Pak. J. Statist.
2020 Vol. 36(3), 225-250

A NEW GENERALIZATION OF RAYLEIGH DISTRIBUTION: PROPERTIES AND APPLICATIONS

A. A. Bhat§ and S.P. Ahmad
Department of Statistics, University of Kashmir, Srinagar, India.
§Corresponding author Email: bhatajaz.msst@gmail.com

ABSTRACT

In this research article, we formulate a new lifetime probability model, named Power Rayleigh distribution (PRD). Numerous significant properties of PRD are acquired including moments, moment generating function, hazard rate, mean residual life, order statistics and quantiles. The stochastic ordering of random variables has also been established. The expression for four different measures of entropy viz., Shannon entropy, Renyi entropy, beta entropy and Mathai and Haubold entropy are also obtained. Maximum likelihood estimation procedure is employed to estimate the unknown parameters. In addition, the practical importance of PRD is illustrated by means of two real data sets.

KEYWORDS

Rayleigh Distribution, Power Rayleigh Distribution, Statistical Properties, Stochastic Ordering, Order Statistics, Entropy, Maximum Likelihood Estimation.

1. INTRODUCTION

Rayleigh distribution is the well-known probability distribution named after Lord Rayleigh (1880) and has significant applications for modeling data in engineering and medical sciences. Siddiqui (1962) studied the genesis and other features of this model. Due to its importance in diverse fields, numerous authors has carried out work on this model, namely, Howlader and Hossain (1995), Voda (2005), Ahmad et al. (2014), Kundu and Raqab (2005), Merovci (2013), Ahmad et al. (2017) , Gazal and Hasaballah (2017), Ajami and Jahanshahi (2017), Ateeq et al. (2019) and Sofi et al. (2019).

Statistical distributions have received a broad attention in order to unfold flexible models for modeling diverse data sets. Since, the classical distributions lack superiority in modeling data sets with variable nature. This leads to the growth and expansion of generalized probability models. Designing a new probability model from the formerly constructed models by using different approaches has got enormous scope in the recent years. One such approach used by different researchers is power transformation technique by which an extra parameter is added to the parent distribution. Induction of an extra parameter in the parent model usually provides greater flexibility and improves the goodness of fit. There are plethora of researchers who worked on power generalization of probability models, among them are Meniconi and Barry (1996), Ghitany et al. (2013), Zaka and Akhter (2013), Rady et al. (2016), Krishnarani (2016) and Shukla and Shanker (2018).

Let the random variable X follows Rayleigh distribution with scale parameter θ, then its probability density function (p.d.f) and cumulative distribution function (c.d.f) takes the form

$$f(x;\theta) = \frac{x}{\theta^2} exp\left(-\frac{x^2}{2\theta^2}\right);\ x > 0, \theta > 0,$$

$$F(x;\theta) = 1 - exp\left(-\frac{x^2}{2\theta^2}\right).$$

2. POWER RAYLEIGH DISTRIBUTION (PRD)

The main aim of this research article is to enhance the flexibility of Rayleigh model by formulating an extended version of the model based on power transformation technique. Let X is a variable follows Rayleigh distribution with parameter θ, then the transformed variable $V = X^{\frac{1}{\eta}}$ will follow Power Rayleigh distribution with parameters η and θ.

The density function and distribution function of transformed distribution i.e., Power Rayleigh distribution with two parameters η (shape parameter) and θ (scale parameter) is obtained as:

$$f(v;\eta,\theta) = \frac{\eta}{\theta^2} v^{2\eta-1} exp\left(-\frac{v^{2\eta}}{2\theta^2}\right);\ v > 0, \eta, \theta > 0, \tag{1}$$

$$F(v;\eta,\theta) = 1 - exp\left(-\frac{v^{2\eta}}{2\theta^2}\right). \tag{2}$$

2.1 Shape Behavior of Density Function

We will discuss the shape characteristics of density function $f(v;\eta,\theta)$ given in (1) of PRD. The behavior of pdf at $v = 0$ and $v = \infty$ respectively are given as

$$f(0) = \begin{cases} \infty, if\ \eta < 0.5 \\ \frac{0.5}{\theta^2}, if\ \eta = 0.5; \\ 0, if\ \eta > 0.5 \end{cases} \qquad f(\infty) = 0$$

Theorem 2.1.1:

The pdf $f(v;\eta,\theta)$ in (1) of the PRD is

(i) Decreasing if $\{0 < \eta \leq 0.5, \theta > 0\}$;
(ii) Unimodal if $\{\eta > 0.5, \theta > 0\}$.

Proof:

Since, the first order derivative of $f(v;\eta,\theta)$ will yield

$$f'(v;\eta,\theta) = \frac{g(v;\eta,\theta)}{\theta^2 v} f(v;\eta,\theta),$$

where,

$$g(v;\eta,\theta) = (2\eta - 1)\theta^2 - \eta v^{2\eta}.$$

Case (i)

For $0 < \eta \leq 0.5, g(v;\eta,\theta) < 0$. Hence $f'(v;\eta,\theta) < 0$ which clearly shows that $f(v;\eta,\theta)$ is decreasing function.

Case (ii)

For $\eta > 0.5, f'(v;\eta,\theta) = 0$ iff $g(v;\eta,\theta) = 0$ which implies that $f(v;\eta,\theta)$ has mode at v_0 as given by

$$v_0 = \left[\frac{(2\eta - 1)\theta^2}{\eta}\right]^{\frac{1}{2\eta}}.$$

Since, the second order derivative w.r.t v yields the solution as

$$f''(v;\eta,\theta) = -\frac{2\eta^2 v^{2\eta-1}}{\theta^2 v_0} f(v_0;\eta,\theta) < 0.$$

This implies that $f(v;\eta,\theta)$ has local maximum at v_0.

The density function and distribution function plots of the PRD at different parameter values are displayed in figure 1 and figure 2 respectively.

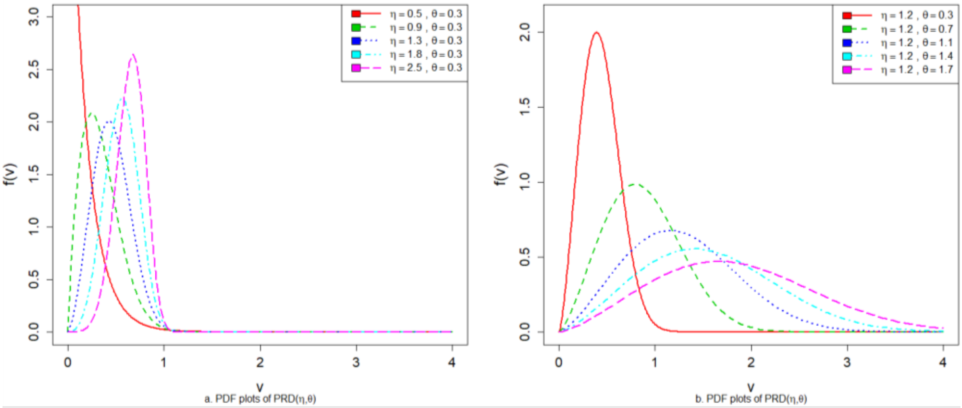

Figure 1: Density Plots of PRD for Different Values of Shape and Scale Parameters

It is evident from the density function plots that the proposed model is unimodal, decreasing, symmetric and right skewed in nature.

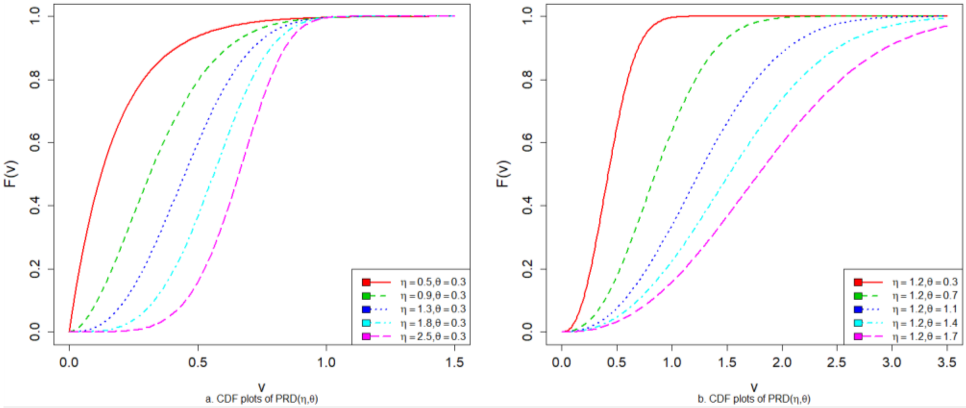

Figure 2: Distribution Function Plots of PRD for different Values of Shape and Scale Parameters

3. RELIABILITY ANALYSIS OF THE PRD

This section is devoted to acquire the expression for reliability (Survival) function, hazard (failure) rate, reverse failure rate, cumulative failure rate and Mills ratio of PRD.

The expression for reliability or survival function of PRD is expressed as:

$$S(v;\eta,\theta) = 1 - F(v;\eta,\theta) = exp\left(-\frac{v^{2\eta}}{2\theta^2}\right). \tag{3}$$

The Hazard rate assess the ability of a lifetime component to fail or to expire depending on the life completed and thus has wide variety of applications in differentiating lifetime distributions.

