{\infty} e^{tz} \left[\left(1 - \sigma B\left(\sigma, \frac{z^{-2}}{\mu} + 1 \right) \right) - \left(1 - \sigma B\left(\sigma, \frac{(z+1)^{-2}}{\mu} + 1 \right) \right) \right]$$

$$M_z(t) = \sum_{z=0}^{\infty} e^{tz} [\psi(z;\sigma,\mu) - \psi(z+1;\sigma,\mu)$$

where $\psi(z;\sigma,\mu) = 1 - \sigma\left(\sigma, \frac{z^{-2}}{\mu} + 1 \right)$

$$M_z(t) = \begin{pmatrix} \psi(0:\sigma,\mu) + e^t\psi(1;\sigma,\mu) + e^{2t}\psi(2;\sigma,\mu) + e^{3t}\psi(3;\sigma,\mu) \\ + \ldots\ldots - \left\{ \psi(0;\sigma,\mu) + e^t\psi(1;\sigma,\mu) + e^{2t}\psi(2;\sigma,\mu) + \ldots\ldots \right\} \end{pmatrix}$$

$$M_z(t) = \psi(0:\sigma,\mu) + \left(e^t - 1\right)\psi(1;\sigma,\mu) + \left(e^{2t} - e^t\right)\psi(2;\sigma,\mu) + \left(e^{3t} - e^{2t}\right)\psi(3;\sigma,\mu)$$

$$M_z(t) = 1 + \sum_{i=1}^{\infty} \left(e^{zt} - e^{(z-1)t} \right) \psi(z;\sigma,\mu) \tag{5.1}$$

Differentiating equation 5.1 with respect to t

$$M_z^{(r)}(t) = \sum_{z=1}^{\infty} \left(z^r e^{zt} - (z-1)^r e^{(z-1)t} \right) \psi(z;\sigma.\mu)$$

The first four moments of IRMD are given by

$$\mu_1{}' = \sum_{z=1}^{\infty} \psi(z;\sigma,\mu)$$

$$\mu_2{}' = \sum_{z=1}^{\infty} (2z-1)\,\psi(z;\sigma,\mu)$$

$$\mu_3{}' = \sum_{z=1}^{\infty} \left(3z^2 - 3z + 1\right)\psi(z;\sigma,\mu)$$

$$\mu_4{}' = \sum_{z=1}^{\infty} \left(4z^3 - 6z^2 + 4z - 1\right)\psi(z;\sigma,\mu)$$

(b) PGF of IRMD Iis given as

$$H_{[z]}(t)=\sum_{z=0}^{t} t^{z}\, p(z)$$

$$H_{[z]}(t)=\sum_{z=0}^{t} t^{z}\left(\sigma\left[B\left(\sigma,\frac{(z+1)^{-2}}{\mu}+1\right)-B\left(\sigma,\frac{z^{-2}}{\mu}+1\right)\right]\right)$$

$$H_{[z]}(t)=\sum_{i=1}^{\infty} t^{z}\left[\psi(z;\sigma,\mu)-\psi(z+1:\sigma,\mu)\right]$$

where $\psi(z;\sigma,\mu)=\left\{1-\sigma B\left(\sigma,\frac{z^{-2}}{\mu}+1\right)\right\}$

$$H_{[z]}(t)=\psi(0,\sigma,\mu)+(t-1)\psi(1,\sigma,\mu)+t(t-1)\psi(2,\sigma,\mu)+t^{2}(t-1)\psi(3,\sigma,\mu)+...$$

$$H_{z}(t)=1+(t-1)\sum_{z=1}^{\infty} t^{z-1}\psi(z;\sigma,\mu)$$

Differentiating $H_{[z]}(t)$ with respect t

$$H_{z}(t)=\sum_{i=1}^{\infty}\left((t-1)(z-1)t^{z-2}+t^{z-1}\right)\psi(z;\sigma,\mu)$$

$$H'_{z}(t)=\sum_{i=1}^{\infty}\left(t^{z-2}(zt-z+1)\right)\psi(z;\sigma,\mu)$$

$$H''_{z}(t)=\sum_{i=1}^{\infty}(z-1)t^{z-3}\left\{(t-1)(z-2)+2t\right\}\psi(z;\sigma,\mu)$$

