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A NEW EXPONENTIATED DISTRIBUTION APPLICABLE TO SURVIVAL TIMES

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ABSTRACT

We developed three parameter Pranav distribution known as Exponentiated Two Parameter Pranav Distribution by using exponentiated technique. Expressions for statistical characteristics along with reliability measures have been derived for proposed model. Expression for order statistics is also derived. The unknown shape, scale and exponentiated parameters of proposed model are estimated by using maximum likelihood estimation method. For examining suitability of proposed model over its related models, we fitted the proposed model and its related models to two real life data sets by computing Kolmogorov statistic, p-value, AIC, BIC, AICC, HQIC values.

KEYWORDS

Two Parameter Pranav Distribution, Exponentiated Two Parameter Pranav Distribution, Exponentiated Technique, Structural Properties, Maximum Likelihood Estimation, Application.

1. INTRODUCTION

For understanding the nature of lifetime data generated from many fields of real life we have to analyze the data by fitting appropriate probability model to that data. A lot of probability models have been fitted by researchers over the years to lifetime data. Numbers of parameters in a probability model fitted to any kind of data play a significant role in providing the better fit to the data. More are the parameters in the model more will be the amount of variation captured by model from the data. Also for obtaining more and more flexibility in applying probability models to the data a model with more parameters is preferred than model with less number of parameters. Exponentiation technique is one of the techniques which we can use to add extra parameters to the classical or existing models. Researchers have shown that exponentiated technique finds greater applicability in the real life in many situations as exponentiated models fit better to real life data as compared to other existing models. Nadarajah (2011) studied exponentiated exponential distribution with properties and applications in real life [1]. Gupta & Kundu (1999) introduced generalized exponential distributions [2]. Dey, Kumar, Ramos & Louzada (2017) formulated Exponentiated Chen distribution and obtained its vital properties [3]. Ashour & Eltehiwy (2015) introduced Exponentiated Power Lindley distribution and applied it to real life [4]. Haq (2016) studied transmuted exponentiated inverse Rayleigh distribution and obtained its various properties [5]. Shukla (2018) obtained Pranav distribution by mixing gamma distribution shape parameter 4 and exponential distribution by taking appropriate mixing proportions and obtained its applications in real life [6]. Hassan, Wani & Para (2018) developed three parameter Quasi Lindley distribution by weighting technique and studied various properties of that model [7]. Hassan, Wani, Shafi and Sheikh (2020) introduced Lindley-Quasi Xgamma Distribution (LQXD) and studied its applications along with properties [8].

Generalizing probability distributions by exponentiated technique to obtain more flexible probability distributions is practically needed in many situations. Here we have incorporated an extra parameter to Two Parameter Pranav distribution which is useful life time model introduced by Umeh & Ibenegbu (2019) [9].

A continuous r.v Y follows Two Parameter Pranav Distribution (TPPD) if its p.d.f $g(y)$ is of the form

$$g(y)=\frac{\theta^4}{(\alpha\theta^4+6)}\left(\alpha\theta+y^3\right)e^{-\theta y} \quad y>0,\theta>0,\alpha>0 \tag{1.1}$$

with the corresponding c.d.f $G(y)$ given below

$$G(y)=\left\{1-\left[1+\frac{\theta y(3\theta y+\theta^2 y^2+6)}{(\alpha\theta^4+6)}\right]e^{-\theta y}\right\} \quad y>0,\theta>0,\alpha>0 \tag{1.2}$$

Here we have used exponentiated technique to generalize Two Parameter Pranav Distribution.

The c.d.f of exponentiated distribution is obtained by from baseline distribution by using the below given relation

$$F(y)=\left(G(y)\right)^{\beta}$$

where $F(y)$ is the c.d.f of exponentiated model , β is the exponentiated parameter and $G(y)$ is the c.d.f of base model.

2. EXPONENTIATED TWO PARAMETER PRANAV DISTRIBUTION

A non-negative r v Y follows Exponentiated Two Parameter Pranav Distribution (ETPPD) with parameters $\theta(scale),\alpha(shape)\,\&\,\beta$ *(exponentiated)* if its cumulative distribution function (c.d.f) $F\left(y\right)$ is obtained as

$$F(y)=\left(G(y)\right)^{\beta} \tag{2.1}$$

$$F(y)=\left\{1-\left[1+\frac{\theta y(3\theta y+\theta^2 y^2+6)}{(\alpha\theta^4+6)}\right]e^{-\theta y}\right\}^{\beta} \quad y>0,\theta>0,\alpha>0,\beta>0 \tag{2.2}$$

where $G(y)$ is the c.d.f of Two Parameter Pranav Distribution given in (1.2)

The graphs of c.d.f plot for different values of parameters are plotted in Figures 1(a) & 1(b).

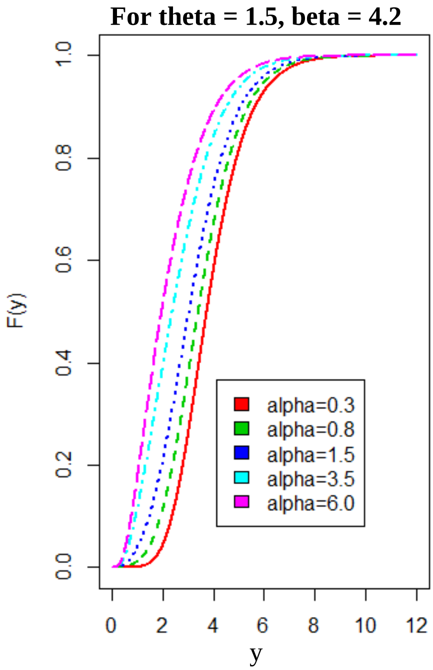

Figure 1(a): Graph of Cumulative Distribution Function

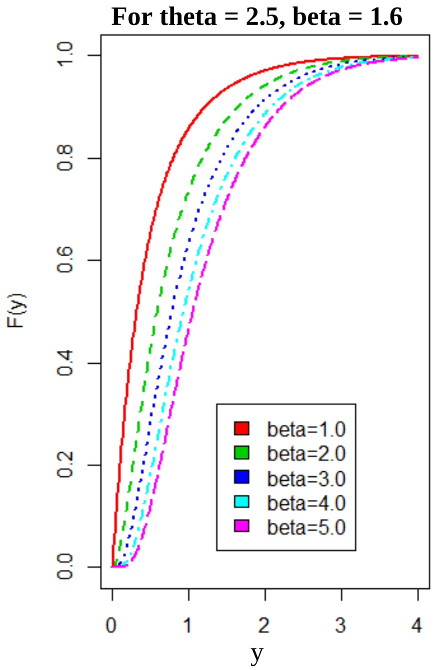

Figure 1(b): Graph of Cumulative Distribution Function

The corresponding p.d.f $f\left(y\right)$ of Exponentiated Two Parameter Pranav Distribution is obtained by differentiating (2.1) with respect to y on both sides and using (1.1) and (1.2) as

