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EFFICIENT PLUG-IN ESTIMATORS OF THE TOPP-LEONE DISTRIBUTION SHAPE PARAMETER

Ahmad Zghoul
Department of Mathematics, College of Science
The University of Jordan, Amman, 11942, Jordan
Email: a.zghoul@ju.edu.jo

ABSTRACT

This paper addresses estimation of the shape parameter of the Topp-Leone distribution which is known for its applicability in lifetime data. For fixed scale parameter, four plug-in estimators are suggested to estimate the shape parameter. The properties of these estimators are examined and their performances against the maximum likelihood estimator (MLE) are evaluated for a range of the shape parameter values. Based on the second moment, an estimator, which has an algebraic solution, is derived. It turns out that this proposed estimator outperforms the estimator based on the first moment that has no algebraic solution. A plug-in estimator based on solving the estimation equation mapping the empirical joint distribution function to the joint distribution function of a random sample is also derived and proven to constantly outperform the maximum likelihood estimator for all values of the shape parameter.

KEYWORDS

Topp-Leone distribution; MLE; Plug-in estimators; MSE; Efficiency; Relative efficiency.

Subject code: 62F10; 62F12.

1. INTRODUCTION

Topp-Leone distribution (TL) is one of the researchers' interests in recent years. Nadarajah and Kotz (2003) drew attention to this distribution which was first introduced by Topp and Leone (1955). They discussed its relevance to the analysis of lifetime data, provided its moments and characteristic function, and derived the maximum likelihood estimator (MLE) of its shape parameter. Since then, many researchers have contributed to the distribution; Ghitany et al. (2005) studied some reliability measures of the distribution and investigated their stochastic orderings. Van Dorp and Kotz (2006) applied the distribution, with some modification, to income data. The kurtosis of the distribution was studied by Kotz and Seier (2007). Vicaria et al. (2008) introduced and investigated two-sided generalized (TL) distributions. Order statistics and record values from the (TL) distribution were studied by Zghoul (2010, 2011) and Genç (2012). Goodness-of–fit tests were considered by Al-Zahrani (2012). Genç (2013) estimated the reliability of a system, $P(X > Y)$, where X (strength) and Y (stress) are independent random variables from Topp-Leone distribution. Sindhu et al. (2013) considered Bayesian estimates based o trimmed samples. Bayoud (2015, 2016) estimated the TL shape parameter based on Type I and Type II censored samples. Bayoud (2016), also derived Bayes estimators for the shape parameter of TL distribution. A class of continuous distributions bases on TL distribution called “the Topp–Leone odd log-logistic family” was introduced by Brito et al. (2017). Kumaraswamy (1980) proposed a more general distribution to model hydrological random variables such as daily rainfall, daily stream flow, etc. John (2009), compared Kumaraswamy distribution to the beta distribution, and he further derived the maximum likelihood estimator of Kumaraswamy distribution parameters. Rezaei et al. (2017) introduced a two-parameter generalization of TL distribution and estimated both of its parameters by the maximum likelihood method.

Despite the growing interest in the distribution of TL and its generalizations, few research articles have dealt with procedures for estimating its parameters. As far as I know, only the MLE of the shape parameter was derived; Nadaraja and Kotz (2003) for the basic distribution of TL, John (2009) for the distribution of Kumaraswamy and Rezaei et al. (2017) for what they call “TL generated” distribution. The objective of this article is to apply and suggest different plug-in procedures to estimate the shape parameter of the TL distribution. The estimators will then be evaluated and their distributional properties will be studied. We will also conduct performance comparisons in terms of bias and MSE with the MLE.

Since some of the proposed methods only have tractable properties if the scale parameter is known, we limit this article only to the study of this case. However, given its wide spectrum of shapes (Figure 1), there are still large applications that can be fitted to the one parameter TL model. For example, data with predetermined upper limits, such as the time to complete a task with a specified time limit or scores of known ceiling value, could be modeled with this distribution.

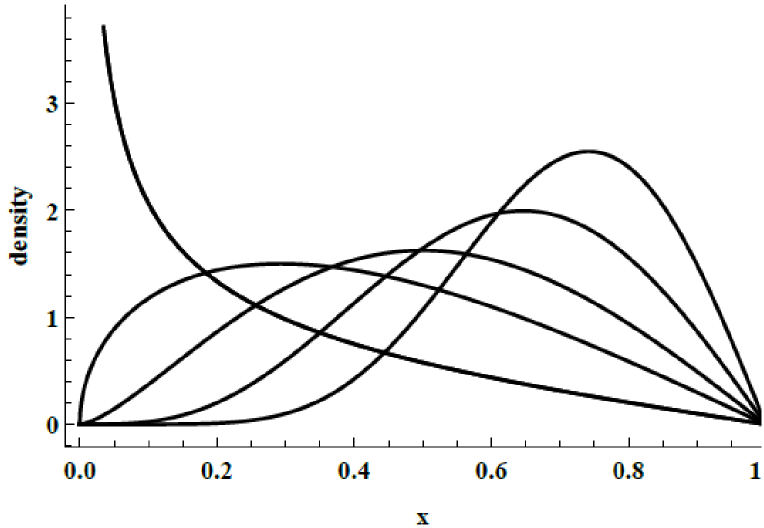

Figure 1: Different Shapes of the Topp-Leone Distribution for θ=0.5, 1.5, 2.5, 4.5, and 8