At $t=1$, $H'_{[z]}(t),H''_{[z]}(t)$ gives first and second factorial moments

$$E(z)=\sum_{i=1}^{\infty}\psi(z;\sigma,\mu)$$

$$E\left(z^{2}\right)=\sigma\sum_{i=1}^{\infty}(2z-1)\psi(z;\sigma,\mu)\ .$$

6. ORDER STATISTICS

Let $Z_{(1)},Z_{(2)},Z_{(3)}....,Z_{(n)}$ be the ordered statistics of the random sample $Z_1,Z_2,Z_3,....Z_n$ drawn from the IRMD with cdf $F_Z\left(z\right)$ and pmf $P_Z\left(z\right)$, then the pmf of rth order statistics $Z_{(r)}$ is given by:

$$f_{z(r)}\left(z,\mu,\sigma\right)=\frac{n!}{\left(r-1\right)!\left(n-r\right)!}P\left(z\right)\left[F\left(z\right)\right]^{r-1}\left[1-F\left(z\right)\right]^{n-r}.$$

$$r = 1, 2, 3,\ldots, n \qquad (6.1)$$

Put the values cdf and pmf in equation (6.1) we get the rth statistics of IRMD is given as

$$f_{(r)}\left(z,\mu,\sigma\right)=\frac{n!}{(r-1)!(n-r)!}\sigma\left[B\left(\sigma,\frac{(z+1)^{-2}}{\mu}+1\right)-B\left(\sigma,\frac{z^{-2}}{\mu}+1\right)\right]$$

$$\left[\sigma B(z+1;\mu,\sigma)\right]^{r-1}\left[1-\sigma B(z+1;\mu,\sigma)\right]^{n-r}$$

Then, the pmf of first order $Z_{(1)}$ IRMD is given by:

$$f_1\left(Z;\mu,\sigma\right)=n\sigma\left[B\left(\sigma,\frac{(z+1)^{-2}}{\mu}+1\right)-B\left(\sigma,\frac{z^{-2}}{\mu}+1\right)\right]\left[1-\sigma B(z+1;\mu,\sigma)\right]^{n-1}.$$

And the pmf of nth order $Z_{(n)}$ IRMD is given as:

$$f_{(n)}\left(z,\mu,\sigma\right)=n\,\sigma\left[B\left(\sigma,\frac{(z+1)^{-2}}{\mu}+1\right)-B\left(\sigma,\frac{z^{-2}}{\mu}+1\right)\right]\left[\sigma B(z+1;\mu,\sigma)\right]^{n-1}.$$

7. ESTIMATION OF PARAMETERS

In this section, we estimate the parameters of the IRMD by using methods of maximum likelihood estimation.

7.1 Method of Maximum Likelihood Estimation

This is one of the most efficient and simple method for estimating the different parameters of the distribution. Let $Z_1, Z_2, Z_3, ..., Z_n$ be the random sample of size n draw from IRMD, then the likelihood function of IRMD is given as

$$\log L = n\log\sigma - +\sum_{i=1}^{n}\log\left(\left[B\left(\sigma,\frac{(z+1)^{-2}}{\mu}+1\right)-B\left(\sigma,\frac{z^{-2}}{\mu}+1\right)\right]\right)$$

$$\frac{\delta}{\delta\mu}\log L=\sum_{i=1}^{n}\left[\frac{\frac{\delta}{\delta\mu}\left[\frac{\Gamma(\sigma)\Gamma\left(\frac{(z+1)^{-2}}{\mu}+1\right)}{\Gamma\left(\sigma+\frac{(z+1)^{-2}}{\mu}+1\right)}-\frac{\Gamma(\sigma)\Gamma(\frac{z^{-2}}{\mu}+1)}{\Gamma\left(\sigma+\frac{(z)^{-2}}{\mu}+1\right)}\right]}{\left[\frac{\Gamma(\sigma)\Gamma\left(\frac{(z+1)^{-2}}{\mu}+1\right)}{\Gamma\left(\sigma+\frac{(z+1)^{-2}}{\mu}+1\right)}-\frac{\Gamma(\sigma)\Gamma(\frac{z^{-2}}{\mu}+1)}{\Gamma\left(\sigma+\frac{(z)^{-2}}{\mu}+1\right)}\right]}\right]=0$$