$$\frac{\partial}{\partial y}F(y)=\frac{\partial}{\partial y}\left(G(y)\right)^{\beta}$$

$$f(y)=\frac{\beta\theta^4}{(\alpha\theta^4+6)}\left\{1-\left[1+\frac{\theta y(3\theta y+\theta^2y^2+6)}{(\alpha\theta^4+6)}\right]e^{-\theta y}\right\}^{\beta-1}\left(\alpha\theta+y^3\right)e^{-\theta y}$$

$$y>0,\theta>0,\alpha>0,\beta>0 \quad (2.3)$$

The graphs of p.d.f for different parameter values are plotted in Figures 2(a) & 2(b) indicating that higher is the value of θ more peaked is the curve and more skewed is the curve.

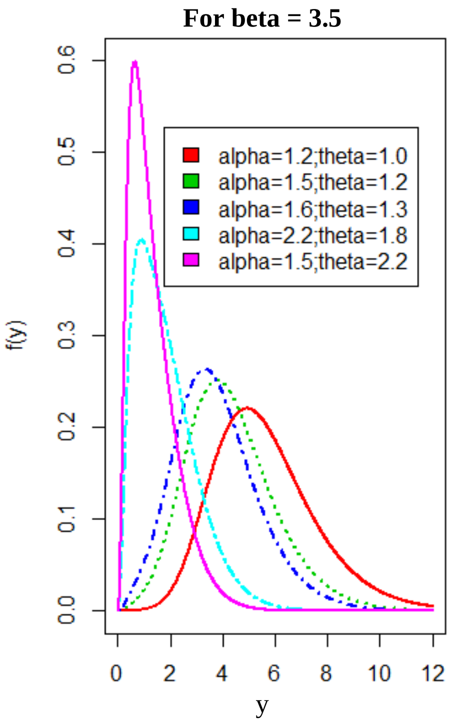

Figure 2(a): Graph of Density Function

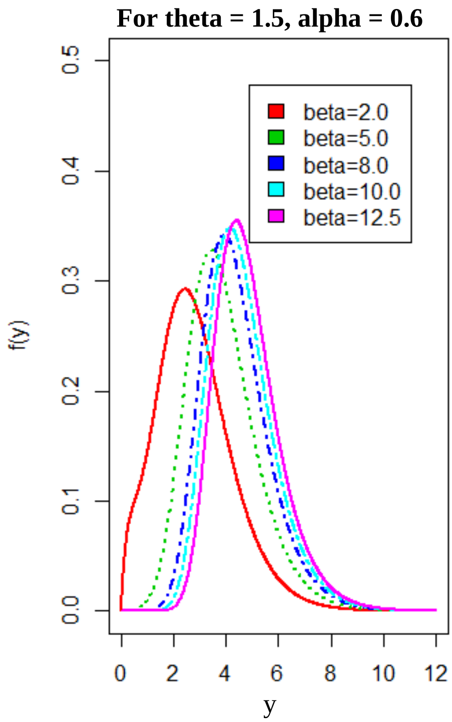

Figure 2(b): Graph of Density Function

3. RELIABILITY ANALYSIS

We derived expressions for various reliability measures of proposed exponentiated two parameter Pranav distribution in this section of paper.

3.1 Survival Function R(y)

The survival analysis $R(y,\alpha,\theta,\beta)$ gives the numerical value of chance of surviving a system with lifetime Y beyond a specified time 't'. The survival function of exponentiated two parameter Pranav distribution is found as:

$$R(y,\alpha,\theta,\beta)=1-F(y)$$

$$R(y,\alpha,\theta,\beta)=1-\left\{1-\left[1+\frac{\theta y(3\theta y+\theta^2 y^2+6)}{(\alpha\theta^4+6)}\right]e^{-\theta y}\right\}^{\beta}$$

The below graphs represent survival function of Exponentiated Two Parameter Pranav Distribution.

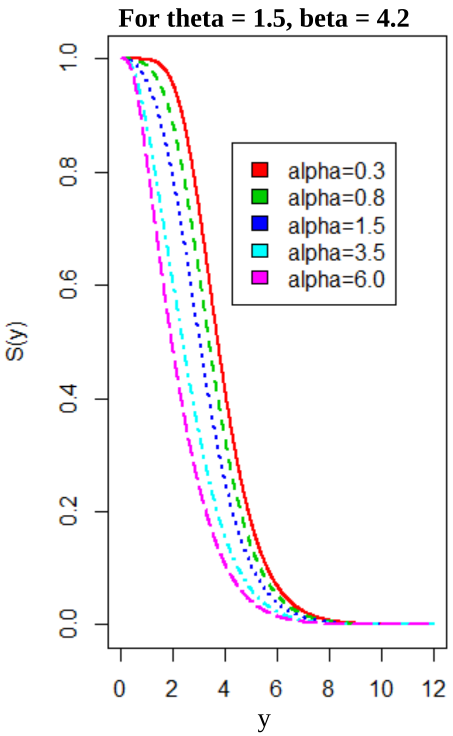

Figure 3(a): Graph of Survival Function

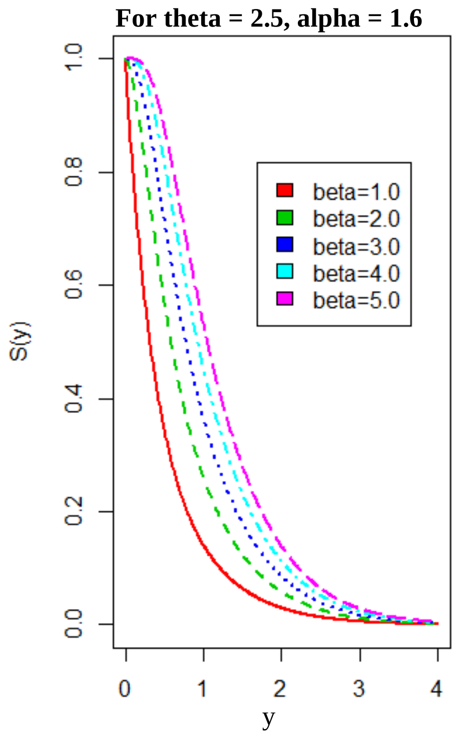

Figure 3(b): Graph of Survival Function

3.2 Hazard Function

The hazard function of exponentiated two parameter Pranav distribution is given as:

$$H.R = h\left(y;\alpha,\theta,\beta\right) = \frac{f(y)}{R\left(y,\theta,\alpha,\beta\right)}$$

$$= \frac{\dfrac{\beta\theta^4}{(\alpha\theta^4+6)}\left\{1-\left[1+\dfrac{\theta y(3\theta y+\theta^2 y^2+6)}{(\alpha\theta^4+6)}\right]e^{-\theta y}\right\}^{\beta-1}\left(\alpha\theta+y^3\right)e^{-\theta y}}{1-\left\{1-\left[1+\dfrac{\theta y(3\theta y+\theta^2 y^2+6)}{(\alpha\theta^4+6)}\right]e^{-\theta y}\right\}^{\beta}}$$

The below graphs represent hazard function of Exponentiated Two Parameter Pranav Distribution.

