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RANDOM MATRICES AND APPLICATION TO THE PARIS STOCK MARKET PRICES

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ABSTRACT

The spectral law of random matrices developed by Wigner and Wishart converges to a deterministic law when the dimension of the matrix tends to infinity. In this paper, the objective is to apply this theory in the first degree on the returns of companies from the Paris stock exchange. This allows the theory to be approved through the use of real cases. Also, this study attempts to compare the empirical distribution of the eigenvalues resulting from the minimization of the covariance matrix (Markowitz's theory) with the law of random matrices.

KEYWORDS

Random matrices, Stock market prices of Paris; R software; Markowitz Theory of Portfolio.

INTRODUCTION

One of the most interesting areas of probability theory is the random matrix theory. It has recently been shown that ideas and techniques of random matrices can be applied to key problems.

The first appearance of Random Matrices started with the statistician Wishart who introduced random matrices in 1928. Wishart [12] wanted to study a large table of browsed data, the main question was to identify the real information (Principal Component Analysis). Suggested to observe the eigenvalues to analyze the vector axes corresponds to the large eigenvalue.

The second appearance was with Wigner who proposed in 1956 to approximate the hamilltonian of heavy nuclei by a random matrix, the spectrum represents the energy levels. We are looking to study in particular the spacings and find the Hamiltonian class of heavy nuclei.

Montgomery in 1973 [13], made the link between the random matrices and the Zeta function, The Montgomery conjecture makes it possible to link certain behaviors of random matrices, with that of the zeros of the Riemann Zeta function.

1. THEORIES OF RANDOM MATRICES

1.1 Gaussian Sets of Random Matrices

There are several classic sets of random matrices [2], the study of which has been very thorough:

- G.U.E, Gaussian Unitary Ensemble, based on Hermitian matrices.
- G.O.E, Gaussian Orthogonal Ensemble, based on real symmetric matrices.
- G.S.E, Gaussian Symplectic Ensemble, based on real quaternion matrices.

These sets were introduced by Wigner [5] for the study of the theory of nuclear spectra: The G.O.E. was introduced for time invariant systems, the G.U.E. for non-invariant time inversion systems, and the G.S.E. for systems with spin. In his case study, Wishart was led to study random Hermitian matrices, as well as his eigenvaluest the case of the G.U.E.

1.2 Wigner's Theorem

Let $X_N = \left(\frac{X_{ij}}{\sqrt{N}}\right)$ be a random matrix of size N by N whose sub-diagonal entries are independent and identically distributed and such that $Var\left(X_{ij}\right) = 1$. The distribution of eigenvalues converges towards the semi-circular distribution, for any continuous function f we will have:

$$\lim_{N\to+\infty} \frac{1}{N}\sum_{i=1}^{N} f(\lambda i) = \frac{1}{2\pi}\int_{-2}^{2} f(t)\sqrt{4-t^2}\,dt \tag{1}$$

Case study:

We consider the continuous and bounded function f defined by:

$$\begin{cases} f(x) = 1\ x\ \in [-2,2] \\ f(x) = 0\ else \end{cases}$$

We use the parity of the function to have:

$$\int_{-2}^{2} \sqrt{4-x^2}dx = 2\int_{-2}^{2} \sqrt{4-x^2}dx. \tag{2}$$

We change the variable $\sin(\theta) = \frac{x}{2}$:

$$\frac{1}{\pi}\int_{-2}^{2} \sqrt{4-x^2}dx = \frac{1}{\pi}\int_{-2}^{2} \sqrt{4-sin(\theta)^2}\, cos(\theta)\, d\theta \tag{3}$$

$$\begin{aligned} \frac{1}{\pi}\int_{0}^{\frac{\pi}{2}} COS(\theta)^2\ d\theta &= \frac{4}{\pi}\int_{0}^{\frac{\pi}{2}} \frac{(1+COS(2\theta)}{2} d\theta \\ &= \frac{4}{\pi}\int_{0}^{\frac{\pi}{2}} \frac{1}{2} d\theta + \frac{4}{\pi}\int_{0}^{\frac{\pi}{2}} \frac{COS(2\theta)}{2} d\theta \end{aligned} \tag{4}$$

Then

$$\frac{1}{2\pi}\int_{0}^{2} \sqrt{4-x^2}dx = \frac{2}{\pi}\left(\frac{\pi}{2} + sin(\pi)\right) = 1.$$

Finally

$$\frac{1}{N}\sum_{i=1}^{N} f(\lambda i) = \frac{1}{2\pi}\int_{-2}^{2} \sqrt{4-x^2}\,dx. \qquad (5)$$

1.3 The Law of the Semicircle

Wigner's theorem concerns matrices W_N, of dimension (N×N), of which all the inputs (W_{IJ}) on and above the diagonal are real (the theorem extends to complex inputs), independent and identically distributed and summable squares. We complete the entries under the diagonal so that the matrix W_N is symmetrical: $W_N = {W_N}^T$ where ${W_N}^T$ is the transposed matrix of W_N. In the case of complex entries, we will impose the condition of symmetry $W_N = {W_N}^*$, where ${W_N}^*$ represents the conjugate transpose of the matrix W_N. We will call such a matrix of Wigner. If the inputs are centered with variance 1, and if we normalize the matrix with $\left(\frac{1}{\sqrt{N}}\right)$. Then the spectrum of W_N is organized around the law of the semicircle [7] in the sense that the histogram of its eigenvalues will follow the law of the semicircle:

$$\forall x \in [-2,2]\ F(x) = \frac{1}{2\pi}\sqrt{4-x^2}.$$

1.4 Marchenko Pastur's Theorem

Marchenko-Pastur's theorem [4] concerns large covariance matrices. Let X be a matrix $(T \times N)$ whose all inputs are real or complex (i.i.d), centered with variance σ^2 and summable square. We consider the matrix $\frac{1}{N}XX^T$, when $T,N \to \infty$ and $\frac{\mathrm{T}}{\mathrm{N}} = Q \geq 1$ fixed, the Marchenko-Pasturle distribution spectrum of the matrix (respectively of the matrix $\frac{1}{N}XX^{*}$ in the case of complex inputs) goes organize according to the distribution of Marcenko-Pastur whose density is given by:

$$\forall x \in [\lambda\min, \lambda\max]\ F_{MP}(x) = \frac{\sqrt{(\lambda_{\max}-x)(x-\lambda_{\min}}}{2\pi c x}$$

$$\lambda_{\max} = \sigma^2\left(1+\sqrt{c}\right)^2 \text{ and } \lambda_{\min} = \sigma^2\left(1-\sqrt{c}\right)^2.$$

1.5 Tracy Widom's Law

Law of λmax of the largest of the eigenvalues Wigner's theorem allows us to note that the largest of the eigenvalues converge almost surely to 2, for the matrices verifying the hypotheses of the theorem. What we would like is to have the speed of convergence of the largest of the eigenvalues towards 2. The law of the largest of the eigenvalues can be approached in a much more precise way, by to the law of Tracy-Widom [8].

Theorem:

Let $Y \in M_{m,n}$ be the matrix composed of independent and identically distributed elements with zero means and unit variance. Let max be the largest eigenvalue of $\frac{1}{N}YY^T$ such that:

$$C = \frac{\mathrm{m}}{\mathrm{n}}, b = (1+\sqrt{\mathrm{c}})2, \sigma = (1+\sqrt{\mathrm{c}})\mathrm{c}^{\frac{4}{3}},$$

then for all x belongs to a compact set:

$$P\left(m^{\frac{2}{3}} \times \frac{(\lambda max - b)}{\sigma} \geq x\right) \xrightarrow[\mathrm{n}\to+\infty]{} F_{TW}(x).$$

$F_{TW}(x)$: represents the distribution function of the law of Tracy-widom [11] defined as follows:

$$F_{TW}(x). = exp\left(-\int_t^{+\infty} q(x)(x-t)^2\, dx\right)$$

With q(.) The function solution of the differential equation below:

$$q''(x) = xq\ddot{(x)} + 2q(x)^2. \tag{6}$$

$$q(x) \sim A_i(x). \tag{7}$$

$$A_i(x) = \frac{1}{2\pi}\int \exp\left(ixt + \frac{it^3}{3}\right) dt. \tag{8}$$

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1.7 Relationship between the Law of the Semicircle and the Number of Catalan

The Catalan numbers [1] form a sequence of natural numbers that appear and help solve counting problems. This nomination is in honor of the Belgian Catalan mathematician. He also represents the number of oriented trees rooted with n stops. The even moments of Wigner's law on [-2, 2] are given by the Catalan numbers. The calculation indeed gives us:

$$\forall \mathrm{n} \geq 0 M_{2n} = C_n = \frac{C_{2n}^n}{(n+1)} = \frac{2n!}{(n+1)!\,n!} \text{ and } M_{2n+1} = 0.$$

Demonstration:

$$M_{2n} = \frac{1}{2\pi}\int_{-2}^{2} x^{2n}\sqrt{(4-x^2}\,dx = \frac{1}{\pi}\int_0^2 x^{2n}\sqrt{(4-x^2}\,dx$$

(because $x \to x^{2n}\sqrt{(4-x^2}$ is an even function).

$$M_{2n} = \frac{1}{\pi}\int_0^2 2x^{2n}\sqrt{(1-\frac{x^2}{2}}\,dx$$

We change the variable $\sin(\theta) = \frac{x}{2}$:

$$M_{2n} = \frac{2^{2n+2}}{\pi}\int_0^{\frac{\pi}{2}} \sin(\theta)^{2n+2}\,d\theta.$$

We perform integration by part we find:

$$M_{2n} = \frac{2^{2n+2}}{\pi(2n+1)} \int_0^{\frac{\pi}{2}} \sin(\theta)^{2n}\, d\,\theta - \frac{2^{2n+2}}{\pi(2n+1)} \int_0^{\frac{\pi}{2}} \sin(\theta)^{2n} \cos(\theta)^2 d\theta.$$

Then

$$M_{2n} = \frac{2(2n-1)}{n+1} M_{2n-2}.$$

Catalan numbers satisfy the same recurrence relation:

$$C_n = \frac{1}{n+1} = \frac{(2n)!}{n! \times n!}$$

On another side $C_n = \frac{2(2n-1)}{(n+1)}$ Cn-1 and $C_0 = 1$.

The Catalan numbers satisfy the recurrence relation then:

$$M_{2n} = C_n = \frac{(2n)}{(n+1)! \times n!}$$

We now place ourselves in the case of odd moments:

$$M_{2n+1} = \frac{1}{2\pi} \int_{-2}^{2} x^{2n+1} \sqrt{(4-x^2}\, dx$$

$$M_{2n+1} = \frac{1}{2\pi} \left(\int_{-2}^{0} x^{2n+1} \sqrt{(4-x^2}\, dx + \frac{1}{2\pi} \int_0^2 x^{2n+1} \sqrt{(4-x^2}\, dx \right)$$

(because $x \rightarrow x^{2n+1}\sqrt{(4-x^2}$ is an odd function).