Using (1) and (3), the hazard rate function of PRD is obtained as

$$h(v;\eta,\theta) = \frac{f(v;\eta,\theta)}{1 - F(v;\eta,\theta)} = \frac{\eta}{\theta^2}v^{2\eta-1}.$$

The expression for reversed failure rate $r(v;\eta,\theta)$, cumulative hazard rate $\Lambda_{PRD}(v;\eta,\theta)$ and Mills Ratio of PRD are respectively obtained as

$$r(v;\eta,\theta) = \frac{f(v;\eta,\theta)}{F(v;\eta,\theta)} = \frac{\eta v^{2\eta-1} exp\left(-\frac{v^{2\eta}}{2\theta^2}\right)}{\theta^2\left\{1 - exp\left(-\frac{v^{2\eta}}{2\theta^2}\right)\right\}}.$$

$$\Lambda_{PRD}(v;\eta,\theta) = -logS(v;\eta,\theta) = \frac{v^{2\eta}}{2\theta^2}.$$

$$M.R. = \frac{F(v;\eta,\theta)}{1 - F(v;\eta,\theta)} = \frac{1 - exp\left(-\frac{v^{2\eta}}{2\theta^2}\right)}{exp\left(-\frac{v^{2\eta}}{2\theta^2}\right)}.$$

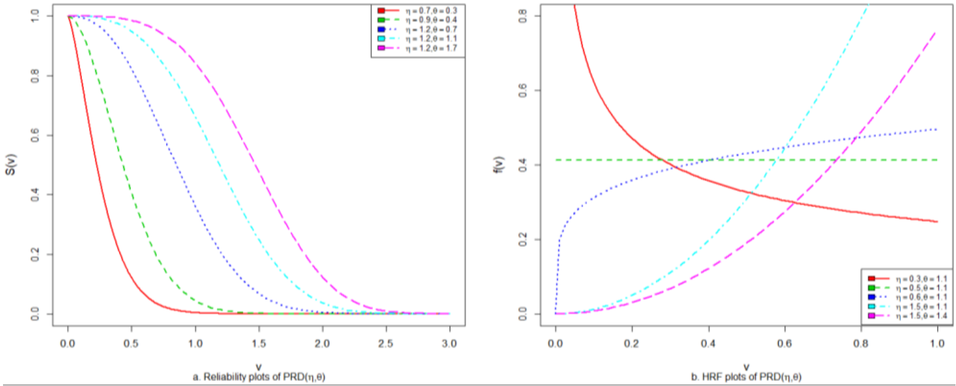

Figure 3: Reliability and Hazard Rate Plots of PRD for Selected Values of Shape and Scale Parameters

3.1 Residual and Reversed Residual Life Functions

This part is devoted to explore some related statistical properties, such as survival function, hazard rate, density function and mean related to residual and reversed residual life functions of PRD.

3.1.1 Residual Life Function

The conditional random variable $R_{(t)} = (V - t/V > t); t \geq 0$ is used to explain the residual life of a lifetime component. The survival function of residual lifetime $R_{(t)}, t \geq 0$ for PRD is defined as

$$S_{R_{(t)}}(v; \eta, \theta) = \frac{S(v+t)}{S(v)} = exp\left(\frac{t^{2\eta} - (v+t)^{2\eta}}{2\theta^2}\right).$$

The cdf and pdf of residual lifetime random variable $R_{(t)}, t \geq 0$ are respectively given as

$$F_{R_{(t)}}(v; \eta, \theta) = 1 - exp\left(\frac{t^{2\eta} - (v+t)^{2\eta}}{2\theta^2}\right).$$

$$f_{R_{(t)}}(v; \eta, \theta) = \frac{\eta}{\theta^2}(v+t)^{2\eta-1} exp\left(\frac{t^{2\eta} - (v+t)^{2\eta}}{2\theta^2}\right).$$

Accordingly, the associated failure rate of $R_{(t)}, t \geq 0$ is given as

$$h_{R_{(t)}}(v; \eta, \theta) = \frac{f_{R_{(t)}}(v; \eta, \theta)}{S_{R_{(t)}}(v; \eta, \theta)} = \frac{\eta}{\theta^2}(v+t)^{2\eta-1}.$$

3.1.2 Mean Residual Life Function (MRL)

The MRL function is defined as the expected life of an item to survive after the age t. It is a conditional concept and provides information about the whole interval in which the item will survive is to be believed. MRL denoted by $m_1(t) = E(V - t \,/\, V > t)$ is defined as

$$m_1(t) = \frac{1}{S(t)} \int_t^{\infty} v f(v; \eta, \theta) dv - t;\; S(t) > 0.$$

Thus, MRL for PRD is obtained as

$$m_1(t) = \frac{\eta}{\theta^2} exp\left(\frac{t^{2\eta}}{2\theta^2}\right) \int_t^{\infty} v^{2\eta} exp\left(-\frac{v^{2\eta}}{2\theta^2}\right) dv - t.$$

$$m_1(t) = (2\theta^2)^{\frac{1}{2\eta}} exp\left(\frac{t^{2\eta}}{2\theta^2}\right) \Gamma\left(\left(1 + \frac{1}{2\eta}\right), \frac{t^{2\eta}}{2\theta^2}\right) - t.$$

3.1.3 Reversed Residual Life Function

The conditional random variable $\bar{R}_{(t)} = (t - V/V \leq t), t \geq 0$ is used to explain the reversed residual life of a lifetime component. The survival function of reversed residual lifetime $\bar{R}_{(t)}$ for PRD is defined as

$$S_{\bar{R}_{(t)}}(v; \eta, \theta) = \frac{F(t - v)}{F(t)} = \frac{1 - exp\left(-\frac{(t - v)^{2\eta}}{2\theta^2}\right)}{1 - exp\left(-\frac{t^{2\eta}}{2\theta^2}\right)};\; 0 \leq v \leq t.$$

The cdf and pdf of reversed residual lifetime random variable $\bar{R}_{(t)}, t \geq 0$ are respectively given as

$$F_{\bar{R}_{(t)}}(v; \eta, \theta) = \frac{exp\left(-\frac{(t - v)^{2\eta}}{2\theta^2}\right) - exp\left(-\frac{t^{2\eta}}{2\theta^2}\right)}{1 - exp\left(-\frac{t^{2\eta}}{2\theta^2}\right)}.$$

$$f_{\bar{R}_{(t)}}(v; \eta, \theta) = \frac{\eta (t - v)^{2\eta - 1} exp\left(-\frac{(t - v)^{2\eta}}{2\theta^2}\right)}{\theta^2 \left\{1 - exp\left(-\frac{t^{2\eta}}{2\theta^2}\right)\right\}}.$$

Accordingly, the associated failure rate of $\bar{R}_{(t)}, t \geq 0$ is given as

$$h_{\bar{R}_{(t)}}(v; \eta, \theta) = \frac{f_{\bar{R}_{(t)}}(v; \eta, \theta)}{S_{\bar{R}_{(t)}}(v; \eta, \theta)} = \frac{\eta (t - v)^{2\eta - 1} exp\left(-\frac{(t - v)^{2\eta}}{2\theta^2}\right)}{\theta^2 \left\{1 - exp\left(-\frac{(t - v)^{2\eta}}{2\theta^2}\right)\right\}}.$$

3.1.4 Mean Reversed Residual Life Function (MRRL) OR Mean Inactivity Time (MIT)

The MRRL or MIT denoted by $M_1(t) = E(t - V \,/\, V \leq t)$ is defined as

$$M_1(t) = t - [F(t)]^{-1} \int_0^t v f(v;\eta,\theta)dv.$$

$$M_1(t) = t - \left[1 - exp\left(-\frac{t^{2\eta}}{2\theta^2}\right)\right]^{-1} (2\theta^2)^{\frac{1}{2\eta}} \gamma\left(\left(1 + \frac{1}{2\eta}\right), \frac{t^{2\eta}}{2\theta^2}\right).$$

4. ORDINARY MOMENTS AND RELATED MEASURES OF PRD

In this section, moments and other related measures of the proposed model have been obtained viz., mean, variance, index of dispersion, co-efficient of skewness and kurtosis respectively.

4.1 Raw Moments

The s^{th} moment about origin μ_s' (raw moment) is generally defined as

$$\mu_s' = \int_0^\infty v^s f(v;\eta,\theta)dv.$$

Thus, the s^{th} moment about origin of PRD is obtained as

$$\mu_s' = \frac{\eta}{\theta^2}\int_0^\infty v^{s+2\eta-1} exp\left(-\frac{v^{2\eta}}{2\theta^2}\right)dv$$

$$\mu_s' = (2\theta^2)^{\frac{s}{2\eta}}\Gamma\left(\frac{s}{2\eta}+1\right). \qquad (4)$$

Using (4) and substituting $s = 1,2,3$ and 4, we obtain the first four moments about origin of PRD as

$$\mu_1' = (2\theta^2)^{\frac{1}{2\eta}}\Gamma\left(\frac{1}{2\eta}+1\right).$$

This is the mean of our model.

$$\mu_2' = (2\theta^2)^{\frac{1}{\eta}}\Gamma\left(\frac{1}{\eta}+1\right), \qquad \mu_3' = (2\theta^2)^{\frac{3}{2\eta}}\Gamma\left(\frac{3}{2\eta}+1\right)$$

$$\mu_4' = (2\theta^2)^{\frac{2}{\eta}}\Gamma\left(\frac{2}{\eta}+1\right).$$

4.2 Moments About Mean (Central Moments)

The moments about mean also known as central moments of PRD are obtained as

$$\mu_2 = \mu_2' - (\mu_1')^2 = (2\theta^2)^{\frac{1}{\eta}}\left[\Gamma\left(\frac{1}{\eta}+1\right) - \left\{\Gamma\left(\frac{1}{2\eta}+1\right)\right\}^2\right].$$