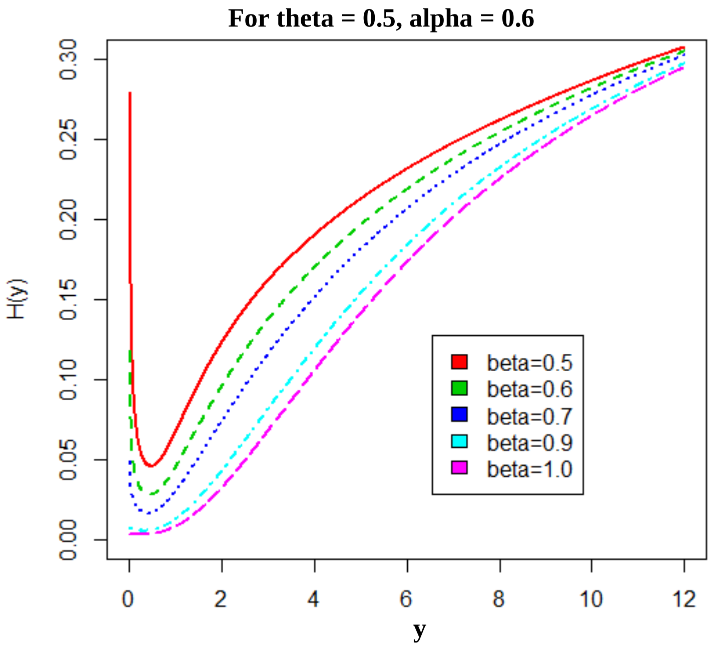

Figure 4: Grpah of Hazard Function

The graph of hazard rate of ETPPD reveals that proposed model increasing as well as decreasing hazard rate for different values of exponentiated parameter. In the beginning graph is decreasing then it is increasing after a short period and goes on increasing as value of y increases and for large values of y it becomes constant which is a common phenomenon in most real life situations.

3.3 Reverse Hazard Rate (R.H.R)

The R.H.R of the exponentiated two parameter Pranav distribution is specified as:

$$R.H.R = h_r\left(y,\alpha,\theta,\beta\right) = \frac{f(y)}{F(y)}$$

$$= \frac{\frac{\beta\theta^4}{(\alpha\theta^4+6)}\left\{1-\left[1+\frac{\theta y(3\theta y+\theta^2 y^2+6)}{(\alpha\theta^4+6)}\right]e^{-\theta y}\right\}^{\beta-1}\left(\alpha\theta+y^3\right)e^{-\theta y}}{\left\{1-\left[1+\frac{\theta y(3\theta y+\theta^2 y^2+6)}{(\alpha\theta^4+6)}\right]e^{-\theta y}\right\}^{\beta}}.$$

4. STATISTICAL PROPERTIES

Here we have obtained the different structural properties of the proposed exponentiated two parameter Pranav distribution. These comprise of moments, m.g.f and characteristic function.

4.1 Moments about origin of Exponentiated Two Parameter Pranav Distribution

Suppose Y is a random variable following ETPPD with parameters θ, α and β. Then the r^{th} moment about origin of ETPPD is found as

$$\mu_r' = E\left(Y^r\right) = \int_0^\infty y^r f(y)dy$$

$$= \int_0^\infty y^r \frac{\beta\theta^4}{(\alpha\theta^4+6)}\left\{1-\left[1+\frac{\theta y(3\theta y+\theta^2 y^2+6)}{(\alpha\theta^4+6)}\right]e^{-\theta y}\right\}^{\beta-1}\left(\alpha\theta+y^3\right)e^{-\theta y}dy$$

$$= \left\{\begin{array}{l} \int_0^\infty y^r \frac{\alpha\theta\beta\theta^4}{(\alpha\theta^4+6)}\left\{1-\left[1+\frac{\theta y(3\theta y+\theta^2 y^2+6)}{(\alpha\theta^4+6)}\right]e^{-\theta y}\right\}^{\beta-1} e^{-\theta y}dy \\ +\int_0^\infty y^{r+3} \frac{\beta\theta^4}{(\alpha\theta^4+6)}\left\{1-\left[1+\frac{\theta y(3\theta y+\theta^2 y^2+6)}{(\alpha\theta^4+6)}\right]e^{-\theta y}\right\}^{\beta-1} e^{-\theta y}dy \end{array}\right\} \quad (4.1.1)$$

Using the binomial expansion

$$\left\{1-\left[1+\frac{\theta y(3\theta y+\theta^2 y^2+6)}{(\alpha\theta^4+6)}\right]e^{-\theta y}\right\}^{\beta-1}$$

$$= \sum_{v=0}^{\infty}\binom{\beta-1}{v}(-1)^v\left[1+\frac{\theta y(3\theta y+\theta^2 y^2+6)}{(\alpha\theta^4+6)}\right]^v e^{-\theta v y} \quad (4.1.2)$$

And further using the binomial expansion

$$\left[1+\frac{\theta y(3\theta y+\theta^2 y^2+6)}{(\alpha\theta^4+6)}\right]^v = \sum_{j=0}^{\infty}\binom{v}{j}\left(\frac{\theta y(3\theta y+\theta^2 y^2+6)}{(\alpha\theta^4+6)}\right)^j$$

$$= \sum_{j=0}^{\infty}\binom{v}{j}\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}\frac{\theta^j y^j}{(\alpha\theta^4+6)^j}3^k 2^l \theta^{2j-k-l} y^{2j-k-l} \quad (4.1.3)$$