Let $X_1, \dots, X_n$ be a random sample from some distribution function F that belongs to a family of distributions $\{F_\theta, \theta \in \Theta\}$, where Θ is a given parameter space, and let $\tilde{F}$ be an estimate of F based on the given random sample. Consider a function g that maps F_θ onto ℝ; $g: F_\theta \to \mathbb{R}$. Then, as stated in Bickel and Doksum (2015), $g(\tilde{F})$ is the plug-in estimate of $g(F)$. For instance, if $X_1, \dots, X_n$ are i.i.d. (independent and identically distributed) random variables with common distribution F, then the empirical distribution function $\hat{F}(x) \equiv \frac{1}{n}\sum_{i=1}^{n} I(X_i \leq x), \forall x \in \mathbb{R}$, is the natural estimate of F and $g(\hat{F})$ is the plug-in estimate of $g(F)$.

If $E\left|[h(X)]^j\right| < \infty, j = 1,2, \dots$ then the jth moment of *h(X)* is a function of F given by $E[h(X)]^j = \int_{\mathbb{R}} [h(x)]^j dF(x)$ and the jth sample moment of *h(X)* is $\int_{\mathbb{R}} [h(x)]^j d\hat{F}(x) = \frac{1}{n}\sum_{i=1}^{n} [h(X_i)]^j$ and, hence, the plug-in estimate of $E[h(X)]^j$ is $\frac{1}{n}\sum_{i=1}^{n} [h(X_i)]^j$. This particular plug-in estimate is called the *method of moment estimate*.

A quantile estimate and in particular the *median estimate* is another plug-in estimate that will be applied and investigated in this article. For an absolutely continuous distribution function F of X, the αth quantile is X_α that satisfies $F(X_\alpha) = \alpha, 0 < \alpha < 1$. A particular quantile is the median $X_{0.5} \equiv \tilde{\mu}$ that satisfies $F(\tilde{\mu}) = 0.5$.

A third Plug-in estimate will be obtained by equating the joint empirical sample distribution to the corresponding joint population distribution; viz.

$$\prod_{j=1}^{n} \hat{F}(X_j) = \prod_{j=1}^{n} F(X_j)$$

The least property that an estimator should satisfy is consistency; the Plug-in estimators are consistent as proved in Bickel and Doksum (2015).

This article is organized as follows. In Section 2, we will introduce four plug-in methods of estimation for the shape parameter when the scale is fixed, study their properties, evaluate their relative efficiency and derive their distributions and/or asymptotic distributions. Comparisons of estimators are conducted in Section 3. An application is given in Section4, and in Section 5, we will summarize the conclusions of this article.

2. ESTIMATION OF THE SHAPE PARAMETER

Assuming the scale parameter β is known and, without loss of generality, equals to one, the distribution function and the probability density function of the TL distribution of shape parameter θ are, respectively, given by

$$\begin{aligned} F(x|\theta) &= [1-(1-x)^2]^\theta, 0 < x < 1, \theta > 0, \\ f(x|\theta) &= 2\theta(1-x)[1-(1-x)^2]^{\theta-1}, 0 < x < 1, \theta > 0. \end{aligned} \quad (1)$$

The mean of a random variable X from the family (1), Nadarajah and Kotz (2003), is

$$E(X) = 1 - 4^\theta \frac{\Gamma^2(\theta+1)}{\Gamma(2\theta+2)} = 1 - 4^\theta B(\theta+1, \theta+1) \quad (2)$$

where $B(\theta+1, \theta+1)$ is the usual beta function.

In this section, we will propose four methods of estimation of the parameter θ and study their properties.

2.1 Moments Estimators

Suppose $X_1, \ldots, X_n$ are i.i.d. random variables from the family of distributions given in (1), and $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is its sample mean. A moment estimator for θ is obtained by equating the sample mean to the population mean and then solving for θ,

$$\bar{X} = 1 - 4^{\theta} B(\theta + 1, \theta + 1). \tag{3}$$

Analytic solution of (2) might not be possible, so numerical solution, to be denoted $\hat{\theta}_{MM1}$, will be obtained. That is; $\hat{\theta}_{MM1}$ is the estimator satisfies

$$\bar{X} = 1 - 4^{\hat{\theta}_{MM1}} B(\hat{\theta}_{MM1} + 1, \hat{\theta}_{MM1} + 1). \tag{4}$$

To show that (3) has exactly one root, rewrite it as $4^{\theta} B(\theta + 1, \theta + 1) + \bar{X} - 1 = 0$, and set $d(\theta) = 4^{\theta} B(\theta + 1, \theta + 1) + \bar{X} - 1$. Then

$$d(\theta) = \int_0^1 [4x(1 - x)]^{\theta}\, dx + \bar{X}.$$

Differentiating $d(\theta)$ with respect to θ, we have

$$d'(\theta) = \int_0^1 [4x(1 - x)]^{\theta} \log[4x(1 - x)]\, dx. \tag{5}$$

The term $\log[4x(1 - x)]$ is negative for all $0 < x < 1$ and, hence, the integrand in (5) is also negative, so $d'(\theta) < 0$ for all $\theta > 0$, implying $d(\theta)$ is strictly monotone. Therefore, equation (3) has exactly one solution. Since explicit solution of (4) may not be possible, numerical methods can be applied to solve for θ.