This is the variance of our formulated model.

$$\mu_3 = (2\theta^2)^{\frac{3}{2\eta}}\left[\Gamma\left(\frac{3}{2\eta}+1\right) - 3\Gamma\left(\frac{1}{2\eta}+1\right)\Gamma\left(\frac{1}{\eta}+1\right) + 2\left\{\Gamma\left(\frac{1}{2\eta}+1\right)\right\}^3\right].$$

$$\mu_4 = (2\theta^2)^{\frac{2}{\eta}}\left[\Gamma\left(\frac{2}{\eta}+1\right) - 4\Gamma\left(\frac{1}{2\eta}+1\right)\Gamma\left(\frac{3}{2\eta}+1\right) + 6\left\{\Gamma\left(\frac{1}{2\eta}+1\right)\right\}^2\Gamma\left(\frac{1}{\eta}+1\right) - 3\left\{\Gamma\left(\frac{1}{2\eta}+1\right)\right\}^4\right].$$

Thus, the measure of Skewness, Kurtosis, co-efficient of variation and index of dispersion can be obtained from these expressions respectively as

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{\left[\Gamma\left(\frac{3}{2\eta}+1\right) - 3\Gamma\left(\frac{1}{2\eta}+1\right)\Gamma\left(\frac{1}{\eta}+1\right) + 2\left\{\Gamma\left(\frac{1}{2\eta}+1\right)\right\}^3\right]^2}{\left[\Gamma\left(\frac{1}{\eta}+1\right) - \left\{\Gamma\left(\frac{1}{2\eta}+1\right)\right\}^2\right]^3}.$$

Since, the nature of skewnesss can't be calculated with this relation. So, we need another measure which depends on the sign of third central moment and is given as

$$\gamma_1 = \frac{\mu_3}{\mu_2^{3/2}} = \frac{\Gamma\left(\frac{3}{2\eta}+1\right) - 3\Gamma\left(\frac{1}{2\eta}+1\right)\Gamma\left(\frac{1}{\eta}+1\right) + 2\left\{\Gamma\left(\frac{1}{2\eta}+1\right)\right\}^3}{\left[\Gamma\left(\frac{1}{\eta}+1\right) - \left\{\Gamma\left(\frac{1}{2\eta}+1\right)\right\}^2\right]^{3/2}}.$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{\Gamma\left(\frac{2}{\eta}+1\right) - 4\Gamma\left(\frac{1}{2\eta}+1\right)\Gamma\left(\frac{3}{2\eta}+1\right) + 6\left\{\Gamma\left(\frac{1}{2\eta}+1\right)\right\}^2\Gamma\left(\frac{1}{\eta}+1\right) - 3\left\{\Gamma\left(\frac{1}{2\eta}+1\right)\right\}^4}{\left[\Gamma\left(\frac{1}{\eta}+1\right) - \left\{\Gamma\left(\frac{1}{2\eta}+1\right)\right\}^2\right]^2}.$$

$$\gamma_2 = \frac{\Gamma\left(\frac{2}{\eta}+1\right) - 4\Gamma\left(\frac{1}{2\eta}+1\right)\Gamma\left(\frac{3}{2\eta}+1\right) + 6\left\{\Gamma\left(\frac{1}{2\eta}+1\right)\right\}^2\Gamma\left(\frac{1}{\eta}+1\right) - 3\left\{\Gamma\left(\frac{1}{2\eta}+1\right)\right\}^4}{\left[\Gamma\left(\frac{1}{\eta}+1\right) - \left\{\Gamma\left(\frac{1}{2\eta}+1\right)\right\}^2\right]^2} - 3.$$

$$C.V = \frac{\sigma}{\mu_1'} = \frac{\left[\Gamma\left(\frac{1}{\eta}+1\right) - \left\{\Gamma\left(\frac{1}{2\eta}+1\right)\right\}^2\right]^{1/2}}{\Gamma\left(\frac{1}{2\eta}+1\right)}.$$

$$\gamma = \frac{\sigma^2}{\mu_1'} = \frac{\Gamma\left(\frac{1}{\eta}+1\right) - \left\{\Gamma\left(\frac{1}{2\eta}+1\right)\right\}^2}{\Gamma\left(\frac{1}{2\eta}+1\right)}.$$

Table 1
Brief Description of PRD for Various Values of Parameter Combinations

θ	η	$Mean$ (μ)	Var (σ^2)	Mode	$I.D$ (γ)	$c.v.$	$Skew$ (γ_1)	$Kurt$ (γ_2)
0.5	0.5	0.500	0.2500	0	0.5000	1.000	2.000	6.000
0.5	0.9	0.605	0.1210	0.434	0.1999	0.575	0.779	0.557
0.5	1.5	0.709	0.0664	0.693	0.0936	0.363	0.168	-0.271
0.5	3.0	0.827	0.0256	0.864	0.0310	0.194	-0.373	-0.035
0.5	10.0	0.940	0.0034	0.963	0.0036	0.062	-0.868	1.267
1.5	10.0	1.050	0.0042	1.075	0.0040	-	-	-
3.0	10.0	1.125	0.0048	1.153	0.0043	-	-	-
5.0	10.0	1.184	0.0054	1.213	0.0045	-	-	-
10.0	10.0	1.269	0.0062	1.300	0.0049	-	-	-

Table 1 reveals that scale parameter θ doesn't affect c.v., skewness and kurtosis of PRD although mean, variance, mode and index of dispersion increases with increase in θ. Contrary to this, all characteristics decrease with increase in shape parameter η except mean and mode which increases and kurtosis which shows diverse nature. Consequently, PRD can be used for modeling both positively and negatively skewed data sets. Also, from table 1, we conclude that the proposed model is uni-modal.

4.3 Incomplete Moments

The sth incomplete moment of PRD about origin, say $\tau_s(q) = \int_0^q v^s f(v;\eta,\theta)dv$, is obtained as

$$\tau_s(q) = \frac{\eta}{\theta^2}\int_0^q v^{s+2\eta-1} exp\left(-\frac{v^{2\eta}}{2\theta^2}\right)dv,$$

$$\tau_s(q) = (2\theta^2)^{\frac{s}{2\eta}}\gamma\left(\left(\frac{s}{2\eta}+1\right),\frac{q^{2\eta}}{2\theta^2}\right).$$

4.4 Harmonic Mean

The Harmonic mean of PRD is obtained as

$$\frac{1}{H} = \int_0^\infty \frac{1}{v} f(v;\eta,\theta)dv.$$

$$\frac{1}{H} = \frac{\eta}{\theta^2}\int_0^\infty v^{2(\eta-1)} exp\left(-\frac{v^{2\eta}}{2\theta^2}\right)dv.$$

$$H = \left[\frac{1}{(2\theta^2)^{\frac{1}{2\eta}}}\Gamma\left(1-\frac{1}{2\eta}\right)\right]^{-1}.$$

4.5 Moment Generating Function, Characteristic Function and Cumulant Generating Function

The moment generating function of a random variable V is defined as

$$M_v(t) = E[exp(tv)] = \int_0^\infty exp(tv) f(v;\eta,\theta) dv.$$

Thus, the moment generating function of PRD is obtained as

$$M_v(t) = \frac{\eta}{\theta^2} \int_0^\infty exp(tv) v^{2\eta-1} exp\left(-\frac{v^{2\eta}}{2\theta^2}\right) dv$$

Using the expansion of $exp(tv) = \sum_{j=0}^{\infty} \frac{t^j v^j}{j!}$ the above equation can be rewritten as

$$M_v(t) = \frac{\eta}{\theta^2} \int_0^\infty \sum_{j=0}^{\infty} \frac{t^j v^j}{j!} v^{2\eta-1} exp\left(-\frac{v^{2\eta}}{2\theta^2}\right) dv$$

$$M_v(t) = \frac{\eta}{\theta^2} \sum_{j=0}^{\infty} \frac{t^j}{j!} \int_0^\infty v^{j+2\eta-1} exp\left(-\frac{v^{2\eta}}{2\theta^2}\right) dv$$

$$M_v(t) = \sum_{j=0}^{\infty} \frac{t^j}{j!} (2\theta^2)^{\frac{j}{2\eta}} \Gamma\left(\frac{j}{2\eta} + 1\right).$$

Using the relation $\varphi_v(t) = M_v(it)$ the characteristic function of PRD is obtained as

$$\varphi_v(t) = \sum_{j=0}^{\infty} \frac{(it)^j}{j!} (2\theta^2)^{\frac{j}{2\eta}} \Gamma\left(\frac{j}{2\eta} + 1\right).$$

Also, the Cumulant generating function $K_v(t) = log M_v(t)$ of PRD is obtained as

$$K_v(t) = log \sum_{j=0}^{\infty} \frac{t^j}{j!} (2\theta^2)^{\frac{j}{2\eta}} \Gamma\left(\frac{j}{2\eta} + 1\right).$$

The cumulants can be calculated by using the relation

$$k_n = \mu'_n - \sum_{r=0}^{n-1} \binom{n-1}{r-1} k_1 \mu'_{n-r},$$

Where, $k_1 = \mu'_1, k_2 = \mu'_2 - (\mu'_1)^2, k_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2(\mu'_1)^3$ etc.