Using equation (4.1.3) in equation (4.1.2) we have

$$\left\{1-\left[1+\frac{\theta y(3\theta y+\theta^2 y^2+6)}{(\alpha\theta^4+6)}\right]e^{-\theta y}\right\}^{\beta-1}$$

$$= \sum_{v=0}^{\infty}\binom{\beta-1}{v}(-1)^v\sum_{j=0}^{\infty}\binom{v}{j}\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}\frac{\theta^j y^j}{(\alpha\theta^4+6)^j}3^k 2^l \theta^{2j-k-l} y^{2j-k-l} e^{-\theta v y} \quad (4.1.4)$$

Use equation (4.1.4) in (4.1.1), we get

$$E\left(Y^r\right)=\frac{\alpha\theta^5\beta}{(\alpha\theta^4+6)}\sum_{v=0}^{\infty}\binom{\beta-1}{v}(-1)^v$$
$$\sum_{j=0}^{\infty}\binom{v}{j}\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}\frac{\theta^j}{(\alpha\theta^4+6)^j}3^k 2^l\,\theta^{2j-k-l}\int_0^{\infty} y^{r+3j-k-l}e^{-\theta y(v+1)}dy$$
$$+\frac{\theta^4\beta}{(\alpha\theta^4+6)}\sum_{v=0}^{\infty}\binom{\beta-1}{v}(-1)^v$$
$$\sum_{j=0}^{\infty}\binom{v}{j}\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}\frac{\theta^j}{(\alpha\theta^4+6)^j}3^k 2^l\,\theta^{2j-k-l}\int_0^{\infty} y^{r+3j-k-l+3}e^{-\theta y(v+1)}dy$$

Using the relation

$$\int_0^{\infty} y^n e^{-\alpha y}dy=\frac{\Gamma(n+1)}{\alpha^{n+1}} \text{ in } E\left(Y^r\right) \text{ we get}$$

$$E\left(Y^r\right)=\frac{\alpha\theta^5\beta}{(\alpha\theta^4+6)}\sum_{v=0}^{\infty}\binom{\beta-1}{v}(-1)^v$$
$$\sum_{j=0}^{\infty}\binom{v}{j}\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}\frac{\theta^j}{(\alpha\theta^4+6)^j}3^k 2^l\,\theta^{2j-k-l}\theta^{-(r+3j-k-l+1)}$$
$$\left(\frac{(r+3j-k-l)!}{(r+1)^{r+3j-k-l+1}}\right)+\frac{\theta^4\beta}{(\alpha\theta^4+6)}\sum_{v=0}^{\infty}\binom{\beta-1}{v}(-1)^v$$
$$\sum_{j=0}^{\infty}\binom{v}{j}\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}\frac{\theta^j}{(\alpha\theta^4+6)^j}3^k 2^l\,\theta^{2j-k-l}\theta^{-(r+3j-k-l+4)}$$
$$\left(\frac{(r+3j-k-l+3)!}{(r+1)^{r+3j-k-l+4}}\right)$$
$$=\sum_{v=0}^{\infty}\binom{\beta-1}{v}(-1)^v\sum_{j=0}^{\infty}\binom{v}{j}\sum_{k=0}^{j}\binom{j}{k}$$
$$\sum_{l=0}^{k}\binom{k}{l}\frac{\alpha\beta\theta^{-r+4}}{(\alpha\theta^4+6)^{j+1}}3^k 2^l\left(\frac{(r+3j-k-l)!}{(r+1)^{r+3j-k-l+1}}\right)$$
$$+\sum_{v=0}^{\infty}\binom{\beta-1}{v}(-1)^v\sum_{j=0}^{\infty}\binom{v}{j}\sum_{k=0}^{j}\binom{j}{k}$$
$$\sum_{l=0}^{k}\binom{k}{l}\frac{\theta^{-r}\beta}{(\alpha\theta^4+6)^{j+1}}3^k 2^l\left(\frac{(r+3j-k-l+3)!}{(r+1)^{r+3j-k-l+4}}\right) \quad (4.1.5)$$

Equation (4.1.5) gives the r^{th} moment about origin of ETPPD.

4.2 Moment generating function and Characteristic function of Exponentiated Two Parameter Pranav Distribution (ETPPD)

Theorem 1.1:

If Y follows ETPPD$\left(\alpha,\theta,\beta\right)$, then the m.g.f $M_Y(t)$ and c.g.f $\varphi_Y\left(t\right)$ are

$$M_Y(t)=\sum_{v=0}^{\infty}\binom{\beta-1}{v}(-1)^v\sum_{j=0}^{\infty}\binom{v}{j}\sum_{l=0}^{j}\binom{j}{l}\sum_{m=0}^{l}\binom{l}{m}\sum_{h=0}^{\infty}\left(\frac{t}{\theta}\right)^h\frac{\alpha\beta 3^l 2^m\theta^4}{h!(\alpha\theta^4+6)^{j+1}}\cdot\frac{(h+3j-l-m)!}{(v+1)^{h+3j-l-m+1}}+\sum_{v=0}^{\infty}\binom{\beta-1}{v}(-1)^v\sum_{j=0}^{\infty}\binom{v}{j}\sum_{l=0}^{j}\binom{j}{l}\sum_{m=0}^{l}\binom{l}{m}\sum_{h=0}^{\infty}\left(\frac{t}{\theta}\right)^h\frac{\beta 3^l 2^m}{h!(\alpha\theta^4+6)^{j+1}}\frac{(h+3j-l-m+3)!}{(v+1)^{h+3j-l-m+4}}$$

and

$$\varphi Y(t)=\sum_{v=0}^{\infty}\binom{\beta-1}{v}(-1)^v\sum_{j=0}^{\infty}\binom{v}{j}\sum_{l=0}^{j}\binom{j}{l}\sum_{m=0}^{l}\binom{l}{m}\sum_{h=0}^{\infty}\left(\frac{it}{\theta}\right)^h\frac{\alpha\beta 3^l 2^m\theta^4}{h!(\alpha\theta^4+6)^{j+1}}\cdot\frac{(h+3j-l-m)!}{(v+1)^{h+3j-l-m+1}}+\sum_{v=0}^{\infty}\binom{\beta-1}{v}(-1)^v\sum_{j=0}^{\infty}\binom{v}{j}\sum_{l=0}^{j}\binom{j}{l}\sum_{m=0}^{l}\binom{l}{m}\sum_{h=0}^{\infty}\left(\frac{it}{\theta}\right)^h\frac{\beta 3^l 2^m}{h!(\alpha\theta^4+6)^{j+1}}\frac{(h+3j-l-m+3)!}{(v+1)^{h+3j-l-m+4}}$$

respectively.