In practice, estimators without explicit representations may not be favorable, so one may look for possible alternatives. Based on the second moment, an estimator with analytic solution can be obtained by equating the mean of $(1 - X)^2$ to its corresponding sample mean. We have,

$$E(1 - X)^2 = \int_0^1 2\theta (1 - x)^3 (1 - (1 - x)^2)^{\theta - 1} dx \tag{6}$$

Integrating (6) by parts setting $u = (1 - x)^2$ and $dv = 2\theta(1 - x)(1 - (1 - x)^2)^{\theta - 1}$, one has

$$\int_0^1 2\theta (1 - x)^3 (1 - (1 - x)^2)^{\theta - 1} dx$$

$$= (1 - x)^2 (1 - (1 - x)^2)^{\theta} \Big|_0^1 - \int_0^1 2(1 - x)(1 - (1 - x)^2)^{\theta} dx$$

$$= \frac{(1 - (1 - x)^2)^{\theta + 1}}{\theta + 1}\Bigg|_0^1 = \frac{1}{\theta + 1}$$

Thus,

$$E(1-X)^2 = \frac{1}{\theta+1} \tag{7}$$

Replacing the left-hand side of (6) by its sample counterpart, we get

$$\frac{1}{n}\sum_{i=1}^{n}(1-X_i)^2 = \frac{1}{\theta+1} \tag{8}$$

Solving (8) for θ, we obtain the estimator

$$\hat{\theta}_{MM2} = \left(\frac{n}{\sum_{i=1}^{n}(1-X_i)^2} - 1\right) \tag{9}$$

As it is clear from (8), $T_n \equiv \frac{1}{n}\sum_{i=1}^{n}(1-X_i)^2$ is an unbiased estimator of $(\theta+1)^{-1}$. To compute its variance, we have

$$E[(1-X)^4] = \int_0^1 2\theta(1-x)^5(1-(1-x)^2)^{\theta-1}dx$$

Setting $u = (1-x)^2$, we obtain

$$E[(1-X)^4] = \theta\int_0^1 u^2(1-u)^{\theta-1}du$$
$$= \theta Beta(3,\theta) = \frac{2}{(\theta+1)(\theta+2)} \tag{10}$$

Therefore, (6) and (10) imply

$$Var[(1-X)^2] = \frac{2}{(\theta+2)(\theta+1)} - \frac{1}{(\theta+1)^2} = \frac{\theta}{(\theta+1)^2(\theta+2)}.$$

Therefore, $Var(T_n) = \frac{1}{n^2}\sum_{i=1}^{n}Var[(1-X_i)^2] = \frac{\theta}{n(\theta+1)^2(\theta+2)}$.

It is straightforward to show that $U = (1-X)^2$ has a power function distribution, which is a $Beta(1,\theta)$ distribution, with density function $f(u) = \theta(1-u)^{\theta-1}, 0 < u < 1$. So the distribution of $\hat{\theta}_{MM2}$ is proportional to the distribution of the reciprocal of sum of independent and identically $Beta(1,\theta)$ random variables. And this may not have a nice closed form distribution. However, applying the central limit theorem, an asymptotic distribution can be obtained.

T_n is a sum of iid random variables with common mean $(\theta+1)^{-1}$ and common variance $\theta(\theta+1)^{-2}(\theta+2)^{-1}$, so by the central limit theorem we have

$$\sqrt{n}(T_n - (\theta+1)^{-1}) \xrightarrow{D} N(0, \theta(\theta+1)^{-2}(\theta+2)^{-1}).$$

To obtain the distribution of $\hat{\theta}_{MM2} = T_n^{-1}$, we apply the delta criterion with $g(\theta) = (1-\theta)/\theta$, hence $g((\theta+1)^{-1}) = \theta$.

Upon differentiating g and evaluating the derivative at $(\theta+1)^{-1}$, we get

$g'((\theta+1)^{-1}) = -(\theta+1)^2$ and $(g'((\theta+1)^{-1})^2 = (\theta+1)^4$.

Therefore,

$$\sqrt{n}\left(\hat{\theta}_{MM2} - \theta\right) \xrightarrow{D} N\left(0, \frac{(\theta+1)^2\theta}{(\theta+2)}\right). \quad (11)$$

Obviously, $\hat{\theta}_{MM2}$ is a consistent estimator of θ with asymptotic variance

$[(1+\theta)^2\theta]/n(2+\theta)$.