5. SIMULATION AND QUANTILES OF PRD

Let V denotes the random variable with pdf given in (1). The quantile function, say $Q(u)$ is defined as $F[Q(u)] = u$

$$Q(u) = F^{-1}(u)$$

Here u is the uniform random variable defined on the unit interval (0,1).

Thus, for PRD the quantile function is obtained as

$$Q(u) = [-2\theta^2 log(1-u)]^{\frac{1}{2\eta}} \tag{5}$$

Accordingly, the quartiles of PRD can be calculated from the above expression and are given as

$$Q\left(\frac{1}{4}\right) = F^{-1}\left(\frac{1}{4}\right) = [1.204\theta^2]^{\frac{1}{2\eta}}$$

$$Q\left(\frac{1}{2}\right) = F^{-1}\left(\frac{1}{2}\right) = [0.602\theta^2]^{\frac{1}{2\eta}}$$

$$Q\left(\frac{3}{4}\right) = F^{-1}\left(\frac{3}{4}\right) = [0.248\theta^2]^{\frac{1}{2\eta}}$$

Since, the limitations of classical measure of skewness and kurtosis are well-known when the moments don't exist for any distribution. To overcome these shortcomings the analysis of variability of the skewness and kurtosis can be investigated based on quantile measures. The Bowley measure of skewness based on quantiles is given as

$$B = \frac{Q(3/4) - 2Q(1/2) + Q(1/4)}{Q(3/4) - Q(1/4)}.$$

Similarly, the Moors measure of kurtosis based on octiles is given as

$$M = \frac{Q(3/8) - Q(1/8) + Q(7/8) - Q(5/8)}{Q(6/8) - Q(2/8)}.$$

Where, $Q(u)$ is given in (5).

6. ORDER STATISTICS OF PRD

Suppose $V_{(s;m)}$ be the s^{th} ordered statistics of the random sample $v_{(1)}, v_{(2)}, \dots, v_{(m)}$ obtained from PRD with pdf $f(v;\eta,\theta)$ and cdf $F(v;\eta,\theta)$. Then the pdf and cdf of $v_{(s;m)}$ say $f_{(s;m)}(v)$ and $F_{(s;m)}(v)$ is respectively expressed as

$$f_{(s;m)}(v) = \frac{m!}{(s-1)!\,(m-s)!}[F(v;\eta,\theta)]^{s-1} [1-F(v;\eta,\theta)]^{m-s} f(v;\eta,\theta) \tag{6}$$

$$F_{(s;m)}(v) = \sum_{j=s}^{m} \binom{m}{j} [F(v;\eta,\theta)]^j [1-F(v;\eta,\theta)]^{m-j} \tag{7}$$

Using (1) and (2) in (6) and (7), we obtain the pdf and cdf of s^{th} ordered statistics for PRD as given by

$$f_{(s;m)}(v)=\frac{m!}{(s-1)!\,(m-s)!}\left[1-exp\left(-\frac{v^{2\eta}}{2\theta^2}\right)\right]^{s-1}\left[exp\left(-\frac{v^{2\eta}}{2\theta^2}\right)\right]^{m-s}\frac{\eta}{\theta^2}v^{2\eta-1}exp\left(-\frac{v^{2\eta}}{2\theta^2}\right) \quad (8)$$

$$F_{(s;m)}(v)=\sum_{j=s}^{m}\binom{m}{j}\left[1-exp\left(-\frac{v^{2\eta}}{2\theta^2}\right)\right]^{j}\left[exp\left(-\frac{v^{2\eta}}{2\theta^2}\right)\right]^{m-j}.$$

The expression for pdf of smallest (minimum) order statistics $v_{(1)}$ and largest (maximum) order statistics $v_{(m)}$ of PRD are respectively obtained by letting $s=1$ and m in (8) and are given as

$$f_{(1;m)}(v)=\frac{m\eta}{\theta^2}v^{2\eta-1}\left[exp\left(-\frac{v^{2\eta}}{2\theta^2}\right)\right]^{m}.$$

$$f_{(m;m)}(v)=\frac{m\eta}{\theta^2}v^{2\eta-1}\left[1-exp\left(-\frac{v^{2\eta}}{2\theta^2}\right)\right]^{m-1}exp\left(-\frac{v^{2\eta}}{2\theta^2}\right).$$

Similarly, the pdf of median order statistics $V_{(m+1)}$ for PRD can be written as

$$f_{((m+1);m)}(v)=\frac{(2m+1)!}{m!\,m!}[F(v)]^m[1-F(v)]^m f(v)$$

$$f_{((m+1);m)}(v)=\frac{(2m+1)!}{m!\,m!}\left[1-exp\left(-\frac{v^{2\eta}}{2\theta^2}\right)\right]^{m}\left[exp\left(-\frac{v^{2\eta}}{2\theta^2}\right)\right]^{m}\frac{\eta}{\theta^2}v^{2\eta-1}exp\left(-\frac{v^{2\eta}}{2\theta^2}\right).$$

7. STOCHASTIC ORDERING OF PRD

A random variable V_1 is considered to be smaller than random variable V_2 in the

(A) Stochastic order ($V_1 \le_{st} V_2$) if $F_1(v) \ge F_2(v)\ \forall\ v$.
(B) Hazard rate order ($V_1 \le_{hr} V_2$) if $h_1(v) \ge h_2(v)\ \forall\ v$.
(C) Mean residual life order ($V_1 \le_{mrl} V_2$) if $m_1(v) \le m_2(v)\ \forall\ v$.
(D) Likelihood ratio order ($V_1 \le_{lr} V_2$) if $\frac{F_1(v)}{F_2(v)}$ decreases in v.

The following interrelationship exists among the above said properties of the distribution

(a) $V_1 \le_{lr} V_2 \implies V_1 \le_{hr} V_2 \implies V_1 \le_{mrl} V_2$
(b) $V_1 \le_{lr} V_2 \implies V_1 \le_{hr} V_2 \implies V_1 \le_{st} V_2$

The PRD is ordered with respect to the strongest 'likelihood ratio ordering' due to which other ordering of distribution follows:

Theorem 7.1:

Suppose $V_1 \sim PRD(\eta_1, \theta_1)$ and $V_2 \sim PRD(\eta_2, \theta_2)$. If for $\theta_1 > \theta_2$ and $\eta_1 = \eta_2 = \eta$ or ($\theta_1 = \theta_2 = \theta$ and $\eta_1 > \eta_2$), we have $V_1 \leq_{lr} V_2$ then $V_1 \leq_{hr} V_2, V_1 \leq_{mrl} V_2$ and $V_1 \leq_{st} V_2$.

Proof:

Since we have

$$f_1(v;\eta,\theta) = \frac{\eta_1}{\theta_1^2} v^{2\eta_1 - 1} exp\left(-\frac{v^{2\eta_1}}{2\theta_1^2}\right)$$

$$f_2(v;\eta,\theta) = \frac{\eta_2}{\theta_2^2} v^{2\eta_2 - 1} exp\left(-\frac{v^{2\eta_2}}{2\theta_2^2}\right)$$

$$\frac{f_1(v;\eta,\theta)}{f_2(v;\eta,\theta)} = \left(\frac{\eta_1\theta_2^2}{\eta_2\theta_1^2}\right) v^{2(\eta_1-\eta_2)} exp\left(\frac{v^{2\eta_2}}{2\theta_2^2} - \frac{v^{2\eta_1}}{2\theta_1^2}\right) \tag{9}$$

Differentiate (9) partially w.r.to v, we get

$$\frac{\partial}{\partial v}\left[\frac{f_1(v;\eta,\theta)}{f_2(v;\eta,\theta)}\right] = \left(\frac{\eta_1\theta_2^2}{\eta_2\theta_1^2}\right)\left[\begin{array}{c} v^{2(\eta_1-\eta_2)} exp\left(\frac{v^{2\eta_2}}{2\theta_2^2} - \frac{v^{2\eta_1}}{2\theta_1^2}\right)\left(\frac{\eta_2 v^{2\eta_2-1}}{\theta_2^2} - \frac{\eta_1 v^{2\eta_1-1}}{\theta_1^2}\right) \\ +2exp\left(\frac{v^{2\eta_2}}{2\theta_2^2} - \frac{v^{2\eta_1}}{2\theta_1^2}\right)(\eta_1-\eta_2)v^{2(\eta_1-\eta_2)-1}\end{array}\right].$$

From the above expression, it is clear that for both cases $\theta_1 > \theta_2$ and $\eta_1 = \eta_2 = \eta$ (or $\theta_1 = \theta_2 = \theta$ and $\eta_1 > \eta_2$) $\frac{f_1(v;\eta,\theta)}{f_2(v;\eta,\theta)}$ is decreasing function i.e., $\frac{\partial}{\partial v}\left[\frac{f_1(v;\eta,\theta)}{f_2(v;\eta,\theta)}\right] < 0$. This clearly shows that $V_1 \leq_{lr} V_2$ and hence $V_1 \leq_{hr} V_2, V_1 \leq_{mrl} V_2$ *and* $V_1 \leq_{st} V_2$. Thus PRD follows the strongest likelihood ratio ordering.