Proof:

Suppose Y is a random variable following exponentiated two parameter Pranav distribution with parameters θ, α and β .Then the moment generating function for a given probability distribution is given by

$$M_Y(t)=E\left(e^{tY}\right)=\int_0^{\infty}e^{ty}f(y)dy$$

$$=\int_0^{\infty}e^{ty}\frac{\beta\theta^4}{(\alpha\theta^4+6)}\left\{1-\left[1+\frac{\theta y(\theta^2y^2+3\theta y+6)}{(\alpha\theta^4+6)}\right]e^{-\theta y}\right\}^{\beta-1}\left(\alpha\theta+y^3\right)e^{-\theta y}dy$$

$$=\left\{\begin{array}{l}\int_0^{\infty}e^{ty}\frac{\alpha\theta\beta\theta^4}{(\alpha\theta^4+6)}\left\{1-\left[1+\frac{\theta y(\theta^2y^2+3\theta y+6)}{(\alpha\theta^4+6)}\right]e^{-\theta y}\right\}^{\beta-1}e^{-\theta y}dy\\+\int_0^{\infty}y^3e^{ty}\frac{\beta\theta^4}{(\alpha\theta^4+6)}\left\{1-\left[1+\frac{\theta y(\theta^2y^2+3\theta y+6)}{(\alpha\theta^4+6)}\right]e^{-\theta y}\right\}^{\beta-1}e^{-\theta y}dy\end{array}\right\}\quad(4.2.1)$$

Using the binomial expansion

$$\left\{1-\left[1+\frac{\theta y(\theta^2 y^2+3\theta y+6)}{(\alpha\theta^4+6)}\right]e^{-\theta y}\right\}^{\beta-1}$$

$$=\sum_{v=0}^{\infty}\binom{\beta-1}{v}(-1)^v\left[1+\frac{\theta y(\theta^2 y^2+3\theta y+6)}{(\alpha\theta^4+6)}\right]^v e^{-\theta v y} \quad (4.2.2)$$

And further using the binomial expansion

$$\left[1+\frac{\theta y(\theta^2 y^2+3\theta y+6)}{(\alpha\theta^4+6)}\right]^v=\sum_{j=0}^{\infty}\binom{v}{j}\left(\frac{\theta y(\theta^2 y^2+3\theta y+6}{\alpha\theta^4+6}\right)^j$$

$$=\sum_{j=0}^{\infty}\binom{v}{j}\sum_{l=0}^{j}\binom{j}{l}\sum_{m=0}^{l}\binom{l}{m}\frac{\theta^j y^j}{(\alpha\theta^4+6)^j}3^l 2^m\theta^{2j-l-m}y^{2j-l-m} \quad (4.2.3)$$

Using equation (4.2.3) in equation (4.2.2) we have

$$\left\{1-\left[1+\frac{\theta y(\theta^2 y^2+3\theta y+6)}{(\alpha\theta^4+6)}\right]e^{-\theta y}\right\}^{\beta-1}$$

$$=\sum_{v=0}^{\infty}\binom{\beta-1}{v}(-1)^v\sum_{j=0}^{\infty}\binom{v}{j}\sum_{l=0}^{j}\binom{j}{l}\sum_{m=0}^{l}\binom{l}{m}\frac{\theta^j y^j}{(\alpha\theta^4+6)^j}3^l 2^m\theta^{2j-m-l}y^{2j-m-l}e^{-\theta v y}$$

Also $e^{ty}=\sum_{h=0}^{\infty}\frac{(ty)^h}{h!}$

Substituting these values in we get

$$M_Y(t)=\sum_{v=0}^{\infty}\binom{\beta-1}{v}(-1)^v\sum_{j=0}^{\infty}\binom{v}{j}\sum_{l=0}^{j}\binom{j}{l}\sum_{m=0}^{l}\binom{l}{m}$$

$$\sum_{h=0}^{\infty}\frac{t^h}{h!}\frac{\alpha\beta}{(\alpha\theta^4+6)^{j+1}}3^l 2^m\theta^{3j-m-l+5}\int_0^{\infty}y^{h+3j-l-m}e^{-\theta y(v+1)}dx$$

$$+\sum_{v=0}^{\infty}\binom{\beta-1}{v}(-1)^v\sum_{j=0}^{\infty}\binom{v}{j}\sum_{l=0}^{j}\binom{j}{l}\sum_{m=0}^{l}\binom{l}{m}$$

$$\sum_{h=0}^{\infty}\frac{t^h}{h!}\frac{\beta}{(\alpha\theta^4+6)^{j+1}}3^l 2^m\theta^{3j-m-l+4}\int_0^{\infty}y^{h+3j-l-m+3}e^{-\theta y(v+1)}dx$$

$\int_0^{\infty}y^n e^{-\alpha y}dy=\frac{\Gamma(n+1)}{\alpha^{n+1}}$ in $M_X(t)$ we get

$$M_Y(t) = \sum_{v=0}^{\infty}\binom{\beta-1}{v}(-1)^v \sum_{j=0}^{\infty}\binom{v}{j}\sum_{l=0}^{j}\binom{j}{l}$$
$$\sum_{m=0}^{l}\binom{l}{m}\sum_{h=0}^{\infty}\left(\frac{t}{\theta}\right)^h \frac{\alpha\beta 3^l 2^m \theta^4}{h!(\alpha\theta^4+6)^{j+1}}\cdot\frac{(h+3j-l-m)!}{(v+1)^{h+3j-l-m+1}}$$
$$+\sum_{v=0}^{\infty}\binom{\beta-1}{v}(-1)^v \sum_{j=0}^{\infty}\binom{v}{j}\sum_{l=0}^{j}\binom{j}{l}\sum_{m=0}^{l}\binom{l}{m}$$
$$\sum_{h=0}^{\infty}\left(\frac{t}{\theta}\right)^h \frac{\beta 3^l 2^m}{h!(\alpha\theta^4+6)^{j+1}}\frac{(h+3j-l-m+3)!}{(v+1)^{h+3j-l-m+4}}$$

Also we know that $\varphi_Y(t) = M_Y(it)$

Therefore,

$$\varphi Y(t) = \sum_{v=0}^{\infty}\binom{\beta-1}{v}(-1)^v \sum_{j=0}^{\infty}\binom{v}{j}\sum_{l=0}^{j}\binom{j}{l}$$
$$\sum_{m=0}^{l}\binom{l}{m}\sum_{h=0}^{\infty}\left(\frac{it}{\theta}\right)^h \frac{\alpha\beta 3^l 2^m \theta^4}{h!(\alpha\theta^4+6)^{j+1}}\cdot\frac{(h+3j-l-m)!}{(v+1)^{h+3j-l-m+1}}$$
$$+\sum_{v=0}^{\infty}\binom{\beta-1}{v}(-1)^v \sum_{j=0}^{\infty}\binom{v}{j}\sum_{l=0}^{j}\binom{j}{l}\sum_{m=0}^{l}\binom{l}{m}$$
$$\sum_{h=0}^{\infty}\left(\frac{it}{\theta}\right)^h \frac{\beta 3^l 2^m}{h!(\alpha\theta^4+6)^{j+1}}\frac{(h+3j-l-m+3)!}{(v+1)^{h+3j-l-m+4}} \quad (4.2.4)$$

which is the characteristic function of exponentiated two parameter Pranav distribution.