2.2 Quantile Estimator

A quantile estimator for θ can be obtained by equating the αth quantile, $0 < \alpha < 1$, of the random variable X to the corresponding sample quantile. There is no known criterion to choose α that leads to best (minimum MSE) estimator. It is, however, advised to choose a value of α that is not near 0 or 1. Choosing $\alpha = 0.5$, an estimator of θ is obtained as a solution of the estimate equation matches the population and sample medians.

Denote the population median by $\tilde{\mu}$ and the sample median by $\tilde{X}$, then $\tilde{\mu}$ satisfies $0.5 = F(\tilde{\mu}) = (1-(1-\tilde{\mu})^2)^\theta$ or equivalently $\tilde{\mu} = F^{-1}(0.5) = 1 - \sqrt{1-2^{-1/\theta}}$. Therefore the median estimator of θ, to be denoted $\hat{\theta}_{Med}$, is the solution of $\tilde{X} = 1 - \sqrt{1-2^{-1/\theta}}$ given by

$$\hat{\theta}_{Med} = (-\log 2)(\log[1-(1-\tilde{X})^2])^{-1} \quad (12)$$

From which the first and second moments, and hence the bias and MSE, can be obtained.

The asymptotic distribution of the sample median of a random sample of size n from a distribution of density $f(x)$, see for example Serfling (2002), is

$$\sqrt{n}\left(\tilde{X} - \tilde{\mu}\right) \xrightarrow{D} N\left(0, \frac{1}{4f^2(\tilde{\mu})}\right).$$

So for a random sample from the TL(θ) distribution, we have

$$\sqrt{n}\left(\tilde{X} - \tilde{\mu}\right) \xrightarrow{D} N\left(0, \frac{1}{4(2^{2/\theta} - 2^{1/\theta})\theta^2}\right) \quad (13)$$

To obtain the asymptotic distribution of $\hat{\theta}_{Med}$, we apply the delta method with, $g(\theta) = -\log 2/log[1-(1-\theta)^2]$, hence,

$$g'(\theta) = \frac{2(1-\theta)Log2}{(1-(1-\theta)^2)(Log[1-(1-\theta)^2])^2}.$$

Thus, $g(\tilde{X}) = \hat{\theta}_{Med}$, $g(\tilde{\mu}) = \theta$, and $g'(\tilde{\mu}) = 2\theta^2\sqrt{2^{2/\theta} - 2^{1/\theta}}/Log2$.

By (13), the asymptotic variance of $\tilde{X}$ is $[4(2^{2/\theta} - 2^{1/\theta})\theta^2]^{-1}$, therefore,

$$\sqrt{n}(\hat{\theta}_{Med} - \theta) \xrightarrow{D} N(0, (\theta/log2)^2). \quad (14)$$

2.3 Empirical Distribution Function Estimator

As mentioned in the introduction, the empirical distribution function is a natural estimate of the distraction function, Bickel and Doksum (2015). Equating the joint distribution function of a random sample from TL(θ) to the corresponding joint empirical distribution function, one has

$$\prod_{j=1}^{n} \hat{F}(X_j) = \left(\prod_{j=1}^{n} [X_j(2 - X_j)]\right)^{\theta}. \tag{15}$$

Solving (15) for θ, we obtain

$$\hat{\theta}_{EDF} = \frac{\sum_{j=1}^{n} Log(\hat{F}(X_{j:n}))}{\sum_{j=1}^{n} Log\left[X_j(2 - X_j)\right]} = \frac{\sum_{j=1}^{n} Log\left(j/n\right)}{\sum_{j=1}^{n} Log\left[X_j\left(2 - X_j\right)\right]}, \tag{16}$$

where $X_{j:n}$ is the jth order statistic of a sample of size n.

To find the mean and variance of $\hat{\theta}_{EDF}$, we first derive the distribution of $-Log[X_j(2 - X_j)], j = 1,..,n$. We have

$$\begin{aligned} P(-Log[X(2 - X)] \leq u) &= P(X(2 - X) \geq e^{-u}) \\ &= P(1 - (1 - X)^2 \geq e^{-u}) \\ &= P((1 - X)^2 \leq 1 - e^{-u}) \\ &= P\left(X \geq 1 - \sqrt{1 - e^{-u}}\right) \\ &= 1 - F\left(1 - \sqrt{1 - e^{-u}}\right) \\ &= 1 - e^{-\theta u}, u > 0. \end{aligned}$$

That is, $-Log[X_j(2 - X_j)], j = 1,..,n$ is distributed exponential with mean θ^{-1}, and hence $-\sum_{j=1}^{n} Log[X_j(2 - X_j)]$ is distributed *Gamma(n,θ)*.