8. MEAN DEVIATIONS (MD) OF PRD

The mean deviation about mean (μ) and mean deviation about median (M) can be calculated respectively by using the following expressions

$$MD(\mu) = \int_0^\infty |v-\mu| f(v;\eta,\theta)dv \text{ and } MD(M) = \int_0^\infty |v-M| f(v;\eta,\theta)dv$$

where $\mu = E(v) = (2\theta^2)^{\frac{1}{2\eta}}\Gamma\left(\frac{1}{2\eta}+1\right)$ and $M = Median(v) = [0.602\theta^2]^{\frac{1}{2\eta}}$

These two measures can also be calculated by using the following relations

$$MD(\mu) = 2[\mu F(\mu) - \tau(\mu)] \text{ and } MD(M) = \mu - 2\tau(M) \tag{10}$$

For the PRD $\tau(q) = \int_0^q v f(v)dv$ can be easily calculated as

$$\tau(q) = \frac{\eta}{\theta^2}\int_0^q v^{2\eta} exp\left(-\frac{v^{2\eta}}{2\theta^2}\right)dv$$

$$\tau(q) = (2\theta^2)^{\frac{1}{2\eta}}\gamma\left(\left(\frac{1}{2\eta}+1\right),\frac{q^{2\eta}}{2\theta^2}\right)$$

Using the value of $\mu, F(\mu)$ and $\tau(q)$ in (10), the mean deviation about mean and mean deviation about median of PRD takes the following forms

$$MD(\mu) = 2(2\theta^2)^{\frac{1}{2\eta}}\left[\Gamma\left(\frac{1}{2\eta}+1\right)\left(1-exp\left(-\frac{\mu^{2\eta}}{2\theta^2}\right)\right)-\gamma\left(\left(\frac{1}{2\eta}+1\right),\frac{\mu^{2\eta}}{2\theta^2}\right)\right].$$

$$MD(M) = (2\theta^2)^{\frac{1}{2\eta}}\left[\Gamma\left(\frac{1}{2\eta}+1\right)-2\gamma\left(\left(\frac{1}{2\eta}+1\right),\frac{M^{2\eta}}{2\theta^2}\right)\right].$$

The expression $\tau(q) = (2\theta^2)^{\frac{1}{2\eta}}\gamma\left(\frac{1}{2\eta}+1,\frac{q^{2\eta}}{2\theta^2}\right)$ can be used to construct the Bonferroni and Lorenz curves which are very vital in finance, demography, insurance, reliability and other fields.

The Bonferroni curve of random variable V takes the form

$$B(p) = \frac{1}{p\mu}\tau(q)$$

Here μ is the mean of the distribution.

$$B(p) = \frac{1}{p\Gamma\left(\frac{1}{2\eta}+1\right)}\gamma\left[\left(\frac{1}{2\eta}+1\right),\frac{q^{2\eta}}{2\theta^2}\right].$$

The Lorenz curve developed by Max O. Lorenz is given as

$$L(p) = \frac{1}{\mu}\tau(q)$$

Accordingly for PRD, the expression of Lorenz curve is obtained as

$$L(p) = \frac{1}{\Gamma\left(\frac{1}{2\eta}+1\right)}\gamma\left[\left(\frac{1}{2\eta}+1\right),\frac{q^{2\eta}}{2\theta^2}\right].$$

9. CHARACTERIZATION OF PRD

Theorem 9.1:

Suppose $v_1, v_2, \dots, v_m$ are m independently and identically distributed random samples taken from PRD with sample mean $\bar{v}_m$ and sample variance s_m^2, then

$$\lim_{m\to\infty} E\left(\frac{s_m^2}{\bar{v}_m^2}\right) = \left(\frac{\sigma}{\mu}\right)^2 = (c.v.)^2.$$

Proof:

We know that

$$E(\bar{v}_m) = \mu \text{ and } Var(\bar{v}_m) = \frac{\sigma^2}{m}$$

Since, $E(\bar{v}_m)^2 = Var(\bar{v}_m) + [E(\bar{v}_m)]^2$

$$E(\bar{v}_m)^2 = \frac{(2\theta^2)^{\frac{1}{\eta}}}{n}\left[\Gamma\left(\frac{1}{\eta}+1\right) - \left\{\Gamma\left(\frac{1}{2\eta}+1\right)\right\}^2\right] + \left[(2\theta^2)^{\frac{1}{2\eta}}\Gamma\left(\frac{1}{2\eta}+1\right)\right]^2$$

Also, $E(s_m^2) = \sigma^2$

$$E(s_m^2) = (2\theta^2)^{\frac{1}{\eta}}\left[\Gamma\left(\frac{1}{\eta}+1\right) - \left\{\Gamma\left(\frac{1}{2\eta}+1\right)\right\}^2\right]$$

Now,

$$E\left(\frac{s_m^2}{\bar{v}_m^2}\right) = \frac{(2\theta^2)^{\frac{1}{\eta}}\left[\Gamma\left(\frac{1}{\eta}+1\right) - \left\{\Gamma\left(\frac{1}{2\eta}+1\right)\right\}^2\right]}{\frac{(2\theta^2)^{\frac{1}{\eta}}}{m}\left[\Gamma\left(\frac{1}{\eta}+1\right) - \left\{\Gamma\left(\frac{1}{2\eta}+1\right)\right\}^2\right] + \left[(2\theta^2)^{\frac{1}{2\eta}}\Gamma\left(\frac{1}{2\eta}+1\right)\right]^2}$$

Applying $\lim_{m\to\infty}$ from both sides, we obtain

$$\lim_{m\to\infty} E\left(\frac{s_m^2}{\bar{v}_m^2}\right) = \frac{(2\theta^2)^{\frac{1}{\eta}}\left[\Gamma\left(\frac{1}{\eta}+1\right) - \left\{\Gamma\left(\frac{1}{2\eta}+1\right)\right\}^2\right]}{\left[(2\theta^2)^{\frac{1}{2\eta}}\Gamma\left(\frac{1}{2\eta}+1\right)\right]^2}$$

$$\lim_{m\to\infty} E\left(\frac{s_m^2}{\bar{v}_m^2}\right) = \left(\frac{\sigma}{\mu}\right)^2 = (c.v.)^2.$$

10. INFORMATION MEASURES OF PRD

The existence of uncertainty or variation in a random variable is measured by statistical entropy. The greater value of entropy reveals the greater uncertainty in the data. This section is devoted to obtain the expression for different entropy measures of PRD.

10.1 Shannon's Measure of Entropy

The Shannon's measure of entropy (1948) for a continuous random variable V is mathematically defined as

$$H(v;\eta,\theta) = E\big(-logf(v;\eta,\theta)\big) = -\int_{-\infty}^{+\infty}[logf(v;\eta,\theta)]\,f(v;\eta,\theta)dv$$

Thus, for PRD the Shannon's measure of entropy is calculated as

$$H(v;\eta,\theta) = -log\left(\frac{\eta}{\theta^2}\right) - (2\eta - 1)E\big(log(v)\big) + \frac{1}{2\theta^2}E(v^{2\eta}) \tag{11}$$

Now,

$$E\big(log(v)\big) = \int_0^\infty log(v) f(v;\eta,\theta)dv$$

$$E\big(log(v)\big) = \frac{\eta}{\theta^2}\int_0^\infty log(v) v^{2\eta-1}\, exp\left(-\frac{v^{2\eta}}{2\theta^2}\right)dv$$

Solving the above expression will yield

$$E\big(log(v)\big) = \frac{1}{2\eta}[log(2\theta^2) + \Gamma'(1)] \tag{12}$$

Where,

$$\Gamma'(v) = \int_0^\infty z^{v-1}(logz)exp(-z)dz$$

Also,

$$E(v^{2\eta}) = \frac{\eta}{\theta^2}\int_0^\infty v^{4\eta-1}\, exp\left(-\frac{v^{2\eta}}{2\theta^2}\right)dv$$

$$E(v^{2\eta}) = 2\theta^2 \tag{13}$$

Thus, the Shannon's measure of entropy for PRD is obtained by using (12) and (13) in (11) and is given as

$$H(v;\eta,\theta) = 1 - log\left(\frac{\eta}{\theta^2}\right) - \frac{(2\eta - 1)}{2\eta}[log(2\theta^2) + \Gamma'(1)].$$

10.2 Renyi Entropy

This uncertainty measure given by Alfred Renyi in 1960 is mathematically expressed as

$$I_\delta(v) = (1-\delta)^{-1} log \int_{-\infty}^{\infty} f^\delta(v;\eta,\theta)dv,$$

Where, $\delta > 0$ and $\delta \neq 1$.

Thus for PRD, Renyi entropy is obtained as

$$I_\delta(v) = (1-\delta)^{-1} log\left(\frac{\eta}{\theta^2}\right)^\delta \int_0^\infty \left[v^{2\eta-1} exp\left(-\frac{v^{2\eta}}{2\theta^2}\right)\right]^\delta dv\,.$$

Solving the integral after making the proper substitution has yielded the solution as follows

$$I_{\delta}(v)=(1-\delta)^{-1}log\left[\frac{1}{2\eta}\left(\frac{\eta}{\theta^2}\right)^{\delta}\left(\frac{2\theta^2}{\delta}\right)^{\frac{(2\eta-1)\delta+1}{2\eta}}\Gamma\left(\frac{(2\eta-1)\delta+1}{2\eta}\right)\right].$$

10.3 $\boldsymbol{\beta}$ Entropy

This entropy measure given by Harvda and Charvat (1967) is defined as

$$H_{\beta}(v)=(\beta-1)^{-1}\left[1-\int_{0}^{\infty}f^{\beta}(v;\eta,\theta)dv\right],$$

Where, $\beta>0$ and $\beta\neq 1$.