5. ORDER STATISTICS OF EXPONENTIATED TWO PARAMETER PRANAV DISTRIBUTION

Consider $Y_{(1)}, Y_{(2)}, Y_{(3)}....,Y_{(n)}$ to be an ordered statistic of randomly selected sample $y_1, y_2, y_3,....y_n$ obtained from the ETPPD with c.d.f $F(y)$ and p.d.f $f(y)$, then the probability density function of p^{th} order statistics $Y_{(p)}$ is specified as:

$$f_{(p)}(y,\alpha,\theta,\beta) = \frac{n!}{(p-1)!\,p!} f(y)\left[F(y)\right]^{p-1}\left[1-F(y)\right]^{n-p}. \; r=1, 2, 3\ldots n$$

By making use of equations (2.2) and (2.3), the p.d.f of p^{th} order statistics of ETPPD is specified as:

$$f_{(p)}\left(y,\alpha,\theta,\beta\right)=\left[\begin{array}{l}\frac{n!}{\left(p-1\right)!\left(n-p\right)!}\frac{\beta\theta^4}{(\alpha\theta^4+6)} \\ \quad\left\{1-\left[1+\frac{\theta y(\theta^2y^2+3\theta y+6)}{(\alpha\theta^4+6)}\right]e^{-\theta y}\right\}^{\beta-1}\left(\alpha\theta+y^3\right)e^{-\theta y} \\ \left[\left\{1-\left[1+\frac{\theta y(\theta^2y^2+3\theta y+6)}{(\alpha\theta^4+6)}\right]e^{-\theta y}\right\}^{\beta}\right]^{p-1} \\ \quad\left[1-\left\{1-\left[1+\frac{\theta y(\theta^2y^2+3\theta y+6)}{(\alpha\theta^4+6)}\right]e^{-\theta y}\right\}^{\beta}\right]^{n-p}\end{array}\right].$$

Then, the pdf of first order $Y_{(1)}$ ETPPD is specified by:

$$f_{(1)}\left(x,\alpha,\theta,\beta\right)=\left\{\begin{array}{l}n\frac{\beta\theta^4}{(\alpha\theta^4+6)}\left\{1-\left[1+\frac{\theta y(\theta^2y^2+3\theta y+6)}{(\alpha\theta^4+6)}\right]e^{-\theta y}\right\}^{\beta-1}\left(\alpha\theta+y^3\right)e^{-\theta y} \\ \left[1-\left\{1-\left[1+\frac{\theta y(\theta^2y^2+3\theta y+6)}{(\alpha\theta^4+6)}\right]e^{-\theta y}\right\}^{\beta}\right]^{n-1}.\end{array}\right\}$$

and the pdf of n^{th} order $Y_{(n)}$ statistic of ETPPD is specified as:

$$f_{(n)}\left(x,\alpha,\theta,\beta\right)=\left\{\begin{array}{l}n\frac{\beta\theta^4}{(\alpha\theta^4+6)}\left\{1-\left[1+\frac{\theta y(\theta^2y^2+3\theta y+6)}{(\alpha\theta^4+6)}\right]e^{-\theta y}\right\}^{\beta-1}\left(\alpha\theta+y^3\right)e^{-\theta y} \\ \left[\left\{1-\left[1+\frac{\theta y(\theta^2y^2+3\theta y+6)}{(\alpha\theta^4+6)}\right]e^{-\theta y}\right\}^{\beta}\right]^{n-1}\end{array}\right\}.$$

6. ESTIMATION OF PARAMETERS OF EXPONENTIATED TWO PARAMETER PRANAV DISTRIBUTION

We have made use of method of maximum likelihood estimation method for estimating parameters of ETPPD.

Consider $x_1, x_2, x_3.....x_n$ to be the randomly selected sample of size n drawn from ETPPD then the likelihood function of ETPPD is specified as:

$$L\left(x\mid c,\theta\right)=\prod_{i=1}^{n}\left[\frac{\beta\theta^4}{(\alpha\theta^4+6)}\left\{1-\left[1+\frac{\theta y(\theta^2y^2+3\theta y+6)}{(\alpha\theta^4+6)}\right]e^{-\theta y}\right\}^{\beta-1}\left(\alpha\theta+y^3\right)e^{-\theta y}\right]$$

The log likelihood function becomes:

$$\log L=\left\{\begin{array}{l} n\log\beta+4n\log\theta+(\beta-1) \\ \sum_{i=1}^{n}\log\left[1-\left\{1+\frac{\theta y_i(\theta^2y_i^2+3\theta y_i+6)}{(\alpha\theta^4+6)}\right\}e^{-\theta y_i}\right]+\sum_{i=1}^{n}\log(\alpha\theta+y_i^3)-\theta\sum_{i=1}^{n}y_i\end{array}\right\} \tag{6.1}$$

Now we differentiate equation (6.1) w. r. t α, θ, β and equate the result to zero to get the following normal equations.

$$\frac{\partial\log L}{\partial\theta}=\left\{\frac{4n}{\theta}+(\beta-1)\sum_{i=1}^{n}\left[y_ie^{-\theta y_i}\right.\right.$$

$$-\frac{\left\{\frac{(\alpha\theta^4+6)\begin{pmatrix} y_i^3\theta^2e^{-\theta y_i}(3-\theta y_i)+3y_i^2\theta \\ e^{-\theta y_i}(2-\theta y_i)+6y_ie^{-\theta y_i}(1-\theta y_i)\end{pmatrix}-4\alpha\theta^4y_i(\theta^2y_i^2+3\theta y_i+6)}{(\alpha\theta^4+6)^2}\right\}}{\left[1-\left\{1+\frac{\theta y_i(\theta^2y_i^2+3\theta y_i+6)}{(\alpha\theta^4+6)}\right\}e^{-\theta y_i}\right]}$$

$$\left.\left.-\sum_{i=1}^{n}y_i+\sum_{i=1}^{n}\left(\frac{\alpha}{\alpha\theta+y_i^3}\right)\right]\right]=0. \tag{6.2}$$

$$\frac{\partial\log L}{\partial\alpha}=\left\{(\beta-1)\sum_{i=1}^{n}\left[\frac{\left\{-\frac{\theta^5y_i(\theta^2y_i^2+3\theta y_i+6)}{(\alpha\theta^4+6)^2}\right\}}{\left[1-\left\{1+\frac{\theta y_i(\theta^2y_i^2+3\theta y_i+6)}{(\alpha\theta^4+6)}\right\}e^{-\theta y_i}\right]}\right]+\sum_{i=1}^{n}\left(\frac{\theta}{\alpha\theta+y_i^3}\right)\right\}=0 \tag{6.3}$$

$$\frac{\partial\log L}{\partial\beta}=\frac{n}{\beta}+\sum_{i=}^{n}\left\{\log\left[1-\left\{1+\frac{\theta y_i(\theta^2y_i^2+3\theta y_i+6)}{(\alpha\theta^4+6)}\right\}e^{-\theta y_i}\right]\right\} \tag{6.4}$$