Then

$$M_{2n+1} = \frac{1}{2\pi} \left(\int_0^2 y^{2n+1} \sqrt{(4-y^2}\, dy - \frac{1}{2\pi} \int_0^2 y^{2n+1} \sqrt{(4-y^2}\, dx \right)$$

(we ask the change of variable y = –x).

Finally M_{2n+1}=0.

So the number of Catalans satisfies the relationship:

$$M_{2n} = C_n = \frac{C_{2n}^{n}}{(n+1)} = \frac{2n!}{(n+1)!\, n!} \text{ and } M_{2n+1} = 0.$$

2. APPLICATION OF THE IMPORTANT LAWS OF RANDOM MATRICES ON TH RETURNS OF THE PARIS STOCK EXCHANGE

In this part, we will to work proceed the closing price of 400 companies over a period of 3 years, then we will calculate the companies' returns, then we will normalize the returns by subtracting the average and dividing by the standard deviation, to have $X_1^* \ldots X_N^*$. Which follows a Normal law of expected value 0 and variance 1. To apply the theorems of random matrices all the calculations were made with the R language, the codes are available in this paper. All the data come from the site of the Stock Exchange from Paris.

2.1 Treatment of Missing Values

Missing observations can be a problem in analysis, and sometimes series measures cannot be calculated if values are missing in the series. Sometimes the value of a given observation is simply not known. Missing data at the start or end of a series is fine, it just reduces the useful length of the series. Holes in the middle of a series (integrated missing data) can be a much more serious problem. The extent of the problem depends on the analysis procedure used. In our case we chose to replace the missing values by the average of the neighboring points by the tool replace the missing values on the SPSS Software. This software allows us to replace the missing values by:

- The Average of the Series.
- The Average of Neighbor Points.
- Linear trend in Point.
- Median of missing values.
- Linear interpolation.

2.2 Definition of Return

We assume that we can invest in N risky assets, we start investing at t = 0 and we look at the gains at t = 1. We note P(t_i) the closing price of the asset (t: time) t ∈[0, 1]. We then define the return on the asset by:

$$R_i = \frac{P_i(1)}{P_i(0)} - 1 \text{ and } Yi = R_i + 1.$$

By noting = $(Y_1 \ldots . Y_N)$ we suppose that the universe on which we invest is defined by the following parameters, known at t = 0: $\mu = E[Y]$, $\Omega = Var(Y)$.

2.3 Application of Marchenko Pastur's Theory on the Corporate Returns

Let N be real random variables $X_1 \ldots .: X_N$ centered and of variance σ^2. We observe T realizations of these random variables, which are summarized by M ∈ $M_{N,T}(R)$. The classical estimator of the covariance variance matrix is: $C = \frac{1}{T}$ M M^T.

In our case we work on reduced centered returns, which amount to wanting to estimate the correlation matrix which we will denote by C. We have in our case (σ^2= 1), we have 400 actions, over a period of 400 days which gives us dates.

The Figure represents the distribution of the Marchenko-Pastur density as a function of the eigenvalues for the case (Q = 1) and (Q = 4), (Q = 10):

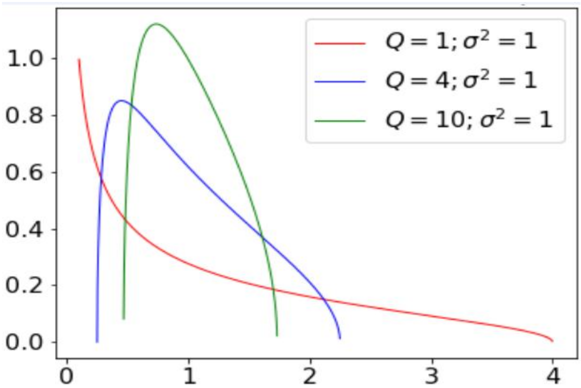


Figure 1: The Distribution of the Marchenko-Pastur Density for Different Values of Q

Figure 1 represents the law of Marchenko-Pastur for different values of Q, in order to illustrate this phenomenon. The more the magnitude of $\frac{1}{Q}$ approaches 0, that is to say that Q tends towards infinity, the closer the distribution is to 1. Thus it is possible to obtain a probability distribution with deterministic support bounded for their eigenvalues; the support depends only on the quantity Q and the variance.

When T=N we'll have:

$Q = 1, \lambda_{\max} = 4$ and $\lambda_{\min} = 0.$

$$F_{MP}(x) = \frac{\sqrt{(4-x)x}}{2\pi x} \quad if \ \ \forall x \in [0,4]$$

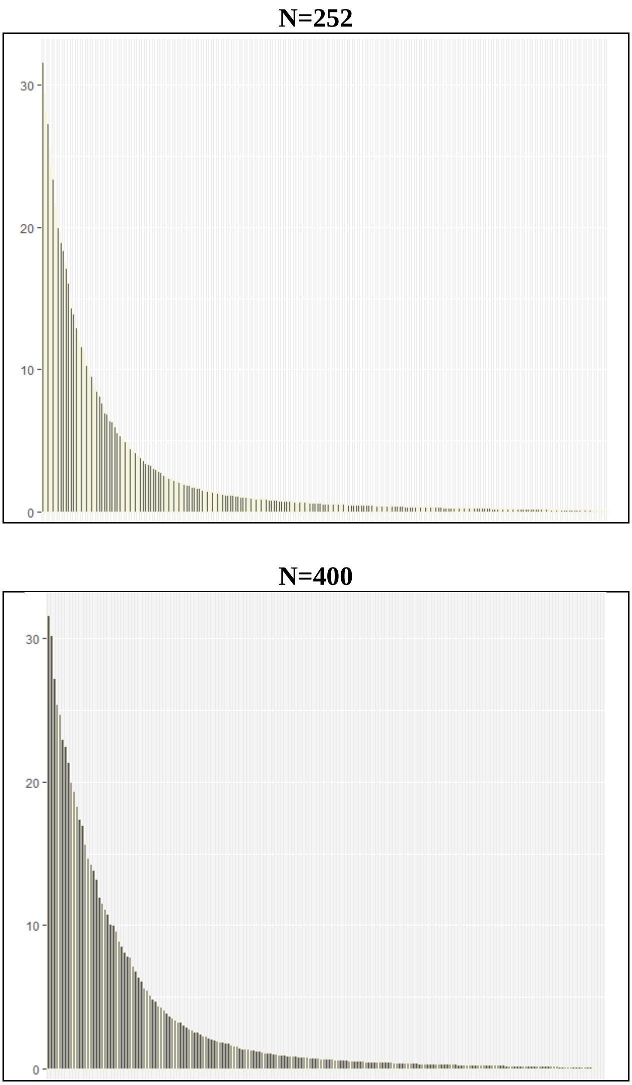


Figure 2: The Distribution of the Density of the Semicircle according to the Eigenvalues of 252 and 400 Companies

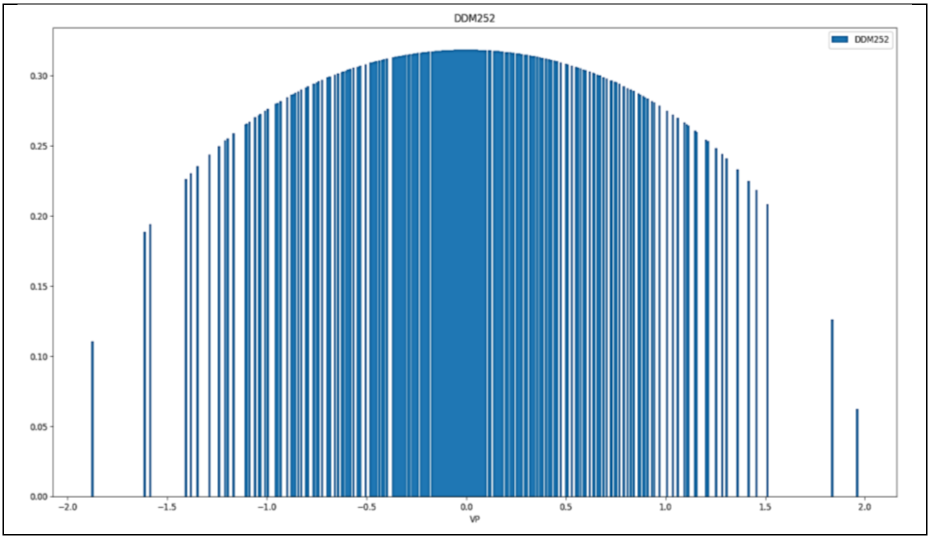
Figure 2 shows for the same value of $Q = 1$, the different histograms obtained for the eigenvalues of the returns of zero mean and unit variance The eigenvalues of the matrix $\frac{MM^T}{\sqrt{N}}$ are distributed approximately like the law by Marchenko-Pastur and this is all the more apparent as the dimension n is large.

2.4 Application of the Semicircle Theory on Corporate Returns

The law of the semicircle in the sense that the histogram of its eigenvalues follows the law of the semicircle such that:

$$F(x) = \frac{1}{2\pi}\sqrt{4 - x^2} \text{ if } x \in [-2,2].$$

N=252



N=400

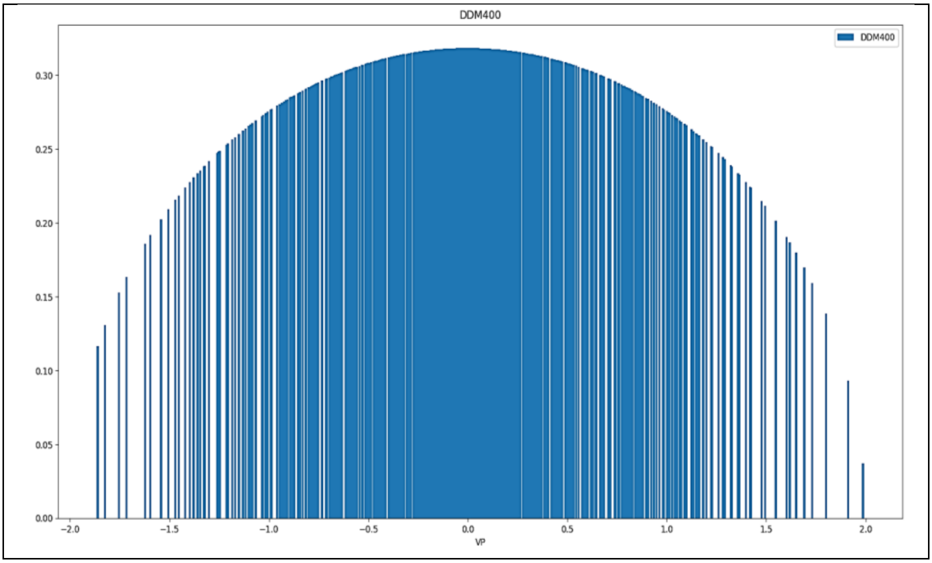


Figure 3: The Distribution of the Density of the Semicircle according to the Eigenvalues of 252 and 400 Companies