$$\frac{\delta}{\delta\sigma}=\frac{n}{\sigma}+\sum_{i=1}^{n}\left[\frac{\frac{\delta}{\delta\sigma}\left[\frac{\Gamma(\sigma)\Gamma\left(\frac{(z+1)^{-2}}{\mu}+1\right)}{\Gamma\left(\sigma+\frac{(z+1)^{-2}}{\mu}+1\right)}-\frac{\Gamma(\sigma)\Gamma(\frac{z^{-2}}{\mu}+1)}{\Gamma\left(\sigma+\frac{(z)^{-2}}{\mu}+1\right)}\right]}{\left[\frac{\Gamma(\sigma)\Gamma\left(\frac{(z+1)^{-2}}{\mu}+1\right)}{\Gamma\left(\sigma+\frac{(z+1)^{-2}}{\mu}+1\right)}-\frac{\Gamma(\sigma)\Gamma(\frac{z^{-2}}{\mu}+1)}{\Gamma\left(\sigma+\frac{(z)^{-2}}{\mu}+1\right)}\right]}\right]=0$$

The above equations can be solved numerically by using R software (3.5.2).

8. APPLICATIONS OF INVERSE RAYLEIGH MINIMAX DISTRIBUTION

In this section, we fit our proposed distribution to a dataset which are already in statistical literature**.**so, as to illustrate our claim that the proposed model fits well when compared to other competing models. The data set is given in Table 1:

Table 1
Dataset Representing Mammalian Cytogenetic Dosimetry Lesions in Rabbit Lymphoblast induced by Streptonigrin (NSC-45383), Exposure-90 $\mu g\,/\,kg$

Counts	0	1	2	3	4	5	6
Actual	155	83	33	14	11	3	1

We have fitted Inverse Rayleigh Minimax (IRMD), Zero Inflated Poisson (ZIPD), Poisson distribution (PD), Negative Binomial (NB), Discrete Rayleigh (DR), Discrete Weibull (DW), Poisson Weighted Lindley distribution (PWLD), Generalized Poisson Lindley distribution (GPLD) and Discrete Generalized Inverse Weibull Distribution (DGIWD) to the dataset given in Table 1. The calculated figures are given in Table 2. Based on the chi-square (p-value), we observe that Inverse Rayleigh Minimax distribution provides a satisfactorily better fit for the data set as compared to other competitive models. The parameters are estimated by using the ML method. We have analyzed the data using R software (3.5.2). Parameter estimates along and model function of the fitted distributions are given in Table 2.

Table 2
Fitted Proposed Distribution and other Competing Models to a Data Set given in Table 1

Z	Observed Frequency	PD	NB	ZIP	DWD	DR	PWLD	DGIW	GPLD	IRMD
0	155	127.8	155.1	155	155	82	155.9	153	155.3	155
1	83	109	80.6	71.9	80.75	134.35	80	94.5	80.1	85.6
2	33	46.5	36.7	45.65	36.8	66.7	36.7	26	36.9	29.65
3	14	13.2	15.9	19.3	16	15.15	15.9	10.45	16	12.25
4	11	2.8	6.7	6.15	6.75	1.7	6.7	5.25	6.7	6
5	3	0.5	2.8	1.55	2.8	0.09	2.7	3	2.8	3.4
6	1	0.2	2.2	0.45	1.9	0.01	2.1	7.8	2.2	8.1
Total	300									
Chi-square		24.96	1.60	10.87	1.91	108.8	1.78	4.77	1.69	1.06
d.f		2	2	2	2	2	2	1	2	2
Parameter Estimate		$\hat{\mu} = 0.85$	$\hat{a} = 1.56$ $\hat{b} = 1.33$	$\hat{\mu} = 1.26$ $\hat{\sigma} = 0.32$	$\hat{q} = 0.48$ $\hat{g} = 1.08$	$\hat{p} = 0.72$	$\hat{\mu} = 1.82$ $\hat{\sigma} = 1.16$	$\hat{a} = 1.8$ $\hat{b} = 0.43$ $\hat{t} = 0.04$	$\hat{\alpha} = 1.80$ $\hat{\beta} = 1.18$	$\hat{\mu} = 0.87$ $\hat{\sigma} = 0.82$
p-value		0.00	0.44	0.004	0.38	0	0.41	0.02	0.42	0.587