Since MLEs of α, θ, β cannot be obtained by solving above equations as these equations are not in closed form. So we use Fishers scoring method to obtain MLEs of α, θ, β through R software.

7. APPLICATIONS OF EXPONENTIATED TWO PARAMETER PRANAV DISTRIBUTION

Exponentiated Two Parameter Pranav Distribution and its two sub models are fitted to two survival time data sets for observing the importance and applicability of proposed model over its sub models in real life.

Data Set 1: Data set which is given in table 1 represents the survival times of one hundred twenty one patients having breast cancer and has been taken from Oguntunde et al. [10].

Table 1
Survival Times of One Hundred Twenty One Patients having Breast Cancer

0.3	0.3	4.0	5.0	5.6	6.2	6.3	6.6	6.8	7.4	7.5	8.4
8.4	10.3	11.0	11.8	12.2	12.3	13.5	14.4	14.4	14.8	15.5	15.7
16.2	16.3	16.5	16.8	17.2	17.3	17.5	17.9	19.8	20.4	20.9	21.0
21.0	21.1	23.0	23.4	23.6	24.0	24.0	27.9	28.2	29.1	30.0	31.0
31.0	32.0	35.0	35.0	37.0	37.0	37.0	38.0	38.0	38.0	39.0	39.0
40.0	40.0	40.0	41.0	41.0	41.0	42.0	43.0	43.0	43.0	44.0	45.0
45.0	46.0	46.0	47.0	48.0	49.0	51.0	51.0	51.0	52.0	54.0	55.0
56.0	57.0	58.0	59.0	60.0	60.0	60.0	61.0	62.0	65.0	65.0	67.0
67.0	68.0	69.0	78.0	80.0	83.0	88.0	89.0	90.0	93.0	96.0	103.0
105.0	109.0	109.0	111.0	115.0	117.0	125.0	126.0	127.0	129.0	129.0	139.0
154.0											

Data Set 2: The data set which is presented in table 2 is about the failure times in hours in an accelerated life test of 59 conductors without any censored observation and has been obtained from Lawless [11].

Table 2
Failure Times of Fifty Nine Conductors

2.997	4.137	4.288	4.531	4.700	4.706	5.009	5.381	5.434	5.459	5.589
5.640	5.807	5.923	6.033	6.071	6.087	6.129	6.352	6.369	6.476	6.492
6.515	6.522	6.538	6.545	6.573	6.725	6.869	6.923	6.948	6.956	6.958
7.024	7.224	7.365	7.398	7.459	7.489	7.495	7.496	7.543	7.683	7.937
7.945	7.974	8.120	8.336	8.532	8.591	8.687	8.799	9.218	9.254	9.289
9.663	10.092	10.491	11.038							

For analyzing data sets 1 & 2 we made use of R software version 3.3.4. We fitted Exponentiated Two Parameter Pranav distribution and its sub models (OPPD, TPPD) to these two real life data sets. The summary statistic of data sets 1 & 2 is given in table 3. The estimates of the parameters, model functions are displayed in table 4 for these two data sets. The goodness of fit measures which include Kolmogorov statistic, p-value, log-likelihood values, AIC, AICC, HQIC, BIC & Shannon's entropy are given in table 5 & 6 for data sets 1 & 2 respectively.

Table 3
Summary Statistic of Data Sets 1 & 2

Data Set	No. of Observations	Min.	First Quartile	Median	Mean	Third Quartile	Max.
1	121	0.30	17.50	40.00	46.33	60.00	154.00
2	59	2.997	6.052	6.923	6.980	7.941	11.038

Table 4
ML Estimates, Standard Error of Estimates in Parenthesis, Model Function of Sub Models and Proposed Model for Data Sets 1 & 2

Data Set	Distribution	ML Estimates with Standard Errors	Model Function
1	**Exponentiated Two Parameter Pranav Distribution (ETPPD)**	$\hat{\alpha}=0.39963405$ $\hat{\theta}=0.05267800$ (0.00487525) $\hat{\beta}=0.34995491$ (0.04168481)	$\frac{\beta\theta^4}{(\alpha\theta^4+6)}\left\{1-\left[1+\frac{\theta y(\theta^2y^2+3\theta y+6)}{(\alpha\theta^4+6)}\right]e^{-\theta y}\right\}^{\beta-1}\left(\alpha\theta+y^3\right)e^{-\theta y}$
	One Parameter Pranav Distribution (OPPD)	$\hat{\theta}=0.0866$ (0.0039)	$\frac{\theta^4}{(\theta^4+6)}\left(\theta+y^3\right)e^{-\theta y}$
	Two Parameter Pranav Distribution (TPPD)	$\hat{\alpha}=1.289168\times10^5$ $\hat{\theta}=6.695151\times10^2$	$\frac{\theta^4}{(\alpha\theta^4+6)}\left(\alpha\theta+y^3\right)e^{-\theta y}$
2	**Exponentiated Two Parameter Pranav Distribution (ETPPD)**	$\hat{\alpha}=8.29324487$ (15.17002004) $\hat{\theta}=1.04142636$ (0.07425552) $\hat{\beta}=22.77703610$ (26.06190957)	$\frac{\beta\theta^4}{(\alpha\theta^4+6)}\left\{1-\left[1+\frac{\theta y(\theta^2y^2+3\theta y+6)}{(\alpha\theta^4+6)}\right]e^{-\theta y}\right\}^{\beta-1}\left(\alpha\theta+y^3\right)e^{-\theta y}$
	One Parameter Pranav Distribution (OPPD)	$\hat{\theta}=$ 0.56391429 (0.03584635)	$\frac{\theta^4}{(\theta^4+6)}\left(\theta+y^3\right)e^{-\theta y}$
	Two Parameter Pranav Distribution (TPPD)	$\hat{\alpha}=0.0010$ $\hat{\theta}=$ 0.5730509 (0.01684118)	$\frac{\theta^4}{(\alpha\theta^4+6)}\left(\alpha\theta+y^3\right)e^{-\theta y}$