For PRD, Beta entropy is obtained as

$$H_{\beta}(v)=(\beta-1)^{-1}\left[1-\left(\frac{\eta}{\theta^2}\right)^{\beta}\int_{0}^{\infty}\left\{v^{2\eta-1}exp\left(-\frac{v^{2\eta}}{2\theta^2}\right)\right\}^{\beta}dv\right].$$

$$H_{\beta}(v)=(\beta-1)^{-1}\left[1-\frac{1}{2\eta}\left(\frac{\eta}{\theta^2}\right)^{\beta}\left(\frac{2\theta^2}{\beta}\right)^{\frac{(2\eta-1)\beta+1}{2\eta}}\Gamma\left(\frac{(2\eta-1)\beta+1}{2\eta}\right)\right].$$

10.4 Mathai and Haubold Entropy

This entropy measure is calculated by replacing δ by $\delta-2$ and is defined as

$$I_{MH}(v)=(\delta-1)^{-1}\int_{0}^{\infty}f^{\delta-2}(v;\eta,\theta)dv,$$

where, $\delta>0$ and $\delta\neq 1$.

Thus, for PRD the Mathai and Haubold entropy is obtained as

$$I_{MH}(v)=(\delta-1)^{-1}\left(\frac{\eta}{\theta^2}\right)^{\delta-2}\int_{0}^{\infty}\left[v^{2\eta-1}exp\left(-\frac{v^{2\eta}}{2\theta^2}\right)\right]^{\delta-2}dv$$

On solving the above integral, we get the solution as

$$I_{MH}(v)=(\delta-1)^{-1}\left(\frac{\eta}{\theta^2}\right)^{\delta-2}\left(\frac{2\theta^2}{\delta-2}\right)^{\frac{(\delta-2)(2\eta-1)+1}{2\eta}}\Gamma\left(\frac{(\delta-2)(2\eta-1)+1}{2\eta}\right).$$

11. ENTROPY ESTIMATION OF PRD

For model selection, Akaike Information Criteria and Schwarz Information Criteria are well known. These criterions are mathematically given as

$$AIC=2M-2l$$

$$SIC=Mlogm-2l$$

Where M is the total of parameters in the model, m is the total observations used in the data set and l is the log likelihood of the pdf given by

$$l = mlog(\eta) - 2mlog(\theta) + (2\eta - 1)\sum_{i=1}^{m} log(v_i) - \frac{1}{2\theta^2}\sum_{i=1}^{m} {v_i}^{2\eta}$$

$$-\frac{l}{m} = -log\left(\frac{\eta}{\theta^2}\right) - (2\eta - 1)E(logv) + \frac{1}{2\theta^2}E(v^{2\eta}) \tag{14}$$

Equating (11) and (14) will results

$$H(v;\eta,\theta) = -\frac{l}{m}$$

$$l = -mH(v;\eta,\theta)$$

Consequently,

$$AIC = 2M + 2mH(v;\eta,\theta)$$
$$SIC = Mlogm + 2mH(v;\eta,\theta).$$

12. ESTIMATION OF PARAMETERS

This section presents the estimation procedures such as, Maximum likelihood estimation procedure and Method of Moments estimation procedure for estimating the unknown parameters θ and η of the PRD.

12.1 Method of Moment Estimation (MME)

The moment estimates $\hat{\eta}_{MME}$ and $\hat{\theta}_{MME}$ for the unknown parameters η and θ are obtained by equating the moments of population to the corresponding moments of sample. Therefore, the system of equations that are required to obtain the moment estimates are expressed as

$$(2\theta^2)^{\frac{1}{2\eta}}\Gamma\left(\frac{1}{2\eta} + 1\right) = \frac{1}{m}\sum_{i=1}^{m} v_i,$$

$$(2\theta^2)^{\frac{1}{\eta}}\Gamma\left(\frac{1}{\eta} + 1\right) = \frac{1}{m}\sum_{i=1}^{m} v_i^2.$$

The above system of equations is complex and solving by simple mathematical calculations seems very difficult. Hence, by using R-software we can obtain the estimates $\hat{\eta}_{ME}$ and $\hat{\theta}_{ME}$.

12.2 Maximum Likelihood Estimation (MLE)

Let $v_1, v_2, \dots, v_m$ are m random samples drawn from PRD(η, θ) with parameter vector $\zeta(\eta, \theta)$. Then, the likelihood function of m observations is given as

$$L(\zeta) = \prod_{i=1}^{m}\left\{\frac{\eta}{\theta^2}v^{2\eta-1}exp\left(-\frac{v^{2\eta}}{2\theta^2}\right)\right\}$$

Thus, the maximum likelihood estimates $\hat{\eta}_{MlE}$ and $\hat{\theta}_{MLE}$ of the unknown parameters η and θ are obtained by maximizing the log-likelihood function

$$l(\zeta) = mlog(\eta) - 2mlog(\theta) + (2\eta - 1)\sum_{i=1}^{m} log(v_i) - \frac{1}{2\theta^2}\sum_{i=1}^{m} {v_i}^{2\eta},$$

The MLE's of these unknown parameters can be also be obtained by solving the system of non-linear equations:

$$\frac{m}{\eta} + 2\sum_{i=1}^{m} ln(v_i) - \frac{1}{\theta^2}\sum_{i=1}^{m} v_i^{2\eta} ln(v_i) = 0,$$

$$\frac{2m}{\theta} - \frac{1}{\theta^3}\sum_{i=1}^{m} v_i^{2\eta} = 0.$$

These two non-linear equations are difficult to solve analytically because they cannot be expressed in closed form. So, $\hat{\eta}_{MlE}$ and $\hat{\theta}_{MLE}$ will be obtained by maximizing the log-likelihood function numerically by using Newton-Raphson technique which is most powerful technique of solving equations iteratively and numerically. Since, all the second order derivatives exist. Therefore, for interval estimation of the parametric vector $\Theta = (\eta, \theta)^T$, the 2x2 Fisher's information matrix required is calculated as follows

$$I^{-1}(\Theta) = -E\begin{pmatrix} I_{\eta\eta} & I_{\eta\theta} \\ I_{\theta\eta} & I_{\theta\theta} \end{pmatrix}$$

$$I^{-1}(\Theta) = -E\begin{bmatrix} \frac{\partial^2 l(v;\eta,\theta)}{\partial\eta^2} & \frac{\partial^2 l(v;\eta,\theta)}{\partial\eta\partial\theta} \\ \frac{\partial^2 l(v;\eta,\theta)}{\partial\theta\partial\eta} & \frac{\partial^2 l(v;\eta,\theta)}{\partial\theta^2} \end{bmatrix} \tag{15}$$

where,

$$\frac{\partial^2 l(v;\eta,\theta)}{\partial\eta^2} = -\frac{m}{\eta^2} - \frac{2}{\theta^2}\sum_{i=1}^{m} v_i^{2\eta}\,(log(v_i))^2,$$

$$\frac{\partial^2 l(v;\eta,\theta)}{\partial\eta\partial\theta} = \frac{\partial^2 l(v;\theta,\eta)}{\partial\theta\partial\eta} = \frac{2}{\theta^3}\sum_{i=1}^{m} v_i^{2\eta}\, log(v_i),$$

$$\frac{\partial^2 l(v;\eta,\theta)}{\partial\theta^2} = \frac{2m}{\theta^2} - \frac{3}{\theta^4}\sum_{i=1}^{m} v_i^{2\eta}.$$

Under certain regularity conditions (Gomez Deniz 2010), the MLE vector $\hat{\Theta} = (\hat{\eta}, \hat{\theta})^T$ is consistent and asymptotically normal; that is, $\sqrt{m}\left[(\hat{\eta}, \hat{\theta})^T - (\eta, \theta)^T\right]$ converges in distribution to a normal distribution with the (vector) mean zero and variance-covariance matrix $I^{-1}(\Theta)$. That is, $\sqrt{m}(\hat{\Theta} - \Theta) \to N_2\left(0, I^{-1}(\hat{\Theta})\right)$. Therefore, approximate $100(1-\alpha)\%$ confidence intervals for η and θ can be respectively obtained as $\hat{\eta} \pm Z_{\frac{\alpha}{2}}\sqrt{var(\hat{\eta})}$ and $\hat{\theta} \pm Z_{\frac{\alpha}{2}}\sqrt{var(\hat{\theta})}$, where the diagonal elements of $I^{-1}(\Theta)$ represents the variances of the model parameters and $Z_{\frac{\alpha}{2}}$ is the $\left(1-\frac{\alpha}{2}\right)$ quantile of standard normal distribution.

13. SIMULATION ILLUSTRATION

This section is devoted to examine the performance of ML estimates. The inverse CDF method discussed in section (5) is used to carry out simulation study by generating random samples of sizes: 20, 50, 75, 100 and 200. The process is repeated 500 times and four different cobinations of parameters η, θ are choosen as (0.7,0.9), (1.2,1.0), (2.0,1.3) and (1.2,1.5). The average bias, variance and mean square error of the ML estimates are calculated.

Table 2 presents the bias, variance and MSE of the ML estimates .We observe from table 2 that the agreement between theory and practice improves as the sample size *n* increases. MSE and Variance of the estimators suggest that the estimators are consistent and the maximum likelihood estimator of the parameters perform quite well and the results are precise and accurate.