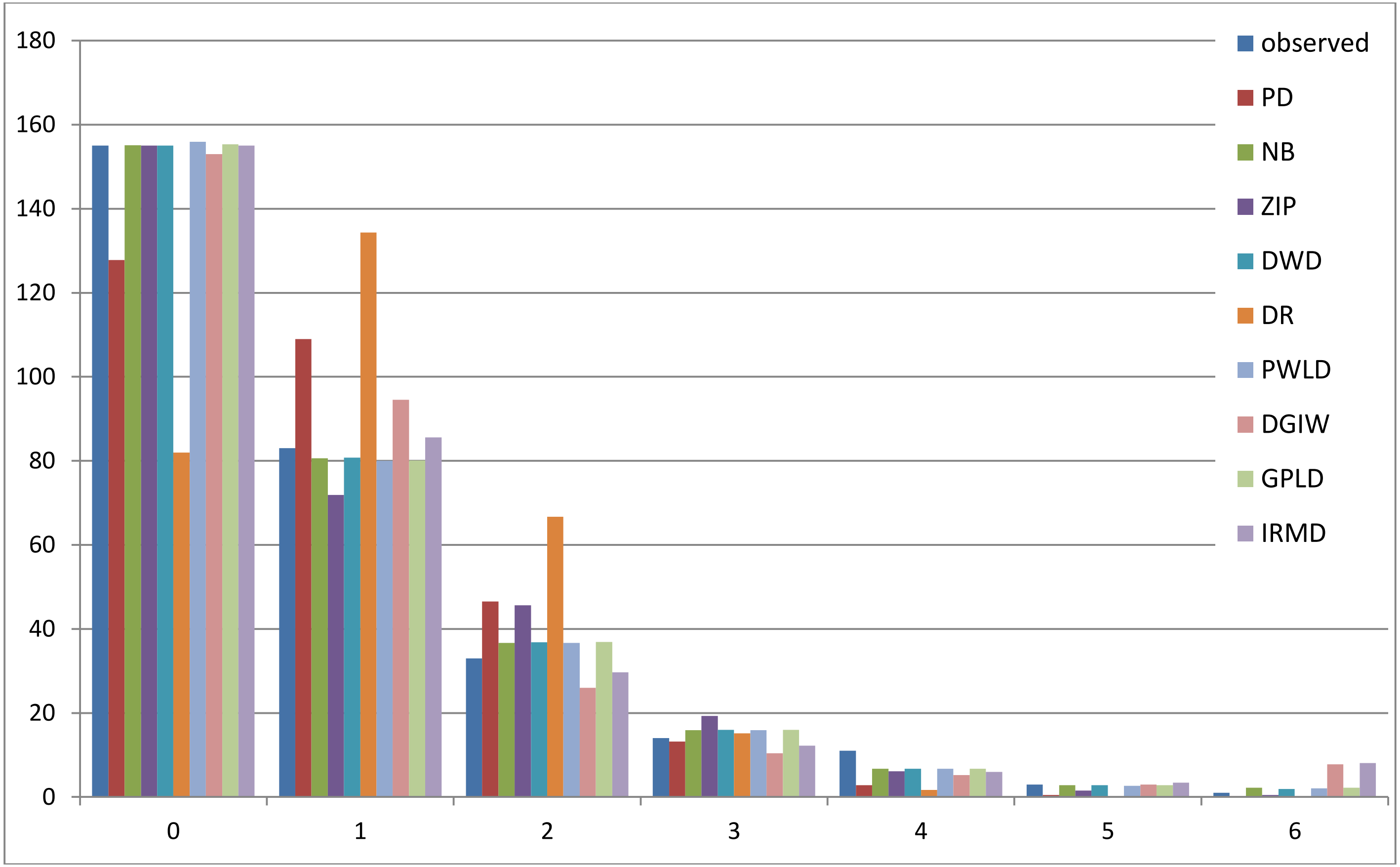

Figure 5: Graphical Overview of the Fitted Data Models given in Table 2

CONCLUSION

A new probability distribution is introduced using compounding technique. Statistical properties of the proposed model are studied and application in handling count dataset representing Mammalian Cytogenetic dosimetry Lesions in Rabbit Lymphoblast counts is analyzed.

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