Therefore, $\hat{\theta}_{EDF}$ is distributed *inverse gamma* with shape parameter n and scale θ. The mean and variance of inverse gamma (α,σ); α is shape and σ is scale, are, respectively, σ/(α-1), α>1, and $\frac{\sigma^2}{(\alpha-1)^2(\alpha-2)}, \alpha > 2$. Accordingly, the mean and variance of $\hat{\theta}_{EDF}$, respectively, are

$$E[\hat{\theta}_{EDF}] = \frac{-(\sum_{i=1}^{n} Log(j/n))\theta}{n - 1}, n > 1, \text{and} \tag{17}$$

$$Var[\hat{\theta}_{EDF}] = \frac{(\sum_{i=1}^{n} Log(j/n))^2\theta^2}{(n - 1)^2(n - 2)}, n > 2. \tag{18}$$

Consequently, the bias and the MSE of $\hat{\theta}_{EDF}$ are

$$bias\left(\hat{\theta}_{EDF}\right) = \frac{(1 - n - \sum_{j=1}^{n} Log(j/n))\theta}{n - 1} = \left(-1 + \frac{\sum_{j=1}^{n} Log(n/j)}{n - 1}\right)\theta,$$

$$MSE\left(\hat{\theta}_{EDF}\right) = \left[\frac{(\sum_{i=1}^{n} Log(j/n))^2}{(n - 1)^2(n - 2)} + \frac{\left(1 - n - \sum_{i=1}^{n} Log(j/n)\right)^2}{(n - 1)^2}\right]\theta^2. \tag{19}$$

To find the asymptotic distribution of $\hat{\theta}_{EDF}$, we first find $\lim_{n\to\infty}\sqrt{n}\,\mathrm{Var}[\hat{\theta}_{EDF}]$, then we have

$$\sum_{i=1}^{n} Log(j/n) = \sum_{j=1}^{n} \log(\mathrm{j}) - n\log(n) = \log(n!) - n\log(n).$$

Applying Stirling's approximation, $\log(n!) = n\log(n) - n + O(\log n)$, then as $n \to \infty$, we obtain

$$\frac{1}{n}\sum_{j=1}^{n} \log(\frac{\mathrm{j}}{\mathrm{n}}) \sim -1 + \frac{O(logn)}{n} \to -1.$$

Therefore,

$$\lim_{n\to\infty}\left(\frac{\left(\sum_{i=1}^{n} Log(j/n)\right)^2}{(n-2)(n-1)^2} + \frac{\left(1-n-\sum_{i=1}^{n} Log(j/n)\right)^2}{(n-1)^2}\right) = 1,$$

and, hence, $\lim_{n\to\infty}\sqrt{n}\mathrm{Var}[\hat{\theta}_{EDF}] = \theta^2$.

The central limit theorem implies that the asymptotic distribution of $\hat{\theta}_{EDF}$ is

$$\sqrt{n}\left(\hat{\theta}_{EDF} - \theta\right) \to N(0,\theta^2). \qquad (20)$$

2.4 The Maximum Likelihood Estimator

The density of the TL(θ) given in (1) belongs to the exponential family of distributions. Based on a random sample of size n from this density, we can express its likelihood function as

$$L(\theta|x_1,\ldots,x_n) \propto Exp\left\{\theta\sum_{i=1}^{n} \log(1-(1-X_j)^2) + n\log(\theta)\right\}$$

So, the statistic $T \equiv \sum_{i=1}^{n} \log(1-(1-X_j)^2)$ is sufficient for θ and the MLE of θ is proportional to T. Differentiating $\log(L(\theta)$ with respect to θ and equating the result to zero, then solving the obtained likelihood equation for θ, the MLE of θ, $\hat{\theta}_{ML}$, is

$$\hat{\theta}_{ML} = -\frac{n}{\sum_{i=1}^{n} \log(1-(1-X_j)^2)}. \qquad (21)$$

Comparing $\hat{\theta}_{EDF}$ and $\hat{\theta}_{ML}$ in (16) and (21), we notice that they only differ by a multiplicative factor. Therefore, based on (17) and (18), it is straightforward to conclude that

$$E[\hat{\theta}_{ML}] = \frac{n\theta}{n-1}, n > 1, \qquad (22)$$

and

$$Var[\hat{\theta}_{ML}] = \frac{n^2\theta^2}{(n-1)^2(n-2)}, n > 2. \qquad (23)$$

Consequently, the bias and the MSE of $\hat{\theta}_{ML}$ are,

$$bias(\hat{\theta}_{ML}) = \frac{\theta}{n-1}, \tag{24}$$

and

$$MSE(\hat{\theta}_{ML}) = \frac{(n+2)\theta^2}{(n-1)(n-2)}, \tag{25}$$

respectively.

We have, $E\left(\frac{\partial^2 \log f(x,\theta)}{\partial\theta^2}\right) = -1/\theta^2$. Therefore the Cramer-Rao lower bound of any unbiased estimator of θ is θ^2/n.