The eigenvalues of the matrix $\left(\frac{M+M^t}{\sqrt{N}}\right)$ are distributed approximately like the law of the semicircle on the interval [-2, 2], and this is all the more clear as the dimension n is large. This law to recognize that, that the large eigenvalues will be rarer, and that on the contrary it melts more numerous close to zero.

```
➢ Dataf<-read.csv ("C:/Users/ACCENT/Desktop/GLO.csv",sep=";").
➢ View (dataf) .
➢ Dim (dataf).
➢ Tdataf=t(dataf).
➢ MM=(1/sqrt(400))*(data+ Tdataf).
➢ ValeurPropres=eigen(MM)$values.        VecteurPropre=eigen(MM)$vectors.

➢ ggplot (data=dataf, aes(x=VP, y=DDM))+  geom_bar(stat="identity").
➢ write.csv(dataf, file = "data.csv",row.names=FALSE, na="").
```

Figure 4: R Code of the Density of the Semicircle

2.5 Application of Tracy Widom's Theory on Corporate Returns

In the limit $N \to \infty$, the cumulative of the maximum eigenvalue max of a matrix (N×N) Gaussian converges to Tracy Widom's law when studying the regime onboard with an appropriate scale:

$$P\left(\mathrm{n}^{\frac{1}{6}}\left(\lambda_{\max} - 2\sqrt{\mathrm{n}}\right) \leq x\right) \xrightarrow[\mathrm{n}\to+\infty]{} \mathrm{F}_{\beta}(x).$$

where the F_β distributions are expressed as a function of the unique solution $q(x)$, called a solution of Hastings-McLeod, from the Painleve equation type II, we can show that these distributions are written explicitly for Gaussian cases β= 1; 2; 4:

$$F_2(x) = \exp\left(-\int_x^{+\infty} (u-x)q(u)^2\, du\right). \tag{9}$$

$$F_1(x) = \exp\left(-\frac{1}{2}\int_x^{+\infty} q(u)du\right)\sqrt{F_2(x)}. \tag{10}$$

$$F_4(x) = \cosh\left(-\frac{1}{2}\int_x^{+\infty} q(u)du\right)\sqrt{F_2(x)}. \tag{11}$$

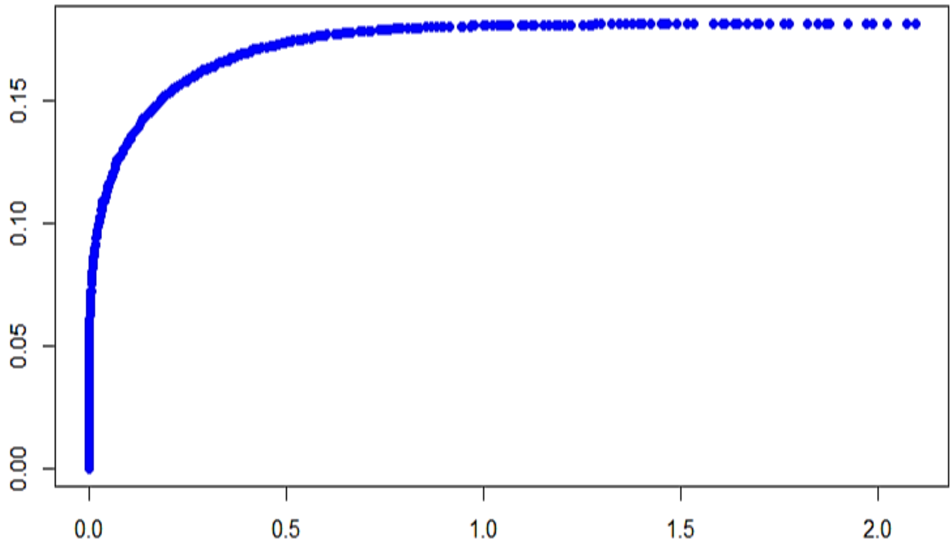


Figure 5: The Distribution of the Law of Tracy Widom according to the Eigenvalues of 400 Companies

The figure represents the distribution of the law of Tracy-Widom for the case (β = 1) according to the eigenvalues. We notice that the density value increases during the interval and becomes constant from the value (λ = 2). We can conclude that the density of the law of Tracy Widom is maximum for my case of $\lambda_{\max} = 2$.

```
Y<-read.csv("C:/Users/ACCENT/Desktop/rendement Bourse .csv", sep=";") σi
> Dim(Y)
> TY=t(Y)
> MPY=(1/252)*as.matrix(Y)%*%as.matrix(t(Y))
> VPMPY=eigen(MPY)$values
> dtw(VPMPY, beta=1, log = FALSE)
> dtw(VPMPY, beta=2, log = FALSE)          >dtw(VPMPY, beta=4, log = FALSE)
➢ plot(X1,Z1, type="b", pch=10, col="blue", xlab="Valeur Propre", ylab="Densite de Tracy
  Widom(B=1)")
```

Figure 6: Code R of the Law of Tracy-Widom

3. COMPARISON BETWEEN MARKOVIAN MINIMIZATION AND THE LAWS OF RANDOM MATRICES

3.1 Formulation of the Problem

The objects of this part is to present the fundamental principles of Markowitz theory of Portfolio [3]. More precisely on the optimization of a portfolio of minimal variance, then to carry out a comparison between the spectral distribution resulting from Markovian minimization and the laws of the theory of Random Matrices. We note that the variance-covariance matrix is symmetrical, that is to say $\sum = \sum^T$.