Table 5
Model Comparison of Proposed Model and its related Models for Data Set 1

Distribution	$-\log L$	AIC	BIC	AICC	HQIC	Shannon Entropy *H(X)*	Kolmogorov Statistic (D)	P-value
Exponentiated Two Parameter Pranav Distribution (ETPPD)	579.842	1165.68	1174.07	1165.88	1169.09	4.792	0.06491	0.6878
One Parameter Pranav Distribution (OPPD)	625.391	1252.782	1255.578	1252.816	1253.91	5.16	0.20086	0.000115
Two Parameter Pranav Distribution (TPPD)	583.434	1170.869	1176.46	1170.97	1173.14	4.82	0.073285	0.5342

Table 6
Model Comparison of Proposed Model and its Sub Models for Data Set 2

Distribution	$-\log L$	AIC	BIC	AICC	HQIC	Shannon Entropy *H(X)*	Kolmogorov Statistic (D)	P-value
Exponentiated Two Parameter Pranav Distribution (ETPPD)	112.8360	231.672	237.904	232.108	234.104	1.91	0.08489	0.7567
One Parameter Pranav Distribution (OPPD)	134.9950	271.989	274.067	272.060	272.800	2.288	0.25164	0.0008856
Two Parameter Pranav Distribution (TPPD)	134.1249	272.249	276.404	272.464	273.871	2.27	0.25285	0.0008215

In order to compare the proposed model with its sub models we computed loss of information criteria's AIC, AICC, BIC & HQIC which represent the loss of information resulting from fitting probability models to data. The better model possesses lesser AIC, AICC, BIC & HQIC values. Also we computed the Shannon's entropy *(H(X))* which represents the average uncertainty. The better model possesses lesser Shannon's entropy value.

$$\text{AIC} = 2\text{k}-2\log L \quad \text{AICC} = \text{AIC}+\frac{2k(k+1)}{n-k-1}$$

$$\text{BIC} = \text{k} \log n-2\log L \quad \text{HQIC}=2\text{k} \log(\log(\text{n}))+2 \log L$$

$$H(X) = -\frac{\text{logL}}{\text{n}}$$

where

k= number of parameters in model
n= size of the sample (number of observations in data set)
logL= value of likelihood function of model

From Tables 5 & 6, it has been observed that the Exponentiated two parameter Pranav distribution possesses lesser AIC, AICC BIC, HQIC and *H(X)* values as compared to one parameter Pranav distribution and two parameter Pranav distribution for data sets 1 & 2 respectively. Hence we can conclude that proposed model leads to better fit than OPPD and TPPD for data sets 1 & 2 respectively.

Also for testing the goodness of fit of proposed model and its sub models to data sets 1 & 2 we computed Kolmogorov statistic and p- value. The model which possesses lesser Kolmogorov statistic value and higher p-value fits better to data set. From Table 5 & 6 it is observed that proposed ETPPD possesses lesser Kolmogorov statistic value and higher p-value than OPPD and TPPD. Hence proposed model fits better to both the data sets.

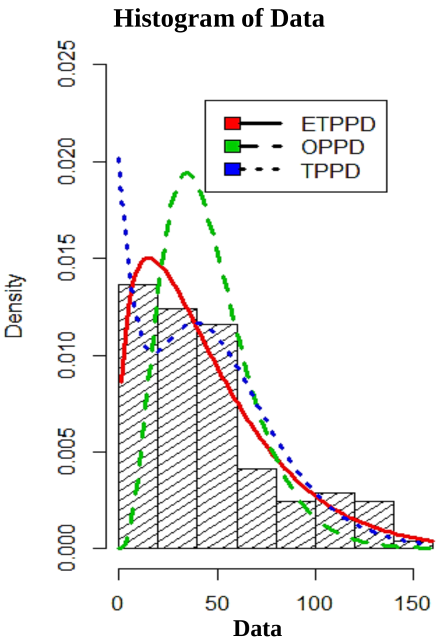

Figure 5: Graph of Data Set 1 Fitted by Proposed Model and related Models

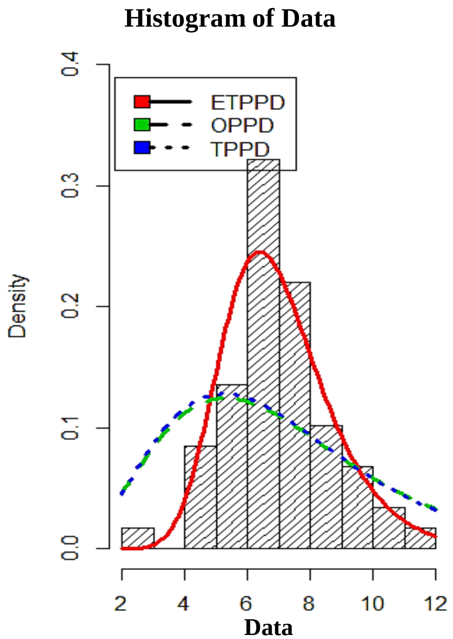

Figure 6: Graph of Data Set 1 Fitted by Proposed Model and related Models

8. SPECIAL CASES OF EXPONENTIATED TWO PARAMETER PRANAV DISTRIBUTION

Case I:

If we put $\beta = 1$, then Exponentiated Two Parameter Pranav distribution (2.3) reduces to two parameter Pranav distribution with probability density function as:

$$f(y) = \frac{\theta^4}{(\alpha\theta^4 + 6)}\left(\alpha\theta + y^3\right)e^{-\theta y} \quad y > 0, \alpha > 0, \theta > 0$$

Case II:

For $\beta = 1, \alpha = 1$, Exponentiated Two Parameter Pranav distribution (2.3) reduces to one parameter Pranav distribution with probability density function given as

$$f(y) = \frac{\theta^4}{(\theta^4 + 6)}\left(\theta + y^3\right)e^{-\theta y} \quad y > 0, \theta > 0\,.$$

9. CONCLUSION

We incorporated Exponentiated Two Parameter Pranav distribution by using exponentiated technique from two parameter Pranav distribution. We obtained crucial properties of our proposed model. We also obtained the estimates of proposed model by using maximum likelihood method of estimation. Finally we fitted our model and its related models to two real life data sets and computed various goodness of fit measures, hence concluded that our model gives better fit to these data sets as compared to its related models and hence our model finds greater applicability in real life.

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