2.5 Comparisons of Estimators

In this section, we compare the methods proposed and discussed in the previous section according to their bias and their MSE. Exact algebraic forms of bias and MSE are available for each of the MLE ($\hat{\theta}_{MLE}$) and the EDF ($\hat{\theta}_{EDF}$) estimators and also for the second moment based estimator ($\hat{\theta}_{MM2}$), but not for the first moment based estimator ($\hat{\theta}_{MM1}$) or the median based estimator ($\hat{\theta}_{MED}$). In the latter two cases, Monte Carlo simulations are used to compute the bias and MSE. The simulation procedure is as follows. First, a sample of specific size n for a particular value of θ is generated, then, the *FindRoot* Mathematica code is used to solve (3) for $\hat{\theta}_{MM1}$. For the same sample, $\hat{\theta}_{MED}$ will be calculated using (12). This process is repeated 10,000 times for each θ and n. The mean and the variance of the set of the 10,000 estimates based on (3) are, respectively, approximations of the mean and the variance of $\hat{\theta}_{MM1}$ and the 10,000 estimates based on (12) are, respectively, approximations of the mean and the variance of $\hat{\theta}_{MED}$. Thus, bias and MSE of both of the estimators $\hat{\theta}_{MM1}$ and $\hat{\theta}_{MED}$ are readily computed. The range of the parameter θ will be from 0.5 to 5 with increments of 0.5 covering various shapes of the distribution, and sample sizes of 10 and 30 will be considered.

The absolute bias of each of the five considered estimators, when $n = 10$ and $n = 30$, are shown in Figures. 2 and 3, respectively. We see that there are no big differences between all estimates for the two sample sizes However, larger relative differences between the estimates for $n = 30$ than for $n = 10$ can be observed.

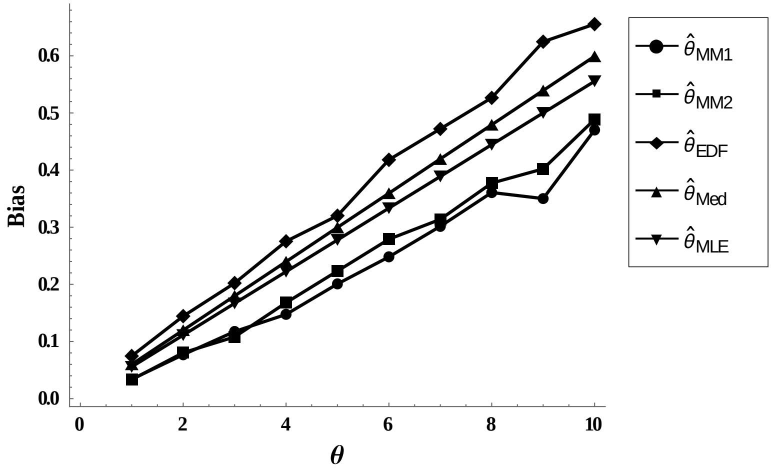

Figure 2: Bias of the Five Considered Estimators for n=10 when θ ranges from 0.5 to 5

It should also be noted that $\hat{\theta}_{MM1}$ and $\hat{\theta}_{MM2}$ have the lowest absolute bias, although the bias of $\hat{\theta}_{MM1}$ is slightly lower. We can also see that the bias in $\hat{\theta}_{EDF}$ is close to the bias in $\hat{\theta}_{MLE}$, with $\hat{\theta}_{MLE}$ bias being slightly higher than that of $\hat{\theta}_{EDF}$. Further, we observe that he median based estimator $\hat{\theta}_{Med}$ has the largest bias among all the estimators considered. Comparing Figures 2 and 3, we notice the considerable impact that sample size has on bias values.

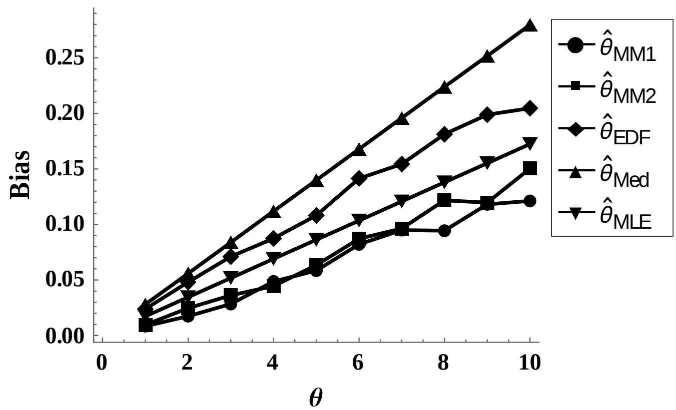

Figure 3: Bias of the Five Considered Estimators for n=30 when θ ranges from 0.5 to 5

The MSEs of the five estimators for n=10 and n=30 are displayed in Figures 4 and 5, respectively. Simulations as described before, were used to compute the MSEs for the estimators $\hat{\theta}_{MM1}$, $\hat{\theta}_{MM2}$ and $\widehat{\theta}_{Med}$, whereas (19) and (25) were used to compute the MSEs for $\hat{\theta}_{Med}$ and $\hat{\theta}_{MLE}$.