We have on the diagonal the variance of each title, it is possible to build a Portfolio $P(w)$: W: Represents the weight of each active title in the Portfolio, the sum of which must always be equal to 1 such as: $W^T U = 1$.

Such that:

$$= (1, \dots, 1)^T, W = (W_1, \dots, W_n)^T \text{ and } \sum = \begin{pmatrix} \sigma_1^2 & \cdots & \sigma_{1.n} \\ \vdots & \ddots & \vdots \\ \sigma_{n.1} & \cdots & \sigma_n^2 \end{pmatrix}$$

3.2 Portfolio Optimization

The optimization is done in the context of the mean-variance, we maximize the mean of the returns under the variance constraint, we then have the problem:

$$\begin{cases} Max\ E(R) \\ Var(x) = W^T \sum W \\ W^T U = 1 \end{cases}$$

$$E(R) = \sum_{i=1}^{N} W_i\, E(R_i), Var(R) = \sum_{i=1}^{N} \sum_{j=1}^{N} W_i W_j\, Cov(R_i, R_j) \text{ and } \sum_{i=1}^{N} W_i = 1$$

$$L(W_i, \lambda) = \sum_{i=1}^{N} W_i E(R_i) - \sum_{i=1}^{N} \sum_{j=1}^{N} W_i W_j\, Cov(R_i, R_j) - \lambda \left(\sum_{i=1}^{N} W_i - 1 \right)$$

The first-order necessary conditions are $\frac{dL_i}{dW_i} = 0$, and $\frac{dL_i}{d\lambda} = 0$ then:

$$E(R_i) - 2 \sum_{i=1}^{N} W_j Cov(R_i, R_j) - \lambda = 0$$

$$\sum_{i=1}^{N} W_i = 1$$

The solution of the system is written in the matrix form:

$$\begin{pmatrix} \sigma_{12} & . & .2\sigma_{1n} & 1 \\ . & . & . & . & . \\ . & . & . & . & 1 \\ 1 & 1 & . & 1 & 0 \end{pmatrix} \begin{pmatrix} W_1 \\ . \\ W_N \\ \lambda \end{pmatrix} = \begin{pmatrix} E(R_1) \\ . \\ E(R_N) \\ 1 \end{pmatrix}$$

3.3 Minimum Variance Portfolio

The portfolio of minimum variance [10] corresponds to the lowest level of risk. It is a question of optimizing the following quadratic program:

$$L(W_i,\lambda) = \sum_{i=1}^{N}\sum_{j=1}^{N} W_i W_j \, Cov\left(R_i, R_j\right) - \lambda\left(\sum_{i=1}^{N} W_i - 1\right)$$

where λ is the Lagrange multiplier, The first-order necessary conditions:

$$\frac{dL_i}{dW_i} = 0{:}\, 2\sum_{i=1}^{N} W_j Cov\left(R_i, R_j\right) - \lambda = 0 \tag{12}$$

$$\frac{dL_i}{d\lambda} = 0{:} \sum_{i=1}^{N} W_i = 1 \tag{13}$$

We will first draw W in equation (4.1), after which we will replace it in equation (4.2) by its expression:

$$2\sum_{i=1}^{N} W_j Cov\left(R_i, R_j\right) = \lambda\,, \qquad W^T U = 1, \ \text{then: } \lambda = \frac{2}{U^t \Sigma^{-1} U}$$

We are going to put this last expression of λ in the expression of w, which will give:

$$W = \frac{\Sigma^{-1} U}{U^t \Sigma^{-1} U}$$

Example:

We consider a portfolio of securities made up of three risky securities whose Correlation matrix is as follows:

Table 1
The Correlation Table of the 3 Companies

Titres	A.S.T Groupe	AB Science	ABEO
A.S.T Groupe	1	-0,008371	-0,066
AB Science	-0,008371	1	-0,049
ABEO	-0,066	-0,049	1

We are going to determine the least risky portfolio of this universe: For that, we start with the calculation of the covariance matrix Σ. The covariance matrix Σ is then given by the following product:

$$\sigma_{ij} = \rho_{ij}\sigma_i\sigma_j$$

Now the covariance are symmetrical $\Sigma_{ij} = \Sigma_{ji}$ in our case we have 3 covariance variables ($\sigma_i = 1$).

Then:

$$M = \begin{pmatrix} 1 & -0,008371 & -0,066 \\ 0,008371 & 1 & -0,049 \\ -0,066 & -0,049 & 1 \end{pmatrix}$$

The least risky portfolio will be the solution to the following program:

$$\begin{cases} Min\, Var(R) = W^T \sum W \\ W_1 + W_2 + W_3 = 1 \end{cases}$$

According to part (3.3) the vector of the proportions of the portfolio of minimum variance is written:

$$W = \frac{\sum^{-1} U}{U^t \sum^{-1} U}$$

Numerically in our case:

$$\sum^{-1} U = \begin{pmatrix} -0,142 \\ -0,144 \\ 0,983 \end{pmatrix} \text{ and } U^t \sum^{-1} U = 0,696$$