Table 2
Bias, Variance and Mean Square Error for the Parameters η and θ

Sample Size n	Parameters	$\eta = 0.7, \theta = 0.9$			$\eta = 1.2, \theta = 1.0$		
		Bias	Variance	MSE	Bias	variance	MSE
20	η	0.04554	0.01674	0.01882	0.08425	0.05103	0.05813
	θ	0.02513	0.01953	0.02016	0.03572	0.02985	0.03112
50	η	0.02130	0.00687	0.00732	0.03391	0.02058	0.02173
	θ	0.00541	0.00597	0.00601	0.01079	0.00936	0.00947
75	η	0.01758	0.00449	0.00480	0.02760	0.01342	0.01412
	θ	0.00821	0.00446	0.00453	0.00846	0.00718	0.00725
100	η	0.01013	0.00317	0.00327	0.02369	0.00873	0.00929
	θ	0.00489	0.00296	0.00299	0.00932	0.00402	0.00411
200	η	0.00545	0.00148	0.00151	0.01379	0.00468	0.00487
	θ	0.00431	0.00162	0.00164	0.00779	0.00220	0.00226
		$\eta = 2.0, \theta = 1.3$			$\eta = 1.2, \theta = 1.5$		
20	η	0.13646	0.14647	0.16509	0.09233	0.05262	0.06114
	θ	0.07231	0.08866	0.09388	0.12679	0.13727	0.15335
50	η	0.05766	0.06276	0.06609	0.03622	0.02166	0.02297
	θ	0.03486	0.03258	0.03379	0.03978	0.04598	0.04756
75	η	0.04843	0.03714	0.03949	0.02077	0.01418	0.01462
	θ	0.02307	0.01662	0.01715	0.03645	0.03191	0.03324
100	η	0.02808	0.02665	0.02744	0.02358	0.01011	0.01067
	θ	0.01417	0.01346	0.01366	0.03360	0.02226	0.02338
200	η	0.01521	0.01201	0.01224	0.00804	0.00473	0.00480
	θ	0.00731	0.00547	0.00552	0.01509	0.01040	0.01063

14. DATA ANALYSIS

This section is devoted to demonstrate the importance, flexibility and appropriateness of the PRD by means of two real data sets. For illustrating the significance and the potentiality of the proposed probability model, we compare the goodness-of-fit of the proposed model with the following lifetime models:

Rayleigh model with pdf

$$f(v) = \frac{v}{\theta^2} exp\left(-\frac{v^2}{2\theta^2}\right),$$

Exponentiated Rayleigh model with pdf

$$f(v) = 2\eta\theta v exp(-\theta v^2)\left(1 - exp(-\theta v^2)\right)^{\eta-1},$$

Weighted Rayleigh model with pdf

$$f(v) = \frac{v^{\beta+1} exp\left(-\frac{v^2}{2\theta^2}\right)}{\theta^{\beta+2} 2^{\frac{\beta}{2}} \Gamma\left(\frac{\beta}{2}+1\right)},$$

Weibull Rayleigh model with pdf

$$f(v) = \frac{\eta v}{\lambda\theta^2}\left(\frac{v^2}{2\lambda\theta^2}\right)^{\eta-1} exp\left(-\frac{v^2}{2\lambda\theta^2}\right)^{\eta},$$

Transmuted Rayleigh model with pdf

$$f(v) = \frac{v}{\theta^2} exp\left(-\frac{v^2}{2\theta^2}\right)\left\{1 - \alpha + 2\alpha\, exp\left(-\frac{v^2}{2\theta^2}\right)\right\}.$$

For collation purposes, the various criterions of goodness-of-fit such as $-2ln(l)$, AIC, SIC, AICC, HQIC and KS distance along with corresponding p-values are used here. The statistic with smaller value along with large p-value is considered to be the best fit. For analysis purposes, the numerical results are obtained using R software.

The descriptive statistics of data set 1 and data set 2 are given in tables 2 and 5, whilst the ML estimates are shown in tables 3 and 6. The numerical values of goodness-of-fit statistics for the two data sets are listed in tables 4 and 7 respectively. Also, the plots of the fitted models are displayed in figure x.

Data Set I^st^

This data set represents the breaking stress of carbon fibres of 50 mm length (GPa) and has been already used by Al-Aqtash et al. (2014) to demonstrate the appropriateness of Gumbell-Weibull distribution. The data set is presented as follows:

0.39, 0.85, 1.08, 1.25, 1.47, 1.57, 1.61, 1.61, 1.69, 1.80, 1.84, 1.87, 1.89, 2.03, 2.03, 2.05, 2.12, 2.35, 2.41, 2.43, 2.48, 2.50, 2.53, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.79, 2.81, 2.82, 2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 3.09, 3.11, 3.11, 3.15, 3.15, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33, 3.39, 3.39, 3.56, 3.60, 3.65, 3.68, 3.70, 3.75, 4.20, 4.38, 4.42, 4.70, 4.90.

The asymptotic variance-covariance matrix of the ML estimates under PRD for data set I[st] is calculated as

$$I^{-1}(\Theta) = \begin{pmatrix} 0.0274 & -0.1643 \\ -0.1643 & 1.0753 \end{pmatrix}$$

Thus, the variances of the ML estimates $\hat{\eta}$ and $\hat{\theta}$ are computed as $var(\hat{\eta}) = 0.0274$ and $var(\hat{\theta}) = 1.0753$. Accordingly, the 95% C.I. for η and θ are given as (1.396,2.045) and (2.818,6.883) respectively.

Table 3
Descriptive Statistics for Breaking Stress of 66 Carbon Fibres of 50 mm Length Data

Min.	Max.	Range	I[st] Qu.	Med.	Mean	3[rd] Qu.	Var.	S.D	Skew.	Kurt.
0.39	4.90	4.51	2.178	2.835	2.760	3.278	0.795	0.891	-0.135	0.338

Table 4
The ML Estimates of the Unknown Parameters for Data Set 1[st]

Model	ML Estimates				
	$\hat{\eta}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\alpha}$
PRD(η,θ)	1.721	4.850	-	-	-
RD(θ)	-	2.049	-	-	-
ERD(η,θ)	2.348	0.192	-	-	-
WRD(θ,β)	-	1.355	2.573	-	-
WBRD(η,θ,λ)	0.810	1.883	-	1.184	-
TRD(θ,α)	-	1.696	-	-	-0.959

Table 5
Goodness-of-Fit Statistics for Data Set 1[st]

Model	AIC	BIC	AICC	HQIC	$-2ln(l)$	K-S	*p*-value
PRD(η,θ)	176.14	180.51	176.33	177.87	172.14	0.08	0.763
RD(θ)	198.42	200.61	198.48	199.28	196.42	0.23	0.002
ERD(η,θ)	181.27	185.65	181.46	183.00	177.28	0.12	0.293
WRD(θ,β)	179.71	184.09	179.90	181.44	175.72	0.11	0.396
WBRD(η,θ,λ)	199.28	205.84	199.66	201.87	193.28	0.18	0.026
TRD(θ,α)	181.75	186.13	181.94	183.48	177.74	0.14	0.145

Data Set 2nd

As a second application, we examined a data set originally reported by Badar and Priest (1982) and represent the strength data measured in GPa, of 69 single carbon fibres tested under tension at gauge lengths of 20 mm. For illustrative purpose, we are considering the same transformed data set as taken by Raqab and Kundu (2005). The transformed data set is presented as follows:

0.312, 0.314, 0.479, 0.552, 0.700, 0.803, 0.861, 0.865, 0.944, 0.958, 0.966, 0.977, 1.006, 1.021, 1.027, 1.055, 1.063, 1.098, 1.140, 1.179, 1.224, 1.240, 1.253, 1.270, 1.272, 1.274, 1.301, 1.301, 1.359, 1.382, 1.382, 1.426, 1.434, 1.435, 1.478, 1.490, 1.511, 1.514, 1.535, 1.554, 1.566, 1.570, 1.586, 1.629, 1.633, 1.642, 1.648, 1.684, 1.697, 1.726, 1.770, 1.773, 1.800, 1.809, 1.818, 1.821, 1.848, 1.880, 1.954, 2.012, 2.067, 2.084, 2.090, 2.096, 2.128, 2.233, 2.433, 2.585, 2.585.

The asymptotic variance-covariance matrix of the ML estimates under PRD for data set 2nd is calculated as

$$I^{-1}(\Theta) = \begin{pmatrix} 0.0235 & -0.0219 \\ -0.0219 & 0.0292 \end{pmatrix}$$

Thus, the variances of the ML estimates $\hat{\eta}$ and $\hat{\theta}$ are computed as $var(\hat{\eta}) = 0.0235$ and $var(\hat{\theta}) = 0.0292$. Accordingly, the 95% C.I. for η and θ are given as (1.323,1.923) and (1.208,1.877) respectively.