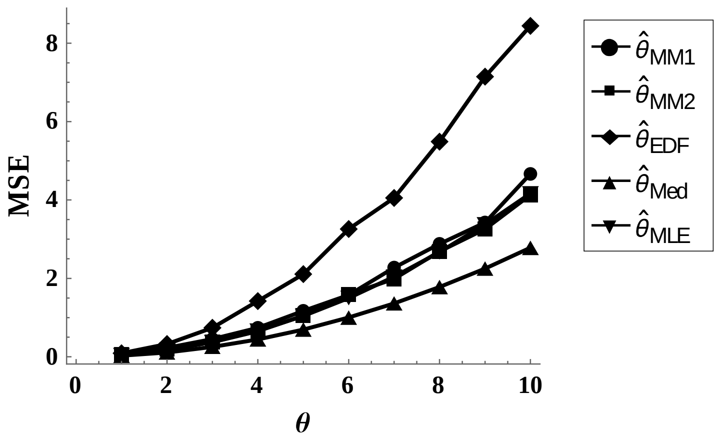

Figure 4: MSE of the Five Considered Estimators for n=10 when θ ranges from 0.5 to 5

In Figure 4, we see that the empirical distribution function based estimator outperforms all other estimators. It can also be concluded that $\hat{\theta}_{MLE}$ and $\hat{\theta}_{MM1}$ and $\hat{\theta}_{MM2}$ have approximately a similar MSE. It should also be noted that the MSE of $\hat{\theta}_{Med}$ for both sample sizes n = 10 and n = 30 differs significantly from the other four estimators. For the larger sample n = 30, all estimators with the exception of the median estimator are compatible. When comparing Figures 4 and 5, we see that the MSE of all estimators is significantly smaller at n = 30 than at n = 10.

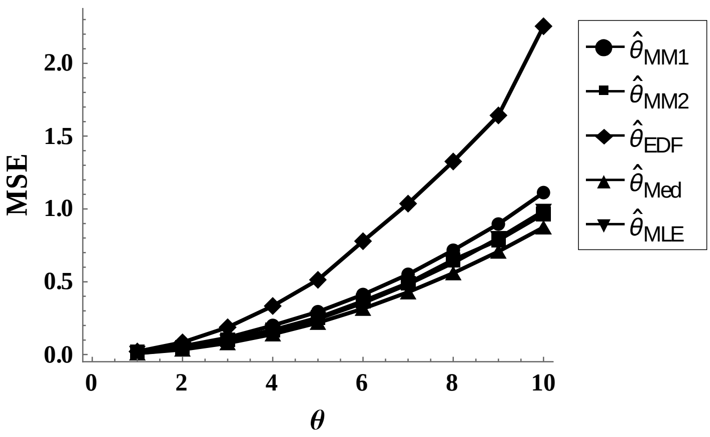

Figure 5: MSE of the Five Considered Estimators for n=30 when θ ranges from 0.5 to 5

To see how each of the estimators is compared to the MLE, we compute the efficiency of the MLE relative to each of the other estimators as

$$RE(\hat{\theta}_{ML}, \hat{\theta}) = \frac{MSE(\hat{\theta})}{MSE(\hat{\theta}_{MLE})},$$

and then we display the results in Figure 6.

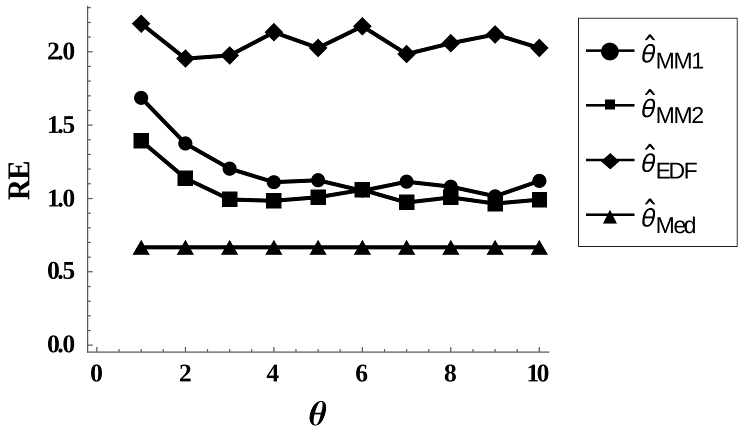

Figure 6: Relative Efficiency of $\hat{\theta}_{MLE}$ to each of $\hat{\theta}_{MM1}$, $\hat{\theta}_{MM2}$, $\hat{\theta}_{MED}$, and $\hat{\theta}_{EDF}$ when n=10 and θ ranges from 0.5 to 5

Obviously, as it can be seen in Figure 6, the median based estimator MSE is approximately twice as large as the maximum likelihood estimator MSE. Moments based estimators have a larger MSE than the MLE for small values of θ, but they approach when θ increases. And surprisingly, the plug-in estimator based on the empirical distribution function is consistently more efficient than the MLE for all values of θ.

Simulations of the coverage probabilities of confidence intervals and expected lengths of the proposed estimators are presented in Table 1. Confidence intervals with nominal 95% were constructed based on the asymptotic distributions of the estimators. Samples of sizes 30, 50, and 100 were simulated from TL(θ) with various values of θ. We can see that the EDF method gives the shortest expected length but is permissive. All other methods recover the nominal value to within one point. The MED gives the longest expected interval length.