And so we find the following vector of weights:

$$W = \begin{pmatrix} -0,204 \\ -0,206 \\ 1,411 \end{pmatrix}$$

Then

$$Var(R) = W^T \sum W = 1,435.$$

```
➢ data<-read.csv("C:/Users/ACCENT/Desktop/M.csv",sep=";").
➢ Matrice.covariance=cov(M).
➢ Inverse.Matrice.covariance=solve(M).
➢ U=matrix(c(1,1,1),nrow=3,ncol=1)        Transpose.U=t(U).
➢ A=(Inverse.Matrice.covariance%*%U)  .
➢ B=transpose.U%*% Inverse.Matrice.covariance%*% U.
➢ W= (1/B)*A .
➢ Var= (t(w)%*% Matrice.covariance%*%W).
```

Figure 7: Code R of the Example

3.4 Comparison between Markowitz Theory and Random Matrix Theory

A Portfolio of 400 stocks with weights W_i and the variance-covariance matrix of returns. Markowitz's theory aims to minimize the daily variance:

$$R^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} W_i\, W_j Cov(Ri, Rj)$$

Thus the variance is minimal on the vector associated with the smallest eigenvalues of $\sum$. Taking X the matrix (400×400) of the returns observed over a period of 400 Days, and the empirical Covariance matrix, by comparing the spectrum from the Variance-covariance matrix R, with the spectrum of the matrix $\frac{1}{T}XX^T$, almost the same values are obtained. The graph represents the distributions of eigenvalues from the Variance-covariance matrix and the matrix $\left(\omega = \frac{1}{T}XX^T\right)$.

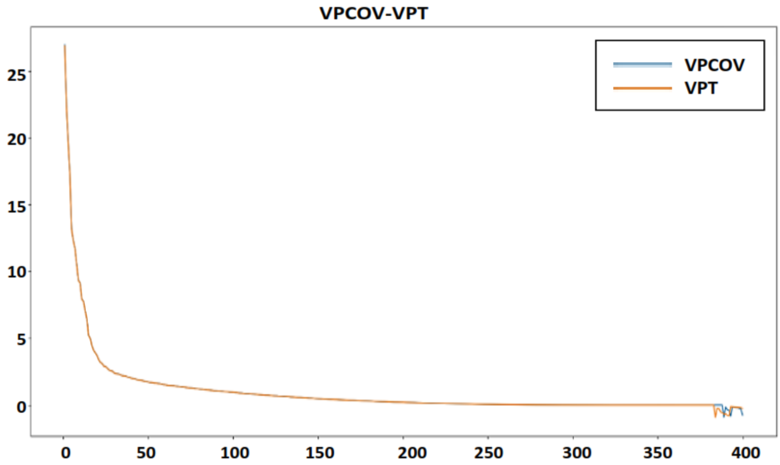


Figure 8: The Distribution of the Eigenvalues of the Covariance Matrix and Matrix ω

The distribution of the eigenvalues of the empirical covariance matrix and of the matrix $\frac{1}{T}XX^T$, are almost equal in the case of large dimensions, which illustrates the link between the spectrum of the empirical covariance matrix and the law of Marchenko-Pastur.

Existence of a minimal difference between the distribution of the eigenvalues of the covariance matrix and the matrix means that there is a linear regression. The linear regression model [14] is often estimated by the least-squares method, but there are also many other methods for estimating this model. We can for example, estimate the model by maximum likelihood or even by Bayesian inference. This technique is linear modeling that allows estimates to be made in the future based on information from the past. In this linear regression model, we have the explanatory variable and the other which is an explained variable.

To visualize the difference between the two spectra, we define the error E_i which represents the difference between the eigenvalues from the covariance matrix and those from the matrix $\left(\omega = \frac{1}{T}XX^T\right)$.

$VP_{I\omega}$: The i^{th} eigenvalue of the matrix ω. and
VP_{COV}: The i^{th} eigenvalue of the covariance matrix.

∀i ∈ [1; 400] $Error_i = VP_{i\omega} - VP_{iCOV}$

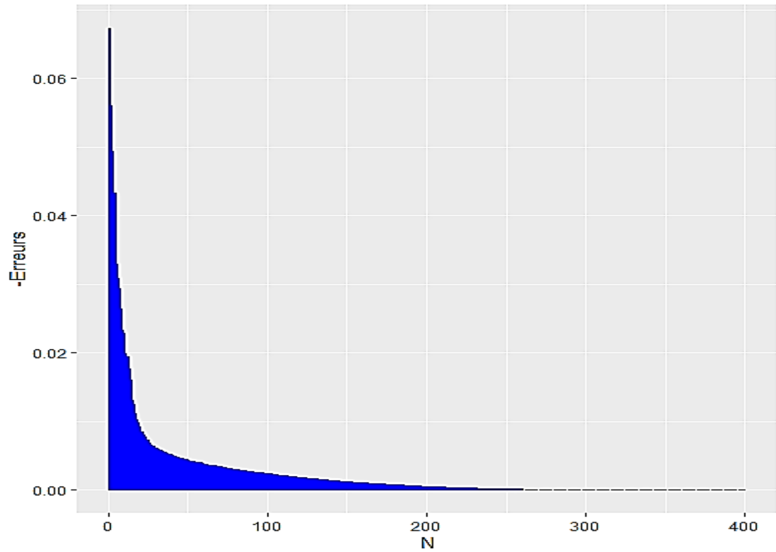


Figure 9: The Distribution of Errors of the Eigenvalues

Figure 9 shows that the error value averages 0.0025 and varies between 0 and 0.067 as the minimum and maximum value respectively, which proves that the spectral distributions are similar for large dimensions. Thus we can minimize the variance of a portfolio P of N actions by identifying the eigenvectors associated with the smallest eigenvalues associated with the covariance matrix of returns (Markowitz theory), or by observing the minimum eigenvalues associated with the matrix $\frac{1}{T}XX^T$.