Table 6
Descriptive Statistics for Strengths of 69 Single Carbon Fibres of 10 mm Length Data

Min.	Max.	Range	Ist Qu.	Med.	Mean	3rd Qu.	Var.	S.D	Skew.	Kurt.
0.312	2.585	2.273	1.098	1.478	1.451	1.773	0.245	0.495	-0.029	0.023

Table 7
The ML Estimates of the Unknown Parameters for Data Set 2nd

Model	ML Estimates				
	$\hat{\eta}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\alpha}$
PRD(η, θ)	1.623	1.543	-	-	-
RD(θ)	-	1.083	-	-	-
ERD(η, θ)	2.175	0.662	-	-	-
WRD(θ, β)	-	0.746	2.221	-	-
WBRD(η, θ, λ)	0.780	1.090	-	1.038	-
TRD(θ, α)	-	0.895	-	-	-0.961

Table 8
Goodness-of-Fit Statistics for Data Set 2nd

Model	AIC	BIC	AICC	HQIC	$-2ln(l)$	K-S	p-value
PRD(η,θ)	102.07	106.53	102.25	103.84	98.06	0.04	0.999
RD(θ)	120.84	123.07	120.90	121.72	118.84	0.19	0.008
ERD(η,θ)	105.81	110.28	105.99	107.58	101.80	0.08	0.829
WRD(θ,β)	104.64	109.11	104.82	106.41	100.64	0.07	0.921
WBRD(η,θ,λ)	120.98	127.69	121.35	123.64	114.98	0.16	0.066
TRD(θ,α)	105.91	110.37	106.09	107.68	101.90	0.09	0.649

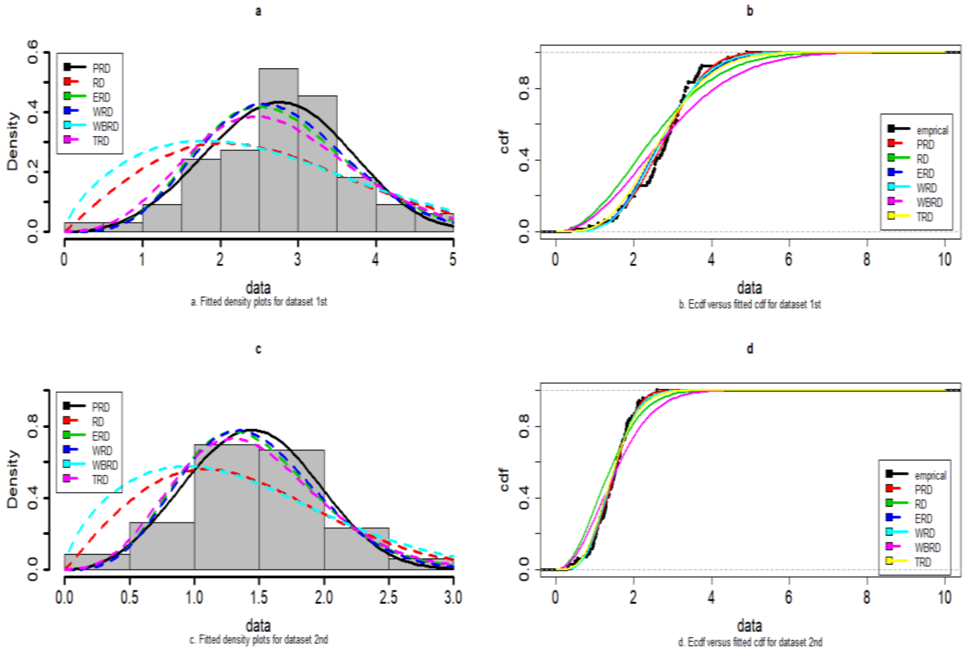

Figure 4: The Estimated Density and Distribution Plots of PRD and other Fitted Models for the Data set 1st and Data Set 2nd

Based on the goodness-of-fit criteria in tables 4 and 7, we conclude that PRD provides a superior fit than the other fitted models using the two data sets. Also, from the density and distribution plots it is clear that PRD provides a close fit to the two data sets.

CONCLUDING REMARKS

This research article formulates a new lifetime probability model, called the Power Rayleigh distribution, which extends the Rayleigh distribution via Power transformation technique. The shape behavior shows that the proposed model is uni-modal, decreasing and constant depending upon different values of parameter combinations. Numerous significant properties of PRD are acquired including ordinary moments, moment generating function, hazard rate, mean residual life, order statistics and quantiles. The stochastic ordering of random variables has also been established. The expression for four different measures of entropy viz., Shannon entropy, Renyi entropy, beta entropy and Mathai and Haubold entropy are also obtained. Maximum likelihood estimation procedure is employed to estimate the unknown parameters. In addition, the practical importance of PRD is demonstrated by means of two real data sets. We prospect that the proposed model will draw wider applications in statistics.

REFERENCES

1. Abdous, B. and Berred, A. (2005). Mean residual life estimation. *Journal of Statistical Planning and Inference*, 132, 3-19.
2. Ahmad, A., Ahmad, S.P. and Ahmed, A. (2017). Characterization and estimation of Weibull-Rayleigh distribution with applications to lifetime data. *Applied Mathematics and Information Sciences Letters*, 5(2), 71-79.
3. Ahmad, A., Ahmad, S.P. and Ahmed, A. (2014). Transmuted Inverse Rayleigh distribution. *Mathematical Theory and Modeling*, 4(7), 90-98.
4. Ajami, M. and Jahanshahi, S.M.A. (2017). Parameter estimation in Weighted Rayleigh distribution. *Journal of Modern Applied Statistical Methods*, 16(2), 256-276.
5. Akaike, H. (1974). A new look at the statistical model identification. Selected papers of Hirotugu: *Springer*, 215-222.
6. Al-Aqtash, R., Lee, C. and Famoye, F. (2014). Gumbel-Weibull distribution: properties and applications. *Journal of Modern Applied Statistical Methods*, 13, 201-225.
7. Ateeq, K., Qasim, B.T. and Alvi, R.A. (2019). An extension of Rayleigh distribution and applications. *Cogent Mathematics and Statistics*, 6(1), 1-16.
8. Badar, M.G. and Priest, A.M. (1982). Statistical aspects of fiber and bundle strength in hybrid composites. Hayashi, T., Kawata, K. and Umekawa, S. (eds). *Progress in Science and Engineering Composites*, ICCM-IV, Tokyo, 1129-1136.
9. Bazdogan, H. (1987). Model selection and Akaike's information criterion: The general theory and its analytical extensions. *Psychometrika*, 52, 345-370.
10. Ghazal, M.G.M. and Hasaballah, H.M. (2017). Exponentiated Rayleigh distribution: A Bayes study using MCMC approach based on unified hybrid censored data. *Journal of Advances in Mathematics*, 12(12), 6863-6880.
11. Ghitany, M.E., Al-Mutairi, D.K., Balakrishanan, N. and Al-Enezi, L.J. (2013). Power Lindley distribution and associated inference. *Computational Statistics and Data Analysis*, 64, 20-33.
12. Gomez-Deniz, E. (2010). Another generalization of the geometric distribution. *Test*. 19, 399-415.
13. Hannan, E.J. and Quinn, B.G. (1979). The determination of the order of an autoregression. *Journal of the Royal Statistical Society: Series B*, 41, 190-195.
14. Harvda, J. and Charvat, F. (1967). Quantification method in classification processes: concept of structural entropy. *Kybernetika*, 3, 30-35.

15. Howlader, H.A. and Hossain, A. (1995). On bayesian estimation and prediction from Rayleigh distribution based on Type-II censored data. *Communications in Statistics-Theory and Methods*, 24(9), 2249-2259.
16. Krishnarani, S.D. (2016). On a Power Transformation of Half-Logistic distribution. *Journal of Probability and Statistics*, 5, 1-10.
17. Kundu, D. and Raqab, M.Z. (2005). Generalized Rayleigh distribution: Different methods of estimations. *Computational Statistics and Data Analysis,* 49, 187-200.
18. Mathai, A.M. and Haubold, H.J. (2007). Pathway model, Superstatistics, Tsallis statistics and generalized measure of entropy. *Physica A: Statistical Mechanics and its Applications*, 375(1), 110-122.
19. Meniconi, M. and Barry, D.M. (1996). The Power Function distribution: A useful and simple distribution to assess electrical component reliability. *Microelectronics Reliability*, 36(9), 1207-1212.
20. Merovci, F. (2013). Transmuted Rayleigh distribution. *Austrian Journal of Statistics*, 42(1), 21-31.
21. Mudasir, S., Jan, U. and Ahmad, S.P. (2019). Weighted Rayleigh distribution revisited via informative and non-informative priors. *Pakistan Journal of Statistics*, 35(4), 321-348.
22. Rady, E.H.A., Hassanein, W.A. and Elhaddad, T.A. (2016). The Power Lomax distribution with an application to bladder cancer data. *Springer Plus*, 5(2016).
23. Rayleigh, J. (1980). On the resultant of a large number of vibrations of the same pitch and of arbitrary phase. *Philosophical Magazine, 5th Series*, 10(60), 73-78.
24. Raqab, M.Z. and Kundu, D. (2005). Comparison of different estimators of P(Y<X) for a scaled Burr-type X distribution. *Communication in Statistics*, *Simulation and Computation*, 34(2), 465-483.
25. Renyi, A. (1960). On measures of entropy and information. *Berkeley Symposium on Mathematical Statistics and Probability*, 1(1), 547-561.
26. Schwarz, G. (1978). Estimating the dimensions of a model. *The Annals of Statistics*, 6, 461-464.
27. Shannon, E. (1948). A mathematical theory of communication. *Bell System Technical Journal*, 27(3), 379-423.
28. Shukla, K.K. and Shanker, R. (2018). Power Ishita distribution and its application to model lifetime data. *Statistics in Transition*, 19(1), 135-148.
29. Siddiqui, M.M. (1962). Some problems connected with Rayleigh distributions. *Journal of Research, National Bureau of Standards*, 60D, 167-174.
30. Voda, V.G. (2007). A new generalization of Rayleigh distribution. *Reliability: Theory and Applications,* 2.
31. Zaka, A. and Akhter, A.S. (2013). Methods for estimating the parameters of Power Function distribution. *Pakistan Journal of Statistics and Operation Research*, 9, 213-224.