Table 1
Coverage Probabilities and Expected Interval Lengths of Confidence Intervals Constructed based on the Suggested Estimators

Sample Size	θ	Method of Estimation							
		MM2		MED		EDF		MLE	
		CP	EL	CP	EL	CP	EL	CP	EL
30	0.5	0.936	0.486	0.939	0.516	0.893	0.338	0.952	0.370
50		0.952	0.376	0.941	0.400	0.916	0.268	0.961	0.284
100		0.956	0.265	0.941	0.283	0.937	0.192	0.948	0.199
30	1.0	0.939	0.846	0.930	1.033	0.883	0.676	0.951	0.741
50		0.948	0.648	0.939	0.800	0.922	0.534	0.953	0.567
100		0.953	0.455	0.948	0.566	0.925	0.382	0.956	0.395
30	1.5	0.948	1.197	0.932	1.549	0.887	1.013	0.952	1.110
50		0.953	0.915	0.946	1.200	0.905	0.794	0.952	0.843
100		0.947	0.645	0.934	0.848	0.919	0.574	0.952	0.593
30	2.5	0.953	1.916	0.931	2.581	0.900	1.689	0.959	1.851
50		0.939	1.467	0.937	1.999	0.906	1.332	0.953	1.413
100		0.951	1.034	0.943	1.414	0.928	0.960	0.949	0.992
30	3.5	0.950	2.654	0.923	3.614	0.900	2.376	0.954	2.603
50		0.953	2.023	0.952	2.799	0.916	1.869	0.955	1.983
100		0.936	1.418	0.936	1.979	0.917	1.341	0.942	1.385
30	5.0	0.946	3.747	0.927	5.163	0.885	3.389	0.953	3.713
50		0.952	2.861	0.956	3.999	0.916	2.670	0.952	2.833
100		0.959	2.006	0.950	2.828	0.937	1.917	0.958	1.980

3. AN APPLICATION

The actual energy production divided by the maximum possible energy production from an energy source over a period of time is called the capacity factor, Neill and Hashemi (2018). We apply the above estimation procedures to unit capacity factor data given in Caramanis et al. (1983) and Mazumdar and Gaver (1984). The data set presents capacity factor values of 23 units produced by what they called SC16 algorithm.

There are many test methods available for attributing a set of data to a probability model, here we are implementing the Anderson-Darling (A^2) test which as a goodness-of-fit test has proven to be superior to many other tests. To compute A^2, the parameter θ must first be estimated from the sample. Using the MLE method, the estimate is calculated at $\hat{\theta}_{MLE} = 0.5943$. The A^2 test rejection region cut point is calculated to be

1.36. This value was obtained based on 20,000 simulated samples from TL(0.5943) each of size 23. The calculated value of A^2 based on the data is 1.288 which gives a p-value of 0.18 supporting that TL(0.5943) is appropriate fit. The Q-Q plot shown in Figure 7 further supports a reasonable compliance of the data with the model.

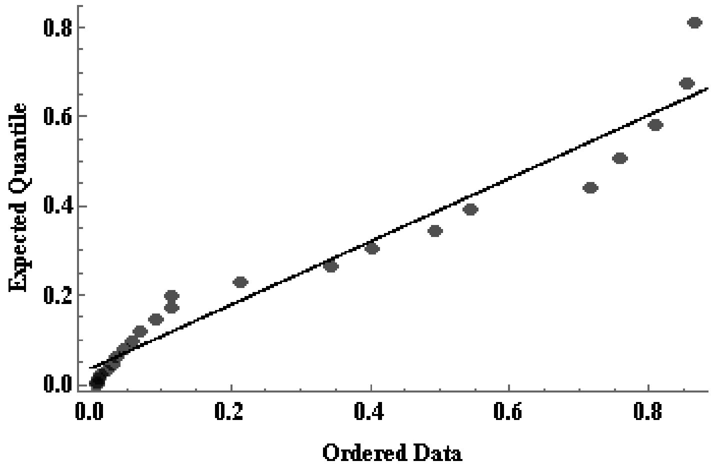

Figure 7: The Q-Q Plot of Capacity Factor Data Produced by 23 Power Source Units

Table 2 shows the estimates of θ obtained using the four estimation methods discussed in this article along with their approximate standard errors. We can notice that EDF estimate has the smallest standard error and the MM2 method has the largest standard error.

Table 2
Estimates and their Standard Errors of θ using four Methods of Estimation

Method	Estimate of θ	Standard Error of Estimate
MM2	0.6567	0.1717
MED	0.4558	0.1371
EDF	0.5299	0.1078
MLE	0.5943	0.1356

4. CONCLUSIONS

We have suggested four plug-in estimators for the TL distribution shape parameter; two are moment based estimators ($\hat{\theta}_{MM1}$ and $\hat{\theta}_{MM2}$), one quantile estimator ($\hat{\theta}_{Med}$), and one is empirical distribution function based estimator ($\hat{\theta}_{EDF}$). In addition to that $\hat{\theta}_{MM2}$ has a closed algebraic form, while $\hat{\theta}_{MM1}$ has none, $\hat{\theta}_{MM2}$ has somewhat smaller MSE than that of $\hat{\theta}_{MM1}$. We also conclude that $\hat{\theta}_{EDF}$outperforms all other estimators including the MLE. Finally, all estimators are consistent estimates and have a positive bias, with the exception of the quantile estimator, which has a negative bias.

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