Thus the Value of the minimum error and the strong linear correlation (0.88) which exists between the eigenvalues of the matrix T and the covariance matrix pushed us to carry out a simple linear regression, by explaining the eigenvalues of the matrix T by the eigenvalues of the covariance matrix, the results obtained are summarized in the following model.

```
Call:
lm(formula = VPT ~ VPCOV, data = data)

Residuals:
       Min         1Q     Median         3Q        Max
-3.587e-07 -3.910e-08 -2.932e-08  1.777e-08  8.379e-07

Coefficients:
             Estimate Std. Error   t value Pr(>|t|)
(Intercept) 3.405e-08  5.693e-09 5.981e+00 4.95e-09 ***
VPCOV       9.975e-01  1.988e-09 5.018e+08  < 2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 1.067e-07 on 398 degrees of freedom
Multiple R-squared:      1,     Adjusted R-squared:      1
F-statistic: 2.518e+17 on 1 and 398 DF,  p-value: < 2.2e-16
```

Figure 10: Linear Regression Model VP_T as a Function VP_{COV}

Linear regression analysis is used to predict the value of one variable based on the value of another variable. The variable whose value you want to predict is the dependent variable. The variable we use to predict the value of the other variable is the independent variable. In our case the independent variable is VP_{COV}: (the eigenvalues of the covariance matrix) is significant (P-value < 0,05). This proves that there is a strong Correlation relation between the distributions of the two spectra. The equation of the linear line is written in the form:

$$\forall i \in [1; 400] \qquad \mathrm{VP}_{\omega\mathrm{i}} = 3,405\, e^{-8} + (9,975\, e^{-1}\, \mathrm{VP}_{\mathrm{COVi}}).$$

```
➢ Library(ggplot2)        library(Matrix)
➢ data<-read.csv("C:/Users/ACCENT/Desktop/data.csv",sep=";").
➢ M.Covariance=cov(data)
➢ M.T=(/400)*(as.matrix(data)%*%t(data))
➢ MV.covariance=eigen(M.covariance)$values   (La matrice des valeurs Propres  de MV)
➢ MV.T=eigen(M.T)$values                      (La matrice des valeurs Propres  de MT)
➢ Vect.MV.T=matrix (data= MV.T,ncol=1,nrow=400).
➢ Vect.MV.covariance=matrix (data= MV.covariance.T,ncol=1,nrow=400).
➢ Erreur= (Vect.MV.T- Vect.MV.covariance).
➢ ggplot (data=, aes(x=N, y=Erreur))+  geom_bar(stat="identity",color="blue").

➢ > VPCOV=data[,1]
➢ > VPT=data[,2]
➢ > MODEL.VALUE<-lm(VPT~VPCOV,data=data)
➢ > summary(MODEL.VALUE)
```

Figure 11: Code *R* of the Part (4.4)

CONCLUSION

This present work concerns a field of application of random matrices, which are adopted and developed within the framework of several fields of scientific studies, namely: nuclear physics, chaos theory, or number theory. Our work comes in the context of giving a real example of the application of fundamental laws on the prices of the Paris stock exchange. This is to visualize the distribution of Marchensko-Pastur's laws and the semicircle law on reduced centered returns.

Subsequently, the observation of the distribution of the spectrum relative to the covariance matrix and the matrix $\frac{1}{T}XX^T$ showed that the spectra are similar to an error that varies between 0 and 0.06. Thus the vector associated with the smallest eigenvalue represents the vector minimizing the covariance matrix which illustrates the Markowitz theory for the case of reduced centered returns.

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APPENDIX

Table 2
Comparison between the First 10 Eigenvalues of the Covariance Matrix and of Matrix ω

The Eigenvalues of the Covariance Matrix	The Eigenvalues of the Covariance Matrix	Erros
26,86891998	26,93625938	-0,008371
22,37461772	22,4306932	0,056075482
19,69431738	19,74367598	0,049358598
17,28684182	17,33016674	0,043324914
13,15306969	13,18603466	0,032964977
12,30515864	12,33599841	0,030839773
11,70937134	11,73871753	0,029346189
10,51174627	10,5380915	0,026345224
9,311546171	9,33488317	-0,023336999
9,132473458	9,155361754	0,022888296

Table 3
Comparison between the last 10 Eigenvalues of the Covariance Matrix and of Matrix ω

The Eigenvalues of the Covariance Matrix	The Eigenvalues of the Covariance Matrix	Erros
-7,47e^{-17}	-3,46e^{-17}	4,01e^{-17}
-8,26e^{-17}	-3,91e^{-17}	4,35e^{-17}
-1,15e^{-17}	-8,33e^{-17}	3,2e^{-17}
-1,32e^{-16}	-1,39e^{-16}	-6,81e^{-18}
-1,5e^{-16}	-1,68e^{-16}	-1,77e^{-17}
-1,56e^{-16}	-1,86e^{-16}	-2,94e^{-17}
-1,64e^{-16}	-1,96e^{-16}	-3,16e^{-17}
-1,8e^{-16}	-2,38e^{-16}	-5,76e^{-17}
-1,9e^{-16}	-2,94e^{-16}	-1,04e^{-16}
-2,39e^{-16}	-7,45e^{-16}	-5,05e^{-16}